Solvability of three point boundary value problems for second order differential equations with deviating arguments

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Received 3 January 2005
Available online 25 April 2005
Submitted by S. Heikkilä

Abstract

The monotone iterative technique is used to boundary problems for second order ordinary differential equations with deviating arguments. Corresponding results are formulated when the problem has extremal solutions or weakly coupled extremal quasi-solutions.

Keywords: Monotone iterative technique; Equations and inequalities with deviating arguments; Existence of extremal solutions or weakly coupled extremal quasi-solutions

1. Introduction

It is well known that the monotone iterative technique offers an approach for obtaining approximate solutions of nonlinear differential equations. It can be used both initial and boundary problems also with impulses (for details, see for example books [13,15] and the references therein). There exists a vast literature devoted the application of this technique for ordinary differential equations (see, for example, [1–17]). There are only a few papers when the monotone iterative technique is used to first order differential problems with
delayed arguments (see [7,10,11,16,17]), and with advanced arguments [9]. Usually, it is assumed that a function $f$ appearing on the right-hand side of a differential equation satisfies a one-sided Lipschitz condition with corresponding constant coefficients. When we have a differential equation with deviating arguments, then it is better to discuss such problems when constants coefficients are replaced by corresponding functions because in this case we obtain a less restrictive condition for the existence of solutions in comparing with the corresponding one when functions are replaced by constants. For the first time such assumption with functional coefficients (instead of constants ones from a one-sided Lipschitz condition) appeared in papers [9] and [10].

The purpose of this paper is to apply this method for a class of second order ordinary differential equations with deviating arguments subject to boundary conditions of the form

$$\begin{cases}
  x''(t) = f(t, x(t), x(\alpha(t))) \equiv Fx(t), & t \in J = [0, T], \quad T < \infty, \\
  x(0) = 0, & x(T) = rx(\gamma), \quad 0 < \gamma < T,
\end{cases}$$

(1)

where $f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $r$ and $\gamma$ are fixed numbers, $r \in \mathbb{R}$. The argument $\alpha \in C(J, J)$; for example, it can be defined by $\alpha(t) = \bar{\alpha} t$, $t \in J$, for fixed $\bar{\alpha} \in (0, 1)$, or $\alpha(t) = \sqrt{t}$, $t \in [0, 1]$. To use the monotone iterative technique for problems of type (1) some discussion devoted of second order differential inequalities with deviating arguments is necessary. If $r \geq 0$, we use lower and upper solutions for (1) assuming that $f$ satisfies one-sided Lipschitz condition with corresponding functional coefficients $M, N$ (see Assumptions $H_2, H_3$) and we can show that problem (1) has extremal solutions in a segment. Problem (1) is also discussed when $r < 0$. In this case, we formulate a result when (1) has a weakly coupled extremal quasi-solution. It is important to add that in both cases a one-sided Lipschitz condition on $f$ is assumed. In the last part of this paper, we discuss a problem when we have more deviating arguments to obtain related results. Some examples satisfying the assumptions are presented.

There are some papers devoted the application of the monotone iterative technique for second order differential equations with boundary conditions when function $f$ depends on $x$ and $x'$ without deviating arguments (see, for example, [1–4,12]). In the above cited papers, boundary conditions have one of the forms: $x(0) = x(T), x'(0) = x'(T)$, or $x(0) = x'(T) = 0$.

2. Lemmas

To apply the monotone iterative method to problems of type (1), we need a fundamental result on differential inequalities.

Lemma 1. Assume that:

$H_1$: $\alpha \in C(J, J), M, N \in C(J, [0, \infty)), M(t) > 0, t \in (0, T), M(0) \geq 0, M(T) \geq 0,$

$H_2$: $\rho \equiv \max\{\int_0^T \left( \int_s^T [M(t) + N(t)] \, dt \right) \, ds, \int_0^T \left( \int_0^t [M(t) + N(t)] \, dt \right) \, ds \} \leq 1.$
Let \( p \in C^2(J, \mathbb{R}) \) and
\[
\begin{align*}
p''(t) & \geq M(t)p(t) + N(t) \min[p(\alpha(t)), 0], \quad t \in J, \\
p(0) & \leq 0, \quad p(T) \leq 0.
\end{align*}
\]
Then \( p(t) \leq 0 \) on \( J \).

**Proof.** Some ideas are taken from the proof of Theorem 2.2 in [11].

**Case 1.** Suppose that \( p(t) \geq 0, \ p(t) \neq 0 \) on \( J \). Then \( p(0) = p(T) = 0 \). Let
\[
p(t_0) = \max_{t \in J} p(t) = d > 0.
\]
Note that \( p''(t_0) \leq 0, \ t_0 \in (0, T) \), and
\[
0 \geq p''(t_0) \geq M(t)d > 0.
\]
It is a contradiction.

**Case 2.** There exist \( t_1, t_2 \in J \) such that \( p(t_1) > 0 \) and \( p(t_2) < 0 \). Then there exist \( t_3 \in (0, T) \) and \( \xi \in J \) such that
\[
p(t_3) = \max_{t \in J} p(t) > 0 \quad \text{and} \quad p'(t_3) = 0, \quad p(\xi) = \min_{t \in J} p(t) < 0.
\]

**Case 2.1.** Assume that \( \xi \in [0, t_3) \). Then
\[
p''(t) \geq M(t)p(\xi) + N(t) \min[p(\alpha(t)), 0] \geq [M(t) + N(t)]p(\xi), \quad t \in J.
\]
Integrating the above inequality from \( s \) to \( t_3 \), we get
\[
-p'(s) = p'(t_3) - p'(s) \geq p(\xi) \int_s^{t_3} [M(t) + N(t)] \, dt.
\]
Next, we integrate the above inequality from \( \xi \) to \( t_3 \) to obtain
\[
p(\xi) > -p(t_3) + p(\xi) \geq p(\xi) \int_{\xi}^{t_3} \left( \int_s^{t_3} [M(t) + N(t)] \, dt \right) ds.
\]
Dividing by \( p(\xi) \), we finally get
\[
1 < \int_0^T \left( \int_s^T [M(t) + N(t)] \, dt \right) ds \leq \rho,
\]
since \( p(\xi) < 0 \). It is a contradiction.
Case 2.2. Assume that $\xi > t_3$. Then

$$p''(t) \geq [M(t) + N(t)] p(\xi), \quad t \in J.$$ 

Integrating the above inequality from $t_3$ to $s$, we have

$$p'(s) = p'(s) - p'(t_3) \geq p(\xi) \int_{t_3}^{s} [M(t) + N(t)] dt.$$ 

Next, we integrate the above inequality from $t_3$ to $\xi$ obtaining

$$p(\xi) > p(\xi) - p(t_3) \geq p(\xi) \int_{t_3}^{\xi} [M(t) + N(t)] dt.$$ 

Hence

$$\frac{1}{\int_{0}^{T} \left( \int_{0}^{s} [M(t) + N(t)] dt \right) ds} \leq \rho.$$ 

