Periodic solutions of asymptotically linear delay differential systems via Hamiltonian systems

Chun-gen Liu

School of Mathematics and LPMC, Nankai University, Tianjin 300071, People’s Republic of China

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1. Introduction and main results

The goal of this paper is to consider the existence and multiplicity of periodic solutions for some asymptotically linear delay differential systems and some asymptotically linear delay Hamiltonian systems via the methods of Hamiltonian systems and index theory. For this purpose, we first consider in general the so-called $\mathcal{M}$-boundary problem of a Hamiltonian system.

1.1. $\mathcal{M}$-boundary problem of a Hamiltonian system

For a skew-symmetric non-degenerate $2N \times 2N$ matrix $\mathcal{J} = (a_{ij})$, it can define a symplectic structure on $\mathbb{R}^{2N}$ by

$$\omega(v, w) = v^T \cdot \mathcal{J}^{-1} \cdot w$$
or \( \omega = \frac{1}{2} \sum_{i,j} a_{ij} \, dx_i \wedge dx_j \) with \( J^{-1} = (a_{ij}) \). A \( 2N \times 2N \) matrix \( M \) is called \( J \)-symplectic if there holds \( M^T J^{-1} M = J^{-1} \). We denote the set of all \( J \)-symplectic matrices by \( \text{Sp}_J(2N) \). The usual symplectic group \( \text{Sp}(2N) \) is the special case of \( J = J_N = \left( \begin{smallmatrix} 0 & -I_N \\ I_N & 0 \end{smallmatrix} \right) \). Here \( I_N \) is the \( N \times N \) identity matrix. We will write \( J \) for \( J_N \) if the dimension 2N is clear from the text. For a \( J \)-symplectic matrix \( M \) with \( M^k = I_{2N} \), and a function \( H \in C^2(\mathbb{R} \times \mathbb{R}^{2N}) \) with \( H(t + \tau, Mz) = H(t, z) \), we consider \( k \tau \)-periodic solution of the following \( M \)-boundary value problem

\[
\begin{cases}
\dot{z}(t) = J \nabla H(t, z(t)), \\
z(\tau) = Mz(0).
\end{cases}
\]  

(1.1)

The corresponding functional is defined in \( E = W^{1/2,2}(S^1, \mathbb{R}^{2N}) \) with \( S^1 = \mathbb{R}/(k \tau \mathbb{Z}) \) by

\[
\varphi(z) = \frac{1}{2} \int_0^{k \tau} (J^{-1} \dot{z}(t), z(t)) \, dt - \int_0^{k \tau} H(t, z(t)) \, dt.
\]  

(1.2)

The critical point of \( \varphi \) in \( E \) is a \( k \tau \)-periodic solution of the nonlinear Hamiltonian system in (1.1). In order to solve the problem (1.1), we define a group action \( \sigma \) on \( E \) by

\[ \sigma z(t) = M z(t - \tau). \]

It is clear that \( \sigma^k = \text{id} \) and \( \varphi \) is \( \sigma \)-invariant, i.e., there holds

\[ \varphi(\sigma z) = \varphi(z). \]  

(1.3)

Setting \( E^\sigma = \{ z \in E \mid \sigma z = z \} = \text{fix}(\sigma) \), by the well-known Palais symmetric principal (see [27]), a critical point of \( \varphi \) in \( E^\sigma \) is a solution of the boundary problem (1.1). For \( z \in E^\sigma \), there holds

\[
\varphi(z) = \frac{k}{2} \int_0^\tau (J^{-1} \dot{z}(t), z(t)) \, dt - k \int_0^\tau H(t, z(t)) \, dt.
\]

The linearized system along a solution \( z(t) \) of the nonlinear Hamiltonian system in (1.1) is the following linear Hamiltonian system

\[ \dot{y}(t) = J \mathcal{H}''(t, z(t)) y(t). \]

Its fundamental solution \( \gamma_z(t) \) with \( \gamma_z(0) = I_{2N} \) should satisfy

\[ \gamma_z(t) = J \mathcal{H}''(t, z(t)) \gamma_z(t). \]

The following result is well known and the proof is omitted.

**Lemma 1.1.** \( \gamma_z \) is a \( J \)-symplectic path, i.e., \( \gamma_z(t)^T J^{-1} \gamma_z(t) = J^{-1} \) for all \( t \in \mathbb{R} \).

For simplicity we take \( \tau = 1 \). For a symmetric continuous matrix function \( B(t) \) satisfying \( \mathcal{M}^T B(t + 1) \mathcal{M} = B(t) \), suppose \( \gamma_B(t) \) is the fundamental solution of the linear Hamiltonian system

\[ \dot{y}(t) = J B(t) y(t). \]
Definition 1.2. The \((\mathcal{J}, \mathcal{M})\)-nullity of the symmetric matrix function \(B\) is defined by
\[
\nu_{\mathcal{M}}^{\mathcal{J}}(B) = \dim_{\mathbb{C}} \ker_{\mathbb{C}}(\gamma_B(1) - \mathcal{M}).
\]

For the standard case of \(\mathcal{J} = J_N\) and a matrix \(P \in \text{Sp}_J(2N)\), the \((J, P)\)-nullity and Maslov-type index of a symmetric matrix function \(B\) was defined first in [19] by algebraic method. We will define the \((\mathcal{J}, \mathcal{M})\)-index \(i_{\mathcal{M}}^{\mathcal{J}}(B)\) in Section 2 below via analytic method. So the index pair \((i_{\mathcal{M}}^{\mathcal{J}}(B), \nu_{\mathcal{M}}^{\mathcal{J}}(B)) \in \mathbb{Z} \times \{0, 1, \ldots, 2N\}\) makes sense for all symmetric continuous matrix function \(B(t)\) satisfying \(\mathcal{M}^T B(t + 1) \mathcal{M} = B(t)\).

1.2. Delay differential systems

For simplicity, as in [15] we first consider the \(4\tau\)-periodic solutions of the following delay differential system
\[
\begin{align*}
\dot{x}(t) = \nabla V(t, x(t - \tau)), \quad & (1.4) \\
\end{align*}
\]
where the function \(V \in C^2(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})\) is \(\tau\)-periodic in variable \(t\) and is even in variables \(x\). To find \(4\tau\)-periodic solution \(x(t)\), we only need to find solution with \(x(t + 2\tau) = -x(t)\). If \(x(t)\) is such a solution, let \(x_1(t) = x(t), x_2(t) = x(t - \tau)\) and \(z(t) = (x_1(t), x_2(t))^T\), then there holds
\[
\begin{align*}
x_1'(t) &= \nabla V(t, x_2(t)), \\
x_2'(t) &= -\nabla V(t, x_1(t)).
\end{align*}
\]

Set \(H(t, x_1, x_2) = V(t, x_1) + V(t, x_2)\), then we can rewrite (1.5) as
\[
\dot{z}(t) = J_1 \nabla H(t, z(t)), \quad J_1 = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}. \tag{1.6}
\]

Moreover, if \(z(t) = (x_1(t), x_2(t))^T\) is a \(4\tau\)-periodic solution of (1.6) with \(z(t) = \sigma z(t)\) for the \(4\)-periodic action
\[
\sigma z(t) = J_1 z(t + \tau), \tag{1.7}
\]
then \(x(t) = -x_1(t)\) is a solution of (1.4) with \(x(t + 2\tau) = -x(t)\). The condition (1.7) is equivalent to
\[
z(\tau) = J_1^{-1} z(0). \tag{1.8}
\]

So the problem can be transformed to the problem (1.1) with \(\mathcal{J} = J_1\) and \(\mathcal{M} = J_1^{-1}\).

