1. Introduction

Equations of the form

\[ [r(t)y']' + q(t)y = 0 \]  \quad (1.1)

where \( r \in C^1[\alpha, \infty), r > 0, \) and \( q \in C[\alpha, \infty), \) are classified by the behavior of their real solutions, as oscillatory or nonoscillatory. In the first instance, one, and thereby every, solution vanishes at an infinite number of isolated points in \([\alpha, \infty);\) in the second instance each solution has only a finite number of zeros in \([\alpha, \infty).\) By solution is always meant a function which is not identically zero. A special instance of nonoscillation is the disconjugate case in which every solution has at most one zero in \([\alpha, \infty).\) Although there are many results concerning the classification of equations of the form (1.1) with respect to these properties, no completely satisfactory answer has yet been obtained. The purpose of this paper or survey is to identify the known results, to relate the new results and old results to one another, and to unify some aspects of the known theory. For the sake of completeness, we will mention most of the results included in the excellent survey of Ráb [73; 1959]. There is a further justification of this duplication in that we will develop the known theory in a different manner than did Ráb.

The qualitative study of second order linear equations originated in the classic paper of Sturm [81; 1836]. However, the general importance and usefulness of Sturm’s work was not properly recognized until the end of the 19th and the beginning of the 20th centuries. At that time the work of Bócher [4–7] had a considerable influence in getting recognition of Sturm’s work. For the problem of classifying the solutions of (1.1), Sturm’s main result is his famous comparison theorem:

* This survey was begun while the author was at the 1967 Associated Western Universities Differential Equations Symposium at the University of Colorado. It was partially supported by NASA contract 45-003-038.
Sturm Comparison Theorem. If
\[ q_1 \geq q_2, \ r_1 \leq r_2, \ \text{and} \ \ (r_1 y')' + q_1 y = 0 \]
is nonoscillatory, then \( (r_2 y')' + q_2 y = 0 \) is nonoscillatory.

As an application of this theorem, let
\[ l_0(t) = t, \quad l_n(t) = \log l_{n-1}(t), \quad n = 1, 2, \ldots, \quad (1.2) \]
and
\[ L_n(t) = \prod_{k=0}^{n} l_k(t), \quad n = 0, 1, \ldots. \quad (1.3) \]

Then \( u_n = L_n^{1/2} \) is a nonoscillatory solution to the equation
\[ u'' + q_n u_n = 0, \]
where
\[ q_n(t) = \left[ \sum_{k=0}^{n} L_k^{-2}(t) \right] / 4. \quad (1.4) \]

Hence, if there exists a nonnegative integer \( n \) such that
\[ q(t) - q_n(t) \leq 0, \quad a \leq t < \infty, \quad (1.5) \]
then \( y'' + qy = 0 \) is nonoscillatory by the Sturm Theorem. It happens
that if there exist a nonnegative integer \( n \) and a number \( \epsilon \) such that
\[ [q(t) - q_n(t)] L_n^2(t) \geq \epsilon > 0, \quad a \leq t < \infty, \quad (1.5') \]
then \( y'' + qy = 0 \) is oscillatory. We will give a simple proof of this latter
result in the next section. For \( n = 0 \), these propositions become
\( q(t) \leq \frac{1}{4} t^2 \rightarrow \text{nonoscillation}, \quad q(t) \geq (1 + \epsilon)/4 t^2 \rightarrow \text{oscillation}, \)
which is a result first noted by Kneser [37; 1893]. The general results involving
\( (1.5, 1.5') \) were essentially derived by Riemann and Weber [75; 1912].
They later reappeared in various forms in [26], [31], [42], and [55].

With the exception of two results in Section 5, we will not include
in this survey any details of the many generalizations and ramifications
of the Sturm Comparison Theorem. However, the main references in
this regard are [6], [7], [14], [17], [21], [24], [34–36], [38], [43], [46–49],
[57], [60], [62], [64], [74], [83], [84], [94].

Throughout this survey, \( \int f(s) \, ds \) will denote any absolutely
continuous function \( F \) with the property that \( F'(t) = f(t) \). Whenever \( \int^\infty \)
is written, it is to be assumed that
\[ \int^\infty = \lim_{t \to \infty} \int^t, \]
and that this limit exists in the extended real numbers \([-\infty, \infty]\). Whenever equations of the form (1.1) are considered, it will be implicitly assumed that \(q \in C(a, \infty), r \in C^1(a, \infty),\) and \(r > 0\).

2. Results originating directly from Kummer and Riccati Transformations.

Let
\[
\varphi \in C^4(a, \infty), \quad \varphi' > 0, \quad \psi \in C^2[a, \infty), \quad \psi(t) \neq 0, \quad a \leq t < \infty. \tag{2.1}
\]
The so-called Kummer transformation (cf. Kummer [4]; 1834))*
\[
\tau = \varphi(t), \quad y(t) = \psi(t) x(\tau) \tag{2.2}
\]
transforms (1.1) into
\[
\frac{d}{d\tau} \left[ R(\tau) \frac{dx}{d\tau} \right] + Q(\tau) x = 0, \tag{2.3}
\]
where
\[
R(\tau) = r(t) \varphi'(t) \psi^2(t),
\]
\[
Q(\tau) = [(r(t) \psi'(t))' + q(t) \psi(t)] \psi(t)[\varphi'(\tau)]^{-1}. \tag{2.4}
\]
Equations (1.1) and (2.3) obviously have the same oscillatory behavior, because of the form of (2.2) and the assumptions on \(\varphi\) and \(\psi\). Furthermore, one can always choose
\[
\tau = \varphi(t) = \int_0^t \frac{[\varphi(s)]^{-1}}{[\varphi'(s)]^{-1}} ds, \tag{2.5}
\]
so that (2.3) is of the simpler form
\[
\ddot{x} + p(\tau) x = 0, \quad \varphi(a) \leq \tau < \varphi(\infty)(= d/d\tau), \tag{2.6}
\]
where
\[
p(\tau) = [(r(t) \psi'(t))' + q(t) \psi(t)] \psi^2(t) r(t). \tag{2.7}
\]

* Stäckel [80; 1893] and Lie [51; 1894] showed at about the same time that (2.2) is the only schlichte transformation of \((t, y, y')\) space into \((\tau, x, x')\) space that preserves the form of (1.1). Boruvka [11; pp. 102–105 & 183–186] and Sansone [78; pp. 90–101] discuss the use and history of (2.2) in the classical theory of differential equations. Boruvka [10; 1962] presents a survey of results concerned with the problem of when two given equations \(y'' + q(t)y = 0\) and \(\ddot{x} + p(\tau)x = 0\) can be transformed into one another by a Kummer transformation.
The Kummer transformation with \( \varphi \) chosen as in (2.5) \((r = 1)\) and \( \psi = u_n = L_n^{1/2} \), where \( L_n \) is defined by (1.3), produces

\[
\rho(t) = \left[ u_n''(t) + q(t) u_n(t) \right] u_n^3(t) = [q(t) - q_n(t)] L_n^3(t). \tag{2.8}
\]

Hence, (1.5') and the Sturm Comparison Theorem imply that equation (2.6) in this context, and therefore, the equation \( y'' + qy = 0 \), is oscillatory. This then proves the other half of the Riemann–Weber result mentioned in Section 1.