It is a contradiction too. This proves the lemma. □

Remark 1. Since $p(\alpha(t)) \geq \min\{p(\alpha(t)), 0\}$, then, in view of Lemma 1, we have $p(t) \leq 0$, $t \in J$, if

$$\begin{cases} p''(t) \geq M(t)p(t) + N(t)p(\alpha(t)), & t \in J, \\ p(0) \leq 0, & p(T) \leq 0. \end{cases}$$

Remark 2. Let $M(t) = M > 0$, $N(t) = N \geq 0$ and

$$(M + N)T^2 \leq 2. \quad (2)$$

Then Assumption $H_2$ is satisfied. Indeed, Assumption $H_2$ is less restrictive than condition (2). Put $M(t) = Mt^2$, $N(t) = t^3$ for $J = [0, 1]$. Then $M \leq \frac{16}{5}$, by Assumption $H_2$, and $M \leq 1$, by condition (2).

Lemma 2. Let Assumptions $H_1$, $H_2$ be satisfied. Let $y \in C^2(J, \mathbb{R})$, $\sigma \in C(J, \mathbb{R})$ and

$$\begin{cases} y''(t) = M(t)y(t) + N(t)y(\alpha(t)) + \sigma(t), & t \in J, \\ y(0) = k_1 \in \mathbb{R}, & y(T) = k_2 \in \mathbb{R}. \end{cases} \quad (3)$$

Then problem (3) has at most one solution.

Proof. Suppose problem (3) has two distinct solutions $z, w \in C^2(J, \mathbb{R})$. Put $p = z - w$. Then $p(0) = p(T) = 0$ and $p''(t) = M(t)p(t) + N(t)p(\alpha(t))$ on $J$. In view of Remark 1, $p \leq 0$, so $z(t) \leq w(t)$, $t \in J$. Now putting $p = w - z$, we have $w(t) \leq z(t)$, $t \in J$, by Remark 1. Hence $w(t) = z(t)$, $t \in J$ and Lemma 2 holds. □
3. Integral representation

Let $u \in C^2(J, \mathbb{R})$. We introduce the following operators:

$$L[u](t) = u''(t), \quad U_1[u] = u(0), \quad U_2[u] = u(T).$$

Take the Green function $G$ defined by

$$G(t, s) = -\frac{1}{T} \begin{cases} (T - t)s & \text{if } 0 \leq s \leq t \leq T, \\ (T - s)t & \text{if } 0 \leq t \leq s \leq T. \end{cases}$$

Let $h$ be integrable on $J$ and $\beta \in \mathbb{R}$. Then the problem

$$L[u](t) = h(t), \quad t \in J, \quad U_1[u] = 0, \quad U_2[u] = \beta$$

has a unique solution given by

$$u(t) = \int_0^T G(t, s)h(s)\,ds + \frac{\beta}{T}t, \quad t \in J.$$

**Theorem 1.** Suppose that $\alpha \in C(J, J)$, $M, N \in C(J, \mathbb{R})$. Let $u, v \in C^2(J, \mathbb{R})$, $u(t) \leq v(t)$, $t \in J$ and

$$P[t, y(\alpha(t))] = \begin{cases} u(\alpha(t)) & \text{if } y(\alpha(t)) < u(\alpha(t)), \\ y(\alpha(t)) & \text{if } u(\alpha(t)) \leq y(\alpha(t)) \leq v(\alpha(t)), \\ v(\alpha(t)) & \text{if } y(\alpha(t)) > v(\alpha(t)). \end{cases}$$

Then the problem

$$\begin{cases} y''(t) = M(t)y(t) + N(t)P[t, y(\alpha(t))] + \sigma(t), & t \in J, \quad \sigma \in C(J, \mathbb{R}), \\ y(0) = 0, \quad y(T) = \beta, & \beta \in \mathbb{R}, \end{cases}$$

has a solution $y \in C^2(J, \mathbb{R})$.

**Proof.** Consider the integral equation

$$y(t) = \int_0^T G(t, s)\left\{M(s)y(s) + N(s)P[s, y(\alpha(s))] + \sigma(s)\right\}\,ds + \frac{\beta}{T}t, \quad t \in J,$$

where the Green function $G$ is defined earlier. Denote by $A$ the operator defined by the right-hand side of (5). Consider the Banach space $B = C(J, \mathbb{R})$ with the norm $\|y\| = \max_{t \in J} \|y(t)\|$. We employ Schauder’s fixed point theorem to show that operator $A$ has a fixed point. Let $y \in B$. Note that $P \in C(J, \mathbb{R})$. We see that $M(t)y(t) + N(t)P[t, y(\alpha(t))] + \sigma(t)$ is bounded in $J$, so operator $A : B \to B$ is continuous and bounded. In fact $A$ is a compact map. Let

$$|M(t)y(t) + N(t)P[t, y(\alpha(t))] + \sigma(t)| \leq K, \quad K > 0.$$
Take \( t_1, t_2 \in J, \ t_1 < t_2 \) such that \(|t_1 - t_2| < \frac{\varepsilon}{4KT}\) for \( \varepsilon > 0 \). Then we have
\[
\left| Ay(t_1) - Ay(t_2) \right| = \left| \frac{1}{T} \int_0^T \left( G(t_1, s) - G(t_2, s) \right) \left\{ M(s)y(s) + N(s)P[s, y(\alpha(s))] + \sigma(s) \right\} ds \right|
\]
\[
- t_1 \int_{t_1}^{t_2} (T - s) \left\{ M(s)y(s) + N(s)P[s, y(\alpha(s))] + \sigma(s) \right\} ds
\]
\[
+ (T - t_2) \int_{t_1}^{t_2} \left\{ M(s)y(s) + N(s)P[s, y(\alpha(s))] + \sigma(s) \right\} ds
\]
\[
\leq 4KT |t_1 - t_2| < \varepsilon.
\]
Consequently \( A : B \to B \) is compact. Schauder’s fixed point theorem guarantees that \( A \) has a fixed point in \( B \). In view of (5), we have \( y(0) = 0, \ y(T) = \beta, \) and \( y'' \) exists and \( y'' \in B \). Moreover, \( y \in C^2(J, \mathbb{R}) \) is a solution of problem (4). This ends the proof.

