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Extremal solutions of periodic boundary value problems for first order integro-differential equations of mixed type

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Abstract

This paper investigates the maximal and minimal solutions of periodic boundary value problems for first order nonlinear integro-differential equations of mixed type by establishing a comparison result and using the monotone iterative technique.

Keywords: Integro-differential equation; Periodic boundary value problem; Comparison theorem; Monotone iterative technique; Extremal solution

1. Introduction

In [1–4,7,8], the existence of solutions to periodic boundary value problems for differential equations and integro-differential equations has been investigated. In this paper, we shall study the following periodic boundary value problems (PBVP for brevity) for first order nonlinear integro-differential equations of mixed type

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$$\begin{cases} u' = f(t, u, Tu, Su), \quad t \in J, \\ u(0) = u(1), \end{cases} \tag{1}$$

where $f \in C[J \times R \times R \times R, R]$, $J = [0, 1]$,

$$Tu(t) = \int_0^t k(t,s)u(s)\,ds, \qquad Su(t) = \int_0^1 h(t,s)u(s)\,ds,$$

$k(t,s) \in C[D, R^+]$, $h(t,s) \in C[J \times J, R^+]$, $D = \{(t,s) \in R^2\colon\ 0 \leqslant s \leqslant t \leqslant 1\}$, $R^+ = [0, +\infty)$, $k_0 = \max\{k(t,s)\colon\ (t,s) \in D\}$, $h_0 = \max\{h(t,s)\colon\ (t,s) \in J \times J\}$. In the special case where f dose not contain Su, i.e., (1) is a PBVP of Volterra type, the extremal solutions of (1) have been obtained by means of the monotone iterative technique based on a comparison result (see [1,2,4,8]). But, it is easy to see that the method for obtaining a comparison result is not applicable in the general case. Therefore, in this paper, we shall obtain a comparison result for the general case by a completely different way. And then, using standard monotone iterative technique (see [3–5,7,8]), an existence theorem of minimal and maximal solutions of PBVP (1) is obtained. Finally, we give several examples for applying this existence theorem.

2. Several lemmas

In this section we combine the ideas in [6] together with those in [7] to obtain a new comparison result.

The following comparison results play an important role in this paper.

Lemma 1 (*Comparison theorem*). *Assume that $u = u(t) \in C^1[J, R]$ satisfies*

$$\begin{cases} u'(t) \geqslant -Mu(t) - N\int_0^t k(t,s)u(s)\,ds - L\int_0^1 h(t,s)u(s)\,ds, \quad t \in J, \\ u(0) \geqslant u(1), \end{cases} \tag{2}$$

where $M, N, L \geqslant 0$ are constants and satisfy

$$(Nk_0 + Lh_0)\big(e^{2M} - 1\big) < M^2. \tag{3}$$

Then $u(t) \geqslant 0$, $\forall t \in J$.

Proof. Let $p(t) = u(t)e^{Mt}$, $t \in J$. Thus, by (2) we have that

$$p'(t) \geqslant -N\int_0^t e^{M(t-s)}k(t,s)p(s)\,ds - L\int_0^1 e^{M(t-s)}h(t,s)p(s)\,ds,$$
$$p(0) \geqslant e^{-M}p(1). \tag{4}$$

If $\min\{p(t)\colon\ t \in J\} < 0$, the continuity of $p(t)$ implies that there exists $t_0 \in (0,1)$ and $t_1 \in J$ such that

$$p(t_0) < 0, \quad p(t_1) = \max\big\{p(t)\colon\ t \in J\big\} \equiv \lambda.$$

We now show that $\lambda > 0$.

Assume that $\lambda \leqslant 0$, by (4) we know that $p'(t) \geqslant 0$, $t \in J$, hence, $p(0) \leqslant p(t_0) < 0$. Thus by (4), we have that $p(0) < p(1) < 0$, which contradicts $p(1) \leqslant e^M p(0) < 0$. Hence, we obtain $\lambda > 0$.