If

\[
\int_{-\infty}^{\infty} [r(s) \psi^2(s)]^{-1} ds = \infty, \tag{2.9}
\]

then (2.5) maps the unbounded interval \([a, \infty)\) onto the unbounded interval \([\varphi(a), \infty)\). Kummer transformations with this property are often needed in oscillation theory. Condition (2.9) can be always achieved; for example:

If \( \int_{-\infty}^{\infty} r^{-1}(s) ds = \infty \), let \( \psi(t) = 1 \).

If \( \int_{-\infty}^{\infty} r^{-1}(s) ds < \infty \), let \( \psi(t) = \int_{t}^{\infty} r^{-1}(s) ds \).

These choices of \( \psi \) are especially elegant, because in each case \((r\psi')' = 0\). Hence, (2.7) is particularly simple.

Thus, the oscillation classification problem for equations of the form \((ry')' + qy = 0\) on an unbounded interval is equivalent to the same problem for equations of the form \(y'' + py = 0\) on an unbounded interval. In this survey, we will not be consistent in whether results are stated for \((ry')' + qy = 0\) or \(y'' + py = 0\).

A second useful transformation in oscillation theory is the well known Riccati transformation*:

(i) If \( y \) is a nonvanishing solution of (1.1) on an interval \( I \), then \( u = ry'y^{-1} \) is a solution of

\[
u' + q + r^{-1}u^2 = 0 \tag{2.11}
\]
on \( I \).

* See [85; p. 217] for a history of the Riccati transformation.
(ii) If \( u \) is a solution of (2.11) on \( I \), then
\[
y(t) = \exp \left[ \int_{t}^{t} u(s) r^{-1}(s) \, ds \right],
\]
is a nonvanishing solution of (1.1) on \( I \).

**Theorem 2.1** (Bocher [5; 1900–01]). Equation (1.1) is disconjugate, if and only if, there exists \( u \in C^1[a, \infty) \) such that
\[
u'(t) + q(t) + r^{-1}(t) u^2(t) \leq 0, \quad a < t < \infty. \tag{2.13}
\]

**Proof.** If (1.1) is disconjugate, then the solutions \( y \) satisfying \( y(a) = 0 \), \( y'(a) \neq 0 \) do not vanish in \((a, \infty)\). For such a solution, \( u = ry'y^{-1} \) satisfies (2.11), hence, (2.13).

Conversely, if \( u \) is a solution of (2.13), let \( \psi(t) \) be defined by (2.12) for \( t \geq b > a \) and let \( \phi(t) \) be defined by (2.5) with lower limit \( b \). The resulting Kummer transformation (2.2) takes (1.1) into (2.6) with
\[
\rho(\tau) = [u'(t) + r^{-1}(t) u^2(t) + q(t)] r(t) \phi^4(t).
\]

Hence, (2.13) implies \( \rho(\tau) \leq 0 \), \( \phi(b) \leq \tau < \phi(\infty) \). We conclude from the Sturm Comparison Theorem that \( x' + \rho(\tau) x = 0 \) is disconjugate on \([\phi(b), \phi(\infty))\). Therefore, (1.1) is disconjugate on \([b, \infty)\). Since this is true for all \( b > a \), (1.1) is disconjugate on \([a, \infty)\).

By letting \( u = ry'y^{-1} \) in (2.13), we can reformulate Theorem 2.1 in the following manner: (1.1) is disconjugate, if and only if, there exist \( y \in C^2[a, \infty) \), \( y(t) > 0 \) when \( a < t < \infty \), such that
\[
(ry')' + qy \leq 0. \tag{2.14}
\]

Kondratčev [39; 1957] has given a direct and elementary proof of this result. His proof is based upon the fact that if \( \psi \), where \( \psi \) satisfies (2.14), is substituted into (1.1) for \( y \), then the resulting second order linear differential equation in \( \psi \) has a nonpositive coefficient of \( \psi \) because of (2.14). This coefficient remains nonpositive upon putting the equation in normal form (in the form (1.1)). Hence, the Sturm Comparison Theorem implies that there is a nonvanishing solution for \( \psi \), and, therefore, (1.1) is disconjugate.

If we let
\[
u(t) = \left[ 2 \int_{t}^{\infty} sp(s) \, ds + 1 \right] / (2t),
\]
in Theorem 2.1, then we obtain the following generalization of a result of Hille [31; 1948] and Hartman and Wintner [29; 1948]:

**Corollary 2.1*. If**

\[ \int_a^\infty sp(s) \, ds < \infty, \quad a \leq t < \infty, \]

*then \( y'' + py = 0 \) is nonoscillatory.*

If we apply Corollary 2.1 to (2.6) with \( \psi \) chosen as in (2.10), we obtain the following generalization of various results in [30; 1953], [44; 1949], [58; 1955], [89; 1948], and [90; 1949]:

**Corollary 2.2.** If

\[ \int_a^\infty r^{-1}(s) \, ds < \infty \quad \text{and} \quad \int_a^\infty q(t) \left[ \int_t^\infty r^{-1}(s) \, ds \right] \, dt < \infty \]

or

\[ \int_a^\infty r^{-1}(s) \, ds = \infty \quad \text{and} \quad \int_a^\infty q(t) \left[ \int_t^\infty r^{-1}(s) \, ds \right] \, dt < \infty, \]

*then (1.1) is nonoscillatory.*

Corollary 2.2 is also an obvious consequence of the following result of Zubova [96; 1957]:

**Theorem 2.2.** Equation (1.1) is disconjugate, if and only if, there exist positive functions \( h \in C(a, \infty), f \in C^1(a, \infty) \) such that

\[ h(t)f'(t) + \int_t^a f(s) q(s) \, ds = 0 \quad \text{and} \quad 0 < h(t) \leq r(t), \quad a < t < \infty. \]  

**(2.15)**

*Proof.* The function \( f \), which does not vanish in \( (a, \infty) \), satisfies the equation \((hf')' + qf = 0\). Since \( h \leq r \), the Sturm Comparison Theorem implies that no solutions of \((ry')' + qy = 0\) can vanish more than once in \([a, \infty)\). The converse is obvious.

A result similar to Theorem 2.2, but involving oscillation instead of disconjugacy, can be similarly established by using the contrapositive of the Sturm Comparison Theorem. Zubova lists such a result.

*Zlámal [95; 1950] proved that \( \int_a^\infty s^\sigma p(s) \, ds = \infty \) for some constant \( \sigma < 1 \) implies that \( y'' + py = 0 \) is oscillatory.*
Corollary 2.3. If
\[(r_i y')' + q_i y = 0, \quad i = 1, \ldots, n,\]
are disconjugate and \(c_i\) are nonnegative constants such that
\[\sum_{i=1}^{n} c_i = 1,\]
then
\[
\left( \sum_{i=1}^{n} c_i r_i \right) y' + \sum_{i=1}^{n} c_i q_i y = 0
\]
is disconjugate.

Proof. Theorem 2.1 implies that there exist functions \(u_i \in C^1[a, \infty),\)
\(i = 1, \ldots, n,\) such that in \((a, \infty)\)
\[u_i' + q_i + r_i u_i^2 \leq 0, \quad i = 1, \ldots, n.\]
It follows that the function
\[u = \sum_{i=1}^{n} c_i u_i,\]
satisfies in \((a, \infty)\)
\[u' + \sum_{i=1}^{n} c_i q_i + \left( \sum_{i=1}^{n} c_i r_i \right)^{-1} u^2 \leq 0.\]
Hence, Theorem 2.1 implies that (2.16) is disconjugate.