4. Main results when \( r \geq 0 \)

Let \( r \geq 0 \). A function \( y_0 \in C^2(J, \mathbb{R}) \) is said to be a lower solution of (1) if
\[
y''_0(t) \geq Fy_0(t), \quad t \in J, \quad y_0(0) \leq 0, \quad y_0(T) \leq ry_0(\gamma), \quad 0 < \gamma < T.
\]
A function \( z_0 \in C^2(J, \mathbb{R}) \) is said to be an upper solution of problem (1) if
\[
z''_0(t) \leq Fz_0(t), \quad t \in J, \quad z_0(0) \geq 0, \quad z_0(T) \geq rz_0(\gamma).
\]

**Theorem 2.** Suppose that Assumptions \( H_1 \) and \( H_2 \) are satisfied. Let \( u, v \in C^2(J, \mathbb{R}) \) be lower and upper solutions of problem (1), respectively, and \( u(t) \leq v(t), \ t \in J \). Moreover, assume that:

\( H_3: \ f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R}) \) and
\[
f(t, \bar{u}_1, \bar{v}_1) - f(t, u_1, v_1) \geq -M(t)[u_1 - \bar{u}_1] - N(t)[v_1 - \bar{v}_1]
\]
for \( v(t) \geq u_1 \geq \bar{u}_1 \geq u(t), \ v(\alpha(t)) \geq v_1 \geq \bar{v}_1 \geq \bar{v}(\alpha(t)) \) on \( J \).
Then

(i) the problem
\[
\begin{align*}
y''(t) &= Fu(t) + M(t)[y(t) - u(t)] \\
&\quad + N(t)[P[t, y(\alpha(t))] - u(\alpha(t))], \quad t \in J, \\
y(0) &= 0, \quad y(T) = ru(\gamma), \quad 0 < \gamma < T
\end{align*}
\]
(6)

has a solution \(y \in C^2(J, \mathbb{R})\) and \(u(t) \leq y(t) \leq v(t), \ t \in J\);

(ii) the problem
\[
\begin{align*}
z''(t) &= g(t, u, z), \quad t \in J, \\
z(0) &= 0, \quad z(T) = ru(\gamma), \quad 0 < \gamma < T
\end{align*}
\]
(7)

has a unique solution \(z \in C^2(J, \mathbb{R})\), \(z\) is a lower solution of problem (1) and \(u(t) \leq z(t) \leq v(t), \ t \in J\), where
\[
g(t, a, b) = Fa(t) + M(t)[b(t) - a(t)] + N(t)[b(\alpha(t)) - a(\alpha(t))], \quad t \in J;
\]

(iii) the problem
\[
\begin{align*}
Y''(t) &= Fv(t) + M(t)[Y(t) - v(t)] \\
&\quad + N(t)[P[t, Y(\alpha(t))] - v(\alpha(t))], \quad t \in J, \\
Y(0) &= 0, \quad Y(T) = rv(\gamma), \quad 0 < \gamma < T
\end{align*}
\]
(8)

has a solution \(Y \in C^2(J, \mathbb{R})\) and \(u(t) \leq Y(t) \leq v(t), \ t \in J\);

(iv) the problem
\[
\begin{align*}
Z''(t) &= g(t, v, Z), \quad t \in J, \\
Z(0) &= 0, \quad Z(T) = rv(\gamma), \quad 0 < \gamma < T
\end{align*}
\]
(9)

has a unique solution \(Z \in C^2(J, \mathbb{R})\), \(Z\) is an upper solution of problem (1) and \(u(t) \leq Z(t) \leq v(t), \ t \in J\);

(v) \(z(t) \leq Z(t)\) on \(J\).

**Proof.** First we need to prove assertion (i). Note that problem (6) has a solution \(y\), by Theorem 1. Put \(p(t) = u(t) - y(t), \ t \in J\), so \(p(0) \leq 0, \ p(T) \leq 0\). Moreover,
\[
p''(t) \geq Fu(t) - Fu(t) - M(t)[y(t) - u(t)] - N(t)[P[t, y(\alpha(t))] - u(\alpha(t))]
\]
\[
= M(t)p(t) - N(t)
\]
\[
\begin{cases}
\text{if } p(\alpha(t)) > 0, & u(\alpha(t)) - u(\alpha(t)) \\
\text{if } p(\alpha(t)) \leq 0 \text{ and } y(\alpha(t)) - u(\alpha(t)) & y(\alpha(t)) - u(\alpha(t)) \\
\text{and } y(\alpha(t)) \leq v(\alpha(t)), & y(\alpha(t)) - u(\alpha(t)) \\
\text{and } y(\alpha(t)) > v(\alpha(t)), & v(\alpha(t)) - u(\alpha(t)) \\
\text{and } p(\alpha(t)) < 0 & v(\alpha(t)) - u(\alpha(t))
\end{cases}
\]
\[
\geq M(t)p(t) + N(t) \min[p(\alpha(t)), 0].
\]
Hence \(u(t) \leq y(t), \ t \in J\), by Lemma 1. Now we put \(q(t) = y(t) - v(t), \ t \in J\), so \(q(0) \leq 0, \ q(T) \leq 0\). In view of Assumption \(H_3\), we get
\[ q''(t) \geq f\left(t, u(t), u(\alpha(t))\right) - f\left(t, v(t), v(\alpha(t))\right) + M(t)[y(t) - u(t)] \\
\quad + N(t) \begin{cases} 
  y(\alpha(t)) - u(\alpha(t)) & \text{if } q(\alpha(t)) \leq 0, \\
  v(\alpha(t)) - u(\alpha(t)) & \text{if } q(\alpha(t)) > 0,
\end{cases} \\
\geq -M(t)[v(t) - u(t)] - N(t)[v(\alpha(t)) - u(\alpha(t))] + M(t)[y(t) - u(t)] \\
\quad + N(t) \begin{cases} 
  y(\alpha(t)) - u(\alpha(t)) & \text{if } q(\alpha(t)) \leq 0, \\
  v(\alpha(t)) - u(\alpha(t)) & \text{if } q(\alpha(t)) > 0,
\end{cases} \\
= Mq(t) + N(t) \begin{cases} 
  q(\alpha(t)) & \text{if } q(\alpha(t)) \leq 0, \\
  0 & \text{if } q(\alpha(t)) > 0,
\end{cases} \\
= M(t)q(t) + N(t) \min[q(\alpha(t)), 0].
\]

Hence, by Lemma 1, \( y(t) \leq v(t) \) on \( J \). This proves part (i).

Note that \( P[t, y(\alpha(t))] = y(\alpha(t)) \), because \( u(\alpha(t)) \leq y(\alpha(t)) \leq v(\alpha(t)), \ t \in J \), by (i).

This shows that \( y \) is also a solution of problem (7). This and Lemma 2 assure that problem (7) has a unique solution and denotes it by \( z \). Moreover, \( u(t) \leq z(t) \leq v(t), \ t \in J \).