In general, for a function \(V \in C^2(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})\) with period \(\tau\) in variable \(t\) and even in variables \(x\), we consider the \(2m\tau\)-periodic solutions of the following delay differential system
\[
\dot{x}(t) = \nabla V(t, x(t - \tau)) + \nabla V(t, x(t - 2\tau)) + \cdots + \nabla V(t, x(t - (m - 1)\tau)),
\]
If we get a solution \(x(t)\) with \(x(t - m\tau) = -x(t)\), then by setting \(x_1(t) = x(t), x_2(t) = x(t - \tau), \ldots, x_m(t) = x(t - (m - 1)\tau)\) and \(H(t, x_1, \ldots, x_m) = V(t, x_1) + \cdots + V(t, x_m)\), \(z = (x_1, \ldots, x_m)^T\), we rewrite the system (1.9) as
\[
\dot{z}(t) = A_m \nabla H(t, z(t)), \tag{1.10}
\]
where the $mn \times mn$ skew-symmetric matrix $A_m$ is defined by
\[
A_m = \begin{pmatrix}
0 & I_n & \cdots & I_n \\
-I_n & 0 & \cdots & I_n \\
\vdots & \vdots & \ddots & \vdots \\
-I_n & -I_n & \cdots & 0
\end{pmatrix}.
\]

We see that $\det A_m \neq 0$ if $m \in 2\mathbb{N}$ and $\det A_m = 0$ if $m \in 2\mathbb{N} + 1$. Furthermore, if we take $X = (I_n - I_n, I_n, \cdots, (-1)^{m-1} I_n)$, there holds $XA_m = 0$ when $m \in 2\mathbb{N} + 1$. Set
\[
T_m = \begin{pmatrix}
0 & I_n & 0 & \cdots & 0 \\
0 & 0 & I_n & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-I_n & 0 & 0 & \cdots & 0
\end{pmatrix} = \begin{pmatrix}
0 & I_{n(m-1)} \\
-I_n & 0
\end{pmatrix}.
\]

It is easy to see that $T_m^T = -I_{mn}$, $T_m^{-1} = I_{mn}$ and $(T_m^{-1})^T A_m^{-1} T_m^{-1} = A_m^{-1}$ when $m \in 2\mathbb{N}$, so $G_m = \{ g \mid g = T_k^m, k = 1, 2, \ldots, 2m \}$ is a discrete group. We see also that
\[
T_m^T T_m^{-1} = \begin{pmatrix}
0 & \cdots & 0 & -I_n \\
I_n & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & I_n & 0
\end{pmatrix} = \begin{pmatrix}
0 & -I_n \\
I_{n(m-1)} & 0
\end{pmatrix}.
\]

If $z(t)$ is a solution of (1.10) with $z(\tau) = T_m^{-1}z(0)$ then $x(t) = x_1(t)$ is a $2m\tau$-periodic solution of Eq. (1.9) for $m \in 2\mathbb{N}$. So we also turn our problem into the problem (1.1) with $J = A_m$ and $M = T_m^{-1}$. In order to understand the case of $m \in 2\mathbb{N} + 1$, we first recall the notation about Poisson structure.

1.3. Poisson structure

Any $k \times k$ skew-symmetric matrix $A$ determines a Poisson structure on $\mathbb{R}^k$. For any functions $F, H \in C^\infty(\mathbb{R}^k, \mathbb{R})$, the Poisson bracket is defined by the following properties
\[
\{ F, H \} = (\nabla F)^T \cdot A \cdot \nabla H \in C^\infty(\mathbb{R}^k, \mathbb{R}).
\]

We recall the definition of a Poisson structure $\{ \cdot, \cdot \}$ on a manifold $M$. For any two functions $F, H \in C^\infty(M)$, the Poisson bracket $\{ F, H \} \in C^\infty(M)$ is defined by the following properties

1. $\{ c_1 F_1 + c_2 F_2, H \} = c_1 \{ F_1, H \} + c_2 \{ F_2, H \}$, $\forall c_1, c_2 \in \mathbb{R}$,
2. $\{ F, H \} = -\{ H, F \}$,
3. $\{ [F, H], P \} + \{ [P, F], H \} + \{ [H, P], F \} = 0$,
4. $\{ F, H \cdot P \} = \{ F, H \} P + H(F, P)$.

A differential manifold $M$ with a Poisson structure is called Poisson manifold. Furthermore if there holds

5. $\{ F, H \} = 0$ for any function $F$ implies $H \equiv c$,.
then the Poisson structure \( \{ \cdot , \cdot \} \) determines a symplectic structure, and \( M \) is a symplectic manifold. For example, when \( k = 2m \) and \( A = J_m = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix} \), then it defines a Poisson structure on \( \mathbb{R}^{2m} \) by
\[
\{ F, H \} = \sum_{i=1}^{m} \left( \frac{\partial F}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial H}{\partial q_i} \right).
\]
This Poisson structure on \( \mathbb{R}^{2m} \) is the standard Poisson structure and it determines the standard symplectic structure on \( \mathbb{R}^{2m} \).

Let \( M \) be a Poisson manifold and \( H : M \to \mathbb{R} \) be a smooth function. The Hamiltonian vector field associated with \( H \) is the unique smooth vector field \( V_H \) on \( M \) satisfying
\[
V_H(F) = \{ F, H \}
\]
for every smooth function \( F : M \to \mathbb{R} \). If the Poisson structure on \( \mathbb{R}^k \) is defined by (1.11), the Hamiltonian vector field is
\[
V_H(x) = A \nabla H(x).
\]
The Hamiltonian equation becomes
\[
\dot{x}(t) = A \nabla H(x(t)).
\]
For a time depending function \( H_t(x) = H(t, x) \), the Hamiltonian vector field makes sense as \( V_{H_t}(x) = A \nabla_x H(t, x) \) and the Hamiltonian equation becomes
\[
\dot{x}(t) = A \nabla_x H(t, x(t)).
\]
The following Darboux Theorem will be very useful.

**Darboux Theorem.** Suppose the rank of the \( k \times k \) matrix \( A \) in (1.11) is \( 2m \) with \( k = 2m + l \), then there is a coordinate transformation \( y = Bx \) such that
\[
\{ F, H \}(x) = \sum_{i=1}^{m} \left( \frac{\partial f}{\partial q_i} \frac{\partial h}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial h}{\partial q_i} \right)(y) = \nabla f(y)^T \cdot J' \nabla h(y),
\]
where \( f(y) = F(B^{-1} y), h(y) = H(B^{-1} y) \) and \( y = (p_1, \ldots, p_m, q_1, \ldots, q_m, z_1, \ldots, z_l)^T \). That is to say \( J' = BAB^T = \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix} \).

Particularly, for the matrix \( A_m \) defined in (1.10) with \( m \in 2 \mathbb{N} + 1 \), and the matrix \( B_m \) defined by
\[
B_m = \begin{pmatrix}
I_n & 0 & \cdots & 0 & 0 \\
0 & I_n & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & I_n & 0 \\
I_n & -I_n & \cdots & -I_n & I_n
\end{pmatrix},
\]
there holds
\[
B_mA_mB_m^T = \begin{pmatrix} A_{m-1} & 0 \\ 0 & 0 \end{pmatrix}.
\]
So by choosing matrix $C_{m-1}$ satisfying $C_{m-1}A_{m-1}c_{m-1}^T = J_{n\ell}$ with $\ell = \frac{m-1}{2}$ and taking $B_m'(C_{m-1} 0 \quad 0 \quad \ell_m)$, there holds $B_m'B_mA_m(B_m'B_m)^T = (J_{\ell\ell} 0 \quad 0 \quad 0)$. By taking the coordinate transformation $y = (y_m) = B_mz$ and $h(t, y) = H(t, z)$, the system (1.10) becomes

$$
\dot{y}(t) = A_{m-1}\nabla_y h(t, y), \quad y_m = c.
$$

From the coordinate transformation, we see that $y_i = x_i$ for $1 \leq i \leq m - 1$ and $y_m = x_1 - x_2 + \cdots - x_{m-1} + x_m$. So system (1.12) becomes

$$
\dot{z}(t) = A_{m-1}\nabla \bar{H}(t, z(t)),
$$

where $H(t, z) = H(t, z, c - x_1 + x_2 - \cdots + x_{m-1})$ and $z = (x_1, \ldots, x_{m-1})$. When choosing $c = 0$, any solution $\tilde{z}(t)$ of (1.13) with $z(0) = \tilde{B}_{m-1}\tilde{z}(1)$ determines a solution of system (1.10) with $x_m(t) = -x_1(t) + x_2(t) - \cdots + x_{m-1}(t)$, so $x(t) = x_1(t)$ is a solution of the delay system (1.1), where

$$
\bar{B}_{m-1} = \begin{pmatrix}
0 & I_n & 0 & \cdots & 0 \\
0 & 0 & I_n & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & I_n \\
-I_n & I_n & -I_n & \cdots & I_n
\end{pmatrix}.
$$

It is easy to see that $\bar{B}_{m-1}A_{m-1}(\bar{B}_{m-1})^T = A_{m-1}$ and $(\bar{B}_{m-1})^{2m} = I_{n(m-1)}$. So in this case, we also transform the problem into the problem (1.1) with $\mathcal{J} = A_{m-1}$ and $\mathcal{M} = \bar{B}_{m-1}^{-1}$.