Adamov [1; 1948] established Corollary 2.3 for the special case when
\(n = 2, \quad r_1 = r_2 = 1,\) and \(q_1, q_2\) are periodic of the same period.
Petrovlovskaya [68; 1955] generalized Adamov's result by removing
the periodicity assumption on \(p_1, p_2;\) Markus and Moore [54; 1956]
further removed the condition that \(r_1 = r_2 = 1.\) Finally, Kondratév
[40; 1957] established the result for general \(n,\) but with \(r_i \equiv 1\) for all
values of \(i.\)

Corollary 2.4. (Hartman [27; 1951]). Let \(P \in C^1[a, \infty)\) be any
function such that \(P' = -p.\) If
\[y'' + 4P^2 y = 0\]
is disconjugate, then \(y'' + py = 0\) is disconjugate.
Proof. The disconjugacy of (2.17) implies that there exists \( v \in C^1(a, \infty) \) such that \( v'' + v^3 + 4P^2 \leq 0 \). But then \( u = P + v/2 \) satisfies \( u' + u^2 + p \leq 0 \); and so Theorem 2.1 implies that \( y'' + py = 0 \) is disconjugate.

Other less elegant results that can be obtained by specializing \( u \) in Theorem 2.1 have been obtained by Hartman [26; 1948], Kondratév [40, 1957], Wintner [91; 1951], and Zlámal [95; 1950].

**Theorem 2.3.** The equation \( y'' + py = 0 \) is nonoscillatory, if and only if, there exists \( \psi \in C^2[a, \infty) \), \( \psi > 0 \), such that

\[
\int_0^\infty \psi'(t) [\psi''(t) + p(t) \psi(t)] \, dt < \infty,
\]

(2.18)

where

\[
\Psi(t) = \begin{cases} 
\psi_1(t) & \text{when } \psi_1(t) = \int_t^\infty \psi^{-2}(s) \, ds < \infty \\
\psi_2(t) & \text{when } \psi_2(t) = \int_t^\infty \psi^{-2}(s) \, ds \to \infty \text{ as } t \to \infty.
\end{cases}
\]

(2.19)

Proof. When (2.18) holds, Corollary 2.2, applied to equation (2.3) with \( \varphi' \equiv 1 \) and \( r \equiv 1 \), implies that \( y'' + py = 0 \) is nonoscillatory.

Conversely, when \( y'' + py = 0 \) is nonoscillatory, (2.18) is satisfied by any positive function \( \psi \) that coincides with a solution \( y \) on an interval \([b, \infty)\) in which \( y \) does not vanish. Furthermore, if \( y \) is a maximal solution, then

\[
\int_0^\infty \psi^{-2}(s) \, ds < \infty,
\]

(2.20)

and if \( y \) is a minimal solution, then

\[
\int_0^\infty \psi^{-2}(s) \, ds = \infty.
\]

(2.21)

Theorem 2.3 with \( \psi \) satisfying (2.20) in both directions was first proven by Wintner [87; 1948]. Levin [50; 1965] listed Theorem 2.3 as it is here.

### 3. Oscillation and the Bohl transformation.

The so-called Bohl transformation (Bohl [8; 1906]) can be described as follows:
(i) If \( \lambda \in C^2[a, \infty) \) is a solution to the nonlinear differential equation
\[
(r\lambda')' + q\lambda = (r\lambda^2)^{-1},
\]
then
\[
y(t) = \lambda(t) \sin \left( \int_{t_0}^{t} \left( r(s) \lambda^2(s) \right)^{-1} \, ds \right)
\]
is a solution of (1.1).

(ii) Conversely, if \( y_1 \) and \( y_2 \) are linearly independent solutions of (1.1) and have Wronskian equal to \( 1/r \), then
\[
\lambda = \left( y_1^2 + y_2^2 \right)^{1/2}
\]
is a solution of (3.1).

Ráb [73; 1959] bases his survey of oscillation theory upon the Bohl transformation. For the sake of completeness we will give the highlights of that theory most directly related to the Bohl transformation in this section.

**Theorem 3.1.** (Ráb [73; p. 337]) Equation (1.1) is oscillatory, if and only if, there exists \( \lambda \in C^2[a, \infty), \lambda > 0 \), such that \( \lambda \) is a solution of (3.1) and
\[
\int_{a}^{\infty} \left[ r(s) \lambda^2(s) \right]^{-1} \, ds = \infty.
\]

**Theorem 3.2.** (Ráb [73; p. 339]) If for each function \( P \in C^1[a, \infty) \) such that \( P' = -p \), it is true that
\[
\int_{a}^{\infty} \exp \left[ 2 \int_{t}^{\infty} P(s) \, ds \right] \, dt = \infty,
\]
then \( y'' + py = 0 \) is oscillatory.

**Proof.** Suppose that \( y'' + py = 0 \) is nonoscillatory on \([a, \infty)\). Then there exists \( b \geq a \) and two nonvanishing linearly independent solutions \( y_1, y_2 \) of \( y'' + py = 0 \) on \([b, \infty)\) with the Wronskian of \( y_1, y_2 \) equal to unity. Letting \( y \) denote either \( y_1 \) or \( y_2 \), we obtain by means of a Riccati transformation that
\[
y'(t) y^{-1}(t) \leq C - \int_{b}^{t} p(s) \, ds \equiv P(t), \quad t \geq b,
\]
where
\[
C = \max_{i=1,2} \left[ |y_i'(b)| y_i^{-1}(b) \right].
\]
Hence,
\[ y(t) \leq y(b) \exp \left[ \int_b^t P(s) \, ds \right], \quad t \geq b; \quad (3.6) \]
and (ii) implies that \( \lambda = (y_1'^2 + y_2'^2)^{1/2} \) satisfies \( \lambda'' + p\lambda - \lambda^{-3} \). Furthermore, (3.6) implies that
\[ \lambda^{-2}(t) \geq [y_1'^2(b) + y_2'^2(b)]^{-1} \exp \left[ 2 \int_b^t P(s) \, ds \right], \quad t \geq b. \]
Hence, (3.5) implies that
\[ \int_{-\infty}^\infty \lambda^{-2}(t) \, dt = \infty; \]
and so, Theorem 3.1 implies that \( y'' + py = 0 \) is oscillatory, which is a contradiction.

**Corollary 3.1**. (Wintner [88; 1949]) If
\[ \int_{-\infty}^\infty p(s) \, ds = \infty, \]
then \( y'' + py = 0 \) is oscillatory.

**Corollary 3.2**. (Moore [58; 1955]) Equation (1.1) is oscillatory, if and only if, there exists \( \psi \in C^2[a, \infty) \), \( \psi > 0 \), such that
\[ \int_{-\infty}^\infty [r(s) \psi^2(s)]^{-1} \, ds = \infty \quad (3.7) \]
and
\[ \int_{-\infty}^\infty \{[r(s) \psi'(s)]' \psi(s) + q(s) \psi^2(s)\} \, ds = \infty. \quad (3.8) \]

**Proof.** Assume first that (3.7) and (3.8) hold. If (1.1) is transformed by a Kummer transformation (2.2) with \( \varphi \) defined by (2.5), then the resulting equation \( \ddot{x} + p(\tau) x = 0 \) has
\[ \int_{-\infty}^\infty p(\tau) \, d\tau = \infty, \]
* See the footnote attached to Corollary 2.1.
† Fite [22; 1918] originally proved this result with the additional assumption \( p > 0 \).
‡ Gagliardo [25; 1954] originally proved that (3.7) and (3.8) for the case \( r = 1 \) were sufficient for oscillation.
because of (3.8). Hence, Corollary 3.1 implies that this equation is oscillatory, and so, (1.1) is oscillatory.

