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Gevrey hypoellipticity for linear and non-linear Fokker–Planck equations

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Abstract

This paper studies the Gevrey regularity of weak solutions of a class of linear and semi-linear Fokker–Planck equations.

Keywords: Gevrey regularity; Fokker–Planck equation; Subelliptic estimate

1. Introduction

Much attention has been paid to the study of the spatially homogeneous Boltzmann equation without the angular cut-offs in recent years (see [2,3,8,20] and references therein). These studies demonstrate that the singularity of the collision cross-section improves the regularity on weak solutions for the Cauchy problem. For instance, one can obtain, from these studies, the C^∞ regularity of weak solutions for the spatially homogeneous Boltzmann operator when there are no angular cut-offs. In the local setting, the Gevrey regularity has been first proved in [19] for the initial data that has the same Gevrey regularity. A more general result on the Gevrey regularity is obtained in [15] for the spatially homogeneous linear Boltzmann equation with any initial Cauchy

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data. Hence, one sees a similar smoothness effect for the homogeneous Boltzmann equations as in the case of the heat equation.

The consideration for the inhomogeneous equation seems to be a relatively open field. There is no general result in this study yet. A recent work in [1] investigated a kinetic equation with the diffusion term as a non-linear function of the velocity variable. In [1], making use of the uncertainty principle and microlocal analysis, a C^∞ regularity result was obtained when there is no angular cut-off in the linear spatially inhomogeneous Boltzmann equation.

In this paper, we study the Gevrey regularity of the weak solutions for the following Fokker–Planck operator in $\mathbb{R}^{2n+1}$

$$\mathcal{L} = \partial_t + v \cdot \partial_x - a(t,x,v)\Delta_v, \tag{1.1}$$

where Δ_v is the Laplace operator in the velocity variables v and $a(t,x,v)$ is a strictly positive function in $\mathbb{R}^{2n+1}$.

The motivation of studying such an equation is related to the study of inhomogeneous Boltzmann equation without angular cut-offs, Landau equation (see [14]) and a non-linear Vlasov–Fokker–Planck equation (see [11,12]).

To state our main results, we first recall the definition of Gevrey class functions. Let U be an open subset of $\mathbb{R}^N$ and f be a real function defined in U. We say $f \in G^s(U)$ $(s \geqslant 1)$ if $f \in C^\infty(U)$ and for any compact subset K of U, there exists a constant $C = C_K$, depending only on K, such that for all multi-indices $\alpha \in \mathbb{N}^N$ and for all $x \in K$

$$\left|\partial^\alpha f(x)\right| \leqslant C_K^{|\alpha|+1}\big(|\alpha|!\big)^s. \tag{1.2}$$

Denote by $\bar{U}$ the closure of U in $\mathbb{R}^N$. We say $f \in G^s(\bar{U})$ if $f \in G^s(W)$ for some open neighborhood W of $\bar{U}$. The estimate (1.2) for $x \in K$ is equivalent to the following L^2-estimate (see, for instance, Chen and Rodino [5,6] or Rodino [16]):

$$\left\|\partial^\alpha f\right\|_{L^2(K)} \leqslant C_K^{|\alpha|+1}\big(|\alpha|\big)^{s|\alpha|}.$$

In what follows, we shall use the definition based on the above L^2-estimate for the Gevrey functions.

We say that an operator P is G^s-hypoelliptic in U if for any $u \in \mathcal{D}'$ and $Pu \in G^s(U)$ it then holds that $u \in G^s(U)$. Likewise, we say an operator P is C^∞-hypoelliptic in U if for any $u \in \mathcal{D}'$ and $Pu \in C^\infty(U)$ it then holds that $u \in C^\infty(U)$.

When the operator $\mathcal{L}$ satisfies the well-known Hörmander condition, then a famous result of Hörmander [13] says that $\mathcal{L}$ is C^∞-hypoelliptic. In the aspect of the Gevrey class, Derridj and Zuily [7] studied the G^s-hypoellipticity for the second-order degenerate operators of Hörmander type, and proved that $\mathcal{L}$ is G^s-hypoelliptic when $s > 6$.

In this paper, we first improve the result in [7] for the Fokker–Planck operator (1.1). In fact, similar to the result of [17], we have obtained the following optimal estimate for Gevrey index $s \geqslant 3$:

Theorem 1.1. *For any $s \geqslant 3$, if the positive coefficient $a(t,x,v)$ is in $G^s(\mathbb{R}^{2n+1})$, then the operator $\mathcal{L}$ given in* (1.1) *is G^s-hypoelliptic in $\mathbb{R}^{2n+1}$.*

Remark 1.2. Our proof of Theorem 1.1 actually shows that the result in Theorem 1.1 holds even for the following more general operators:

$$\tilde{\mathcal{L}} = \partial_t + A(v) \cdot \partial_x - \sum_{j,k=1}^{n} a_{jk}(t,x,v)\partial^2_{v_j v_k},$$

defined over a domain U in $\mathbb{R}^{2n+1}$. Here, A is a non-singular $n \times n$ constant matrix, $(a_{jk}(t,x,v))$ is a positive definite matrix over U with all entries being in the $G^s(U)$-class.

Remark 1.3. Our result in Theorem 1.1 is of the local nature. Namely, if there exists a weak solution in $\mathcal{D}'$, then this solution is in the Gevrey class in the interior of the domain. Hence, interior regularity of a weak solution does not depend much on the regularity of the initial Cauchy date.

Our second result is concerned with the Gevrey regularity of a non-linear version of (1.1). We consider the following semi-linear equation:

$$\mathcal{L}u = \partial_t u + v \cdot \nabla_x u - a(t,x,v)\Delta_v u = F(t,x,v,u,\nabla_v u), \tag{1.3}$$

where $F(t,x,v,w,p)$ is a non-linear function of real variables (t,x,v,w,p). We prove the following:

Theorem 1.4. *Let u be a weak solution of Eq.* (1.3). *Assume that $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^{2n+1})$ and $\nabla_v u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^{2n+1})$. Then*

$$u \in G^s\big(\mathbb{R}^{2n+1}\big)$$

for any $s \geqslant 3$, if the positive coefficient $a(t,x,v) \in G^s(\mathbb{R}^{2n+1})$ and the non-linear function $F(t,x,v,w,p) \in G^s(\mathbb{R}^{2n+2+n})$.

Remark 1.5. If the non-linear term $F(t,x,v,w,p)$ is independent of p or F is of the form $\nabla_v G(t,x,v,u)$, then it is enough to suppose in Theorem 1.4 that the weak solution $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^{2n+1})$.

The paper is organized as follows: In Section 2, we obtain a sharp subelliptic estimate for the Fokker–Planck operator $\mathcal{L}$ via a direct computation. We then prove the Gevrey hypoellipticity of $\mathcal{L}$. In Section 3, we prove the Gevrey regularity for the weak solution of the semi-linear Fokker–Planck equation (1.3).

2. Subelliptic estimates

As usual, we write $\|\cdot\|_\kappa, \kappa \in \mathbb{R}$, for the classical Sobolev norm in $H^\kappa(\mathbb{R}^{2n+1})$, and (h,k) for the inner product of h, $k \in L^2(\mathbb{R}^{2n+1})$. For f, $g \in C_0^\infty(\mathbb{R}^{2n+1})$, by the Hölder and Young inequalities, we have that for any $\varepsilon > 0$,

$$\big|(f,g)\big| \leqslant \|h\|_\kappa \|g\|_{-\kappa} \leqslant \frac{\varepsilon \|h\|_\kappa^2}{2} + \frac{\|g\|_{-\kappa}^2}{2\varepsilon}. \tag{2.1}$$

We also recall the following interpolation inequality in the Sobolev space: For any $\varepsilon > 0$ and $r_1 < r_2 < r_3$, it holds that

$$\|h\|_{r_2} \leqslant \varepsilon \|h\|_{r_3} + \varepsilon^{-(r_2-r_1)/(r_3-r_2)} \|h\|_{r_1}. \tag{2.2}$$

Let Ω be an open subset of $\mathbb{R}^{2n+1}$. We denote by $S^m = S^m(\Omega)$, $m \in \mathbb{R}$, the symbol space of the classical pseudo-differential operators and $P = P(t, x, v, D_t, D_x, D_v) \in \mathrm{Op}(S^m)$ a pseudo-differential operator of symbol $p(t, x, v; \tau, \xi, \eta) \in S^m$. If $P \in \mathrm{Op}(S^m)$, then P is a continuous operator from $H_c^\kappa(\Omega)$ to $H_{\mathrm{loc}}^{\kappa-m}(\Omega)$. Here $H_c^\kappa(\Omega)$ is the subspace of $H^\kappa(R^{2n+1})$ consisting of the distributions having their compact support in Ω, and $H_{\mathrm{loc}}^{\kappa-m}(\Omega)$ consists of the distributions h such that $\phi h \in H^{\kappa-m}(\mathbb{R}^{2n+1})$ for any $\phi \in C_0^\infty(\Omega)$. For more properties concerning the pseudo-differential operators, we refer the reader to the book [18]. Observe that if $P_1 \in \mathrm{Op}(S^{m_1})$, $P_2 \in \mathrm{Op}(S^{m_2})$, then $[P_1, P_2] \in \mathrm{Op}(S^{m_1+m_2-1})$.