Evidently, the relationships between t_0 and t_1 must be one of the following two cases:

Case 1: $t_1 < t_0$;
Case 2: $t_0 < t_1$.

Case 1. By (4), we have that

$$\begin{aligned}
0 > p(t_0) &= p(t_1) + \int_{t_1}^{t_0} p'(s)\,ds \\
&\geqslant \lambda - N\int_{t_1}^{t_0} ds \int_0^s e^{M(s-\tau)} k(s,\tau)p(\tau)\,d\tau \\
&\quad - L\int_{t_1}^{t_0} ds \int_0^1 e^{M(s-\tau)} h(s,\tau)p(\tau)\,d\tau \\
&\geqslant \lambda\left[1 - Nk_0\int_{t_1}^{t_0} e^{Ms}\,ds \int_0^s e^{-M\tau}\,d\tau - Lh_0\int_{t_1}^{t_0} e^{Ms}\,ds\int_0^1 e^{-M\tau}\,d\tau\right] \\
&\geqslant \lambda\left[1 - \frac{Nk_0 + Lh_0}{M^2}\cdot\left(e^M - 1\right)\right].
\end{aligned}$$

Thus, $M^2 < (Nk_0 + Lh_0)\cdot(e^M - 1)$, which contradicts (3).

Case 2. By (4), we have that

$$\begin{aligned}
0 > p(t_0) &= p(0) + \int_0^{t_0} p'(s)\,ds \\
&\geqslant p(0) - \int_0^{t_0}\left[N\int_0^s e^{M(s-\tau)}k(s,\tau)p(\tau)\,d\tau\right. \\
&\quad \left. + L\int_0^1 e^{M(s-\tau)}h(s,\tau)p(\tau)\,d\tau\right] ds \\
&\geqslant p(0) - \lambda\int_0^{t_0} e^{Ms}\left[Nk_0\int_0^s e^{-M\tau}\,d\tau + Lh_0\int_0^1 e^{-M\tau}\,d\tau\right] ds
\end{aligned}$$

$$\geqslant p(0) - \lambda \cdot \frac{Nk_0 + Lh_0}{M^2} \cdot \left(e^M - 1\right),$$

i.e.,

$$p(0) \leqslant \lambda \cdot \frac{Nk_0 + Lh_0}{M^2} \cdot \left(e^M - 1\right), \tag{5}$$

$$\begin{aligned}
\lambda \equiv p(t_1) &= p(1) - \int_{t_1}^{1} p'(s)\,ds \\
&\leqslant p(1) + \int_{t_1}^{1} \left[N \int_0^s e^{M(s-\tau)} k(s,\tau) p(\tau)\,d\tau \right. \\
&\quad \left. + L \int_0^1 e^{M(s-\tau)} h(s,\tau) p(\tau)\,d\tau \right] ds \\
&\leqslant p(1) + \lambda \int_{t_1}^{1} e^{Ms} \left[Nk_0 \int_0^s e^{-M\tau}\,d\tau + Lh_0 \int_0^1 e^{-M\tau}\,d\tau \right] ds \\
&\leqslant p(1) + \lambda \cdot \frac{Nk_0 + Lh_0}{M^2} \cdot \left(e^M - 1\right).
\end{aligned}$$

Hence, by (4) and (5), we know that

$$\begin{aligned}
\lambda \cdot \left[1 - \frac{Nk_0 + Lh_0}{M^2} \cdot \left(e^M - 1\right) \right] &\leqslant p(1) \leqslant e^M p(0) \\
&\leqslant \lambda \cdot \frac{Nk_0 + Lh_0}{M^2} \cdot e^M \cdot \left(e^M - 1\right).
\end{aligned}$$

Thus, $M^2 \leqslant (Nk_0 + Lh_0) \cdot (e^{2M} - 1)$, which contradicts (3).