Now we need to show that \( z \) is a lower solution of problem (1). We see that
\[ z(0) = 0, \quad z(T) = ru(\gamma) \leq rz(\gamma) \]
and, in view of Assumption \( H_3 \), we obtain
\[ z''(t) = Fu(t) + M(t)[z(t) - u(t)] + N(t)[z(\alpha(t)) - u(\alpha(t))] - Fz(t) + Fz(t) \\
\geq Fz(t) - M(t)[z(t) - u(t)] - N(t)[z(\alpha(t)) - u(\alpha(t))] + M(t)[z(t) - u(t)] \\
\quad + N(t)[z(\alpha(t)) - u(\alpha(t))] = Fz(t).
\]

The above proves that \( z \) is a lower solution of problem (1). This ends the proof of part (ii).

The proof of parts (iii) and (iv) is similar to the proof of parts (i) and (ii), respectively, and therefore it is omitted.

Now we need to prove part (v). Put \( p(t) = z(t) - Z(t) \), so \( p(0) = 0, p(T) \leq 0 \). Using Assumption \( H_3 \), we get
\[ p''(t) = Fu(t) - Fv(t) + M(t)[z(t) - u(t) - Z(t) + v(t)] \\
\quad + N(t)[z(\alpha(t)) - u(\alpha(t)) - Z(\alpha(t)) + v(\alpha(t))] \\
\geq -M(t)[v(t) - u(t)] - N(t)[v(\alpha(t)) - u(\alpha(t))] \\
\quad + M(t)[z(t) - u(t) - Z(t) + v(t)] \\
\quad + N(t)[z(\alpha(t)) - u(\alpha(t)) - Z(\alpha(t)) + v(\alpha(t))] \\
= M(t)p(t) + N(t)p(\alpha(t)).
\]

Hence, \( z(t) \leq Z(t), \ t \in J \), by Remark 1. This ends the proof. \( \Box \)

**Remark 3.** Note that if \( f \) is nonincreasing with respect to the last two variables, then Assumption \( H_3 \) holds.

**Theorem 3.** Let Assumptions \( H_1, H_2, H_3 \) hold. Let \( y_0, z_0 \in C^2(J, \mathbb{R}) \) be lower and upper solutions of problem (1), respectively, and \( y_0(t) \leq z_0(t) \) on \( J \).
Then problem (1) has, in the segment \([y_0, z_0]\), the minimal and maximal solutions with 
\([y_0, z_0] = \{ w \in C^2(J, \mathbb{R}) : y_0(t) \leq w(t) \leq z_0(t), \ t \in J \} \).

**Proof.** Let
\[
\begin{cases}
y_n''(t) = g(t, y_{n-1}, y_n), \quad t \in J, \\
y_n(0) = 0, \quad y_n(T) = r y_{n-1}(\gamma), \quad 0 < \gamma < T,
\end{cases}
\]  
(10)

\[
\begin{cases}
z_n''(t) = g(t, z_{n-1}, z_n), \quad t \in J, \\
z_n(0) = 0, \quad z_n(T) = r z_{n-1}(\gamma), \quad 0 < \gamma < T,
\end{cases}
\]  
(11)

for \(n = 1, 2, \ldots\). Function \(g\) is defined earlier.

Note that, for \(n = 1\), problems (10) and (11) are well defined, and
\(y_0(t) \leq y_1(t) \leq z_1(t) \leq z_0(t), \ t \in J,\)
by Theorem 2. Also, in view of Theorem 2, \(y_1, z_1\) are lower and upper solutions of problem (1), respectively. By induction in \(n\), we can prove the relation:
\(y_0(t) \leq \cdots \leq y_{n-1}(t) \leq y_n(t) \leq z_n(t) \leq z_{n-1}(t) \leq \cdots \leq z_0(t), \ t \in J, \ n = 1, 2, \ldots.\)

It implies that \(\{y_n\}, \{z_n\}\) are uniformly bounded, so
\(A_1 \leq y_n(t) \leq z_n(t) \leq A_2, \ t \in J, \ n = 0, 1, \ldots.\)

Indeed, \(y_n, z_n\) from (10) and (11) satisfy the integral equations
\[
\begin{cases}
y_n(t) = \int_0^T G(t, s) g(s, y_{n-1}, y_n) \, ds + \frac{r}{T} y_{n-1}(\gamma), \quad t \in J, \\
z_n(t) = \int_0^T G(t, s) g(s, z_{n-1}, z_n) \, ds + \frac{r}{T} z_{n-1}(\gamma), \quad t \in J,
\end{cases}
\]  
(12)

and
\[
\begin{cases}
y_n(0) = 0, \quad y_n(T) = r y_{n-1}(\gamma), \\
z_n(0) = 0, \quad z_n(T) = r z_{n-1}(\gamma),
\end{cases}
\]  
(13)

for \(n = 1, 2, \ldots\). \(G\) is the Green function defined earlier. Note that \(y'_n\) exists and
\[
y_n'(t) = \frac{1}{T} \int_0^t g(s, y_{n-1}, y_n) \, ds - \frac{1}{T} \int_t^T g(s, y_{n-1}, y_n) \, ds.\]

Indeed, \(|g(t, a, b)|\) is bounded by a positive constant \(W\) on \(J \times [A_1, A_2] \times [A_1, A_2]\). Take \(\varepsilon > 0\). For \(t_1, t_2 \in J, |t_1 - t_2| < \frac{\varepsilon W}{T}\), we have
\[
|y_n(t_1) - y_n(t_2)| = \left| \int_{t_2}^{t_1} y_n'(\tau) \, d\tau \right|
\]
\[
= \frac{1}{T} \left| \int_{t_2}^{t_1} \left[ \int_0^\tau g(s, y_{n-1}, y_n) \, ds - \int_\tau^T g(s, y_{n-1}, y_n) \, ds \right] \, d\tau \right|
\]
\[
\leq WT |t_1 - t_2| < \varepsilon.
\]
By a similar way, we have \(|z_n(t_1) - z_n(t_2)| < \varepsilon\). It proves that \(\{y_n\}, \{z_n\}\) are equicontinuous on \(J\). The Arzeli–Ascoli theorem guarantees the existence of subsequences \(\{y_{n_k}\}, \{z_{n_k}\}\) and functions \(\tilde{y}, \tilde{z} \in C(J, \mathbb{R})\) with \(y_{n_k}, z_{n_k}\) converging uniformly on \(J\) to \(\tilde{y}\) and \(\tilde{z}\), respectively, if \(n_k \to \infty\). However, since the sequences \(\{y_n\}, \{z_n\}\) are monotonic, we conclude that the whole sequences \(\{y_n\}, \{z_n\}\) converge uniformly on \(J\) to \(\tilde{y}\) and \(\tilde{z}\), respectively, if \(n \to \infty\). Indeed, \(y_n, z_n\) satisfy the integral equations (12) and conditions (13) and if \(n \to \infty\), then we have

\[
\begin{align*}
\dot{y}(t) &= \int_0^T G(t, s) F\dot{y}(s) \, ds + \frac{r}{T} \tilde{y}(\gamma), \\
\dot{z}(t) &= \int_0^T G(t, s) F\dot{z}(s) \, ds + \frac{r}{T} \tilde{z}(\gamma), \quad t \in J,
\end{align*}
\]