We next prove a sharp subelliptic estimate for the operator $\mathcal{L}$. Our proof is based on the work of Bouchut [4] and Morimoto and Xu [14].

Proposition 2.1. *Let K be a compact subset of $\mathbb{R}^{2n+1}$. Then for any $r \geqslant 0$, there exists a constant $C_{K,r}$, depending only on K and r, such that for any $f \in C_0^\infty(K)$,*

$$\|f\|_r \leqslant C_{K,r}\left\{\|\mathcal{L}f\|_{r-2/3} + \|f\|_0\right\}. \tag{2.3}$$

For brevity, we will write, in this section, C_K for a constant that may be different in a different context. We proceed with the following three lemmas, which establishes the regularity result in the variables v, x and t, respectively.

Lemma 2.2. *For any $r \geqslant 0$, there exists a constant $C_{K,r}$ such that for any $f \in C_0^\infty(K)$,*

$$\|\nabla_v f\|_r \leqslant C_{K,r}\left(\|\mathcal{L}f\|_r + \|f\|_r\right).$$

Moreover, one has

$$\|\nabla_v f\|_r \leqslant C_{K,r}\left(\|\mathcal{L}f\|_{r-\frac{1}{3}} + \|f\|_{r+\frac{1}{3}}\right).$$

Lemma 2.2 indicates the regularity gain of order 1 in the variable v. It can be obtained directly by the positivity of the coefficient a and the compact supported property of f. For the space variable x, we have the following subelliptic estimate:

Lemma 2.3. *There exists a constant C_K such that for any $f \in C_0^\infty(K)$,*

$$\left\|D_x^{2/3} f\right\|_0 \leqslant C_K\left(\|\mathcal{L}f\|_0 + \|f\|_0\right),$$

where $D_x^{2/3} = (-\Delta_x)^{1/3}$.

This result is due to [4]. It follows from the estimates

$$\left\|D_x^{2/3} f\right\|_0 \leqslant C_K \|\Delta_v f\|_0^{1/3} \|\partial_t f + v \cdot \partial_x f\|_0^{2/3}$$

and

$$\|\Delta_v f\|_0 \leqslant C_K\big(\|\mathcal{L}f\|_0 + \|f\|_0\big).$$

For the time variable t, we have the regularity result of order $2/3$, namely, we have the following:

Lemma 2.4. *There exists a constant C_K such that for any $f \in C_0^\infty(K)$,*

$$\|\partial_t f\|_{-1/3} \leqslant C_K\big(\|\mathcal{L}f\|_0 + \|f\|_0\big).$$

In fact, we have

$$\|\partial_t f\|_{-1/3} = \big\|\Lambda^{-1/3}\partial_t f\big\|_0 \leqslant \big\|\Lambda^{-1/3}(\partial_t + v\cdot\partial_x)f\big\|_0 + \big\|\Lambda^{-1/3}v\cdot\partial_x f\big\|_0,$$

where $\Lambda = (1+|D_t|^2+|D_x|^2+|D_v|^2)^{1/2}$. From Lemma 2.3, we have

$$\big\|\Lambda^{-1/3}v\cdot\partial_x f\big\|_0 \leqslant C_K\big\|D_x^{2/3}f\big\|_0 \leqslant C_K\big(\|\mathcal{L}f\|_0 + \|f\|_0\big).$$

The estimate for the term $\|\Lambda^{-1/3}(\partial_t + v\cdot\partial_x)f\|_0$ can be obtained by a direct computation as in [14].

Proof of Proposition 2.1. By Lemmas 2.2–2.4, we have

$$\|f\|_{2/3} \leqslant C_K\big\{\|\mathcal{L}f\|_0 + \|f\|_0\big\}. \tag{2.4}$$

Moreover, choose a function $\psi \in C_0^\infty(\mathbb{R}^{2n+1})$ with $\psi|_K \equiv 1$ and $\operatorname{supp}\psi$ being contained in a neighborhood of K. Then, for any $f \in C_0^\infty(K)$ and $r \geqslant 0$, we have

$$\|f\|_r = \|\psi f\|_r \leqslant C_K\big\{\big\|\psi\Lambda^{r-2/3}f\big\|_{2/3} + \big\|\big[\Lambda^{r-2/3},\psi\big]f\big\|_{2/3}\big\}.$$

By virtue of (2.4) and the interpolation inequality (2.2), we have

$$\begin{aligned}\|f\|_r &\leqslant C_K\big\{\big\|\mathcal{L}\psi\Lambda^{r-2/3}f\big\|_0 + \|f\|_{r-2/3}\big\}\\ &\leqslant C_{\varepsilon,K}\big\{\big\|\mathcal{L}\psi\Lambda^{r-2/3}f\big\|_0 + \|f\|_0\big\} + \varepsilon\|f\|_r.\end{aligned}$$

Letting ε sufficiently small, we get

$$\|f\|_r \leqslant C_K\big\{\|\mathcal{L}f\|_{r-2/3} + \|f\|_0 + \big\|\big[\mathcal{L},\psi\Lambda^{r-2/3}\big]f\big\|_0\big\}.$$

Next, a direct calculation yields

$$\begin{aligned}\big[\mathcal{L},\psi\Lambda^{r-2/3}\big] = \big[\partial_t + v\cdot\partial_x,\psi\Lambda^{r-2/3}\big] - \sum_{j=1}^n\big\{\big[a,\psi\Lambda^{r-2/3}\big]\partial_{v_j}^2\\ + a\big[\partial_{v_j},\big[\partial_{v_j},\psi\Lambda^{r-2/3}\big]\big] + 2a\big[\partial_{v_j},\psi\Lambda^{r-2/3}\big]\partial_{v_j}\big\}.\end{aligned}$$

From Lemma 2.2, it thus follows that

$$\left\|\left[\mathcal{L}, \psi\Lambda^{r-2/3}\right]f\right\|_0 \leqslant C_K\left\{\|f\|_{r-2/3} + \sum_{j=1}^{n}\|\partial_{v_j}f\|_{r-2/3}\right\}$$
$$\leqslant C_K\left\{\|\mathcal{L}f\|_{r-2/3} + \|f\|_{r-2/3}\right\}.$$

From the estimates above, we deduce that

$$\|f\|_r \leqslant C_K\left\{\|\mathcal{L}f\|_{r-2/3} + \|f\|_0 + \|f\|_{r-2/3}\right\}.$$

Applying the interpolation inequality (2.2) again and making ε small enough, we see the proof of Proposition 2.1. □

We next consider the commuting property of $\mathcal{L}$ with differential operators and cut-off functions.

Proposition 2.5. *Let K be a compact subset of $\mathbb{R}^{2n+1}$. Then for any $r \geqslant 0$, there are constants $C_{K,r}$, $C_{K,r,\varphi}$ such that for any $f \in C_0^\infty(K)$, we have*

$$\left\|[\mathcal{L}, D]f\right\|_r \leqslant C_{K,r}\left\{\|\mathcal{L}f\|_{r+1-2/3} + \|f\|_0\right\}$$

and

$$\left\|[\mathcal{L}, \varphi]f\right\|_r \leqslant C_{K,r,\varphi}\left\{\|\mathcal{L}f\|_{r-1/3} + \|f\|_0\right\}.$$

Here $\varphi \in C_0^\infty(\mathbb{R}^{2n+1})$ and D is one of the differential operators: ∂_t, ∂_x or ∂_v.