Conversely, if (1.1) is oscillatory, Theorem 3.1 implies the existence of a function \( \lambda \) which satisfies (3.1) and (3.4). If we let \( \psi \equiv \lambda \), then (3.4) implies (3.7), and (3.1) and (3.4) imply (3.8).

If we let \( \psi \equiv 1 \) in Corollary 3.2, then we obtain a sufficient condition for oscillation of Leighton [45; 1950], namely,

\[
\int_{-\infty}^{\infty} r^{-1}(s) \, ds = \int_{-\infty}^{\infty} q(s) \, ds = \infty. \tag{3.9}
\]

Obviously one can obtain an infinite number of special sufficient conditions for oscillation by specializing \( \psi \) in Corollary 3.2. Some of the more interesting of these involve the functions \( l_n \), \( L_n \), and \( q_n \), which are defined in (1.2), (1.3), and (1.4). Each of the following is a sufficient condition for (1.1) to be oscillatory:

(I) The function

\[
R(t) = \int_t^{\infty} r^{-1}(s) \, ds \tag{3.10}
\]

satisfies \( R(t) \to \infty \) as \( t \to \infty \), and there exist a non-negative integer \( n \) and positive number \( \epsilon \) such that

\[
\int_{-\infty}^{\infty} q(t) \, R^\epsilon(t)[L_n(R(t))]^{-1} \, dt = \infty.
\]

(II) \( R(t) \to \infty \) as \( t \to \infty \) and there exists a non-negative integer \( n \) such that

\[
\int_{-\infty}^{\infty} [q(t) - r^{-1}(t) \, q_n(R(t))] \, L_n(R(t)) \, dt = \infty.
\]

(III) \( R(t) \to \infty \) as \( t \to \infty \) and there exist a non-negative integer \( n \) and positive number \( \epsilon \) such that

\[
\int_{-\infty}^{\infty} [q(t) - r^{-1}(t) \, q_n(R(t))] \, L_{n+1}(R(t))[L_{n+2}(R(t))]^{-(1+\epsilon)} \, dt = \infty.
\]

(IV) The Function

\[
R(t) = \frac{1}{\left[ \int_t^{\infty} r^{-1}(s) \, ds \right]}, \tag{3.11}
\]
is positive and there exist a non-negative integer \( n \) and positive number \( \epsilon \) such that

\[
\int_0^\infty q(t)[L_n(\bar{R}(t)) l_n(\bar{R}(t))]^{-1} \, dt = \infty.
\]

(V) \( \bar{R} > 0 \) and there exists a non-negative integer \( n \) such that

\[
\int_0^\infty [q(t) - r^{-1}(t) \bar{R}^4(t) q_n(\bar{R}(t))] L_n(\bar{R}(t)) \, dt = \infty.
\]

The results in cases (I), (II), (IV), and (V) follow directly from Corollary 3.2 by letting \( \psi^2 \) be \( R^2/[L_n(R) l_n(R), L_n(R), 1/L_n(R) l_n(R), \) and \( L_n(\bar{R}) \), respectively. Moore [58; 1955] established (I) and (IV) for the case \( n = 0 \). Zlámal [95; 1950] established (II) for the case \( n = 1 \). Ráb [72; 1957], [73; pp. 346–351] established (III), which is somewhat different from the other four cases. Case (III) essentially follows from the following result:

**Corollary 3.3.** (Ráb [73; p. 342]) Equation (1.1) is oscillatory, if and only if, there exists \( \psi \in C^2[a, \infty) \), \( \psi > 0 \), such that for each function \( P \in C^1[a, \infty) \) satisfying

\[
P' = (r\psi')' \psi + q\psi^2,
\]

it is true that

\[
\int_0^\infty [r(t) \psi^2(t)]^{-1} \exp \left[ 2 \int_t^\infty [r(s) \psi^2(s)]^{-1} P(s) \, ds \right] \, dt = \infty. \tag{3.12}
\]

**Proof.** If (3.12) holds for all admissible functions \( P \), then Theorem 3.2 implies that (2.6) is oscillatory. Hence, (1.1) is oscillatory. Conversely, (3.12) is a direct consequence of Corollary 3.2.

Ráb [73; pp. 339–344] derives other results similar to Corollary 3.3 and supplies a good discussion of this aspect of oscillation theory. Most of the work of El’sin [16–20], [32], [33] is concerned with various ramifications and applications of Corollary 3.3. El’sin formulates (3.12) in terms of a function \( \theta \), where \( \theta \) and \( \psi \) are related by the formula

\[
\psi(t) = \exp \left[ r^{1/2}(t) \int_a^t \theta(s) \, ds \right].
\]

Other work related to Corollary 3.3 and reported in Ráb’s survey has been done by Boruvka [9; 1957], Gagliardo [25; 1954], Kondrat’ev [40; 1957], Laitoch [42; 1955], and Zlámal [95; 1950].
4. Classification in terms of $\int^\infty p(s) \, ds$.

One of the following four cases must always occur:

(i) $\int^\infty p(s) \, ds = \infty$

(ii) $-\infty < \int^\infty p(s) \, ds < \infty$

(iii) $\int^\infty p(s) \, ds = -\infty$

(iv) $\limsup_{t \to \infty} \int^t p(s) \, ds > \liminf_{t \to \infty} \int^t p(s) \, ds$.

The equation $y'' + py = 0$ is always oscillatory if $p$ satisfies (i). However, both oscillation and nonoscillation are compatible with (ii), (iii), and (iv).

The classical Euler equation

$$y'' + \alpha \tau^2 y = 0 \quad (\alpha \text{ constant}),$$

illustrates both cases for (ii), and the classical Mathieu equation

$$y'' + (\alpha - \beta \cos t) y = 0 \quad (\alpha, \beta \text{ constants}),$$

illustrates both cases for (iii). The complete classification of the Mathieu equation seems to still be an open problem. However, extensive oscillation results have been obtained by Moore [59; 1956] and Zubova [97; 1963] for the Mathieu equation and its generalization, the Hill equation. Also, Magnus and Winkler [52; pp. 56–78] list many results for the Hill equation, and Markus and Moore [54; 1956] have studied

$$y'' + [\alpha - \beta p(t)] y = 0 \quad (\alpha, \beta \text{ constants}),$$

under the assumption that $p$ is almost periodic. Yelchin [93; 1946] proved that $y'' + py = 0$ is oscillatory if $p$ has a Fourier series with zero constant term. Sobol [79; 1951] proved that $y'' + py = 0$ is oscillatory if $\int^t p(s) \, ds$ is almost periodic and not constant. Other sufficient conditions for oscillation in the case of a periodic coefficient $p$ have been obtained by Adamov [I; 1948].