Hence $p(t) \geqslant 0,\ \forall t \in J$ and $u(t) \geqslant 0$ for $t \in J$.

Lemma 1 is proved. □

For any $\sigma(t) \in C[J, R]$ and nonnegative real numbers M, N, L, we consider the linear periodic boundary value problems for first order integro-differential equations of mixed type

$$\begin{cases} u'(t) + Mu(t) + N\int_0^t k(t,s)u(s)\,ds + L\int_0^1 h(t,s)u(s)\,ds = \sigma(t), & t \in J, \\ u(0) = u(1). \end{cases} \tag{6}$$

Lemma 2. *If nonnegative real numbers M, N, L satisfying*

$$(Nk_0 + Lh_0)e^M < M^2, \tag{7}$$

then (6) *has a unique solution in* $C[J, R]$.

Proof. Let $v(t)=u(t)e^{Mt}$, $t\in J$. Thus, by (6) we have that

$$\begin{cases} v'(t)=\sigma_1(t)-N\int_0^t e^{M(t-s)}k(t,s)v(s)\,ds-L\int_0^1 e^{M(t-s)}h(t,s)v(s)\,ds, \\ \quad t\in J, \\ v(0)=e^{-M}\cdot v(1), \end{cases} \tag{8}$$

where $\sigma_1(t)=\sigma(t)e^{Mt}$, $t\in J$.

Obviously, $v(t)$ is a solution of (8) if and only if $u(t)=v(t)e^{-Mt}$ is a solution of (6), and $v(t)$ is a solution of (8) if and only if $v(t)$ satisfies the integral equation

$$\begin{aligned} v(t)=\frac{1}{e^M-1}\int_0^1\bigg[\sigma_1(s)-N\int_0^s e^{M(s-\tau)}k(s,\tau)v(\tau)\,d\tau \\ -L\int_0^1 e^{M(s-\tau)}h(s,\tau)v(\tau)\,d\tau\bigg]ds \\ +\int_0^t\bigg[\sigma_1(s)-N\int_0^s e^{M(s-\tau)}k(s,\tau)v(\tau)\,d\tau \\ -L\int_0^1 e^{M(s-\tau)}h(s,\tau)v(\tau)\,d\tau\bigg]ds \\ \equiv Fv(t). \end{aligned} \tag{9}$$

Obviously, the $v^*(t)$ is a solution of (8) if and only if v^* is a fixed point of the F, i.e., $Fv^*=v^*$.

For any $u,v\in C[J,R]$, by (9) we have that

$$\begin{aligned} \big\|Fu(t)-Fv(t)\big\| &\leqslant \|u-v\|_C \\ &\quad\cdot\bigg[\frac{1}{e^M-1}\int_0^1 e^{Ms}\bigg(Nk_0\int_0^s e^{-M\tau}d\tau+Lh_0\int_0^1 e^{-M\tau}\,d\tau\bigg)ds \\ &\quad+\int_0^t e^{Ms}\bigg(Nk_0\int_0^s e^{-M\tau}\,d\tau+Lh_0\int_0^1 e^{-M\tau}\,d\tau\bigg)ds\bigg] \\ &\leqslant \frac{Nk_0+Lh_0}{M^2}\cdot e^M\cdot\|u-v\|_C, \quad \forall t\in J. \end{aligned}$$

Therefore, we have that

$$\|Fu-Fv\|_C\leqslant\frac{Nk_0+Lh_0}{M^2}\cdot e^M\cdot\|u-v\|_C. \tag{10}$$

By (7) and (10), we know that F is a contraction operator on $C[J,R]$. Consequently, by the contraction-mapping theorem, F has a unique fixed point v^*, obviously, the $v^*(t)$ is a unique solution of (8), i.e., $u^*(t) = v^*(t) \cdot e^{-Mt}$ is a unique solution of (6). Lemma 2 is proved. □

Lemma 3. *$u(t) \in C^1[J, R]$ is a solution of PBVP (1) if and only if $u(t) \in C[J, R]$ and it is a solution of the following integral equation*:

$$u(t) = \frac{e^{-Mt}}{e^M - 1} \cdot \int_0^1 e^{Ms}\left[f\left(s, u(s), Tu(s), Su(s)\right) + Mu(s)\right] ds + e^{-Mt} \cdot \int_0^t e^{Ms}\left[f\left(s, u(s), Tu(s), Su(s)\right) + Mu(s)\right] ds.$$

The proof of Lemma 3 is easy, so we omit it.