We will state the main results about the existence and multiplicity of periodic solutions of delay differential equation (1.9) in Theorem 5.1 and Theorem 5.3 below.

1.4. First order delay Hamiltonian systems

For a function $G \in C^2(\mathbb{R} \times \mathbb{R}^{2n}, \mathbb{R})$ with $G(t+\tau, x) = G(t, x)$, we consider the $2\tau$-periodic solutions of the following first order delay Hamiltonian system

$$
\dot{x}(t) = J_n \nabla G(t, x(t - \tau)).
$$

For a $2\tau$-periodic solution $x(t)$ of (1.14), by setting $x_1(t) = x(t)$, $x_2(t) = x(t - \tau)$ and $z = (x_1, x_2)^T$, the delay Hamiltonian system (1.14) is read as

$$
\dot{z}(t) = \bar{J}_{2n} \nabla H(t, z(t))
$$

with $H(t, z) = H(t, x_1, x_2) = G(t, x_1) + G(t, x_2)$ and $\bar{J}_{2n} = (0 \quad J_n)$. If $z(t)$ is a solution of (1.15) with $z(0) = P_{2n}z(\tau)$, $P_{2n} = (0 \quad J_n)$, then $x(t) = x_1(t)$ is a $2\tau$-periodic solution of (1.14). It is easy to see that $P_{2n}^{-1} = P_{2n} = P_{2n}^T$ and $P_{2n}^T \bar{J}_{2n}^{-1} P_{2n} = \bar{J}_{2n}^{-1}$.

In general, we consider the following first order delay Hamiltonian system

$$
\dot{x}(t) = J_n \left( \nabla G(t, x(t - \tau)) + \nabla G(t, x(t - 2\tau)) + \cdots + \nabla G(t, x(t - (m-1)\tau)) \right).
$$

For an $m\tau$-periodic solution $x(t)$ of (1.16), by setting $x_1(t) = x(t)$, $x_2(t) = x(t - \tau)$, $\ldots$, $x_m(t) = x(t - (m-1)\tau)$ and $z = (x_1, x_2, \ldots, x_m)^T$, the delay Hamiltonian system (1.16) is read as

$$
\dot{z}(t) = J_{n,m} \nabla H(t, z(t))
$$

with $H(t, z) = H(t, x_1, x_2, \ldots, x_m) = G(t, x_1, x_2, \ldots, x_m)$ and $J_{n,m} = J_{n\ell \ell}$.
with $H(t, z) = G(t, x_1) + \cdots + G(t, x_m)$ and

$$J_{n,m} = \begin{pmatrix} 0 & J_n & \cdots & J_n & J_n \\ J_n & 0 & \cdots & J_n & J_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ J_n & J_n & \cdots & J_n & 0 \end{pmatrix}.$$ 

Conversely, if $z(t)$ is a solution of (1.17) with $z(0) = P_{n,m}z(\tau)$, where

$$P_{n,m} = \begin{pmatrix} 0 & I_n & 0 & \cdots & 0 \\ 0 & 0 & I_n & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ I_n & 0 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} 0 & I_{n(m-1)} \end{pmatrix},$$

then $x(t) = x_1(t)$ is an $m\tau$-periodic solution of (1.16). It is easy to see that $P_{n,m}^{-1} = P_{n,m}^T = \begin{pmatrix} 0 & I_n \end{pmatrix}$, $P_{n,m}^{m} = I_{nm}$ and $P_{n,m}J_{n,m} = J_{n,m}P_{n,m}$. In this case, we still turn our problem into problem (1.1) with $\mathcal{J} = J_{n,m}$ and $\mathcal{M} = P_{n,m}^{-1}$.

Theorem 5.4 and Theorem 5.5 below are the main existence and multiplicity results of 2\tau-periodic solution of the first order delay Hamiltonian system (1.16).

**1.5. Second order delay Hamiltonian systems**

For a function $V \in C^2(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ with $V(t + \tau, x) = V(t, x)$, we consider the periodic solutions of the following second order delay Hamiltonian system

$$\ddot{x}(t) + \nabla V(\dot{x}(t), x(t - 2\tau)) = 0.$$ 

We can turn it into a first order delay Hamiltonian system as (1.14) with $H(t, x, y) = -\frac{1}{2}y^2 - V(t, x)$ and $y(t) = \dot{x}(t + \tau)$. In general, we consider the $m\tau$-periodic solutions of the following second order delay system

$$\ddot{x}(t) = -\left[\nabla V(t, x(t - \tau)) + \nabla V(t, x(t - 2\tau)) + \cdots + \nabla V(t, x(t - (m-1)\tau))\right]. \quad (1.18)$$

Let $x_1(t) = x(t)$, $x_2(t) = x(t - \tau)$, $x_m(t) = x(t - (m-1)\tau)$ and $z(t) = (x_1(t), \ldots, x_m(t))^T$, then by $x(t + m\tau) = x(t)$, there holds

$$\dot{z}(t) = -\mathcal{A}_m \nabla H(t, z(t)),$$

where $H(t, z) = V(t, x_1) + \cdots + V(t, x_m)$ and

$$\mathcal{A}_m = \begin{pmatrix} 0 & I_n & \cdots & I_n \\ I_n & 0 & \cdots & I_n \\ \vdots & \vdots & \ddots & \vdots \\ I_n & I_n & \cdots & 0 \end{pmatrix}.$$ 

It is easy to see that $\det \mathcal{A}_m \neq 0$. By taking $y(t) = \mathcal{A}_m^{-1} \dot{z}(t)$ and $w = (z, y)^T$, there holds

$$\dot{w}(t) = J_{mn} \nabla \mathcal{H}(t, w(t)), \quad (1.19)$$
where \( \mathcal{H}(t, w) = -\frac{1}{2} (A_{m}y, y) - H(t, z) \). A solution of this Hamiltonian system with the boundary value condition \( w(0) = \left( \begin{array}{cc} P_{n,m} & 0 \\ 0 & A_{m}^{-1}P_{n,m}A_{m} \end{array} \right) w(\tau) \) determines an \( m\tau \)-periodic solution \( x(t) = x_{1}(t) \) of the second order delay system. We note that \( P_{n,m}A_{m} = A_{m}P_{n,m} \), so there holds

\[
\mathcal{P}_{n,m} := \left( \begin{array}{cc} P_{n,m}A_{m}^{-1} & 0 \\ 0 & P_{n,m} \end{array} \right) \in \text{Sp}(2mn),
\]

\( \mathcal{P}_{n,m} J_{mn} \mathcal{P}_{n,m}^{T} = J_{mn} \), and \( \mathcal{P}_{n,m}^{T} = I_{2mn} \). We also turn our problem into the problem (1.1) with \( \mathcal{J} = J_{mn} \) and \( \mathcal{M} = \mathcal{P}_{n,m}^{-1} \).

Theorem 5.6 and Theorem 5.7 below are the main existence and multiplicity results for the \( 2\tau \)-periodic solutions of second order delay Hamiltonian system (1.18).

We note that \( \mathcal{J} = A_{m} \) in (1.10) when \( m \in 2\mathbb{Z} \), \( \mathcal{J} = J_{n,m} \) and \( \mathcal{J} = J_{mn} \) all satisfy the property \( \mathcal{J} \mathcal{J}^{N} = J_{N}\mathcal{J} \) for suitable \( N \).

1.6. Background and related work

In 1974, Kaplan and Yorke in [15] studied the autonomous delay differential equation as (1.4) and introduced a new technique for establishing the existence of periodic solutions. More precisely, the authors of [15] considered the periodic solutions of the following kind of delay differential equations

\[
\dot{x}(t) = f(x(t - 1))
\]

and

\[
\dot{x}(t) = f(x(t - 1)) + f(x(t - 2))
\]

with odd function \( f \). They turned their problems into the problems of periodic solution of autonomous Hamiltonian system and under some twisted condition on the origin and infinity for the function \( f \), it was proved that there exists an energy surface of the Hamiltonian function containing at least one periodic solution. Since then many papers (see [5,6,11,12,16–18] and the references therein) used Kaplan and Yorke’s original idea to search for periodic solutions of more general differential delay equations of the following form

\[
\dot{x}(t) = f(x(t - 1)) + f(x(t - 2)) + \cdots + f(x(t - m + 1)).
\]

The existence of periodic solutions of the above delay differential equation has been investigated by Nussbaum in [25] using different techniques. For other related works, the readers may refer to the references [9,10] and the references therein. Up to the author’s knowledge, for the periodic solutions of asymptotically linear delay differential equations, this paper is the first one dealt with the non-autonomous cases and the delay differential systems including the delay Hamiltonian systems. The readers can also refer to [13,14,26] for systematic introduction on delay differential equations.