and

\[
\begin{align*}
\dot{y}(0) &= 0, \quad \dot{y}(T) = r\tilde{y}(\gamma), \\
\dot{z}(0) &= 0, \quad \dot{z}(T) = r\tilde{z}(\gamma)
\end{align*}
\]

because \(g\) is continuous. Finding \(y''\), \(z''\) from the above integral equations, we see that

\[
\begin{align*}
y''(t) &= F\ddot{y}(t), \quad t \in J, \quad \ddot{y}(0) = 0, \quad \ddot{y}(T) = r\ddot{y}(\gamma), \\
z''(t) &= F\ddot{z}(t), \quad t \in J, \quad \ddot{z}(0) = 0, \quad \ddot{z}(T) = r\ddot{z}(\gamma),
\end{align*}
\]

so \(\tilde{y}, \tilde{z} \in C^2(J, \mathbb{R})\) are solutions of problem (1), and

\[
y_0(t) \leq \tilde{y}(t) \leq \tilde{z}(t) \leq z_0(t), \quad t \in J.
\]

We need to show now that \((\tilde{y}, \tilde{z})\) are extremal solutions of problem (1) in the segment \([y_0, z_0]\). To prove it we assume that \(\tilde{y}\) is another solution of problem (1), and \(y_{n-1}(t) \leq \tilde{y}(t) \leq z_{n-1}(t), \ t \in J\) for some positive integer \(n\). Put \(p(t) = y_n(t) - \tilde{y}(t), \ q(t) = \tilde{y}(t) - z_n(t), \ t \in J\). Hence \(p(0) = 0, \ p(T) \leq 0, \ q(0) = 0, \ q(T) \leq 0\). This and Assumption \(H_3\) yield

\[
p''(t) = Fy_{n-1}(t) + M(t)[y_n(t) - y_{n-1}(t)] \\
+ N(t)[y_n(\alpha(t)) - y_{n-1}(\alpha(t))] - F\tilde{y}(t) \\
\geq -M(t)[\tilde{y}(t) - y_{n-1}(t)] - N(t)[\tilde{y}(\alpha(t)) - y_{n-1}(\alpha(t))] \\
+ M(t)[y_n(t) - y_{n-1}(t)] + N(t)[y_n(\alpha(t)) - y_{n-1}(\alpha(t))] \\
= M(t) p(t) + N(t) p(\alpha(t)),
\]

\[
q''(t) = F\tilde{y}(t) - Fz_{n-1}(t) - M(t)[z_n(t) - z_{n-1}(t)] \\
- N(t)[\tilde{z}(\alpha(t)) - z_{n-1}(\alpha(t))] \\
\geq M(t) q(t) + N(t) q(\alpha(t)).
\]

By Remark 1, \(y_n(t) \leq \tilde{y}(t) \leq z_n(t), \ t \in J\). If \(n \to \infty\), it yields \(y_0(t) \leq \tilde{y}(t) \leq \tilde{y}(t) \leq \tilde{z}(t) \leq z_0(t), \ t \in J\). It proves that \(\tilde{y}, \tilde{z}\) are extremal solutions of problem (1) in the segment \([y_0, z_0]\). This ends the proof. \(\Box\)

**Example 1.** Consider the problem

\[
\begin{align*}
x''(t) &= Mx(t) \sin t + N x\left(\frac{1}{2} t\right) \cos t - M t \sin t, \quad t \in J = [0, \frac{\pi}{2}], \\
x(0) &= 0, \quad x\left(\frac{\pi}{4}\right) = r x\left(\frac{\pi}{4}\right)
\end{align*}
\]

for \(0 \leq r \leq 2, \ M, N > 0\).
Take \( y_0(t) = 0 \), \( z_0(t) = t \). Note that \( y_0(0) = 0 \), \( y_0\left(\frac{\pi}{4}\right) = 0 = ry_0\left(\frac{\pi}{4}\right) \), \( z_0(0) = 0 \), \( z_0\left(\frac{\pi}{4}\right) - rz_0\left(\frac{\pi}{4}\right) \geq 0 \), and

\[
Fy_0(t) = -Mt \sin t \leq 0 = y_0''(t), \quad Fz_0(t) = \frac{1}{2}Nt \cos t \geq 0 = z_0''(t).
\]

It proves that \( y_0, z_0 \) are lower and upper solutions of problem (14). If we extra assume that 
\[
\max\left[ M + N\left(\frac{\pi}{2} - 1\right), M\left(\frac{\pi}{2} - 1\right) + N \right] \leq 1,
\]
then problem (14) has extremal solutions in the segment \([0, t]\), by Theorem 3. For example, if we take \( M = \frac{1}{2} \), then \( N \leq \frac{6 - \pi}{4} \approx 0.7146 \).

**Example 2.** Let us consider the problem
\[
\begin{aligned}
x''(t) &= \frac{1}{8} \sin x(t) + \beta x(\alpha t) - \frac{1}{8}, \quad t \in [0, T], \\
x(0) &= 0, \quad x(T) = rx\left(\frac{1}{3} T\right),
\end{aligned}
\]
where \( \alpha \in (0, 1) \), \( \beta \geq \frac{9}{8} \), and

\[
\begin{aligned}
(a) \quad 1 &\leq r < 1 + \frac{64}{17 + 72\beta}, \\
(b) \quad \frac{18r - 18}{9 - r} &\leq T^2 \leq \frac{16}{1 + 8\beta}.
\end{aligned}
\]

Put \( y_0(t) = 0 \), \( z_0(t) = t^2 + 2 \). Then \( y_0(0) = 0 \), \( y_0(T) = 0 = ry_0\left(\frac{1}{3} T\right) \), \( z_0(0) = 2 > 0 \), \( z_0(T) - rz_0\left(\frac{1}{3} T\right) = T^2 \left(\frac{9}{9} - r\right) + 2 - 2r \geq 0 \), by condition (b), and

\[
Fy_0(t) = -\frac{1}{8} < 0 = y_0''(t), \quad Fz_0(t) = \frac{1}{8} \sin(t^2 + 2) + \beta[(\alpha t)^2 + 2] - \frac{1}{8}
\]
\[
\geq -\frac{1}{8} + 2\beta - \frac{1}{8} \geq -\frac{1}{4} + \frac{9}{4} = 2 = z_0''(t).
\]

It proves that \( y_0, z_0 \) are lower and upper solutions of problem (15). It is easy to see that Assumption \( H_3 \) holds with \( M(t) = \frac{1}{8} \), \( N(t) = \beta \), \( t \in J \). In view of (b), we obtain

\[
\left(\frac{1}{8} + \beta\right) T^2 \leq \left(\frac{1}{8} + \beta\right) \frac{16}{1 + 8\beta} = 2,
\]
and therefore (15) has, in the segment \([y_0, z_0]\), extremal solutions, by Theorem 3.