3. Main results

In this section we shall use the monotone iterative technique to prove the existence of minimal and maximal solutions of the PBVP (1). Assume that $u_0, v_0 \in C[J, R]$ with $u_0(t) \leqslant v_0(t)$, $\forall t \in J$. Set

$$[u_0, v_0] \equiv \left\{u \in C[J, R]:\ u_0(t) \leqslant u(t) \leqslant v_0(t),\ \forall t \in J\right\},$$
$$\Omega \equiv \left\{(u, v, w):\ u \in [u_0, v_0],\ v \in [Tu_0, Tv_0],\ w \in [Su_0, Sv_0]\right\}.$$

We obtain the existence of extremal solutions for PBVP (1) in the next result.

Theorem. *Let $u_0, v_0 \in C[J, R]$ such that $u_0(t) \leqslant v_0(t)$ in J. Assume that the following conditions hold*:

(H_1)
$$u_0'(t) \leqslant f\left(t, u_0(t), Tu_0(t), Su_0(t)\right), \quad t \in J, \qquad u_0(0) \leqslant u_0(1);$$
$$v_0'(t) \geqslant f\left(t, v_0(t), Tv_0(t), Sv_0(t)\right), \quad t \in J, \qquad v_0(0) \geqslant v_0(1).$$

(H_2) *Whenever $t \in J$ and $u_i, v_i, w_i \in \Omega$ $(i = 1, 2)$ and $u_2 \geqslant u_1, v_2 \geqslant v_1, w_2 \geqslant w_1$,*

$$f(t, u_2, v_2, w_2) - f(t, u_1, v_1, w_1) \geqslant -M(u_2 - u_1) - N(v_2 - v_1) - L(w_2 - w_1),$$

where M, N, L are nonnegative real constants and satisfy

$$\begin{cases} (Nk_0 + Lh_0) \cdot e^M < M^2, & \text{if } 0 < M < ln\frac{1+\sqrt{5}}{2}, \\ (Nk_0 + Lh_0) \cdot (e^{2M} - 1) < M^2, & \text{if } M \geqslant ln\frac{1+\sqrt{5}}{2}. \end{cases} \tag{11}$$

Then PBVP (1) *have the minimal solution u^* and maximal solution v^* in $[u_0, v_0]$. Moreover, there exist monotone iteration sequences $\{u_n(t)\}, \{v_n(t)\} \subset [u_0, v_0]$ such that*

$$u_n(t) \to u^*(t), \quad v_n(t) \to v^*(t), \quad \text{as } n \to \infty \text{ uniformly on } t \in J,$$

where $\{u_n(t)\}, \{v_n(t)\}$ *satisfy*

$$\begin{cases} u_n'(t) = f(t, u_{n-1}(t), Tu_{n-1}(t), Su_{n-1}(t)) - M(u_n - u_{n-1})(t) \\ \qquad - NT(u_n - u_{n-1})(t) - LS(u_n - u_{n-1})(t), \quad t \in J, \\ u_n(0) = u_n(1) \quad (n = 1, 2, 3, \ldots), \end{cases} \tag{12}$$

$$\begin{cases} v_n'(t) = f(t, v_{n-1}(t), Tv_{n-1}(t), Sv_{n-1}(t)) - M(v_n - v_{n-1})(t) \\ \qquad - NT(v_n - v_{n-1})(t) - LS(v_n - v_{n-1})(t), \quad t \in J, \\ v_n(0) = v_n(1) \quad (n = 1, 2, 3, \ldots), \end{cases} \tag{13}$$

and

$$u_0 \leqslant u_1 \leqslant \cdots \leqslant u_n \leqslant \cdots \leqslant u^* \leqslant v^* \leqslant \cdots \leqslant v_n \leqslant \cdots \leqslant v_1 \leqslant v_0. \tag{14}$$

Proof. First, it is easy to see by (11) that (3) and (7) hold.