Proof. By using the positivity of the coefficient a, we have

$$\|\Delta_v f\|_r \leqslant C_K\left\{\|\mathcal{L}f\|_r + \|f\|_{r+1}\right\}.$$

Notice that $[\mathcal{L}, D] = [\partial_t + v\cdot\partial_x, D] - [a, D]\Delta_v$. We have

$$\left\|[\mathcal{L}, D]f\right\|_r \leqslant C_K\left\{\|f\|_{r+1} + \|\Delta_v f\|_r\right\}.$$

The first estimate of Proposition 2.5 is thus deduced by the two inequalities above and the subelliptic estimate (2.3).

To treat $\|[\mathcal{L}, \varphi]f\|_r$, we use the second inequality in Lemma 2.2 and the subelliptic estimate (2.3), which gives

$$\|\nabla_v f\|_r \leqslant C_K\left(\|\mathcal{L}f\|_{r-1/3} + \|f\|_{r+1/3}\right) \leqslant C_K\left(\|\mathcal{L}f\|_{r-1/3} + \|f\|_0\right).$$

Now a simple verification shows that

$$\begin{aligned}\big\|[\mathcal{L},\varphi]f\big\|_r &\leqslant C_K\Bigg\{\|f\|_r+\sum_{j=1}^{n}\|\partial_{v_j}f\|_r\Bigg\}\\ &\leqslant C_{K,r}\big\{\|\mathcal{L}f\|_{r-1/3}+\|f\|_0\big\}.\end{aligned}$$

This completes the proof of Proposition 2.5. □

We are now at a position to prove the Gevrey hypoellipticity of $\mathcal{L}$. We need the following result due to M. Durand [9]:

Proposition 2.6. *Let P be a linear partial differential operator of second order with smooth coefficients in $\mathbb{R}_y^N$ and let ϱ,ς be two fixed positive numbers. If for $r\geqslant 0$, compact subset $K\subseteq\mathbb{R}^N$ and $\varphi\in C^\infty(\mathbb{R}^N)$, there exist constants $C_{K,r}$ and $C_{K,r}(\varphi)$ such that for all $f\in C_0^\infty(K)$, the following conditions are fulfilled*:

(H_1) $$\|f\|_r\leqslant C_{K,r}\big(\|Pf\|_{r-\varrho}+\|f\|_0\big),$$

(H_2) $$\big\|[P,D_j]f\big\|_r\leqslant C_{K,r}\big(\|Pf\|_{r+1-\varsigma}+\|f\|_0\big),$$

(H_3) $$\big\|[P,\varphi]f\big\|_r\leqslant C_{K,r}(\varphi)\big(\|Pf\|_{r-\varsigma}+\|f\|_0\big),$$

where

$$D_j=\frac{1}{i}\frac{\partial}{\partial y_j},\qquad j=1,2,\ldots,N,$$

then for $s\geqslant\max(1/\varsigma,2/\varrho)$, P is $G^s(\mathbb{R}^N)$-hypoelliptic, provided that the coefficients of P are in the class of $G^s(\mathbb{R}^N)$.

Proposition 2.1 shows that the operator $\mathcal{L}$ satisfies condition (H_1) with $\varrho=2/3$. Proposition 2.5 assures the conditions (H_2) and (H_3) with $\varsigma=1/3$. Thus, $\mathcal{L}$ is $G^s(\mathbb{R}^{2n+1})$-hypoelliptic for $s\geqslant 3$. This completes the proof of Theorem 1.1.

3. Gevrey regularity of non-linear equations

Let $u\in L^\infty_{\mathrm{loc}}(\mathbb{R}^{2n+1})$ be a weak solution of (1.3). We will prove $u\in C^\infty(\mathbb{R}^{2n+1})$. To this aim, we need the following non-linear composition result (see for example [21]):

Lemma 3.1. *Let $F(t,x,v,w,p)\in C^\infty(\mathbb{R}^{2n+2+n})$ and $r\geqslant 0$. If $u,\nabla_v u\in L^\infty_{\mathrm{loc}}(\mathbb{R}^{2n+1})\cap H^r_{\mathrm{loc}}(\mathbb{R}^{2n+1})$, then $F(\cdot,u(\cdot),\nabla_v u(\cdot))\in H^r_{\mathrm{loc}}(\mathbb{R}^{2n+1})$ with*

$$\big\|\phi_1 F\big(\cdot,u(\cdot),\nabla_v u(\cdot)\big)\big\|_r\leqslant\bar{C}\big\{\|\phi_2 u\|_r+\|\phi_2\nabla_v u\|_r\big\},\tag{3.1}$$

where $\phi_1,\phi_2\in C_0^\infty(\mathbb{R}^{2n+1})$, $\phi_2=1$ on the support of ϕ_1, and $\bar{C}$ is a constant depending only on r,ϕ_1,ϕ_2.

Remark 3.2. If the non-linear term F is independent of p or in the form of

$$\nabla_v\big(F(t,x,v,u)\big)$$

and if $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^{2n+1}) \cap H^r_{\mathrm{loc}}(\mathbb{R}^{2n+1})$, then it holds that $F(\cdot, u(\cdot), \nabla_v u(\cdot)) \in H^r_{\mathrm{loc}}(\mathbb{R}^{2n+1})$.

Lemma 3.3. *Let $u, \nabla_v u \in H^r_{\mathrm{loc}}(\mathbb{R}^{2n+1})$, $r \geqslant 0$. Then we have*

$$\|\varphi_1 \nabla_v u\|_r \leqslant C\|\varphi_2 u\|_r, \tag{3.2}$$

where $\varphi_1, \varphi_2 \in C_0^\infty(\mathbb{R}^{2n+1})$, $\varphi_2 = 1$ on the support of φ_1, and C is a constant depending only on r, φ_1, φ_2.

Proof. Given φ_1 and φ_2 as required above, we choose three functions $\phi_1, \phi_2, \phi_3 \in C_0^\infty(\mathbb{R}^{2n+1})$, satisfying $\phi_1|_{\operatorname{supp}\varphi_1} \equiv 1$, $\varphi_2|_{\operatorname{supp}\phi_3} \equiv 1$ and $\phi_{j+1}|_{\operatorname{supp}\phi_j} \equiv 1$ for $j = 1, 2$. Thus we have

$$\|\varphi_1 \nabla_v u\|_r \leqslant \big\|[\nabla_v, \varphi_1]u\big\|_r + \|\nabla_v \varphi_1 u\|_r.$$

There are two terms on the right-hand side of the estimate above, the first term is bounded by $C\|\varphi_2 u\|_r$, and we can use the second inequality in Lemma 2.2 and the subelliptic estimate (2.3) to estimate the second term, i.e.

$$\begin{aligned}\|\nabla_v \varphi_1 u\|_r &\leqslant C\big(\|\mathcal{L}\varphi_1 u\|_{r-1/3} + \|\nabla_v \varphi_1 u\|_{r+1/3}\big) \leqslant C\big(\|\mathcal{L}\varphi_1 u\|_{r-1/3} + \|\varphi_1 u\|_r\big)\\ &\leqslant C\big(\|\varphi_1 \mathcal{L}u\|_{r-1/3} + \big\|[\mathcal{L}, \varphi_1]u\big\|_{r-1/3} + \|\varphi_1 u\|_r\big),\end{aligned}$$

where C is used to denote different constants depending only on r, φ_1 and φ_2. Notice that $\mathcal{L}u(\cdot) = F(\cdot, u(\cdot), \nabla_v u(\cdot))$, then the estimate above together with (3.1) implies that

$$\|\nabla_v \varphi_1 u\|_r \leqslant C\big(\|\phi_1 \nabla_v u\|_{r-1/3} + \|\phi_1 u\|_r\big).$$

Hence one has

$$\|\varphi_1 \nabla_v u\|_r \leqslant \big\|[\nabla_v, \varphi_1]u\big\|_r + \|\nabla_v \varphi_1 u\|_r \leqslant C\big(\|\phi_1 \nabla_v u\|_{r-1/3} + \|\varphi_2 u\|_r\big).$$

Similarly, it holds that

$$\|\phi_1 \nabla_v u\|_{r-1/3} \leqslant C\big(\|\phi_2 \nabla_v u\|_{r-2/3} + \|\varphi_2 u\|_r\big)$$

and

$$\|\phi_2 \nabla_v u\|_{r-2/3} \leqslant C\big(\|\phi_3 \nabla_v u\|_{r-1} + \|\varphi_2 u\|_r\big) \leqslant C\|\varphi_2 u\|_r.$$

Combining the estimates above, the estimate (3.2) can be deduced directly, which completes the proof of Lemma 3.3. □

Now we are ready to prove

Proposition 3.4. *Let u be a weak solution of* (1.3) *such that $u, \nabla_v u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^{2n+1})$. Then u is in $C^\infty(\mathbb{R}^{2n+1})$.*