2. (\( \mathcal{J}, \mathcal{M} \))-index theory

For a fixed integer \( k \in \mathbb{N} \), we define \( S^{1} = \mathbb{R}/(k\mathbb{Z}) \). The Hilbert space \( W^{1/2,2}(S^{1}, \mathbb{R}^{2N}) \) consists of all the elements of \( z \in L^{2}(S^{1}, \mathbb{R}^{2N}) \) satisfying

\[
z(t) = \sum_{j \in \mathbb{Z}} \exp \left( \frac{2j\pi t}{k} \right) a_{j}, \quad \sum_{j \in \mathbb{Z}} (1 + |j|) a_{j}^{2} < \infty, \quad a_{j} \in \mathbb{R}^{2N}.
\]
For a non-degenerate skew-symmetric matrix $J$, and a $J$-symplectic matrix $\mathcal{M} \in \text{Sp}_J(2N)$ with $\mathcal{M}^k = \text{Id}$, we define

$$W_\mathcal{M} = \{z \in W^{1/2,2}(S^1, \mathbb{R}^{2N}) \mid z(t + 1) = \mathcal{M}z(t)\}.$$ 

By a direct computation, we see that $z \in W_\mathcal{M}$ iff $z \in W^{1/2,2}(S^1, \mathbb{R}^{2N})$ and $a_0$ is an eigenvector of the eigenvalue 1 of $\mathcal{M}$ and $a_j = \alpha_j + J\beta_j$, $a_{-j} = \alpha_j - J\beta_j$, $\alpha_j, \beta_j \in \mathbb{R}^{2N}$ with $\alpha_j - \sqrt{-1}\beta_j$ being an eigenvector of the eigenvalue $\lambda_j = e^{2j\pi \sqrt{-1}/k}$ of $\mathcal{M}^{-1}$ for $j \in \mathbb{Z}$. We set

$$W_{\mathcal{M},l} = \left\{ z \in W_\mathcal{M} \mid z(t) = \sum_{|j|=(l-1)k+1}^{l} \exp\left(\frac{2j\pi t}{k}\alpha_j\right) a_j \right\}, \quad l \in \mathbb{N}$$

and

$$W^n_\mathcal{M} = \bigoplus_{l=0}^{n} W_{\mathcal{M},l}.$$ 

We denote by $P_n : W_\mathcal{M} \rightarrow W^n_\mathcal{M}$ the projection map. In $W^{1/2,2}(S^1, \mathbb{R}^{2N})$ we define the self-adjoint operator $A : W^{1/2,2}(S^1, \mathbb{R}^{2N}) \rightarrow W^{1/2,2}(S^1, \mathbb{R}^{2N})$ by extending the following operator

$$\langle Ax, y \rangle = \int_0^k (J^{-1} \dot{x}(t), y(t)) \, dt$$

and $A_\mathcal{M} = A|_{W_\mathcal{M}}$. It is easy to see that $\Gamma = \{P_n \mid n = 1, 2, \ldots\}$ is the Galerkin approximation sequence of $A_\mathcal{M}$:

1. $W^n_\mathcal{M} := P_n W_\mathcal{M}$ is finite dimensional for all $n \in \mathbb{N}$,
2. $P_n \rightarrow I$ strongly as $n \rightarrow +\infty$,
3. $P_n A_\mathcal{M} - A_\mathcal{M} P_n \rightarrow 0$ as $n \rightarrow +\infty$.

For a self-adjoint operator $S$, we denote by $M^*(S)$ the eigenspaces of $S$ with eigenvalues belonging to $(0, +\infty)$, $\{0\}$ and $(-\infty, 0)$ with $* = +, 0$ and $* = -$, respectively. We denote $m^*(S) = \dim M^*(S)$. Similarly, we denote by $M^*_d(S)$ the $d$-eigenspaces of $S$ with eigenvalues belonging to $(d, +\infty)$, $(-d, d)$ and $(-\infty, -d)$ with $* = +, 0$ and $* = -$, respectively. We denote $m^*_d(S) = \dim M^*_d(S)$.

Let $Q = A_\mathcal{M} - B_\mathcal{M} : W_\mathcal{M} \rightarrow W_\mathcal{M}$ with $B_\mathcal{M} : W_\mathcal{M} \rightarrow W_\mathcal{M}$ the compact self-adjoint operator defined by

$$\langle B_\mathcal{M} x, y \rangle = \int_0^k (\dot{B}(t)x(t), y(t)) \, dt$$

for a symmetric matrix function $\dot{B} \in C([0, k], \text{L}(\mathbb{R}^{2N}))$ satisfying $\mathcal{M}^T \dot{B}(t + 1) \mathcal{M} = \dot{B}(t)$. We note that $A_\mathcal{M}$ is a bounded self-adjoint operator and $\mathcal{N} := \ker Q$ is finite dimensional subspace of $W_\mathcal{M}$. We denote by $P : W_\mathcal{M} \rightarrow \mathcal{N}$ the orthogonal projection. Set $d = \frac{1}{4} \|Q_{|\mathcal{N}^\perp}\|^{-1}$.

**Lemma 2.1.** There exists $n_0$ such that for all $n \geq n_0$, there holds

$$m^-(P_n(Q + P)P_n) = m^-_d(P_n(Q + P)P_n)$$

(2.1)
and

\[ m^- (P_n(Q + P)P_n) = m^- (P_nQ P_n). \]  

(2.2)

**Proof.** The proof of (2.1) is essentially the same as in [4]. We follow the idea of [20] to prove (2.2). We note that \( \dim \ker (Q + P) = 0. \)

By considering the operators \( Q + sP \) and \( Q - sP \) for small \( s > 0 \), for example \( s < \min\{1, d/2\} \), there exists \( m_1 \in \mathbb{N} \) such that

\[ m^- (P_mQ P_m) \leq m^- (P_m(Q + sP)P_m), \quad \forall m \geq m_1 \]  

(2.3)

and

\[ m^- (P_mQ P_m) \geq m^- (P_m(Q - sP)P_m) - m^0_d (P_mQ P_m), \quad \forall m \geq m_1. \]  

(2.4)

In fact, the claim (2.3) follows from

\[ P_m(Q + sP)P_m = P_mQ P_m + sP_mPP_m \]

and for \( x \in M^- (P_mQ P_m), \)

\[ (P_m(Q + sP)P_m x, x) \leq -d\|x\|^2 + s\|x\|^2 \leq -\frac{d}{2}\|x\|^2. \]

The claim (2.4) follows from that for \( x \in M^- (P_m(Q - sP)P_m), \)

\[ (P_mQ P_m x, x) \leq s(P_mPP_m x, x) < d\|x\|^2. \]

By the Floquet theory, for \( m \geq m_1 \) we have \( m^0_d (P_mQ P_m) = \dim N = \dim \text{Im}(P_mPP_m), \) and by \( \text{Im}(P_mPP_m) \subseteq M^0_d (P_mQ P_m) \) we have \( \text{Im}(P_mPP_m) = M^0_d (P_mQ P_m). \) It is easy to see that \( M^0_d (P_mQ P_m) \subseteq M^- (P_m(Q + sP)P_m) \). By using

\[ P_m(Q - sP)P_m = P_m(Q + sP)P_m - 2sP_mPP_m \]

we have

\[ m^- (P_m(Q - sP)P_m) \geq m^- (P_m(Q + sP)P_m) + m^0_d (P_mQ P_m), \quad \forall m \geq m_1. \]  

(2.5)

Now (2.2) follows from (2.3)–(2.5). \( \square \)

**Lemma 2.2.** The difference of the d-Morse indices

\[ m^- (P_n(A_M - B_M)P_n) - m^- (P_nA_M P_n) \]  