For any $u_{n-1}, v_{n-1} \in C[J, R]$, by Lemma 2, we know that Eqs. (12) and (13) have unique solutions u_n and v_n in $C[J, R]$, respectively.

In the following, we will show by induction that

$$u_{n-1} \leqslant u_n \leqslant v_n \leqslant v_{n-1}, \quad n = 1, 2, 3, \ldots. \tag{15}$$

By (12), (13), and the conditions (H_1) and (H_2), we have that

$$\begin{cases} (u_1 - u_0)'(t) \geqslant -M(u_1 - u_0)(t) - N\int_0^t k(t,s)(u_1 - u_0)(s)\,ds \\ \qquad - L\int_0^1 h(t,s)(u_1 - u_0)(s)\,ds, \quad t \in J, \\ (u_1 - u_0)(0) \geqslant (u_1 - u_0)(1); \end{cases}$$

$$\begin{cases} (v_0 - v_1)'(t) \geqslant -M(v_0 - v_1)(t) - N\int_0^t k(t,s)(v_0 - v_1)(s)\,ds \\ \qquad - L\int_0^1 h(t,s)(v_0 - v_1)(s)\,ds, \quad t \in J, \\ (v_0 - v_1)(0) \geqslant (v_0 - v_1)(1); \end{cases}$$

$$\begin{cases} (v_1 - u_1)'(t) \geqslant -M(v_1 - u_1)(t) - N\int_0^t k(t,s)(v_1 - u_1)(s)\,ds \\ \qquad - L\int_0^1 h(t,s)(v_1 - u_1)(s)\,ds, \quad t \in J, \\ (v_1 - u_1)(0) \geqslant (v_1 - u_1)(1). \end{cases}$$

Thus, by Lemma 1 we have that $u_0 \leqslant u_1 \leqslant v_1 \leqslant v_0$.

Now we assume that (15) is true for $k > 1$, i.e., $u_{k-1} \leqslant u_k \leqslant v_k \leqslant v_{k-1}$, and we prove that (15) is true for $k + 1$ too. In fact, by (12), (13), and the condition (H_2), we have that

$$\begin{cases} (u_{k+1} - u_k)'(t) \geqslant -M(u_{k+1} - u_k)(t) - N\int_0^t k(t,s)(u_{k+1} - u_k)(s)\,ds \\ \qquad - L\int_0^1 h(t,s)(u_{k+1} - u_k)(s)\,ds, \quad t \in J, \\ (u_{k+1} - u_k)(0) = (u_{k+1} - u_k)(1); \end{cases}$$

$$\begin{cases} (v_k - v_{k+1})'(t) \geqslant -M(v_k - v_{k+1})(t) - N\int_0^t k(t,s)(v_k - v_{k+1})(s)\,ds \\ \qquad - L\int_0^1 h(t,s)(v_k - v_{k+1})(s)\,ds, \quad t \in J, \\ (v_k - v_{k+1})(0) = (v_k - v_{k+1})(1); \end{cases}$$

$$\begin{cases} (v_{k+1}-u_{k+1})'(t) \geqslant -M(v_{k+1}-u_{k+1})(t) - N\int_0^t k(t,s)(v_{k+1}-u_{k+1})(s)\,ds \\ \qquad\qquad - L\int_0^1 h(t,s)(v_{k+1}-u_{k+1})(s)\,ds, \quad t\in J, \\ (v_{k+1}-u_{k+1})(0) = (v_{k+1}-u_{k+1})(1). \end{cases}$$

Thus, by Lemma 1 we have that $u_k \leqslant u_{k+1} \leqslant v_{k+1} \leqslant v_k$. So, by induction, (15) holds for all positive integer n.