The equations $y'' + y \sin t = 0^*$ and $y'' + yt \sin t = 0$ are examples showing that (iv) is completely compatible with oscillation. It is some-

* It is interesting to note that the equation $y'' + [\sin t/(2 + \sin t)] y = 0$, however, is nonoscillatory.
what more difficult to show that (iv) is compatible with nonoscillation. In this regard, let $p = v' - v^2$, where $v \in C^1[a, \infty) \cap L^2[a, \infty)$. Then,

$$y = \exp \left[ - \int_a^t v(s) \, ds \right],$$

is a nonvanishing solution of $y'' + py = 0$. Clearly, functions $v$ can be found such that

$$\limsup_{t \to \infty} \int_a^t p(s) \, ds = \limsup_{t \to \infty} v(t) - \int_a^\infty v^2(s) \, ds - v(a),$$

and

$$\liminf_{t \to \infty} \int_a^t p(s) \, ds = \liminf_{t \to \infty} v(t) - \int_a^\infty v^2(s) \, ds - v(a),$$

take on any values desired in $[-\infty, \infty]$, and $[-\infty, \infty)$ respectively.

We will next describe a method recently developed by Coles [12; 1968] and Willett [86; 1968]. One of the advantages of this method is to unify and extend the known results for cases (ii) and (iv). For these cases, a rather extensive classification of equations has been obtained.

Let

$$\mathfrak{F} = \left\{ f : f \text{ measurable on } [a, \infty), f \geq 0, \int_a^\infty f(s) \, ds = \infty \right\},$$

and for $f \in \mathfrak{F}$, $p \in C[a, \infty)$, let

$$A_{fp} \equiv A(s, t) = \int_s^t \left[ f(\tau) \int_s^\tau p(\sigma) \, d\sigma \right] d\tau / \int_s^t f(\tau) \, d\tau. \quad (4.1)$$

We say that a function $p \in C[a, \infty)$ has an averaged integral $P = P_f$ with respect to $\mathfrak{F}$, if there exists $f \in \mathfrak{F}$ such that, for each $t \in [a, \infty)$, $lim A_{fp}(s, t)$, as $s \to \infty$, exists in $[-\infty, \infty]$ and

$$P(t) = \lim_{s \to \infty} A_{fp}(s, t). \quad (4.2)$$

If the limit in (4.2) exists for one value of $t$, then it exists for any value of $t$ in $[a, \infty)$. In fact, an averaged integral $P$ always satisfies the following fundamental relationship:

$$P(t) = P(b) - \int_a^t p(s) \, ds, \quad a \leq b, t < \infty; \quad (4.3)$$

hence,

$$P'(t) = -p(t), \quad a \leq t < \infty.$$
Furthermore, if $\int_t^\infty p(s) \, ds$ exists, then

$$P(t) = \int_t^\infty p(s) \, ds, \quad a \leq t < \infty.$$  

On the other hand, consider the example when $p(t) = \cos t$. Let

$$f(t) = \begin{cases} 1 & \text{when } \sin t \geq 0 \\ 0 & \text{when } \sin t < 0 \end{cases}.$$  

Then, $P_f(0)$ exists and $P_f(0) = 2/\pi$. Hence,

$$P_f(t) = 2\pi^{-1} - \sin t, \quad 0 \leq t < \infty.$$  

For this example, $\int_t^\infty p(s) \, ds$ does not exist.

The set $\mathcal{G}$ is too large for our purposes, and so we introduce the following two sets:

$$\mathcal{G}_0 = \left\{ f \in \mathcal{G} : \lim_{t \to \infty} \frac{\int_t^\infty f^2(s) \, ds}{\left[ \int_t^\infty f(s) \, ds \right]^2} = 0 \right\},$$

$$\mathcal{G}_1 = \left\{ f \in \mathcal{G} : \lim_{t \to \infty} \sup \left( \int_t^\infty f(s) \, ds \right) \left[ \int_t^\infty f^2(s) \, ds \right]^{-1} \right\} > 0 \right\}.$$  

It is easy to show that $\mathcal{G} \supset \mathcal{G}_1 \supset \mathcal{G}_0$, and if $f \in \mathcal{G}$ and $f$ is bounded on $[a, \infty)$, then $f \in \mathcal{G}_0$. On the other hand, all nonnegative polynomials are in $\mathcal{G}_0$, and so $\mathcal{G}_0$ does contain some unbounded functions.

**Theorem 4.1.** If there exists $f \in \mathcal{G}_1$ such that the averaged integral $P_f(a) = \infty$, then $y'' + py = 0$ is oscillatory.

For a proof of Theorem 4.1, see Willett [86; 1968]. Coles [12; 1968] proved a similar theorem using a smaller class of weight functions than $\mathcal{G}_1$. Theorem 4.1 is no longer a true statement if $\mathcal{G}_1$ is replaced by $\mathcal{G}$, because if it would be, then $\lim \sup \int f(s) \, ds = \infty$ would be sufficient to imply that $y'' + py = 0$ is oscillatory. The latter is certainly not the case as some of our previous examples clearly indicated. It remains, however, an interesting question as to what is the largest class of weight functions $f$ for which $P_f(a) = \infty$ implies oscillation.

Corollary 3.1 and the following older results are easy consequences of Theorem 4.1:
Corollary 4.1. (Olech, Opial, Wajewski [65; 1957]) If
\[ \lim_{t \to +\infty} \int_{t_0}^{t} p(s) \, ds = \infty, \]
then \( y'' + py = 0 \) is oscillatory.

Corollary 4.2. (Wintner [88; 1949]) If
\[ \lim_{t \to +\infty} t^{-1} \left[ \int_{t_0}^{t} p(\tau) \, d\tau \right] ds = \infty, \]
then \( y'' + py = 0 \) is oscillatory.

A general theorem similar to Theorem 4.1 but with "higher order" weighted averages has been obtained by Coles and Willett [13; 1968]. Rather than reproduce here the general result, which is notationally rather complicated, we will list two of the more interesting applications.

Theorem 4.2. Let \( P \in C^1[a, \infty) \) be any function such that \( P' = p \). If there exists \( f \in \mathcal{F}_1 \) and positive integer \( n \) such that
\[ \lim_{t_n \to +\infty} \frac{\int_{t_n}^{t_{n-1}} \cdots \int_{t_1}^{t_0} P(t_0) \, dt_0 \cdots dt_{n-2} \, dt_{n-1}}{\int_{t_n}^{t_{n-1}} \cdots \int_{t_1}^{t_0} P(t_0) \, dt_0 \cdots dt_{n-2} \, dt_{n-1}} = \infty, \] (4.4)
then \( y'' + py = 0 \) is oscillatory.

Theorem 4.3. Let \( P \in C^1[a, \infty) \) be any function such that \( P' = p \). If for some positive integer \( n \), \( P \) is not Hölder \((H, n)\)-summable, i.e.,
\[ \lim_{t_n \to +\infty} t_n^{-1} \int_{t_n}^{t_{n-1}} \cdots \int_{t_1}^{t_0} P(t_0) \, dt_0 \cdots dt_{n-2} \, dt_{n-1} = \infty, \]
then \( y'' + py = 0 \) is oscillatory.

The left side of (4.4) can be considered a generalized Riesz mean for \( P \). If \( f = 1 \), then (4.4) is the \( n \)th Cesàro sum of \( P \).