(2.6)

and \( m^0_d (P_n(A_M - B_M)P_n) \) eventually becomes a constant independent of \( n \), and for large \( n \) there holds

\[ m^0_d (P_n(A_M - B_M)P_n) = m^0 (A_M - B_M). \]  

(2.7)

Lemma 2.2 is a direct consequence of Theorem 2.2 in [3] and Lemma 2.1.
Definition 2.3. We define the relative index by
\[ I(\mathcal{A}, \mathcal{A} - \mathcal{B}) = \tilde{m}_n - (P_n(\mathcal{A} - \mathcal{B})P_n) - \tilde{m}_n(\mathcal{A}P_n), \quad n \geq n^*, \]  
where \( n^* > 0 \) is a constant large enough such that the difference in (2.6) becomes a constant independent of \( n \geq n^* \). In this case, we define
\[ i^J_{\mathcal{A}}(\tilde{\mathcal{B}}) = I(\mathcal{A}, \mathcal{A} - \mathcal{B}), \quad \nu^J_{\mathcal{A}}(\tilde{\mathcal{B}}) = m^0(\mathcal{A} - \mathcal{B}). \]

Remark. By the Floquet theory, the two definitions of \( \nu^J_{\mathcal{A}}(\tilde{\mathcal{B}}) \) here and in Definition 1.2 are essentially the same. The \((P, J)\)-index \( (i^J_{\mathcal{A}}(\tilde{\mathcal{B}}), \nu^J_{\mathcal{A}}(\tilde{\mathcal{B}})) \) for \( J = J \) and \( \mathcal{M} = P \in \text{Sp}(2N) \) was defined in [19]. It may be proved that up to a constant, the two definitions are the same. Up to a sign, the relative index \( I(\mathcal{A}, \mathcal{A} - \mathcal{B}) \) is exactly the spectral flow of the operator family \( \mathcal{A} - s\mathcal{B} \) (cf. [24]). Precisely there holds
\[ I(\mathcal{A}, \mathcal{A} - \mathcal{B}) = -sf(\mathcal{A} - s\mathcal{B}). \]

Here \( sf(A) \) is the spectral flow of the bounded self-adjoint operator \( A \). A similar way to define the relative index of two operators was appeared in [3]. A different way to study the relative index theory was appeared in [7]. Since \( \exp(\text{sp}(2N)) \neq \text{Sp}(2N) \), the proofs of Galerkin approximation formula (Theorem 4.3 in [19]) and the saddle point reduction formula (Theorem 5.2 in [19]) are not complete. But they are valid for the case when \( P \in \exp(\text{sp}(2N)) \). We note that \( \text{Sp}(2N) \cap \text{O}(2N) \subseteq \exp(\text{sp}(2N)) \).

Definition 2.4. If \( z(t) \) is a solution of the nonlinear problem (1.1) and \( B_z(t) = H''(t, z(t)) \), we define
\[ i^J_{\mathcal{A}}(z) = i^J_{\mathcal{A}}(B_z), \quad \nu^J_{\mathcal{A}}(z) = \nu^J_{\mathcal{A}}(B_z). \]

3. The saddle point reduction

As in the general setting of problem (1.1), we should consider the functional \( \varphi \) defined in \( E^\sigma \). For simplicity, we choose \( \tau = 1 \) and \( \mathcal{J} = J \), \( \mathcal{M} = P \in \text{Sp}(2N) \). To describe the space \( E^\sigma \), we write the element \( z \) in the Fourier series as
\[ z(t) = a_0 + \sum_{j=1}^{\infty} a_j \cos \frac{2j\pi t}{k} + b_j \sin \frac{2j\pi t}{k}, \quad a_0, a_j, b_j \in \mathbb{R}^{2N}. \]

\( z \in E^\sigma \) iff \( z \in E \) and \( a_0 \) belongs to the eigenspace of the eigenvalue 1 of \( P \) and \( a_j + \sqrt{-1} b_j \) belongs to the eigenspace of the eigenvalue \( \lambda_j = e^{2j\pi \sqrt{-1}/k} \) of \( P^{-1} \). If we write \( z \) as
\[ z(t) = \sum_{j \in \mathbb{Z}} \exp \left( \frac{2j\pi t}{k} J \right) a_j, \quad a_j \in \mathbb{R}^{2N}, \]
then \( z \in E^\sigma \) iff \( z \in E \) and \( a_0 \) is an eigenvector of the eigenvalue 1 of \( P \) and \( a_j = a_{\alpha_j + j\beta_j}, a_{-j} = a_j - j\beta_j \), \( a_j, \beta_j \in \mathbb{R}^{2N} \) with \( a_j - \sqrt{-1}\beta_j \) being an eigenvector of the eigenvalue \( \lambda_j = e^{2j\pi \sqrt{-1}/k} \) of \( P^{-1} \) for \( j \in \mathbb{Z} \).

We suppose the following conditions on \( H \).

(H1) \( H \in C^2(\mathbb{R} \times \mathbb{R}^{2N}) \) and \( H(t + 1, Pz) = H(t, z) \).
(H2) There exists a constant $c(H) > 0$ such that
\[ |H''(t, x)| \leq c(H), \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^{2N}. \]

Let $W = W^{1,2}(S^1, \mathbb{R}^{2N})$ with $S^1 = \mathbb{R}/(k\mathbb{Z})$ and $L^2 = L^2(S^1, \mathbb{R}^{2N})$. The $\sigma$-fixed subset of $W$ is $W^\sigma$ and of $L^2$ is $L^\sigma$. In the Hilbert space $L^\sigma$ we define an operator $Ax = -J_N \dot{x}$. We denote $\varphi^\sigma = \varphi|_{W^\sigma}$. A critical point of $\varphi^\sigma$ is a solution of the problem (1.1). We rewrite $\varphi^\sigma$ as
\[ \varphi^\sigma (x) = \frac{1}{2} \langle Ax, x \rangle_2 - g(x), \quad x \in W^\sigma, \quad (3.2) \]
where
\[ \langle Ax, x \rangle_2 = \int_0^k \left( -J_N \dot{x}(t), x(t) \right) dt = \int_0^1 \left( -J_N \dot{x}(t), x(t) \right) dt \quad (3.3) \]
and
\[ g(x) = \int_0^k H(t, x(t)) dt = \int_0^1 H(t, x(t)) dt. \quad (3.4) \]

The kernel of the linear operator $A : L^\sigma \to L^\sigma$ is $E_0 = \{ v \in \mathbb{R}^{2N} : P v = v \}$ with $\dim E_0 \in 2N$. $\varphi^\sigma$ is a nonlinear functional on $L^\sigma$ with the domain in $W^\sigma$. When $E_0 \neq \{0\}$, we define by $P_0 : L^\sigma \to E_0$ the projection map and
\[ A_0 : L^\sigma \to L^\sigma \]
by
\[ A_0 x = Ax + P_0 x, \quad \forall x \in W^\sigma. \quad (3.5) \]

The dom $A_0 = W^\sigma$ is self-adjoint and invertible, its range is closed, and its resolution is compact. The spectrum $\sigma(A_0)$ of $A_0$ is a point spectrum.

We choose $\beta \notin \sigma(A_0)$ and $\beta > 2(c(H) + 1)$ with the constant $c(H)$ defined in (H2). Denote by $\{E_\lambda\}$ the spectral resolution of the self-adjoint operator $A_0$, we define the projections on the Hilbert space $L$ by
\[ P = \int_{-\beta}^\beta dE_\lambda, \quad P^{+} = \int_{\beta}^{+\infty} dE_\lambda, \quad P^{-} = \int_{-\infty}^{-\beta} dE_\lambda. \quad (3.6) \]

Then the Hilbert space $L^\sigma$ possesses an orthogonal decomposition
\[ L^\sigma = L^\sigma_+ \oplus L^\sigma_- \oplus Z, \quad (3.7) \]
where $Z = P E_\sigma$ is a finite dimensional space, and $L^\pm_{\sigma} = P^\pm L^\sigma$. 
Next let
\[ S^\pm = \int_{-\infty}^{\beta} (\lambda)^{-1/2} dE_\lambda, \quad S^- = \int_{-\infty}^{\beta} (\lambda)^{-1/2} dE_\lambda, \quad R = \int_{-\beta}^{\beta} |\lambda|^{-1/2} dE_\lambda. \]

By noting that \( S^\pm g'_0(v) \) are contraction mappings with \( g_0(z) = g(z) + \frac{1}{2}(P_0 z, z) \), one can solve the equations
\[ u^\pm = \pm S^\pm g'_0(S^+ u^+ + S^- u^- + Rz), \quad \forall z \in Z \]
to get the mappings \( u^\pm = u^\pm(z) \). Define \( w(z) = S^+ u^+(R^{-1} z) + S^- u^-(R^{-1} z) \) and \( u(z) = w(z) + z \).