It is easy to know by (15) that

$$u_0 \leqslant u_1 \leqslant \cdots \leqslant u_n \leqslant \cdots \leqslant v_n \leqslant \cdots \leqslant v_1 \leqslant v_0. \tag{16}$$

By (12), (13), and the condition (H_2), we have that

$$\begin{aligned} &f(t,u_0,Tu_0,Su_0) - 2M(v_0-u_0) - 2NT(v_0-u_0) - 2LS(v_0-u_0) \\ &\quad \leqslant u_n'(t) \leqslant f(t,v_0,Tv_0,Sv_0) + M(v_0-u_0) + NT(v_0-u_0) + LS(v_0-u_0). \end{aligned}$$

Thus, $\{u_n'(t)\}$ is uniformly bounded. Also, similarly to the above we can show that $\{v_n'(t)\}$ is uniformly bounded.

From the above we know that $\{u_n\}$ and $\{v_n\}$ are uniformly bounded and equicontinuous in $[u_0, v_0]$. By Arzela–Ascoli theorem and (16), we can see that the sequences $\{u_n\}$ and $\{v_n\}$ are uniformly convergent in J. Let

$$\lim_{n\to\infty} u_n(t) = u^*(t), \qquad \lim_{n\to\infty} v_n(t) = v^*(t). \tag{17}$$

Obviously, $u^*, v^* \in [u_0, v_0]$ and (14) holds.

Furthermore, by (12) and (13), we have that

$$\begin{aligned} u_n(t) = {} & \frac{e^{-Mt}}{e^M - 1}\cdot \int_0^1 e^{Ms}\big[f\big(s,u_{n-1}(s),Tu_{n-1}(s),Su_{n-1}(s)\big) + Mu_{n-1}(s) \\ & - NT(u_n-u_{n-1})(s) - LS(u_n-u_{n-1})(s)\big]\,ds \\ & + e^{-Mt}\cdot \int_0^t e^{Ms}\big[f\big(s,u_{n-1}(s),Tu_{n-1}(s),Su_{n-1}(s)\big) + Mu_{n-1}(s) \\ & - NT(u_n-u_{n-1})(s) - LS(u_n-u_{n-1})(s)\big]\,ds, \quad t\in J, \end{aligned} \tag{18}$$

$$\begin{aligned} v_n(t) = {} & \frac{e^{-Mt}}{e^M - 1}\cdot \int_0^1 e^{Ms}\big[f\big(s,v_{n-1}(s),Tv_{n-1}(s),Sv_{n-1}(s)\big) + Mv_{n-1}(s) \\ & - NT(v_n-v_{n-1})(s) - LS(v_n-v_{n-1})(s)\big]\,ds \\ & + e^{-Mt}\cdot \int_0^t e^{Ms}\big[f\big(s,v_{n-1}(s),Tv_{n-1}(s),Sv_{n-1}(s)\big) + Mv_{n-1}(s) \\ & - NT(v_n-v_{n-1})(s) - LS(v_n-v_{n-1})(s)\big]\,ds, \quad t\in J. \end{aligned} \tag{19}$$

Taking limits as $n\to\infty$, by (17), we have that

$$u^*(t) = \frac{e^{-Mt}}{e^M - 1} \cdot \int_0^1 e^{Ms}\big[f\big(s, u^*(s), Tu^*(s), Su^*(s)\big) + Mu^*(s)\big]\,ds$$
$$+ e^{-Mt} \cdot \int_0^t e^{Ms}\big[f\big(s, u^*(s), Tu^*(s), Su^*(s)\big) + Mu^*(s)\big]\,ds,$$
$$v^*(t) = \frac{e^{-Mt}}{e^M - 1} \cdot \int_0^1 e^{Ms}\big[f\big(s, v^*(s), Tv^*(s), Sv^*(s)\big) + Mv^*(s)\big]\,ds$$
$$+ e^{-Mt} \cdot \int_0^t e^{Ms}\big[f\big(s, v^*(s), Tv^*(s), Sv^*(s)\big) + Mv^*(s)\big]\,ds.$$