5. Main results when \( r < 0 \)

Let \( r < 0 \). A pair of functions \( y_0, z_0 \in C^2(J, \mathbb{R}) \) are called weakly coupled (w.c.) lower and upper solutions of problem (1) if

\[
\begin{aligned}
y_0''(t) &\geq Fy_0(t), \quad t \in J, \quad y_0(0) \leq 0, \quad y_0(T) \leq rz_0(\gamma), \quad 0 < \gamma < T, \\
z_0''(t) &\leq Fz_0(t), \quad t \in J, \quad z_0(0) \geq 0, \quad z_0(T) \geq ry_0(\gamma).
\end{aligned}
\]
A pair \((U, V)\), \(U, V \in C^2(J, \mathbb{R})\) is called a weakly coupled quasi-solution of problem (1) if
\[
\begin{align*}
U'(t) &= FU(t), \quad t \in J, \quad U(0) = 0, \quad U(T) = rV(\gamma), \\
V'(t) &= FV(t), \quad t \in J, \quad V(0) = 0, \quad V(T) = rU(\gamma).
\end{align*}
\]

A weakly coupled quasi-solution \((\bar{U}, \bar{V})\), \(\bar{U}, \bar{V} \in C^2(J, \mathbb{R})\) is called the weakly coupled minimal and maximal quasi-solution of problem (1) if for any weakly coupled quasi-solution \((U, V)\) of (1) we have \(\bar{U}(t) \leq U(t), V(t) \leq \bar{V}(t)\) on \(J\).

**Theorem 4.** Suppose that Assumptions \(H_1–H_3\) are satisfied. Let \(u, v \in C^2(J, \mathbb{R})\) be w.c. lower and upper solutions of problem (1), and \(u(t) \leq v(t), t \in J\).

Then

(i) the problems
\[
\begin{align*}
y''(t) &= Fu(t) + M[y(t) - u(t)] + N[P[t, y(\alpha(t))] - u(\alpha(t))], \quad t \in J, \\
y(0) &= 0, \quad y(T) = rv(\gamma), \quad 0 < \gamma < T,
\end{align*}
\]

have their solutions \(y, Y \in C^2(J, \mathbb{R})\), respectively and \(u(t) \leq y(t) \leq v(t), u(t) \leq Y(t) \leq v(t), t \in J\);

(ii) the problems
\[
\begin{align*}
z''(t) &= g(t, u, z), \quad t \in J, \\
z(0) &= 0, \quad z(T) = rv(\gamma), \quad 0 < \gamma < T,
\end{align*}
\]

have their unique solutions \(z, Z \in C^2(J, \mathbb{R})\), respectively, \(z, Z\) are w.c. lower and upper solutions of (1) and \(u(t) \leq z(t) \leq v(t), u(t) \leq Z(t) \leq v(t), t \in J\);

(iii) \(z(t) \leq Z(t)\) on \(J\).

**Proof.** The proof of parts (i) and (ii) is similar to the proof of Theorem 2 (parts (i)–(iv)) and therefore it is omitted.

To show part (iii), we put \(q(t) = z(t) - Z(t)\), so \(q(0) = 0, q(T) = r[v(\gamma) - u(\gamma)] \leq 0\). Moreover,
\[
q''(t) = g(t, u, z) - g(t, v, Z) \geq Mq(t) + Nq(\alpha(t))
\]
in view of Assumption \(H_3\). This and Remark 2 prove that part (iii) holds. This ends the proof.

**Theorem 5.** Let Assumptions \(H_1–H_3\) hold. Let \(y_0, z_0 \in C^2(J, \mathbb{R})\) be w.c. lower and upper solutions of problem (1) and \(y_0(t) \leq z_0(t)\) on \(J\).
Then problem (1) has, in the segment \([y_0, z_0]\) the w.c. minimal and maximal quasi-solutions.

**Proof.** Let

\[
\begin{align*}
  y''_n(t) &= g(t, y_{n-1}, y_n), \quad t \in J, \\
  y_n(0) &= 0, \quad y_n(T) = rz_{n-1}(\gamma), \quad 0 < \gamma < T, \\
  z''_n(t) &= g(t, z_{n-1}, z_n), \quad t \in J, \\
  z_n(0) &= 0, \quad z_n(T) = ry_{n-1}(\gamma), \quad 0 < \gamma < T
\end{align*}
\]  

(20)  

for \(n = 1, 2, \ldots\). Note that, for \(n = 1\), problems (20) and (21) are well defined, and

\[
y_0(t) \leq y_1(t) \leq z_1(t) \leq z_0(t), \quad t \in J,
\]

by Theorem 4. Also, in view of Theorem 4, \(y_1, z_1\) are w.c. lower and upper solutions of problem (1). By induction in \(n\), we can prove relation (i). It yields that \((y_n, z_n)\) converge uniformly and monotonically on \(J\) to \((\bar{y}, \bar{z})\). Indeed, functions \(\bar{y}, \bar{z}\) are w.c. quasi-solutions of problem (1), and

\[
y_0(t) \leq \bar{y}(t) \leq \bar{z}(t) \leq z_0(t), \quad t \in J.
\]

We show that \((\bar{y}, \bar{z})\) is w.c. maximal and minimal quasi-solution of problem (1). Let \((\bar{y}, \bar{z})\) be another w.c. quasi-solution of (1) such that \(y_0(t) \leq \bar{y}(t), \bar{z}(t) \leq z_0(t), t \in J\). We have to show that \(\bar{y}(t) \leq \bar{y}(t), \bar{z}(t) \leq \bar{z}(t), t \in J\). To do this we assume that \(y_{m-1}(t) \leq \bar{y}(t), \bar{z}(t) \leq z_{m-1}(t), t \in J\) for some positive \(m\). Put \(p(t) = y_{m-1}(t) - \bar{y}(t), q(t) = \bar{z}(t) - z_{m-1}(t), t \in J\), so \(p(0) = 0, p(T) \leq 0, q(0) = 0, q(T) \leq 0\). Moreover, in view of Assumption \(H_3\), we get

\[p''(t) = g(t, y_{m-1}, y_m) - F\bar{y}(t) \geq M(t)p(t) + N(t)p(\alpha(t)),
\]

\[q''(t) = F\bar{z}(t) - g(t, z_{m-1}, z_m) \geq M(t)q(t) + N(t)q(\alpha(t)).
\]

Hence, by Remark 1, we obtain \(y_n(t) \leq \bar{y}(t), \bar{z}(t) \leq z_n(t), t \in J\), for all \(n\), by mathematical induction. Now if \(n \to \infty\), this shows that \((\bar{y}, \bar{z})\) is w.c. maximal and minimal quasi-solution of problem (1). This ends the proof. \(\square\)