With standard arguments as in [1,2,21], we have the following result.

**Theorem 3.1.** Suppose the function \( H \) satisfies the conditions (H1) and (H2). Then there exist a functional \( a \in C^2(Z, \mathbb{R}) \) and an injection map \( u \in C^1(Z, L_\sigma) \) such that \( u : Z \to W^\sigma \) satisfies the following conditions:

1° The map \( u \) has the form \( u(z) = w(z) + z \), where \( P w(z) = 0 \).
2° The functional \( a \) satisfies
\[
\begin{align*}
   a(z) &= \varphi^\sigma(u(z)), \\
   a'(z) &= A z - P g'(u(z)) = A u(z) - g'(u(z)), \\
   a''(z) &= (A P - P d g'(u(z))) u'(z) = [A - d g'(u(z))] u'(z).
\end{align*}
\]
And \( a' \) is globally Lipschitz continuous.
3° \( z \in Z \) is a critical point of \( a \), i.e., \( a'(z) = 0 \), if and only if \( u(z) \) is a critical point of \( \varphi^\sigma \).
4° If \( g(u) = (B u, u)_L = \int_0^L (B u(t), u(t)) \) for all \( u \in L \), where \( B \) is a constant symmetric matrix defined on \( \mathbb{R}^{2n} \) satisfying \( P^T BP = B \), then \( a(z) = \frac{1}{2} ((A - B) z, z)_L \).
5° \( \dim \ker a''(z) = \nu_P(\gamma) \), where \( \gamma \) is the fundamental solution of the linear Hamiltonian system \( \dot{y} = J H''(t, u(z)(t)) y \).

**Lemma 3.2.** Assume that \( H \) satisfies (H1), (H2) and \( H'(t, 0) = 0 \), then we have
\[ \| u^\pm(z) \| \leq \frac{2 \sqrt{\beta} (c(H) + 1) \tilde{c}}{\beta - 2 (c(H) + 1)} \| z \|, \quad \forall z \in Z, \quad (3.8) \]
where \( \tilde{c} \geq \| R \| \) is independent of \( \beta \). We have
\[ \| (u^\pm)'(z) \| \to 0, \quad \text{as} \ \beta \to +\infty. \quad (3.9) \]
Here the norm \( \| \cdot \| \) is the \( L^2 \) norm.

**Proof.** Note that
\[ u^\pm(z) = \pm S^\pm g'_0(S^+ u^+(z) + S^- u^-(z) + Rz), \]
and \( H'(t, 0) = 0 \) implies \( g'(0) = 0 \). Let \( v(z) = S^+ u^+(z) + S^- u^-(z) + Rz \). Since \( \| S^\pm \| \leq \frac{1}{\sqrt{\beta}} \), we have by (H2) that
\[ \| u^\pm(z) \| \leq \| S^+ g_0'(v(z)) \| + \left\| P_0 v(z) \right\| \leq \frac{1}{\sqrt{\beta}} \left\| g'(v(z)) \right\| + \frac{c(H) + 1}{\sqrt{\beta}} \| v(z) \| \] 

Therefore, 

\[ \| u^+(z) \| + \| u^-(z) \| \leq \frac{2 \sqrt{\beta}(c(H) + 1)\tilde{C}}{\beta - 2(c(H) + 1)} \| z \|. \]

It implies (3.8). Next, since 

\[ (u^\pm)'(z) = \pm S^\pm d g_0'(v(z)) (S^+(u^+)'(z) + S^-(u^-)'(z) + R), \]

we have 

\[ \| (u^+)'(z) \| + \| (u^-)'(z) \| \leq \frac{2 \sqrt{\beta}(c(H) + 1)\tilde{C}}{\beta - 2(c(H) + 1)}, \quad \forall z \in Z. \]

It implies (3.9). □

We set 

\[ W^\sigma_I = \left\{ \sum_{|j|=(l-1)k+1}^{lk} \exp\left( \frac{2j\pi tj}{k} \right) a_j \mid a_j \in W^\sigma \right\}, \quad l \in \mathbb{N}. \]

Then \( \dim W^\sigma_I = 2p \leq 4N \). For \( z = \sum_{|j|=(l-1)k+1}^{lk} \exp\left( \frac{2j\pi tj}{k} \right) a_j \in W^\sigma_I \), there holds 

\[ \langle Az, z \rangle_2 = \sum_{j=(l-1)k+1}^{lk} \frac{8\pi j}{k} (|a_j|^2 - |a_{-j}|^2) = \sum_{j=(l-1)k+1}^{lk} \frac{4\pi j}{k} (\sqrt{-1} J x_j, x_j) \]

with \( a_j = a_j + J\beta_j, a_{-j} = a_j - J\beta_j \) and \( a_j - \sqrt{-1}\beta_j \) being an eigenvector of \( P^{-1} \) for the eigenvalue of \( \lambda_j = e^{2j\pi \sqrt{-1}/k} \) and \( x_j = a_j + \sqrt{-1}\beta_j \). Here in the last summation \( \langle \cdot, \cdot \rangle \) is the standard Hermitian product in \( \mathbb{C}^{2N} \). Note that \( G := \sqrt{-1} J \) is Hermitian since \( J \) is antisymmetric. So \( (Gx, y) = (\sqrt{-1} J x, y) \) is a non-degenerate Hermitian structure in \( \mathbb{C}^{2N} \). Denote by \( E_\alpha = \{ a = \alpha - \sqrt{-1}\beta \mid a \text{ is an eigenvector of } \lambda \text{ of } P^{-1} \} \) the complex eigenspace of the eigenvalue \( \lambda \) of \( P^{-1} \) and \( \lambda \neq \mu \), then \( E_\alpha \) and \( E_\mu \) are \( G \)-orthogonal. We also note that if \( y_j = a_j - \sqrt{-1}\beta_j \in E_\lambda \), there holds \( \tilde{y}_j = a_j + \sqrt{-1}\beta_j \in E_\perp \) and \( (Gy_j, y_j) = -(G\tilde{y}_j, \tilde{y}_j) \). Suppose the total complex dimension of the space of eigenvectors of \( P^{-1} \) is \( p \) and since \( A \mid_{W^\sigma_{I0}} \) is non-degenerate, so the Morse index \( m^\pm(A \mid_{W^\sigma_{I0}}) = p \). We set \( W^\sigma_{I0} = \{ a \in \mathbb{R}^{2N} \cap W^\sigma \} \) \( \ker A \). Then \( W^\sigma = \bigoplus_{I=0}^{\infty} W^\sigma_I \). It is well known that, for a symplectic matrix \( P \), if \( \lambda \in \sigma(P) \), then \( \lambda^{-1}, \tilde{\lambda} \in \sigma(P) \).
Lemma 3.3. Suppose $M \in \text{Sp}(2N)$ and $M^k = \text{Id}$. If $1$ or $-1 \in \sigma(M)$, then it is semi-simple, i.e., the invariant root vector space and the eigenspace of $\pm 1$ are the same. That is

\[ \bigcup_{p \geq 1} \text{Ker}(M - \lambda \text{Id})^p = \text{Ker}(M - \lambda \text{Id}), \quad \lambda = \pm 1. \]

Proof. We prove it for $\lambda = 1$. By Theorem I.1.4.1 of [21, p. 18], there exist a matrix $U \in \text{Sp}(2N)$ and an integer $m \in \mathbb{N}$ such that

\[ U^{-1}MU = M_1 \circ \cdots \circ M_m \circ M_0. \]

Here $M_0 \in \text{Sp}(2N_0)$ with $N_0 \geq 0$ and $1 \notin \sigma(M_0)$. For $1 \leq i \leq m$, each $M_i \in \text{Sp}(2N_i)$ with $N_i \geq 1$ and

\[ M_i = \begin{pmatrix} A_i & B_i \\ 0 & C_i \end{pmatrix} \]

with $A_i, B_i, C_i$ the $N_i \times N_i$ matrices and $1 \in \sigma(A_i) \cap \sigma(C_i)$. $A_i$ is the following matrix

\[ A_i = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}. \]

When $N_i = 1$, $B_i$ should be $\pm 1$ or $0$. $M^k = \text{Id}$ implies $A_i^k = \text{Id}$, so the number $N_i = 1$ and the matrices $B_i = 0$ for $i = 1, \ldots, m$. It means that $M_1 \circ \cdots \circ M_m = \text{Id}$. The proof of the case $\lambda = -1$ is similar by use of Theorem I.1.5.1 in [21, p. 24] and we omit the details. \( \square \)

So $\dim W_0^\sigma = 2q$ is even. We set

\[ W^\sigma.n = \bigoplus_{l=0}^{n} W_l^\sigma. \tag{3.11} \]

Then $\dim W^\sigma.n = 2d_n$ is even.