Theorem 4.4. (Willett [86; 1968]) If there exists \( f \in \mathcal{F}_0 \), such that
\[ \lim_{t \to +\infty} \inf_{t \in (0, \infty)} A_{f\nu}(t, a) > -\infty, \] (4.5)
then either \( y'' + py = 0 \) is oscillatory, or the averaged integral \( P_{\gamma}(t) \) exists and is finite for all \( \gamma \in \mathcal{F}_0 \) and \( a \leq t < \infty \).
Corollary 4.3. (Hartman [28; 1952]) If

\[
\limsup_{t \to \infty} t^{-1} \int_t^\infty \left[ \int_0^s p(\tau) \, d\tau \right] ds > \liminf_{t \to \infty} t^{-1} \int_t^\infty \left[ \int_0^s p(\tau) \, d\tau \right] ds > -\infty,
\]

then \( y'' + py = 0 \) is oscillatory.

Theorems 4.1 and 4.4 indicate that the problem of classifying the equations \( y'' + py = 0 \) can be separated into two parts defined by whether an \( f \in \mathcal{F}_1 \) exists such that (4.5) holds. Condition (4.5) is always satisfied if \( \liminf_{t \to \infty} \int_0^t p(s) \, ds > -\infty \) and is never satisfied if \( \int_{-\infty}^\infty p(s) \, ds = -\infty \). We will present results in the next section which, together with Theorems 4.1 and 4.2, will produce a reasonably complete theory for the case when (4.5) can be satisfied.

When (4.5) is not satisfied for any \( f \in \mathcal{F}_1 \), few specific results seem to be known. One positive feature of this case, however, is that it can be at least theoretically eliminated by substituting \( y = r^{1/2}v \), where

\[
r(t) = \exp \left( -2 \int_0^t \left[ \int_0^s p(\tau) \, d\tau \right] ds \right).
\]

Theorem 4.5. The oscillatory properties of the equations \( y'' + py = 0 \) and \((rz')' + qz = 0\), where \( Y \) is defined in (4.6) and

\[
q(t) = r(t) \left( \int_0^t p(s) \, ds \right)^2,
\]

are equivalent.

Of course, the equation \((rz')' + qz = 0\) in Theorem 4.5 can be transformed to an equation of the form \( w'' + \tilde{p}w = 0 \) by a Kummer transformation. If this is accomplished by the transformation described in (2.10), then \( \tilde{p} \geq 0 \) because \( q \geq 0 \). Hence, for the function \( \tilde{p} \), (4.5) is satisfied by all \( f \in \mathcal{F}_1 \). For some other aspects of using Kummer transformations to transform a given equation into an equation where (4.5) can be satisfied for some \( f \in \mathcal{F}_1 \), see Willett [86; 1968].

Putnam [71; 1955] proved the following result, which might apply to some equations for which (4.5) cannot be satisfied by any \( f \in \mathcal{F}_1 \):

Theorem 4.6. If

\[
\limsup_{t \to \infty} t^{-1} \int_t^\infty \left[ \int_0^s p(\tau) \, d\tau \right] ds = \infty,
\]
and if there exists a constant $c > 0$ such that

$$\int_{t}^{\infty} p(s) \, ds > -e^{ct}, \quad a \leq t < \infty,$$

then $y'' + py = 0$ is oscillatory.

Most of the results in this section and the next section that are stated for the equation $y'' + py = 0$ can be used to generate more general results. This can be accomplished by applying these results to equation (2.6) to obtain a new result involving a nearly arbitrary function $\psi$. For example, Corollary 4.2 applied to (2.6) produces the following result of Gagliardo [25; 1954]:

**Corollary 4.4.** If there exists $\psi \in C^2_{\infty}(a, \infty)$, $\psi > 0$, such that (2.9) and

$$\lim_{t \to \infty} t^{-1} \int_{t}^{\infty} \left[ (r(\tau) \psi'(\tau))' + q(\tau) \psi(\tau) \psi'(\tau) d\tau \right] \, ds = \infty$$

hold, then $(ry')' + qy = 0$ is oscillatory.

Once again we refer to Ráb [73; 1959] for a detailed discussion of some results of this type.

We conclude this section with the following miscellaneous results:

(i) (Potter [69; 1953]) If

$$p \in C^{1}_{\infty}(a, \infty), \quad p \geq 0, \quad (p')^2 p^{-3} \leq k < 16, \quad \text{and} \quad \int_{a}^{\infty} p^{1/2}(t) \, dt = \infty,$$

then $y'' + py = 0$ is oscillatory.

(ii) (Leighton [46; 1952]) If $q > 0$, $(qr')' \leq 0$, and $(ry')' + qy = 0$ is oscillatory, then

$$\int_{a}^{\infty} [q(s) r^{-1}(s)]^{1/2} \, ds = \infty.$$

(iii) (Barrett [2; 1955]) If $q > 0$, $(qr')' \leq 0$, and

$$\lim_{t \to \infty} \left\{ \int_{a}^{t} [q(s) r^{-1}(s)]^{1/2} \, ds + \frac{1}{2} \log[q(t) r(t)] \right\} = \infty,$$

then $(ry')' + qy = 0$ is oscillatory.
5. Classification when \( p \) has a finite averaged integral with respect to \( \mathfrak{F}_0 \).

The classification of equations of the form \( y'' + py = 0 \) when there exists \( f \in \mathfrak{F}_1 \) such that

\[
\lim \inf_{t \to \infty} A_{f,p}(t, a) > -\infty,
\]

has been reduced by Theorems 4.1 and 4.4 to the case when \( P_f(t) \) exists and is finite for all \( g \in \mathfrak{F}_0 \). We now turn our attention to this case.

**Theorem 5.1.** (Willett [86; 1968]) If there exist bounded functions \( f, g \in \mathfrak{F} \) such that \( P_f(a) \) and \( P_g(a) \) exist and \( P_f(a) \neq P_g(a) \), then \( y'' + py = 0 \) is oscillatory.

**Corollary 5.1.** (Olech, Opial, Wazewski [65; 1957]) If

\[
\lim \sup_{t \to \infty} \int^t \rho(s) \, ds > \lim \inf_{t \to \infty} \int^t \rho(s) \, ds,
\]

then \( y'' + py = 0 \) is oscillatory.

**Corollary 5.2.** If \( p \) is bounded on one side and

\[
\infty \geq \lim \sup_{t \to \infty} \int^t \rho(s) \, ds > \lim \inf_{t \to \infty} \int^t \rho(s) \, ds \geq -\infty,
\]

then \( y'' + py = 0 \) is oscillatory.

Zlámal [95; 1950] proved Corollary 5.2 for the case when \( p \) is bounded and the \( \lim \sup \) is \( \infty \); Petropavlovskaya [68; 1955] gave a proof for the case when \( p \) is bounded below; Moore [58; 1955] gave a proof for the case when \( p \) is bounded above and the \( \lim \sup \) is \( \infty \); and Olech, Opial, Wazewski [65; 1957] proved that Corollary 5.1 implies Corollary 5.2.

Theorem 5.1 is particularly useful for the difficult problems when \( p \) is not of constant sign, or when \( p \) oscillates about some value. For example, suppose for some \( \epsilon \geq 0 \), the sets

\[
E^+ = \left\{ t \geq a : \int_a^t \rho(s) \, ds > \epsilon \right\} \quad \text{and} \quad E^- = \left\{ t \geq a : \int_a^t \rho(s) \, ds < -\epsilon \right\}
\]

have infinite measure. Then \( P_f(a) \neq P_g(a) \) for \( f \) and \( g \) equal to the
characteristic functions of $E_+^e$ and $E_-^e$, respectively. For the particular function $p(t) = \cos t$, we obtain $P_+(0) = 2/n$ and $P_-(0) = -2/n$ if $f$ and $g$ are taken to be the characteristic functions of $E_+^0$ and $E_-^0$, respectively. Hence, Theorem 5.1 implies that $y'' + y \cos t = 0$ oscillates.