Proof. In fact, from the subelliptic estimate (2.3) and the fact that $\mathcal{L}u(\cdot) = F(\cdot, u(\cdot), \nabla_v u(\cdot))$, it follows that

$$\|\psi_1 u\|_{r+2/3} \leqslant \bar{C}\big\{\big\|\psi_2 F\big(\cdot, u(\cdot), \nabla_v u(\cdot)\big)\big\|_r + \|\psi_2 u\|_0\big\}, \tag{3.3}$$

where $\psi_1, \psi_2 \in C_0^\infty(\mathbb{R}^{2n+1})$ and $\psi_2 = 1$ on the support of ψ_1. Combining (3.1), (3.2) with (3.3), we have $u \in H^\infty_{\mathrm{loc}}(\mathbb{R}^{2n+1})$ by the standard iteration procedure. This completes the proof of Proposition 3.4. □

Now starting from a smooth solution, we prove the Gevrey regularity. It suffices for us to work on the open unit ball

$$\Omega = \big\{(t, x, v) \in \mathbb{R}^{2n+1}\colon\ t^2 + |x|^2 + |v|^2 < 1\big\}.$$

Set

$$\Omega_\rho = \big\{(t, x, v) \in \Omega\colon\ \big(t^2 + |x|^2 + |v|^2\big)^{1/2} < 1 - \rho\big\}, \quad 0 < \rho < 1.$$

Let U be an open subset of $\mathbb{R}^{2n+1}$. Denote by $H^r(U)$ the space consisting of the functions which are defined in U and can be extended to $H^r(\mathbb{R}^{2n+1})$. Define

$$\|u\|_{H^r(U)} = \inf\big\{\|\tilde{u}\|_{H^s(\mathbb{R}^{n+1})}\colon\ \tilde{u} \in H^s\big(\mathbb{R}^{2n+1}\big),\ \tilde{u}|_U = u\big\}.$$

We denote $\|u\|_{r,U} = \|u\|_{H^r(U)}$ and

$$\big\|D^j u\big\|_r = \sum_{|\beta|=j} \big\|D^\beta u\big\|_r.$$

In order to treat the non-linear term F on the right hand of (1.3), we need the following two lemmas. The first one (see [21] for example) concerns weak solutions, and the second is an analogue of Lemma 1 in [10]. In the sequel, $C_j > 1$ will be used to denote constants depending only on n or the function F.

Lemma 3.5. *Let $r > (2n+1)/2$ and $u_1, u_2 \in H^r(\mathbb{R}^{2n+1})$. Then $u_1 u_2 \in H^r(\mathbb{R}^{2n+1})$, moreover*

$$\|u_1 u_2\|_r \leqslant \tilde{C}\|u_1\|_r\|u_2\|_r, \tag{3.4}$$

where $\tilde{C}$ is a constant depending only on n, r.

Lemma 3.6. *Let M_j be a sequence of positive numbers. Assume that for some $B_0 > 0$, the M_j satisfy the monotonicity condition*

$$\frac{j!}{i!(j-i)!} M_i M_{j-i} \leqslant B_0 M_j \quad (i = 1, 2, \ldots, j;\ j = 1, 2, \ldots). \tag{3.5}$$

Suppose $F(t,x,v,u,p)$ satisfies that for $j, m+l \geqslant 2$,

$$\left\| \left(D^j_{t,x,v} D^l_u D^m_p F\right)\left(\cdot, u(\cdot), \nabla_v u(\cdot)\right) \right\|_{r+n+1,\Omega} \leqslant C_1^{j+l+m} M_{j-2} M_{m+l-2}, \tag{3.6}$$

where r is a real number satisfying $r+n+1 > (2n+1)/2$. Then there exist two constants C_2, C_3 such that for any H_0, H_1 satisfying $H_0, H_1 \geqslant 1$ and $H_1 \geqslant C_2 H_0$, if $u(t,x,v)$ satisfies the following conditions:

$$\left\| D^j u \right\|_{r+n+1,\Omega_{\tilde{\rho}}} \leqslant H_0, \quad 0 \leqslant j \leqslant 1, \tag{3.7}$$

$$\left\| D^j u \right\|_{r+n+1,\Omega_{\tilde{\rho}}} \leqslant H_0 H_1^{j-2} M_{j-2}, \quad 2 \leqslant j \leqslant N, \tag{3.8}$$

$$\left\| D_v D^j u \right\|_{r+n+1,\Omega_{\tilde{\rho}}} \leqslant H_0 H_1^{j-2} M_{j-2}, \quad 2 \leqslant j \leqslant N, \tag{3.9}$$

then for all α with $|\alpha| = N$,

$$\left\| \psi_N D^\alpha \left[F\left(\cdot, u(\cdot), \nabla_v u(\cdot)\right)\right] \right\|_{r+n+1} \leqslant C_3 H_0 H_1^{N-2} M_{N-2}, \tag{3.10}$$

where $\psi_N \in C_0^\infty(\Omega_{\tilde{\rho}})$ is an arbitrary function.

Proof. Denote $p = (p_1, p_2, \ldots, p_n) = \nabla_v u$ and $k = (k_1, k_2, \ldots, k_n)$. From Faà di Bruno' formula, $\psi_N D^\alpha[F(\cdot, u(\cdot), \nabla_v u(\cdot))]$ is the linear combination of terms of the form

$$\frac{\psi_N \partial^{|\tilde{\alpha}|+l+|k|} F}{\partial^{\tilde{\alpha}}_{t,x,v} \partial u^l \partial p_1^{k_1} \cdots \partial p_n^{k_n}} \prod_{j=1}^{l} D^{\gamma_j} u \cdot \prod_{i=1}^{n} \prod_{j_i=1}^{k_i} D^{\beta_{j_i}}(\partial_{v_i} u), \tag{3.11}$$

where $|\tilde{\alpha}| + l + |k| \leqslant |\alpha|$ and

$$\sum_{j=1}^{l} \gamma_i + \sum_{i=1}^{n} \sum_{j_i}^{k_i} \beta_{j_i} = \alpha - \tilde{\alpha}.$$

Choose a function $\tilde{\psi} \in C_0^\infty(\Omega_{\tilde{\rho}})$ such that $\tilde{\psi} = 1$ on $\operatorname{supp} \psi_N$. Notice that $n+1+r > (2n+1)/2$. Applying Lemma 3.5, we have

$$\begin{aligned}
&\left\| \frac{\psi_N \partial^{|\tilde{\alpha}|+l+|k|} F}{\partial^{\tilde{\alpha}}_{t,x,v} \partial u^l \partial p_1^{k_1} \cdots \partial p_n^{k_n}} \prod_{j=1}^{l} D^{\gamma_j} u \cdot \prod_{i=1}^{n} \prod_{j_i=1}^{k_i} D^{\beta_{j_i}}(\partial_{v_i} u) \right\|_{r+n+1} \\
&\quad = \left\| \frac{\psi_N \partial^{|\tilde{\alpha}|+l+|k|} F}{\partial^{\tilde{\alpha}}_{t,x,v} \partial u^l \partial p_1^{k_1} \cdots \partial p_n^{k_n}} \prod_{j=1}^{l} \tilde{\psi} D^{\gamma_j} u \cdot \prod_{i=1}^{n} \prod_{j_i=1}^{k_i} \tilde{\psi} \partial_{v_i} D^{\beta_{j_i}} u \right\|_{r+n+1}
\end{aligned}$$

$$\leqslant \tilde{C}\left\|\psi_N\left(\partial^{|\tilde{\alpha}|+l+|k|}F\right)\right\|_{r+n+1}\cdot\prod_{j=1}^{l}\left\|\tilde{\psi}D^{\gamma_j}u\right\|_{r+n+1}\times\prod_{i=1}^{n}\prod_{j_i=1}^{k_i}\left\|\tilde{\psi}\partial_{v_i}D^{\beta_{j_i}}u\right\|_{r+n+1}$$

$$\leqslant C_0\left\|\left(\partial^{|\tilde{\alpha}|+l+|k|}F\right)\right\|_{r+n+1,\Omega}\cdot\prod_{j=1}^{l}\left\|D^{\gamma_j}u\right\|_{r+n+1,\Omega_{\tilde{\rho}}}\times\prod_{i=1}^{n}\prod_{j_i=1}^{k_i}\left\|\partial_{v_i}D^{\beta_{j_i}}u\right\|_{r+n+1,\Omega_{\tilde{\rho}}}. \quad (3.12)$$