**Example 3.** Now we consider the problem

\[
\begin{align*}
  x''(t) &= \beta_1(t) \sin^2 x(t) + \beta_2(t) \sin^2 x\left(\frac{1}{2} t\right) + \beta_3(t)x\left(\frac{1}{2} t\right) + h(t), \quad t \in J, \\
  x(0) &= 0, \quad x(T) = -x\left(\frac{1}{2} T\right),
\end{align*}
\]

(22)

where \(J = [0, T]\), \(h \in C(J, \mathbb{R})\), \(\beta_i \in C(J, [0, \infty)), i = 1, 2, 3\), \(\beta_1(0) \geq 0, \beta_1(T) \geq 0, \beta_1(t) > 0\) for \(t \in (0, T)\),

\[
\begin{align*}
  (a) \quad & (\beta_1(t) + \beta_2(t))0.709 - \beta_3(t) + h(t) \leq 0, \quad t \in J, \\
  (b) \quad & (\beta_1(t) + \beta_2(t))0.708 + \beta_3(t) + h(t) \geq 0, \quad t \in J,
\end{align*}
\]

\[
\max \left\{ \int_0^T \left( \int_s^T \beta(t) \, dt \right) \, ds, \int_0^T \left( \int_0^s \beta(t) \, dt \right) \, ds \right\} \leq 1
\]

(23)

for \(\beta(t) = \beta_1(t) + \beta_2(t) + \beta_3(t), t \in J\).
Take $y_0(t) = -1, z_0(t) = 1$. Then 

$\begin{align*}
  y_0(0) &= -1 < 0, \\
  y_0(T) + z_0\left(\frac{1}{2}T\right) &= 0, \\
  z_0(T) + y_0\left(\frac{1}{2}T\right) &= 0,
\end{align*}$

and

$\begin{align*}
  y_0(0) &= -1 < 0, \\
  y_0(T) + z_0\left(\frac{1}{2}T\right) &= 0, \\
  z_0(0) &= 1 > 0, \\
  z_0(T) + y_0\left(\frac{1}{2}T\right) &= 0, \\
  \beta_1(t) + \beta_2(t) \sin 2\left(\frac{1}{2}T\right) + \beta_3(t) + h(t) &\leq 0 = y_0''(t), \\
  \beta_1(t) + \beta_2(t) \sin 2\left(\frac{1}{2}T\right) + \beta_3(t) + h(t) &\geq 0 = z_0''(t),
\end{align*}$

by conditions (a) and (b). This shows that $y_0, z_0$ are w.c. lower and upper solutions of problem (22). Note that Assumption $H_3$ holds with $M(t) = \beta_1(t)$, $N(t) = \beta_2(t) + \beta_3(t)$. This and (23) guarantee that $(y_0, z_0)$ is w.c. minimal and maximal quasi-solution of problem (22), by Theorem 5.

For example, if we take $\beta_i(t) = \frac{1}{3}$, $t \in J$, $i = 1, 2, 3$, and $h(t) = -0.3$, then conditions (a), (b) and (23) hold if $T \leq \sqrt{2}$. If we take $\beta_i(t) = e^{-t}$, $t \in J = [0, 0.9]$, $i = 1, 2, 3$, and $-2.416e^{-t} \leq h(t) \leq -0.418e^{-t}$, $t \in J$, then conditions (a), (b) and (23) are satisfied.

6. Generalizations

In this section we consider a boundary-value problem of the form

$\begin{align*}
  \{x'(t) = f(t, x(t), x(\alpha_1(t)), \ldots, x(\alpha_m(t))) \equiv Fx(t), \quad t \in J = [0, T], \\
  x(0) = 0, \quad x(T) = rx(\gamma), \quad 0 < \gamma < T, \quad r \in \mathbb{R}.
\end{align*}$

We formulate only corresponding results using the notions of lower and upper solutions of (24) or w.c. lower and upper solutions of (22). These concepts are the same as before with the operator $F$ defined as in (24).

**Theorem 6.** Let $r \geq 0$. Suppose that Assumptions $H'_1$ and $H'_2$ hold, where

$\begin{align*}
  &H'_1: \ f \in C(J \times \mathbb{R}^{m+1}, \mathbb{R}), \ \alpha_i \in C(J, J) \ \text{for} \ i = 1, 2, \ldots, m; \\
  &H'_2: \ \text{there exist functions} \ M, N_i \in C(J, [0, \infty)), \ i = 1, 2, \ldots, m, \ M(t) > 0, \ t \in (0, T), \\
  &\quad M(0) \geq 0, \ M(T) \geq 0 \ \text{and such that} \\
  &\quad f(t, u_1, \tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_m) - f(t, u_1, v_1, v_2, \ldots, v_m) \\
  &\quad \geq -M(t)[u_1 - \tilde{u}_1] - \sum_{i=1}^{m} N_i(t)[v_i - \tilde{v}_i]
\end{align*}$

for $v(t) \geq u_1 \geq \tilde{u}_1 \geq u(t), \ v(\alpha_i(t)) \geq v_i \geq \tilde{v}_i \geq u(\alpha_i(t))$ on $J$ and

$\max(\rho_1, \rho_2) \leq 1$

for

$\begin{align*}
  \rho_1 &= \int_{0}^{T} \left[ \int_{s}^{T} \left( M(t) + \sum_{i=1}^{m} N_i(t) \right) dt \right] ds, \\
  \rho_2 &= \int_{0}^{T} \left[ \int_{0}^{s} \left( M(t) + \sum_{i=1}^{m} N_i(t) \right) dt \right] ds.
\end{align*}$
Let \( y_0, z_0 \in C^2(J, \mathbb{R}) \) be lower and upper solutions of problem (24), respectively, and \( y_0(t) \leq z_0(t) \) on \( J \).

Let
\[
\begin{align*}
\begin{cases}
y''_n(t) = \bar{g}(t, y_{n-1}, y_n), & t \in J, \\
y_n(0) = 0, & y_n(T) = r y_{n-1}(\gamma), & 0 < \gamma < T,
\end{cases}
\begin{cases}
z''_n(t) = \bar{g}(t, z_{n-1}, z_n), & t \in J, \\
z_n(0) = 0, & z_n(T) = r z_{n-1}(\gamma), & 0 < \gamma < T
\end{cases}
\end{align*}
\]
for \( n = 1, 2, \ldots \).

Function \( \bar{g} \) is defined by
\[
\bar{g}(t, u, v) = F u(t) + M(t)[v(t) - u(t)] + \sum_{i=1}^{m} N_i(t)\big[v(\alpha_i(t)) - u(\alpha_i(t))\big].
\]

Then

(i) \( y_0(t) \leq \cdots \leq y_{n-1}(t) \leq y_n(t) \leq z_n(t) \leq z_{n-1}(t) \leq \cdots \leq z_0(t), \ t \in J, \ n = 1, 2, \ldots; \)
(ii) the sequences \( \{y_n, z_n\} \) converge uniformly and monotonically on \( J \) to \( (\bar{y}, \bar{z}) \) if \( n \to \infty \), and \( (\bar{y}, \bar{z}) \) are extremal solutions of problem (24) in the segment \([y_0, z_0]\).