Given $B \in C(S^1, L_2(\mathbb{R}^{2N}))$ with $P^TB(t + 1)P = B(t)$, it induces a symmetric operator on $L^\sigma$ by

\[ \langle Bx, y \rangle_2 = \int_0^k (B(t)x(t), y(t)) \, dt, \quad \forall x, y \in L_\sigma. \]

In this case, we have the following functional

\[ \varphi^\sigma(x) = \frac{1}{2} \langle (A - B)x, x \rangle_2, \quad \forall x \in W^\sigma \subset L^\sigma. \tag{3.12} \]

The functional $\varphi^\sigma : W^\sigma \to \mathbb{R}$ is smooth in the topology of $L^\sigma$. From the definition of $A, B$, it is easy to see that the operator $A - B$ defined on $\text{dom}A$ is linear self-adjoint, has compact resolution, and then its spectrum is a point spectrum. Moreover, $0 \in \sigma(A - B)$ if and only if $\nu_p(B) > 0$. The multiplicity of the eigenvalue $0$ of $A - B$ is precisely $\nu_p(B)$. When $\nu_p(B) = 0$, the operator $A - B$ possesses a bounded inverse.
Using the saddle point reduction method of the above, we obtain a subspace

\[ Z = \bigoplus_{l=0}^{n_0} W_l^\sigma \]

with a sufficiently large \( n_0 \in \mathbb{N} \), an injection map \( u \in C^\infty(Z, L^\sigma) \), and a smooth functional \( a \in C^\infty(Z, \mathbb{R}) \) defined by

\[ a(z) = \varphi^\sigma(u(z)), \quad \forall z \in Z. \]

Let \( 2d_{n_0} = \dim Z \). Note that the origin of \( Z \) as a critical point of \( a \) corresponds to the origin of \( L^\sigma \) as a critical point of \( \varphi^\sigma \). Denote by \( m^+ \), \( m^0 \) and \( m^- \) the positive, null, and negative Morse indices of the functional \( a \) at the origin respectively, i.e., the total multiplicities of positive, zero, and negative eigenvalues of the \( 2d_{n_0} \times 2d_{n_0} \) matrix \( a''(0) \) respectively.

From Lemma 2.2 and Definition 2.3, we have the following result.

**Theorem 3.4.** For sufficiently large \( n_0 \), there holds

\[
\begin{align*}
    m^- &= d_{n_0} - \mu(P) + i_P^J(N)(B), \\
    m^0 &= v_P^J(N)(B), \\
    m^+ &= d_{n_0} + \mu(P) - i_P^J(N)(B) - v_P^J(N)(B),
\end{align*}
\]

where \( \mu(P) \in [0, N] \) is defined by \( 2\mu(P) = \dim \ker(P - \text{id}) \).

**Remark 3.5.** For the problem (1.1), if the skew-symmetric matrix \( J \) furthermore satisfies \( JJ_N = J_NJ \) (see the claim just before Section 1.6), the arguments in this section are valid with the operator \( A \) defined by

\[ \langle Ax, y \rangle = \int_0^1 (J^{-1}x(t), y(t)) \, dt. \]

So the saddle point reduction formulas in Theorem 3.4 are valid for the general index pair \((i_P^M(B), v_P^M(B))\).

### 4. \( M \)-boundary value problem for asymptotically linear Hamiltonian system

Given a non-degenerate skew-symmetric matrix \( J \) with \( JJ_N = J_NJ \) and \( J \)-symplectic matrix \( M \in \text{Sp}_J(2N) \) with \( M^k = \text{id} \), for a function \( H \in C^2(\mathbb{R} \times \mathbb{R}^{2N}, \mathbb{R}) \) with \( H(t + 1, Mx) = H(t, x) \), we consider the following \( M \)-boundary value problem

\[
\begin{align*}
    \dot{x}(t) &= J \nabla H(t, x(t)), \\
    x(1) &= Mx(0).
\end{align*}
\]

We assume the following conditions on the function \( H \).

**(H1)** There exist constants \( a > 0 \) and \( p > 1 \) such that

\[ |H''(t, x)| \leq a(1 + |x|^p), \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^{2N}. \]
(H2) There exists a constant $C > 0$ such that
\[ |H''(t, x)| \leq C, \quad (t, x) \in [0, 1] \times \mathbb{R}^{2N}. \]

(H$_\infty$) There exists a continuous symmetric matrix function $B_\infty(t)$ such that
\[ \mathcal{M}^T B_\infty(t + 1) \mathcal{M} = B_\infty(t), \quad \forall t \in \mathbb{R} \]
and
\[ |H'(t, x) - B_\infty(t) x| = o(|x|) \]
uniformly in $t \in \mathbb{R}$ as $|x| \to \infty$.

We remind that for a matrix function $B = B(t)$ satisfying $\mathcal{M}^T B(t + 1) \mathcal{M} = B(t)$, the index pair $(i^\mathcal{M}_{\mathcal{J}}(B), \nu^\mathcal{M}_{\mathcal{J}}(B))$ is defined by Definition 2.3. In the following result, we suppose $\nu^\mathcal{J}_{\mathcal{M}}(B_{\infty}) = 0$ and $i_\infty = i^\mathcal{M}_{\mathcal{J}}(B_{\infty})$.

**Theorem 4.1.** Suppose $H$ satisfies conditions (H2), (H$_\infty$). Then the problem (4.1) possesses at least one solution $x_0$. Let $B_0(t) = H''(t, x_0(t))$ and $(i_0, \nu_0) = (i^\mathcal{M}_{\mathcal{J}}(B_0), \nu^\mathcal{M}_{\mathcal{J}}(B_0))$. If
\[ i_\infty \notin \{i_0, i_0 + \nu_0\}, \quad (4.2) \]
the problem (4.1) possesses at least two solutions. Furthermore, if $x_0$ is not pseudo-degenerated, and
\[ i_\infty \notin \{i_0 - 2n, i_0 + \nu_0 + 2n\}, \quad (4.3) \]
the problem (4.1) possesses at least three solutions.

**Remark.** $x_0$ is a critical point of $f$ defined by
\[ f(x) = \frac{1}{2} \int_0^1 \langle J^{-1} \dot{x}(t), x(t) \rangle \, dt - \int_0^1 H(t, x(t)) \, dt. \]

Since $\mathcal{J} J_H = J_H \mathcal{J}$, by Remark 3.5, the saddle point reduction arguments in Section 3 are valid. By Theorem 3.1, $z_0 = \mathcal{P} x_0$ is a critical point of $a$. Suppose the critical set of $a$ is isolated. Then $z_0$ is an isolated invariant set of the gradient flow of $a$. By Conley homotopic index theory, we get the Conley homotopic index $h(z_0)$, and its Poincaré polynomial
\[ p(t, h(z_0)) = t^{m^\ast(z_0)} \sum_{j=0}^{m^0(z_0)} a_j t^j, \quad a_j \in \{0, 1, 2, \ldots\}, a_0 = 0 \text{ or } a_{m^0(z_0)} = 0. \]

$m^\ast(z_0), * = 0, \pm$ are defined in Lemma 3.4. We say that $x_0$ is pseudo-degenerated if $p(t, h(z_0)) = 0$ or contains the factor $(1 + t)$. Theorem 4.1 should be compared with the main result of [23] (see also Theorem 7.2.2 of [22] and Theorem 4.1.3 of [2]) where the authors considered the periodic solutions of asymptotically Hamiltonian systems. The main ingredients of the proof are the Maslov-type index theory, the Poincaré polynomial of the Conley homotopic index of isolated invariant set, and the saddle point reduction methods. The Conley homotopic index theory can be used here for the proof of Theorem 4.1. Now by using the index theory and the saddle point methods developed in Section 3, we can prove Theorem 4.1 as done in [22] and [23]. We omit the details.
By the index theory and the Galerkin approximation methods developed in Section 2, similar to [3] and [8], we have the following result.