With (3.7)–(3.9) and (3.12) at our disposal, our consideration is now similar to that in [10]. Indeed, the only difference is that we need to replace the Hölder norm $|u|_j$ by $\|D^j u\|_{r+n+1,\Omega_{\tilde{\rho}}}$ and $\|D_v D^j u\|_{r+n+1,\Omega_{\tilde{\rho}}}$. Hence, the same argument as the proof of Lemma 1 in [10] yields (3.10). This completes the proof of Lemma 3.6. □

Proposition 3.7. *Let $s \geqslant 3$. Suppose $u \in C^\infty(\bar{\Omega})$ is a solution of* (1.3)*, $a(t,x,v) \in G^s(\mathbb{R}^{2n+1})$, $F(t,x,v,w,p) \in G^s(\mathbb{R}^{2n+2+n})$ and $a \geqslant c_0 > 0$. Then there is a constant A such that for any $r \in [0,1]$ and any $N \in \mathbb{N}$, $N \geqslant 3$,*

$$(E)_{r,N} \qquad \left\|D^\alpha u\right\|_{r+n+1,\Omega_\rho}+\left\|D_v D^\alpha u\right\|_{r-1/3+n+1,\Omega_\rho}\leqslant\frac{A^{|\alpha|-1}}{\rho^{s(|\alpha|-3)}}\left(\left(|\alpha|-3\right)!\right)^s(N/\rho)^{sr},$$
$$\forall|\alpha|=N,\ \forall 0<\rho<1.$$

From $(E)_{r,N}$, we immediately obtain

Proposition 3.8. *Under the same assumption as in Proposition* 3.7*, we have $u \in G^s(\Omega)$.*

Proof. In fact, for any compact sunset K of Ω, we have $K \subset \Omega_{\rho_0}$ for some ρ_0 with $0<\rho_0<1$. For any α with $|\alpha| \geqslant 3$, letting $r=0$ in $(E)_{r,N}$, we have

$$\left\|D^\alpha u\right\|_{L^2(K)}\leqslant\left\|D^\alpha u\right\|_{n+1,\Omega_{\rho_0}}\leqslant\frac{A^{|\alpha|-1}}{{\rho_0}^{s(|\alpha|-3)}}\left(\left(|\alpha|-3\right)!\right)^s\leqslant\left(\frac{A}{{\rho_0}^s}\right)^{|\alpha|+1}\left(|\alpha|!\right)^s.$$

This completes the proof of Proposition 3.8. □

The result of Theorem 1.4 can be directly deduced from Propositions 3.4, 3.7 and 3.8.

Proof of Proposition 3.7. We apply an induction argument on N. Assume that $(E)_{r,N-1}$ holds for any r with $0 \leqslant r \leqslant 1$. We will show that $(E)_{r,N}$ still holds for any $r \in [0,1]$. For an α with $|\alpha|=N$, and for a ρ with $0<\rho<1$, choose a function $\varphi_{\rho,N} \in C_0^\infty(\Omega_{\frac{(N-1)\rho}{N}})$ such that $\varphi_{\rho,N}=1$ in Ω_ρ. It is easy to see that

$$\sup\left|D^\gamma\varphi_{\rho,N}\right|\leqslant C_\gamma(\rho/N)^{-|\gamma|}\leqslant C_\gamma(N/\rho)^{|\gamma|},\quad\forall\gamma.$$

We will verify the estimate in $(E)_{r,N}$ by the following lemmas.

Lemma 3.9. *For $r=0$, we have*

$$\left\|D^\alpha u\right\|_{n+1,\Omega_\rho}+\left\|D_v D^\alpha u\right\|_{-1/3+n+1,\Omega_\rho}\leqslant\frac{C_7A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\left(\left(|\alpha|-3\right)!\right)^s,\quad\forall 0<\rho<1.$$

Proof. Write $|\alpha| = |\beta| + 1$. Then $|\beta| = N - 1$. Denote $\frac{N-1}{N}\rho$ by $\tilde{\rho}$. In the sequel, we will often apply the following inequalities:

$$\frac{1}{\rho^{sk}} \leqslant \frac{1}{\tilde{\rho}^{sk}} = \frac{1}{\rho^{sk}} \times \left(\frac{N}{N-1}\right)^{sk} \leqslant \frac{C_4}{\rho^{sk}}, \quad k = 1, 2, \ldots, N-3.$$

Notice that $\varphi_{\rho,N} = 1$ in Ω_ρ. Hence

$$\begin{aligned} \left\| D^\alpha u \right\|_{n+1,\Omega_\rho} &\leqslant \left\| \varphi_{\rho,N} D^\alpha u \right\|_{n+1} \leqslant \left\| \varphi_{\rho,N} D^\beta u \right\|_{1+n+1} + \left\| (D\varphi_{\rho,N}) D^\beta u \right\|_{n+1} \\ &\leqslant C_5 \left\{ \left\| D^\beta u \right\|_{1+n+1,\Omega_{\tilde{\rho}}} + (N/\rho) \left\| D^\beta u \right\|_{n+1,\Omega_{\tilde{\rho}}} \right\}. \end{aligned}$$

Since $(E)_{r,N-1}$ holds by assumption for any r with $0 \leqslant r \leqslant 1$, we have immediately

$$\begin{aligned} &\left\| D^\beta u \right\|_{1+n+1,\Omega_{\tilde{\rho}}} + (N/\rho) \left\| D^\beta u \right\|_{n+1,\Omega_{\tilde{\rho}}} \\ &\quad \leqslant \frac{A^{|\beta|-1}}{\tilde{\rho}^{s(|\beta|-3)}} \big((|\beta| - 3)! \big)^s (N/\tilde{\rho})^s + (N/\rho) \frac{A^{|\beta|-1}}{\tilde{\rho}^{s(|\beta|-3)}} \big((|\beta| - 3)! \big)^s \\ &\quad \leqslant \frac{2A^{|\alpha|-2}}{\tilde{\rho}^{s(|\alpha|-3)}} \big((|\alpha| - 3)! \big)^s \big(N/(N-3) \big)^s \\ &\quad \leqslant \frac{C_6 A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}} \big((|\alpha| - 3)! \big)^s. \end{aligned}$$

Thus

$$\left\| D^\alpha u \right\|_{n+1,\Omega_\rho} \leqslant \frac{C_5 C_6 A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}} \big((|\alpha| - 3)! \big)^s. \tag{3.13}$$

The same argument as above shows that

$$\left\| D_v D^\alpha u \right\|_{-1/3+n+1,\Omega_\rho} \leqslant \frac{C_5 C_6 A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}} \big((|\alpha| - 3)! \big)^s.$$

This along with (3.13) yields the conclusion. □

Lemma 3.10. *For* $0 \leqslant r \leqslant 1/3$, *we have for all* $0 < \rho < 1$

$$\left\| D^\alpha u \right\|_{r+n+1,\Omega_\rho} + \left\| D_v D^\alpha u \right\|_{r-1/3+n+1,\Omega_\rho} \leqslant \frac{C_{35} A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}} \big((|\alpha| - 3)! \big)^s (N/\rho)^{rs}.$$

Proof. We first verify Lemma 3.9 for $r = 1/3$, namely, we first show that for all $0 < \rho < 1$

$$\left\| D^\alpha u \right\|_{1/3+n+1,\Omega_\rho} + \left\| D_v D^\alpha u \right\|_{n+1,\Omega_\rho} \leqslant \frac{C_{35} A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}} \big((|\alpha| - 3)! \big)^s (N/\rho)^{s/3}.$$

We divide our discussions in the following four steps.