**Theorem 7.** Let \( r < 0 \). Let Assumptions \( H'_1 \) and \( H'_2 \) hold. Let \( y_0, z_0 \in C^2(J, \mathbb{R}) \) be w.c. lower and upper solutions of problem (24) and \( y_0(t) \leq z_0(t) \) on \( J \). Let

\[
\begin{align*}
\begin{cases}
y''_n(t) = \bar{g}(t, y_{n-1}, y_n), & t \in J, \\
y_n(0) = 0, & y_n(T) = r y_{n-1}(\gamma), & 0 < \gamma < T,
\end{cases}
\begin{cases}
z''_n(t) = \bar{g}(t, z_{n-1}, z_n), & t \in J, \\
z_n(0) = 0, & z_n(T) = r z_{n-1}(\gamma), & 0 < \gamma < T
\end{cases}
\end{align*}
\]
for \( n = 1, 2, \ldots \).

Then

(i) \( y_0(t) \leq \cdots \leq y_{n-1}(t) \leq y_n(t) \leq z_n(t) \leq z_{n-1}(t) \leq \cdots \leq z_0(t), \ t \in J, \ n = 1, 2, \ldots; \)
(ii) the sequences \( \{y_n, z_n\} \) converge uniformly and monotonically on \( J \) to \( (\bar{y}, \bar{z}) \) if \( n \to \infty \), and \( (\bar{y}, \bar{z}) \) is w.c. extremal quasi-solution of problem (22).

**Example 4.** Consider the problem
\[
\begin{align*}
\begin{cases}
x''(t) = F x(t), & t \in J = [0, T], \\
x(0) = 0, & x(T) = r x\left(\frac{1}{3}T\right)
\end{cases}
\end{align*}
\]
for \( 0 \leq r \leq 1 \), and
\[
F x(t) = -a_1 \cos x(t) - a_2(t) x(t) + a_1 \cos x\left(\frac{1}{2}t\right) - b_1 \sin x\left(\frac{1}{4}t\right)
\]

\[
+ b_2(t) x\left(\frac{1}{3}t\right) - 0.1.
\]
Assume that $a_1 > 0$, $b_1 \geq 0$, $a_2$, $b_2 \in C(J, [0, \infty))$ and
\[
-0.91b_1 + 2(-a_2(t) + b_2(t)) - 0.1 \geq 0, \quad t \in J, \tag{26}
\]
\[
(a_1 + b_1)T^2 + 2\max \left\{ \int_0^T \left( \int_0^T b_2(t) \, dt \right) \, ds, \int_0^T \left( \int_0^T b_2(t) \, dt \right) \, ds \right\} \leq 2. \tag{27}
\]

Take $y_0(t) = 0$, $z_0(t) = 2$. Then
\[
y_0(0) = 0, \quad y_0(T) = 0 = r y_0 \left( \frac{1}{5} T \right),
\]
\[
z_0(0) = 2 > 0, \quad z_0(T) - r z_0 \left( \frac{1}{5} T \right) = 2(1 - r) \geq 0.
\]

Moreover,
\[
F y_0(t) = -a_1 + a_1 - 0.1 < 0 = y_0''(t),
\]
\[
F z_0(t) = -a_1 \cos 2b_2(t) + a_1 \cos 2 - b_1 \sin 2b_2(t) - 0.1
\]
\[
> -0.91b_1 + 2(-a_2(t) + b_2(t)) - 0.1 \geq 0 = z_0''(t),
\]
by condition (26). This shows that $y_0, z_0$ are lower and upper solutions of problem (25), respectively. Moreover, it is easy to see that Assumption $H_2^j$ hold with $M(t) = a_1$, $N_1(t) = 0$, $N_2(t) = b_1$, $N_3(t) = b_2(t)$. In view of Theorem 6, problem (25) has extremal solutions in the segment $[y_0, z_0]$.

For example, if we take $a_1 = a_2(t) = b_1 = 0.1$, $b_2(t) = 0.2$, $t \in J$, then conditions (26) and (27) are satisfied if $T \leq \sqrt{5}$ (similarly if $a_1 = b_1 = 1$, $b_2(t) = 2e^{-t}$, then $T \leq 0.70$).

**Example 5.** Consider the problem
\[
\begin{cases}
x''(t) = M \sqrt{r}x(t) + \frac{1}{2} \sqrt{r}x(\sqrt{r}) + N x \left( \frac{1}{2} t \right) - \frac{1}{2} \sqrt{r} \equiv F x(t), \quad t \in J, \\
x(0) = 0, \quad x(1) = r x \left( \frac{1}{3} \right)
\end{cases} \tag{28}
\]
for $J = [0, 1]$, $0 \leq r \leq 1$. Assume that $M > 0$, $N \geq 0$ and $8M + 15N \leq 26$.

It is easy to verify that $y_0(t) = 0$, $z_0(t) = 1$ are lower and upper solutions of (28), respectively. In view of Theorem 6, problem (28) has extremal solutions in the segment $[y_0, z_0]$.

**Example 6.** Consider the problem
\[
\begin{cases}
x''(t) = M x(t) + x \left( \frac{1}{4} t \right) + N x(\alpha(t)) - t \equiv F x(t), \quad t \in J = [0, 1], \\
x(0) = 0, \quad x(1) = -x \left( \frac{1}{4} \right)
\end{cases} \tag{29}
\]
for a continuous function $\alpha$ defined by
\[
\alpha(t) = \begin{cases}
2t, & 0 \leq t \leq \frac{1}{4}, \\
\sqrt{t}, & \frac{1}{4} \leq t \leq 1.
\end{cases}
\]
Assume that \( M > 0, N \geq 0 \) and \( M + N \leq 1 \). Take \( y_0(t) = -1, z_0(t) = 4t \). Then

\[
\begin{align*}
y_0(0) &= -1 < 0, \quad y_0(1) + z_0\left(\frac{1}{4}\right) = 0, \\
z_0(0) &= 0, \quad z_0(1) + y_0\left(\frac{1}{4}\right) = 3 > 0,
\end{align*}
\]

\[
\begin{align*}
Fy_0(t) &= -M - 1 - N - t < 0 = y_0''(t), \\
Fz_0(t) &= 4Mt + t + 4N\alpha(t) - t \geq 0 = z_0''(t).
\end{align*}
\]

This shows that \( y_0, z_0 \) are weakly coupled lower and upper solutions of problem (29). In view of Theorem 7, problem (29) has extremal quasi-solutions.

References