From the above, by Lemma 3, we know that u^* and v^* are solutions of PBVP (1) in $[u_0, v_0]$.

Next we prove that u^* and v^* are the minimal and maximal solutions of the PBVP (1) in $[u_0, v_0]$, respectively.

In fact, suppose $w \in [u_0, v_0]$ is also a solution of the PBVP (1), i.e.,

$$\begin{cases} w'(t) = f(t, w, Tw, Sw), & t \in J, \\ w(0) = w(1). \end{cases} \tag{20}$$

Using induction, by (12), (13), the condition (H_2) and Lemma 1, it is not difficult to prove that

$$u_n \leqslant w \leqslant v_n, \quad n = 1, 2, 3, \ldots. \tag{21}$$

Thus, letting $n \to \infty$ in (21) and by (17), we have that

$$u^* \leqslant w \leqslant v^*,$$

i.e., u^* and v^* are the minimal and maximal solutions of the PBVP (1) in the interval $[u_0, v_0]$, respectively.

The proof of the theorem is complete. □

4. Examples

Example 1. Consider the PBVP of first order nonlinear integro-differential equations of mixed type:

$$\begin{cases} u'(t) = \frac{2}{15}[t - u(t)]^3 + \frac{1}{625}\big[t^3 - \int_0^t 2tsu(s)\,ds\big]^5 \\ \qquad + \frac{1}{875}\big[t^2 - \int_0^1 3(ts)^2u(s)\,ds\big]^7, \quad t \in J, \\ u(0) = u(1). \end{cases} \tag{22}$$

Conclusion 1. PBVP (22) has the minimal solution $u^*(t)$ and maximal solution $v^*(t)$ such that $0 \leqslant u^*(t) \leqslant v^*(t) \leqslant 1$ for $0 \leqslant t \leqslant 1$ and there exist monotone iteration sequences

$\{u_n(t)\}, \{v_n(t)\} \subset [u_0, v_0]$ such that

$$u_n(t) \to u^*(t), \quad v_n(t) \to v^*(t), \quad \text{as } n \to \infty \text{ uniformly on } t \in J,$$

where $u_0(t) = 0$, $v_0(t) = 1$, $\forall t \in J$.

Proof. Let

$$f(t,u,v,w) = \frac{2}{15}(t-u)^3 + \frac{1}{625}\bigl(t^3 - v\bigr)^5 + \frac{1}{875}\bigl(t^2 - w\bigr)^7,$$

$$M = \frac{2}{5}, \qquad N = L = \frac{1}{125}.$$

Obviously, $u_0(t) \leqslant v_0(t)$, and

$$u_0'(t) \leqslant f\bigl(t, u_0(t), Tu_0(t), Su_0(t)\bigr), \qquad u_0(0) = u_0(1);$$
$$v_0'(t) \geqslant f\bigl(t, v_0(t), Tv_0(t), Sv_0(t)\bigr), \qquad v_0(0) = v_0(1);$$
$$f(t,u,v,w) - f(t,\bar u,\bar v,\bar w) \geqslant -M(u-\bar u) - N(v-\bar v) - L(w-\bar w),$$

where $u_0(t) \leqslant \bar u \leqslant u \leqslant v_0(t)$, $(Tu_0)(t) \leqslant \bar v \leqslant v \leqslant (Tv_0)(t)$, $(Su_0)(t) \leqslant \bar w \leqslant w \leqslant (Sv_0)(t)$, $\forall t \in J$.