Step 1. We claim that

$$\big\|\big[\mathcal{L},\varphi_{\rho,N}D^{\alpha}\big]u\big\|_{-1/3+n+1}\leqslant\frac{C_{19}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big(\big(|\alpha|-3\big)!\big)^{s}(N/\rho)^{s/3}.\tag{3.14}$$

In fact, write $\mathcal{L}=X_0-a\Delta_v$ with $X_0=\partial_t+v\cdot\partial_x$. Then a direct verification shows that

$$\begin{aligned}\big\|\big[\mathcal{L},\varphi_{\rho,N}D^{\alpha}\big]u\big\|_{-1/3+n+1}&\leqslant\big\|\big[X_0,\varphi_{\rho,N}D^{\alpha}\big]u\big\|_{-1/3+n+1}+\big\|a\big[\Delta_v,\varphi_{\rho,N}D^{\alpha}\big]u\big\|_{-1/3+n+1}\\&\quad+\big\|\varphi_{\rho,N}\big[a,D^{\alpha}\big]\Delta_v u\big\|_{-1/3+n+1}\\&=:(I)+(II)+(III).\end{aligned}$$

Denote $[X_0,D^{\alpha}]$ by D^{α_0}. Then $|\alpha_0|\leqslant|\alpha|$ and

$$\begin{aligned}(I)&\leqslant\big\|[X_0,\varphi_{\rho,N}]D^{\alpha}u\big\|_{n+1}+\big\|\varphi_{\rho,N}D^{\alpha_0}u\big\|_{n+1}\\&\leqslant C_8\big\{(N/\rho)\big\|D^{\alpha}u\big\|_{n+1,\Omega_{\tilde\rho}}+\big\|D^{\alpha_0}u\big\|_{n+1,\Omega_{\tilde\rho}}\big\}.\end{aligned}$$

Notice that $s\geqslant 3$. By Lemma 3.9, we have

$$(I)\leqslant C_8(N/\rho+1)\frac{C_7A^{|\alpha|-2}}{\tilde\rho^{s(|\alpha|-3)}}\big(\big(|\alpha|-3\big)!\big)^{s}\leqslant\frac{C_9A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big(\big(|\alpha|-3\big)!\big)^{s}(N/\rho)^{s/3}.\tag{3.15}$$

Next we will estimate (II). It is easy to see that

$$\begin{aligned}&\big\|[\Delta_v,\varphi_{\rho,N}]D^{\alpha}u\big\|_{-1/3+n+1}\\&\quad\leqslant 2\big\|[D_v,\varphi_{\rho,N}]D_vD^{\alpha}u\big\|_{-1/3+n+1}+\big\|\big[D_v,[D_v,\varphi_{\rho,N}]\big]D^{\alpha}u\big\|_{-1/3+n+1}.\end{aligned}\tag{3.16}$$

We first consider the first term on the right-hand side. By Lemma 3.9 again, we have

$$\begin{aligned}\big\|[D_v,\varphi_{\rho,N}]D_vD^{\alpha}u\big\|_{-1/3+n+1}&\leqslant(N/\rho)\big\|D_vD^{\alpha}u\big\|_{-1/3+n+1,\Omega_{\tilde\rho}}\\&\leqslant(N/\rho)\frac{C_7A^{|\alpha|-2}}{\tilde\rho^{s(|\alpha|-3)}}\big(\big(|\alpha|-3\big)!\big)^{s}\\&\leqslant\frac{C_{10}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big(\big(|\alpha|-3\big)!\big)^{s}(N/\rho)^{s/3}.\end{aligned}\tag{3.17}$$

Next we treat $\|[D_v,[D_v,\varphi_{\rho,N}]]D^{\alpha}u\|_{-1/3+n+1}$. We compute

$$\begin{aligned}&\big\|\big[D_v,[D_v,\varphi_{\rho,N}]\big]D^{\alpha}u\big\|_{-1/3+n+1}\\&\quad\leqslant\big\|\big(D^2\varphi_{\rho,N}\big)D^{\beta}u\big\|_{2/3+n+1}+\big\|\big(D^3\varphi_{\rho,N}\big)D^{\beta}u\big\|_{-1/3+n+1}\\&\quad\leqslant C_{11}\big\{(N/\rho)^2\big\|D^{\beta}u\big\|_{2/3+n+1,\Omega_{\tilde\rho}}+(N/\rho)^3\big\|D^{\beta}u\big\|_{n+1,\Omega_{\tilde\rho}}\big\}\end{aligned}$$

$$
\begin{aligned}
&\leqslant C_{11}\left\{(N/\rho)^2\frac{A^{|\beta|-1}}{\tilde{\rho}^{s(|\beta|-3)}}\big((|\beta|-3)!\big)^s(N/\tilde{\rho})^{2s/3}\right.\\
&\quad\left.+(N/\rho)^3\frac{A^{|\beta|-1}}{\tilde{\rho}^{s(|\beta|-3)}}\big((|\beta|-3)!\big)^s\right\}\\
&\leqslant C_{11}\left\{(N/\rho)^2(N/\tilde{\rho})^{-s/3}\frac{A^{|\alpha|-2}}{\tilde{\rho}^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s\right.\\
&\quad\left.+(N/\rho)^3(N/\tilde{\rho})^{-s}\frac{A^{|\alpha|-2}}{\tilde{\rho}^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s\right\}\\
&\leqslant\frac{C_{12}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{s/3}.
\end{aligned}
$$

This along with (3.16) and (3.17) shows that

$$
(II)\leqslant\frac{C_{13}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{s/3}. \tag{3.18}
$$

It remains to treat (III). By Leibniz' formula,

$$
\begin{aligned}
(III)&\leqslant\sum_{0<|\gamma|\leqslant|\alpha|}\binom{\alpha}{\gamma}\big\|\varphi_{\rho,N}\big(D^\gamma a\big)\Delta_v D^{\alpha-\gamma}u\big\|_{-1/3+n+1}\\
&\leqslant\sum_{0<|\gamma|\leqslant|\alpha|}\binom{\alpha}{\gamma}\big\|D^\gamma a\big\|_{n+1,\Omega}\cdot\big\|\varphi_{\rho,N}\Delta_v D^{\alpha-\gamma}u\big\|_{-1/3+n+1}.
\end{aligned}
$$

Since $a\in G^s(\mathbb{R}^{2n+1})$, then

$$
\big\|D^\gamma a\big\|_{n+1,\Omega}\leqslant C_{14}^{|\gamma|-2}\big((|\gamma|-3)!\big)^s,\quad |\gamma|\geqslant 3,
$$

and

$$
\big\|D^\gamma a\big\|_{n+1,\Omega}\leqslant C_{14},\quad |\gamma|=1,2.
$$

Moreover, notice that $|\alpha|-|\gamma|+1\leqslant N$. Applying Lemma 3.9, we have for any γ, $|\gamma|\leqslant|\alpha|-2$,

$$
\begin{aligned}
\big\|\varphi_{\rho,N}\Delta_v D^{\alpha-\gamma}u\big\|_{-1/3+n+1}&\leqslant\big\|D_v D^{\alpha-\gamma+1}u\big\|_{-1/3+n+1,\Omega_{\tilde{\rho}}}\\
&\leqslant\frac{C_7A^{|\alpha|-|\gamma|+1-2}}{\tilde{\rho}^{s(|\alpha|-|\gamma|-2)}}\big((|\alpha|-|\gamma|-2)!\big)^s\\
&\leqslant\frac{C_{15}A^{|\alpha|-|\gamma|+1-2}}{\rho^{s(|\alpha|-|\gamma|-2)}}\big((|\alpha|-|\gamma|-2)!\big)^s.
\end{aligned}
$$

Consequently, we compute

$$
\begin{aligned}
&\sum_{2\leqslant|\gamma|\leqslant|\alpha|-2}\binom{\alpha}{\gamma}\|D^\gamma a\|_{n+1,\Omega}\cdot\|\varphi_{\rho,N}\Delta_v D^{\alpha-\gamma}u\|_{-1/3+n+1}\\
&\quad\leqslant\sum_{2\leqslant|\gamma|\leqslant|\alpha|-2}\binom{\alpha}{\gamma}C_{14}^{|\gamma|-2}\big((|\gamma|-2)!\big)^s\frac{C_{15}A^{|\alpha|-|\gamma|+1-2}}{\rho^{s(|\alpha|-|\gamma|-2)}}\big((|\alpha|-|\gamma|-2)!\big)^s\\
&\quad\leqslant\frac{C_{15}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\sum_{2\leqslant|\gamma|\leqslant|\alpha|-2}\left(\frac{C_{14}}{A}\right)^{|\gamma|-1}|\alpha|!\big((|\gamma|-2)!\big)^{s-1}\big((|\alpha|-|\gamma|-2)!\big)^{s-1}\\
&\quad\leqslant\frac{C_{15}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s\sum_{2\leqslant|\gamma|\leqslant|\alpha|-2}\left(\frac{C_{14}}{A}\right)^{|\gamma|-1}|\alpha|\frac{(|\alpha|-1)(|\alpha|-2)}{(|\alpha|-3)^{s-1}}\\
&\quad\leqslant\frac{C_{16}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{s/3}\sum_{2\leqslant|\gamma|\leqslant|\alpha|-2}\left(\frac{C_{14}}{A}\right)^{|\gamma|-1}.
\end{aligned}
$$

Making A large enough such that $\sum_{2\leqslant|\gamma|\leqslant|\alpha|-2}(\frac{C_{14}}{A})^{|\gamma|-1}\leqslant 1$, then we get

$$
\begin{aligned}
&\sum_{2\leqslant|\gamma|\leqslant|\alpha|-2}\binom{\alpha}{\gamma}\|D^\gamma a\|_{n+1,\Omega}\cdot\|\varphi_{\rho,N}\Delta_v D^{\alpha-\gamma}u\|_{-1/3+n+1}\\
&\quad\leqslant\frac{C_{16}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{s/3}.
\end{aligned}
$$

For $|\gamma|=1$, $|\alpha|-1$ or $|\alpha|$, we can compute directly

$$
\binom{\alpha}{\gamma}\|D^\gamma a\|_{n+1,\Omega}\cdot\|\varphi_{\rho,N}\Delta_v D^{\alpha-\gamma}u\|_{-1/3+n+1}\leqslant\frac{C_{17}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{s/3}.
$$

Combination of the above two inequalities gives that

$$
(III)\leqslant\frac{C_{18}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{s/3}.
$$

This along with (3.15) and (3.18) yields the conclusion (3.14).