**Theorem 5.2.** (Hartman [28; 1952]) If (5.2) holds and if

$$\sup_{0 < v < \infty} \left| \int_u^{u+v} p(t) \, dt \right| (1 + v) \to 0, \quad \text{as} \quad u \to \infty,$$

then $y'' + py = 0$ is oscillatory.

**Theorem 5.3.** (Willett [86; 1968]) Assume that $p$ has a finite averaged integral $P$ with respect to $\mathcal{G}_0$. Then, $y'' + py = 0$ is disconjugate, if and only if, there exists a solution $v \in C^1(a, \infty)$ of

$$v(t) = P(t) + \int_t^\infty v^2(s) \, ds. \quad (5.3)$$

**Proof.** Equation (5.3) implies that $v$ satisfies $v' = -p - v^2$; hence, Theorem 2.1 implies that $y'' + py = 0$ is disconjugate. For the proof of the converse, which is more complicated, see Willett [86].

It is clear in the proof of Theorem 5.3 that (5.3) is sufficient for disconjugacy if $P$ is any function such that $P' = -p$. Furthermore, (5.3) can be replaced in this instance by

$$v(t) \geq P(t) + \int_t^\infty v^2(s) \, ds \geq 0, \quad (5.4)$$

since $u = P + \int_t^\infty v^2(s) \, ds$ would then satisfy

$$u' = -p - v^2 \leq -p - u^2,$$

which also implies that $y'' + py = 0$ is disconjugate by Theorem 2.1.

For $P \in C[a, \infty)$, define

$$F_p = F(t, s) = \exp \left( 2 \int_t^s P(\tau) \, d\tau \right), \quad (5.5)$$

and

$$Q_p = Q(t) = \int_t^\infty P^2(s) \, E(t, s) \, ds. \quad (5.6)$$
Clearly, \[ 0 < E < \infty \quad \text{and} \quad 0 \leq Q \leq \infty. \]

**Theorem 5.4.** (Willett [86; 1968]) Assume that \( p \) has a finite averaged integral \( P \) with respect to \( \mathcal{F}_0 \). Then, \( y'' + py = 0 \) is disconjugate, if and only if, \( Q = Q_p \) is finite and there exists a solution \( v \in C^1(a, \infty) \) of

\[
v(t) \geq Q(t) + \int_t^\infty E(t, s) v^2(s) \, ds \quad (E = E_p). \tag{5.7}
\]

**Proof.** Condition (5.7) implies that the function

\[
u(t) = P(t) + Q(t) + \int_t^\infty E(t, s) v^2(s) \, ds
\]

satisfies \( u' \leq -p - u^2 \). Hence, Theorem 2.1 implies that \( y'' + py = 0 \) is disconjugate. Proof of the converse is more complicated. Theorem 5.3 implies that equation (5.3) has a solution. Let \( u \) be this solution. We can show next from (5.3) that the function

\[
v(t) = \int_t^\infty u^2(s) \, ds,
\]

satisfies

\[
v(t) - Q(t) - \int_t^\infty E(t, s) v^2(s) \, ds = \lim_{\tau \to \infty} \exp \left( 2 \int_t^\tau P(s) \, ds \right) \tag{5.8}
\]

Hence, \( v \) satisfies (5.7).

We can actually show that the limit in (5.8) is zero. This means that Theorem 5.4 remains true if the inequality in (5.7) is replaced by equality. The proof of this fact, when \( P(t) = \int_t^\infty p(s) \, ds \) exists and is finite, is due to Professor J. S. W. Wong. The proof for the general case when \( P \) is an averaged integral is similar and goes as follows:

Suppose the limit in (5.8) is positive for some value of \( t, a \leq t < \infty \). Then there exist \( \epsilon > 0 \) and \( b \geq a \) such that

\[
v(s) \exp \left[ 2 \int_t^s P(\tau) \, d\tau \right] \geq \epsilon \quad \text{for all} \quad s \geq b.
\]

Hence, (5.8) implies that

\[
v(t) \geq \int_t^\infty E(t, s) v^2(s) \, ds \geq \int_t^b E(t, s) v^2(s) \, ds + \epsilon \int_b^\infty E^{-1}(t, s) \, ds.
\]
that is,
\[
\int_b^\infty E^{-1}(t, s) \, ds = \int_b^\infty \exp \left[ -2 \int_t^\infty P(\tau) \, d\tau \right] \, ds < \infty.
\] (5.9)

Condition (5.9) contradicts the following theorem, which is a generalization of results of Wintner [91; 1951] and Hartman [28; 1952]:

**Theorem 5.5.** Assume that \( p \) has a finite averaged integral \( P \) with respect to \( \overline{\mathcal{V}}_0 \). If there exists a constant \( \gamma \), \( 0 < \gamma < 4 \), such that
\[
\int^\infty_0 \exp \left[ -\gamma \int^t P(s) \, ds \right] \, dt < \infty,
\]
then \( y'' + py = 0 \) is oscillatory.

**Proof.** If \( y'' + py = 0 \) is nonoscillatory, then there exists a number \( b > a \) and a function \( v \in C[b, \infty) \) such that \( v \) is a solution to (5.3) for \( b < t < \infty \). The classical proof can be carried from here to the usual contradiction.

**Corollary 5.3.** (Willett [86; 1968]) (i) Let \( P \in C^1[a, \infty) \) be such that \( P' = -p \). If
\[
Q < \infty \quad \text{and} \quad \int^\infty_0 Q^2(s) E(t, s) \, ds \leq Q(t)/4, \quad a \leq t < \infty, \quad (5.10)
\]
\((E := E_p, Q := Q_p)\) then \( y'' + py = 0 \) is disconjugate. (ii) Assume that \( p \) has a finite averaged integral \( P' \equiv P_f \) with respect to \( \overline{\mathcal{V}}_0 \). If \( Q(a) = \infty \) or if there exists \( \epsilon > 0 \) such that
\[
\int^\infty_t Q^2(s) E(t, s) \, ds \geq (1 + \epsilon) Q(t)/4 > 0, \quad a \leq t < \infty, \quad (5.10')
\]
\((Q := Q_p, E := E_p, P \equiv P_f)\) then \( y'' + py = 0 \) is oscillatory.

Condition (5.10) holds if
\[
\int^\infty_0 P^2(s) \, ds \leq P(t)/4, \quad a \leq t < \infty; \quad (5.11)
\]
and condition (5.10') holds for some \( \epsilon > 0 \), if there exists \( \epsilon' > 0 \) such that
\[
\int^\infty_0 P^2(s) \, ds \geq (1 + \epsilon') P(t)/4 > 0, \quad a \leq t < \infty. \quad (5.11')
\]
For other simplifications of (5.10–5.10'), see Willett [86].
Under the assumption that $0 < P(t) = \int_0^t p(s) \, ds < \infty$, Opial [66; 1958] originally proved that (5.11) and (5.11') were sufficient for disconjugacy and oscillation, respectively. Wintner [91; 1951] had recognized earlier that the stronger hypothesis $P^2(t) \leq p(t)/4$ on $[a, \infty)$ implied disconjugacy.