It is easy to compute that $0 < M = \frac{2}{5} < \ln\frac{1+\sqrt5}{2}$, $k_0 = \max\{2ts\colon 0 \leqslant s \leqslant t \leqslant 1\} = 2$, $h_0 = \max\{3(ts)^2\colon t, s \in J\} = 3$ and

$$(Nk_0 + Lh_0)\cdot e^M \leqslant \left(\frac{2}{125} + \frac{3}{125}\right)\cdot e^{2/5} = \frac{1}{25}\cdot e^{0.4} < \frac{4}{25} = M^2.$$

Hence, the PBVP (22) satisfies all conditions of Theorem, it follows by Theorem that our conclusions hold. The proof is complete. □

Example 2. Consider the PBVP of first order nonlinear integro-differential equations of mixed type:

$$\begin{cases} u'(t) = \frac{1}{3}[t-u(t)]^3 + \frac{1}{200}\bigl[t^3 - \int_0^t 2tsu(s)\,ds\bigr]^5 \\ \qquad + \frac{1}{280}\bigl[t^2 - \int_0^1 3(ts)^2 u(s)\,ds\bigr]^7, \quad t \in J, \\ u(0) = u(1). \end{cases} \tag{23}$$

Conclusion 2. PBVP (23) has the minimal solution $u^*(t)$ and maximal solution $v^*(t)$ such that $0 \leqslant u^*(t) \leqslant v^*(t) \leqslant 1$ for $0 \leqslant t \leqslant 1$ and there exist monotone iteration sequences $\{u_n(t)\}, \{v_n(t)\} \subset [u_0, v_0]$ such that

$$u_n(t) \to u^*(t), \quad v_n(t) \to v^*(t), \quad \text{as } n \to \infty \text{ uniformly on } t \in J,$$

where $u_0(t) = 0$, $v_0(t) = 1$, $\forall t \in J$.

Proof. Let

$$f(t,u,v,w) = \frac{1}{3}(t-u)^3 + \frac{1}{200}\bigl(t^3 - v\bigr)^5 + \frac{1}{280}\bigl(t^2 - w\bigr)^7,$$

$$M = 1, \qquad N = L = \frac{1}{40}.$$

Obviously, $u_0(t) \leqslant v_0(t)$, and

$$u_0'(t) \leqslant f\big(t, u_0(t), Tu_0(t), Su_0(t)\big), \qquad u_0(0) = u_0(1);$$
$$v_0'(t) \geqslant f\big(t, v_0(t), Tv_0(t), Sv_0(t)\big), \qquad v_0(0) = v_0(1);$$
$$f(t, u, v, w) - f(t, \bar{u}, \bar{v}, \bar{w}) \geqslant -M(u - \bar{u}) - N(v - \bar{v}) - L(w - \bar{w}),$$

where $u_0(t) \leqslant \bar{u} \leqslant u \leqslant v_0(t)$, $(Tu_0)(t) \leqslant \bar{v} \leqslant v \leqslant (Tv_0)(t)$, $(Su_0)(t) \leqslant \bar{w} \leqslant w \leqslant (Sv_0)(t)$, $\forall t \in J$.

It is easy to compute that $0 < M = 1 > \ln \frac{1+\sqrt{5}}{2}$, $k_0 = \max\{2ts\colon 0 \leqslant s \leqslant t \leqslant 1\} = 2$, $h_0 = \max\{3(ts)^2\colon t, s \in J\} = 3$ and

$$(Nk_0 + Lh_0) \cdot \big(e^{2M} - 1\big) = \frac{1}{8} \cdot \big(e^2 - 1\big) < 1 = M^2.$$

Hence, the PBVP (23) satisfies all conditions of Theorem, it follows by Theorem that our conclusions hold. The proof is complete. □

Remark. In the same way, the similar example can be obtained for $M = \ln \frac{1+\sqrt{5}}{2}$.

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