Step 2. We next claim that

$$
\|\varphi_{\rho,N}D^\alpha[F(\cdot,u(\cdot),\nabla_v u(\cdot))]\|_{-1/3+n+1}\leqslant\frac{C_{21}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{s/3}. \tag{3.19}
$$

We first prove F and u satisfy the conditions in (3.7)–(3.9) for some M_j. By Lemma 3.9, we have

$$
\|D^j u\|_{-1/3+n+1,\Omega_{\tilde\rho}}\leqslant\|D^j u\|_{n+1,\Omega_{\tilde\rho}}\leqslant\frac{C_7A^{j-2}}{\tilde\rho^{s(j-3)}}\big((j-3)!\big)^s,\quad 3\leqslant j\leqslant N, \tag{3.20}
$$

$$
\|D_vD^j u\|_{-1/3+n+1,\Omega_{\tilde\rho}}\leqslant\frac{C_7A^{j-2}}{\tilde\rho^{s(j-3)}}\big((j-3)!\big)^s,\quad 3\leqslant j\leqslant N, \tag{3.21}
$$

and

$$\left\| D^j u \right\|_{-1/3+n+1,\Omega_{\tilde\rho}} \leqslant C_7, \quad 0 \leqslant j \leqslant 2. \tag{3.22}$$

Since $F \in G^s(\mathbb{R}^{2n+1} \times \mathbb{R})$, then for all $k, m+l \geqslant 3$,

$$\left\| \left(D^k_{t,x,v} \partial^l_u D^m_p F\right)\left(\cdot, u(\cdot), \nabla_v u(\cdot)\right) \right\|_{-1/3+n+1,\Omega} \leqslant C_{20}^{k+l} \left((k-3)!\right)^s \left((l-3)!\right)^s. \tag{3.23}$$

Define M_j, H_0, H_1 by setting

$$H_0 = C_7, \qquad H_1 = A, \qquad M_0 = C_7, \qquad M_j = \frac{((j-1)!)^s}{\tilde\rho^{s(j-1)}}, \quad j \geqslant 1.$$

We can choose A large enough such that $H_1 = A \geqslant C_2 H_0$. Then (3.20)–(3.23) can be rewritten as

$$\left\| D^j u \right\|_{-1/3+n+1,\Omega_{\tilde\rho}} \leqslant H_0, \quad 0 \leqslant j \leqslant 1, \tag{3.24}$$

$$\left\| D^j u \right\|_{-1/3+n+1,\Omega_{\tilde\rho}} \leqslant H_0 H_1^{j-2} M_{j-2}, \quad 2 \leqslant j \leqslant |\alpha| = N, \tag{3.25}$$

$$\left\| D_v D^j u \right\|_{-1/3+n+1,\Omega_{\tilde\rho}} \leqslant H_0 H_1^{j-2} M_{j-2}, \quad 2 \leqslant j \leqslant |\alpha| = N, \tag{3.26}$$

$$\left\| \left(D^k_{t,x,v} \partial^l_u D^m_p F\right) \right\|_{-1/3+n+1,\Omega} \leqslant C_{20}^{k+l} M_{k-2} M_{m+l-2}, \quad k, m+l \geqslant 2. \tag{3.27}$$

For each j, notice that $s \geqslant 3$. Hence

$$\begin{aligned}
\frac{j!}{i!(j-i)!} M_i M_{j-i} &= \frac{j!}{i(j-i)} \left((i-1)!\right)^{s-1} \left((j-i-1)!\right)^{s-1} \tilde\rho^{-s(i-1)} \tilde\rho^{-s(j-i-1)} \\
&\leqslant (j!)\left((j-2)!\right)^{s-1} \tilde\rho^{-s(j-1)} \\
&\leqslant \frac{j}{(j-1)^{s-1}} (j-1)!\left((j-1)!\right)^{s-1} \tilde\rho^{-s(j-1)} \\
&\leqslant M_j.
\end{aligned} \tag{3.28}$$

Thus M_j satisfy the monotonicity condition (3.5). In view of (3.24)–(3.28) and making use of Lemma 3.6, we have

$$\begin{aligned}
\left\| \varphi_{\rho,N} D^\alpha \left[F\left(\cdot, u(\cdot)\right)\right] \right\|_{-1/3+n+1} &\leqslant C_3 H_0 H_1^{|\alpha|-2} M_{|\alpha|-2} \\
&\leqslant \frac{C_3 C_7 A^{|\alpha|-2}}{\tilde\rho^{s(|\alpha|-3)}} \left((|\alpha|-3)!\right)^s \\
&\leqslant \frac{C_{21} A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}} \left(\left(|\alpha|-3\right)!\right)^s (N/\rho)^{s/3}.
\end{aligned}$$

This completes the proof of conclusion (3.19).

Step 3. We verify in this step the following:

$$\left\| \mathcal{L} \varphi_{\rho,N} D^\alpha u \right\|_{-1/3+n+1} \leqslant \frac{C_{23} A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}} \left(\left(|\alpha|-3\right)!\right)^s (N/\rho)^{s/3}. \tag{3.29}$$

In fact,

$$\begin{aligned}\|\mathcal{L}\varphi_{\rho,N}D^\alpha u\|_{-1/3+n+1} &\leqslant C_{22}\big\{\big\|[\mathcal{L},\varphi_{\rho,N}D^\alpha]u\big\|_{-1/3+n+1}+\|\varphi_{\rho,N}D^\alpha\mathcal{L}u\|_{-1/3+n+1}\big\}\\&=C_{22}\big\{\big\|[\mathcal{L},\varphi_{\rho,N}D^\alpha]u\big\|_{-1/3+n+1}\\&\quad+\big\|\varphi_{\rho,N}D^\alpha\big[F\big(\cdot,u(\cdot),\nabla_v u(\cdot)\big)\big]\big\|_{-1/3+n+1}\big\}.\end{aligned}$$

This along with (3.14), (3.19) in Steps 1 and 2 yields immediately the conclusion (3.29).

Step 4. We claim that

$$\|\varphi_{\rho,N}D^\alpha u\|_{1/3+n+1}+\|\varphi_{\rho,N}D_v D^\alpha u\|_{1/3-1/3+n+1}\leqslant\frac{C_{31}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{s/3}. \quad (3.30)$$

In fact, applying the subelliptic estimate (2.3), we obtain

$$\|\varphi_{\rho,N}D^\alpha u\|_{1/3+n+1}\leqslant C_{24}\big\{\|\mathcal{L}\varphi_{\rho,N}D^\alpha u\|_{-1/3+n+1}+\|\varphi_{\rho,N}D^\alpha u\|_{n+1}\big\}.$$

Combining Lemma 3.9 and (3.29) in Step 3, we have

$$\|\varphi_{\rho,N}D^\alpha u\|_{1/3+n+1}\leqslant\frac{C_{25}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{s/3}. \quad (3.31)$$

Now it remains to treat $\|\varphi_{\rho,N}D_v D^\alpha u\|_{1/3-1/3+n+1}$, and

$$\|\varphi_{\rho,N}D_v D^\alpha u\|_{1/3-1/3+n+1}\leqslant\|D_v\varphi_{\rho,N}D^\alpha u\|_{n+1}+\big\|[D_v,\varphi_{\rho,N}]D^\alpha u\big\|_{n+1}.$$