Corollary 5.3 implies that the equation

$$y'' + (\mu t^{-1} \sin t) y = 0$$

is oscillatory when $|\mu| > 1/\sqrt{2}$ and nonoscillatory when $|\mu| < 1/\sqrt{2}$. See Willett [86] for other examples.

**Theorem 5.6.** Assume that $y'' + p_1 y = 0$ is disconjugate and that $p_1$ has a finite averaged integral $P_1$ with respect to $\tilde{\mathcal{F}}_0$. Assume that $P_2 \in C_1[a, \infty)$ satisfies $P_2' = -p_2$ and that $Q_2 = Q_{P_2}$ is finite. If

$$Q_1 \geq Q_2 \quad \text{and} \quad P_1 \geq Q_2 \quad P_2 \quad (Q_1 = Q_{P_1}), \quad (5.12)$$

then $y'' + p_2 y = 0$ is disconjugate.

**Proof.** Note first that the disconjugacy of $y'' + p_1 y = 0$ implies that $Q_1$ is finite. Theorem 5.4 implies that there exists $v \in C'[a, \infty)$ such that

$$v(t) \geq Q_1(t) + \int_t^\infty E_1(t, s) v(s) \, ds, \quad a \leq t < \infty \quad (E_1 = E_{P_1}). \quad (5.13)$$

Let

$$u(t) = P_2(t) + Q_2(t) + \int_t^\infty E_1(t, s) v(s) \, ds.$$ 

Because of (5.12) and (5.13), it is now an easy matter to show that $u' + p_2 + u^2 \leq 0$. Hence, Theorem 2.1 implies that $y'' + p_2 y = 0$ is disconjugate.

**Corollary 5.4.** (Taam [82; 1952]) If $y'' + p_1 y = 0$ is disconjugate and

$$\infty > \int_t^\infty p_1(s) \, ds \geq \int_t^\infty p_2(s) \, ds, \quad a \leq t < \infty,$$

then $y'' + p_2 y = 0$ is disconjugate.

Theorem 5.6 for the case when

$$P_i(t) = \int_t^\infty p_i(s) \, ds < \infty, \quad i = 1, 2,$$
was proven by Professor J. S. W. Wong in work not yet published. Corollary 5.4 with the additional assumption that \( p_1 > 0, p_2 \geq 0 \) was proven by Hille [31; 1948]. Corollary 5.4 has been rediscovered by Kondratěv [40; 1957], Wintner [92; 1957], Levin [48; 1960], and Drahlin [15; 1967].

**Theorem 5.7.** Assume that \( p \) is not identically zero on any infinite subinterval of \([a, \infty)\) and that \( p \) has a finite nonnegative averaged integral \( P \) with respect to \( \xi_0 \). Then the equation \( y'' + py = 0 \) is disconjugate, if and only if, for each \( b > a \), the smallest positive eigenvalue \( \lambda \) of the boundary value problem

\[
y'' + \lambda py = 0, \quad y(a) = 0 = y'(b), \quad (5.14)
\]

satisfies \( \lambda > 1 \).

**Proof.** If the eigenvalue condition is satisfied, it is obvious that no solution of \( y'' + py = 0 \) can have more than one zero.

In order to prove the converse, assume that \( y'' + py = 0 \) is disconjugate and that \( z \) is a positive solution on \((a, \infty)\). Furthermore, suppose that there exist \( b > a \) and \( 0 < \lambda \leq 1 \) such that (5.14) has a nontrivial solution \( y \) on \([a, b]\). Let \( w = zy' - yz' \). Then,

\[
0 \leq (1 - \lambda) \int_a^b [y'(s)]^2 \, ds = (1 - \lambda) \int_a^b p(s) y^2(s) \, ds
\]

\[
= \int_a^b z^{-1}(s) y(s) w'(s) \, ds = -z^{-1}(b) y^2(b) z'(b) - \int_a^b z^{-2}(s) w^2(s) \, ds, \quad (5.15)
\]

which implies that \( z'(b) \leq 0 \). Next, Theorem 5.3 implies that \( v = z^{-1}z' \) satisfies

\[
v(t) = P(t) + \int_t^\infty v^2(s) \, ds, \quad a < t < \infty,
\]

where \( P \geq 0 \) by assumption. Hence,

\[
0 \geq v(b) = P(b) + \int_b^\infty v^2(s) \, ds \geq \int_b^\infty v^2(s) \, ds,
\]

which can only occur if \( v(t) = 0 \) for all \( b \leq t < \infty \). Therefore, \( z \) is
constant in \([b, \infty)\), which implies that \(p(t) = 0\) for \(b \leq t < \infty\). This contradicts one of the hypothesis of the theorem.

Theorem 5.7 generalizes results of Nehari [61; 1957] and St. Mary [77; 1968], who assume that \(0 \leq P(t) \equiv \int_t^\infty p(s) \, ds < \infty\). Nehari also assumes that \(p \geq 0\). St. Mary formulates his result as a necessary and sufficient condition for oscillation. We can also generalize this result to averaged integrals as follows:

**Theorem 5.8.** Assume that \(p\) has a finite nonnegative averaged integral \(P\) with respect to \(\mathcal{R}_0\). Then, the equation \(y'' + py = 0\) is oscillatory, if and only if, there exists a sequence of intervals \([a_n, b_n]\) with \(a_n \uparrow \infty\) as \(n \uparrow \infty\) such that the least positive eigenvalue \(\lambda_n\) of the system

\[
y'' + \lambda_n py = 0, \quad y(a_n) = 0 = y'(b_n) \quad (5.16)
\]

satisfies \(\lambda_n \leq 1, \, n = 1, 2, \ldots\).

**Proof.** If \(y'' + py = 0\) is oscillatory, then the eigenvalue condition is satisfied with \(\lambda_n = 1, \, n = 1, 2, \ldots\).

Suppose the eigenvalue condition holds, and assume that \(y'' + py = 0\) has a nonoscillatory solution \(z\). Then, there exists \(b \geq a\) such that \(z\) does not vanish in \([b, \infty)\). Let \(y\) denote the solution of (5.16) which corresponds to \(a_N\), where \(a_N \geq b\). The proof of Theorem 5.7 starting with (5.15) and applied to \(z\) and \(y\) of the present proof once again leads to the contradiction that \(p(t) = 0\) for \(t \geq a_N\). This contradicts the existence of the eigenvalues \(\lambda_n\) with \(n > N\).

Other results involving eigenvalue conditions have been proven by Putnam [70; 1949], Barrett [3; 1959], and St. Mary [77; 1968]. Most of these results are of the comparison type.

Potter [69; 1953] lets \(ry' = z\) in (1.1) to obtain the new equation

\[
(q^{-1}z')' + r^{-1}z = 0. \quad (5.17)
\]

Equation (5.17) is well defined on \([a, \infty)\) if \(q > 0\). (We always assume that \(r > 0\).) Furthermore, (5.17) and (1.1) have the same oscillatory behavior. Hence, with the additional hypothesis \(q > 0\), we can apply most of the results mentioned in this paper to (5.17) to obtain new results for (1.1). Rab [73; pp. 351–352] states specifically some of these results. Barrett [3; 1959] uses this transformation to obtain some new results of the eigenvalue type.
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