First, we treat the first term on the right. By a direct calculation, it follows that

$$\begin{aligned}&\|D_v\varphi_{\rho,N}D^\alpha u\|_{n+1}^2\\&\quad=\mathrm{Re}\big(\mathcal{L}\varphi_{\rho,N}D^\alpha u,a^{-1}\Lambda^{2n+2}\varphi_{\rho,N}D^\alpha u\big)-\mathrm{Re}\big(X_0\varphi_{\rho,N}D^\alpha u,a^{-1}\Lambda^{2n+2}\varphi_{\varepsilon,k\varepsilon}D^\alpha u\big)\\&\quad=\mathrm{Re}\big(\mathcal{L}\varphi_{\rho,N}D^\alpha u,a^{-1}\Lambda^{2n+2}\varphi_{\rho,N}D^\alpha u\big)-\frac{1}{2}\big(\varphi_{\rho,N}D^\alpha u,\big[a^{-1}\Lambda^{2n+2},X_0\big]\varphi_{\rho,N}D^\alpha u\big)\\&\qquad-\frac{1}{2}\big(\varphi_{\rho,N}D^\alpha u,\big[\Lambda^{2n+2},a^{-1}\big]X_0\varphi_{\rho,N}D^\alpha u\big)\\&\quad\leqslant C_{26}\big\{\|\mathcal{L}\varphi_{\rho,N}D^\alpha u\|_{-1/3+n+1}^2+\|\varphi_{\rho,N}D^\alpha u\|_{1/3+n+1}^2\big\}.\end{aligned}$$

This along with (3.29) and (3.31) shows that

$$\|D_v\varphi_{\rho,N}D^\alpha u\|_{r-1/3+n+1}\leqslant\frac{C_{27}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{s/3}.$$

Moreover Lemma 3.9 yields

$$\begin{aligned}\left\|[D_v,\varphi_{\rho,N}]D^\alpha u\right\|_{n+1} &\leqslant C_{28}(N/\rho)\left\|D^\alpha u\right\|_{n+1,\Omega_{\tilde\rho}}\\ &\leqslant \frac{C_{28}C_7A^{|\alpha|-2}}{\tilde\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{s/3}\\ &\leqslant \frac{C_{29}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{s/3}.\end{aligned}$$

From the above two inequalities, we have

$$\left\|\varphi_{\rho,N}D_vD^\alpha u\right\|_{1/3+n+1} \leqslant \frac{C_{30}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{s/3}.$$

This completes the proof of Step 4.

It is clear for any ρ, $0<\rho<1$,

$$\begin{aligned}&\left\|D^\alpha u\right\|_{1/3+n+1,\Omega_\rho}+\left\|D_vD^\alpha u\right\|_{1/3-1/3+n+1,\Omega_\rho}\\ &\quad\leqslant \left\|\varphi_{\rho,N}D^\alpha u\right\|_{1/3+n+1}+\left\|\varphi_{\rho,N}D_vD^\alpha u\right\|_{1/3-1/3+n+1}.\end{aligned}$$

It thus follows from Step 4 that the conclusion in Lemma 3.10 is true for $r=1/3$.

Moreover for any $0<r<1/3$, using the interpolation inequality (2.2), we have

$$\begin{aligned}\left\|D^\alpha u\right\|_{r+n+1,\Omega_\rho} &\leqslant \left\|\varphi_{\rho,N}D^\alpha u\right\|_{r+n+1}\\ &\leqslant \varepsilon\left\|\varphi_{\rho,N}D^\alpha u\right\|_{1/3+n+1}+\varepsilon^{-r/(1/3-r)}\left\|\varphi_{\rho,N}D^\alpha u\right\|_{n+1}\\ &\leqslant \varepsilon\frac{C_{31}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{s/3}+\varepsilon^{-r/(1/3-r)}\frac{C_{32}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s.\end{aligned}$$

Taking $\varepsilon=(N/\rho)^{s(r-1/3)}$, then

$$\left\|D^\alpha u\right\|_{r+n+1,\Omega_\rho} \leqslant \frac{C_{33}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{rs}.$$

Similarly,

$$\left\|D_vD^\alpha u\right\|_{r-1/3+n+1,\Omega_\rho} \leqslant \frac{C_{34}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{rs}.$$

This completes the proof of Lemma 3.10. □

Inductively, we have the following

Lemma 3.11. *For any r with $1/3\leqslant r\leqslant 2/3$, we have for all $0<\rho<1$*

$$\left\|D^\alpha u\right\|_{r+n+1,\Omega_\rho}+\left\|D_vD^\alpha u\right\|_{r-1/3+n+1,\Omega_\rho} \leqslant \frac{C_{38}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{sr}. \quad (3.32)$$

Moreover, the above inequality still holds for any r with $2/3\leqslant r\leqslant 1$.

Proof. Repeating the proof of Lemma 3.10, we have (3.32) for $1/3 \leqslant r \leqslant 2/3$. When $2/3 \leqslant r \leqslant 1$, the consideration is a little different. The conclusion in Step 1 in the above proof still holds for $r = 1$. For the corresponding Step 2, we have to make some modification to prove

$$\left\|\varphi_{\rho,N} D^\alpha\left[F\left(\cdot, u(\cdot), \nabla_v u(\cdot)\right)\right]\right\|_{1/3+n+1} \leqslant \frac{C_{36} A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}} \left(\left(|\alpha|-3\right)!\right)^s (N/\rho)^s.$$

From (3.32) with $1/3 \leqslant r \leqslant 2/3$, it follows that for $3 \leqslant j \leqslant N$

$$\left\|D^j u\right\|_{1/3+n+1,\Omega_{\tilde\rho}} \leqslant \frac{C_{37} A^{j-2}}{\tilde\rho^{s(j-3)}} \left((j-3)!\right)^s (j/\tilde\rho)^{s/3},$$

$$\left\|D_v D^j u\right\|_{1/3+n+1,\Omega_{\tilde\rho}} \leqslant \left\|D_v D^j u\right\|_{2/3-1/3+n+1,\Omega_{\tilde\rho}} \leqslant \frac{C_{37} A^{j-2}}{\tilde\rho^{s(j-3)}} \left((j-3)!\right)^s (j/\tilde\rho)^{2s/3},$$

and that

$$\left\|D^j u\right\|_{1/3+n+1,\Omega_{\tilde\rho}} \leqslant C_{37}, \quad 0 \leqslant j \leqslant 2.$$

Hence we need to define a new sequence $\bar M_j$ by setting

$$\bar M_0 = C_{37}, \qquad \bar M_j = \frac{((j-1)!)^s}{\tilde\rho^{s(j-1)}} \left((j+2)/\tilde\rho\right)^{2s/3}, \quad j \geqslant 1.$$

For each j, notice that $s \geqslant 3$. Hence a direct computation shows that for $0 < i < j$,

$$\begin{aligned}
\frac{j!}{i!(j-i)!} \bar M_i \bar M_{j-i} &= \frac{j!}{i(j-i)} \left((i-1)!\right)^{s-1} \left((j-i-1)!\right)^{s-1} \\
&\quad \times (i+2)^{2s/3} (j-i+2)^{2s/3} \tilde\rho^{-s(j-2)} \tilde\rho^{-4s/3} \\
&\leqslant 4(j!)\left((j-2)!\right)^{s-1} (j+2)^{2s/3-1} (j+1)^{2s/3-1} \tilde\rho^{-s(j-1)} \tilde\rho^{-2s/3} \tilde\rho^{s-2s/3} \\
&\leqslant \frac{4j(j+1)^{2s/3-1}}{(j-1)^{s-1}} (j-1)!\left((j-1)!\right)^{s-1} \tilde\rho^{-s(j-1)} \left((j+2)/\tilde\rho\right)^{2s/3} \\
&\leqslant C_{39} \bar M_j.
\end{aligned}$$

In the last inequality, we used the fact that $s - 1 \geqslant 2s/3$. Thus $\bar M_j$ satisfy the monotonicity condition (3.5). Now the remaining argument is identical to that in the proof of Lemma 3.10. Thus (3.32) holds for $r = 1$ and thus for $2/3 \leqslant r \leqslant 1$ by the interpolation inequality (2.2). This completes the proof of Lemma 3.11. □

Recall C_7, C_{35} and C_{35} are the constants appearing in Lemmas 3.9, 3.10 and 3.11. Now make A sufficiently large such that $A \geqslant \max\{C_7, C_{35}, C_{38}\}$. Then, by the above three lemmas, we see that the estimate in $(E)_{r,N}$ holds for any $r \in [0, 1]$. This completes the proof of Proposition 3.7. □

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