**Theorem 4.2.** Suppose $H$ satisfies conditions (H1), (H∞) and:

(H3) There exists continuous symmetric matrix function $B_0(t)$ such that for all $t \in \mathbb{R}$, $\mathcal{M}^T B_0(t + 1) \mathcal{M} = B_0(t)$ and

$$H'(t, x) = B_0(t)x + o(|x|) \quad \text{as } |x| \to 0 \text{ uniformly in } t. \quad (4.4)$$

(H4) For $h(t, x) = H(t, x) - \frac{1}{2}(B_\infty(t)x, x)$ with $(t, x) \in \mathbb{R} \times \mathbb{R}^{2N}$, either

$$h(t, x) \to 0, \quad |h'(t, x)| \to 0 \quad \text{as } |x| \to +\infty \text{ uniformly in } t \quad (4.5)$$

or

$$h(t, x) \to \pm \infty, \quad |h'(t, x)| = 0 \quad \text{as } |x| \to \infty \text{ uniformly in } t. \quad (4.6)$$

Then the problem (4.1) possesses a nontrivial solution, provided that

$$\left[ i_{\mathcal{M}}^J(B_0), i_{\mathcal{M}}^J(B_0) + v_{\mathcal{M}}^J(B_0) \right] \cap \left[ i_{\mathcal{M}}^J(B_\infty), i_{\mathcal{M}}^J(B_\infty) + v_{\mathcal{M}}^J(B_\infty) \right] = \emptyset. \quad (4.7)$$

The above two results were proved in [19] for the special case: $\mathcal{J} = J$ and the orthogonal symplectic matrix $\mathcal{M}$. After developing the Galerkin approximation theorem and the saddle point reduction theorem with the index pair $(i_{\mathcal{M}}^J(B), v_{\mathcal{M}}^J(B))$ in general setting, the proofs of the above results are formal as in [22] and [3], we omit the details here.

5. Asymptotically linear delay differential systems and Hamiltonian systems

5.1. Asymptotically linear delay differential systems

For function $V \in C^2(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ with 1-period in variable $t$ and even in variables $x$, we consider the $2m$-periodic solutions of the following delay differential system

$$x'(t) = \nabla V(t, x(t - 1)) + \nabla V(t, x(t - 2)) + \cdots + \nabla V(t, x(t - (m - 1))). \quad (5.1)$$

**Theorem 5.1.** Suppose $V$ satisfies the following conditions

(V1) There exists a constant $C > 0$ such that

$$|V''(t, x)| \leq C, \quad (t, x) \in [0, 1] \times \mathbb{R}^n.$$

(V∞) There exists a continuous symmetric matrix function $C_\infty(t)$ such that

$$V'(t, x) = C_\infty(t)x + o(|x|)$$

uniformly in $t \in \mathbb{R}$ as $|x| \to \infty$. 
Let $C_0(t) = V''(t, 0)$. For $m \in 2\mathbb{N}$

\[
B_0(t) = \begin{pmatrix}
C_0(t) & 0 & \cdots & 0 \\
0 & C_0(t) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & C_0(t)
\end{pmatrix},
\]

and

\[
B_{\infty}(t) = \begin{pmatrix}
C_{\infty}(t) & 0 & \cdots & 0 \\
0 & C_{\infty}(t) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & C_{\infty}(t)
\end{pmatrix}.
\]

Set $(i_0, v_0) = (i_{\mathcal{J}}(B_0), v_{\mathcal{J}}(B_0))$, $(i_{\infty}, v_{\infty}) = (i_{\mathcal{J}}(B_{\infty}), v_{\mathcal{J}}(B_{\infty}))$ with $\mathcal{J} = \mathcal{A}_m$, $\mathcal{M} = T_m^{-1}$ defined in Section 1.

For $m \in 2\mathbb{N} + 1$,

\[
B_0(t) = \begin{pmatrix}
2C_0(t) & -C_0(t) & C_0(t) & \cdots & C_0(t) & -C_0(t) \\
-C_0(t) & 2C_0(t) & -C_0(t) & \cdots & -C_0(t) & C_0(t) \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-C_0(t) & C_0(t) & -C_0(t) & \cdots & -C_0(t) & 2C_0(t)
\end{pmatrix},
\]

and

\[
B_{\infty}(t) = \begin{pmatrix}
2C_{\infty}(t) & -C_{\infty}(t) & C_{\infty}(t) & \cdots & C_{\infty}(t) & -C_{\infty}(t) \\
-C_{\infty}(t) & 2C_{\infty}(t) & -C_{\infty}(t) & \cdots & -C_{\infty}(t) & C_{\infty}(t) \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-C_{\infty}(t) & C_{\infty}(t) & -C_{\infty}(t) & \cdots & -C_{\infty}(t) & 2C_{\infty}(t)
\end{pmatrix}.
\]

Set $(i_0, v_0) = (i_{\mathcal{J}}(B_0), v_{\mathcal{J}}(B_0))$, $(i_{\infty}, v_{\infty}) = (i_{\mathcal{J}}(B_{\infty}), v_{\mathcal{J}}(B_{\infty}))$ with $\mathcal{J} = \mathcal{A}_{m-1}$, $\mathcal{M} = \tilde{B}_{m-1}^{-1}$ defined in Section 1.

If

\[
i_{\infty} \notin [i_0, i_0 + v_0],
\]

the delay differential system (5.1) possesses at least one nontrivial $2m\tau$-periodic solution $x$ with $x(t - m\tau) = -x(t)$. Furthermore, if the trivial solution $z_0 = 0$ of problem (4.1), with $H(t, z) = V(t, x_1) + \cdots + V(t, x_m)$ for even $m$ and $H(t, z) = V(t, x_1) + \cdots + V(t, x_{m-1}) + V(t, -x_1 + x_2 - \cdots + x_{m-1})$ for odd $m$, is not pseudo-degenerated, and

\[
i_{\infty} \notin [i_0 - 2N, i_0 + v_0 + 2N],
\]

the delay differential system (5.1) possesses at least two nontrivial $2m\tau$-periodic solutions as above.

**Proof.** Since $V(t, -x) = V(t, x)$, $z_0 = 0$ is the trivial solution of problem (4.1). Here $H(t, z) = V(t, x_1) + \cdots + V(t, x_m)$, $z = (x_1, \ldots, x_m)$ for even $m$ and $H(t, z) = V(t, x_1) + \cdots + V(t, x_{m-1}) + V(t, -x_1 + x_2 - \cdots + x_{m-1})$, $z = (x_1, \ldots, x_{m-1})$ for odd $m$. 


It is clear that the condition (V1) implies (H2) in Theorem 4.1. For \( m \in 2\mathbb{N} \),
\[
\frac{|H'(t, z) - B_\infty(t)z|}{|z|} = \sum_{j=1}^{m} \frac{|V'(t, x_j) - C_\infty(t)x_j| |x_j|}{|z|}
\]
We only need to deal with all \( x_j \neq 0 \). Since \( V \in C^2 \), (V1) implies that \( \frac{|V'(t, x_j) - C_\infty(t)x_j|}{|x_j|} \) is bounded. So when \( \frac{|V'(t, x_j) - C_\infty(t)x_j|}{|x_j|} \to 0 \), then \( \frac{|x_j|}{|z|} \to 0 \) as \( |z| \to \infty \). Since \( \frac{|x_j|}{|z|} \leq 1 \), there holds
\[
\frac{|H'(t, z) - B_\infty(t)z|}{|z|} \to 0, \quad \text{as } |z| \to 0.
\]
The case \( m \in 2\mathbb{N} + 1 \) is similar, we note that \( |z| \to \infty \) if and only if \( (x_1, \ldots, x_{m-1}, -x_1 + x_2 - \cdots + x_{m-1}) \to \infty \). So the condition \( (V_\infty) \) implies \( (H_\infty) \) in Theorem 4.1. Now the result follows from Theorem 4.1. \( \square \)

**Remark 5.2.** (i) Since \( V(t, -x) = V(t, x) \), so the solutions of Eq. (5.1) appear in pairs. That is to say if \( x \) is a solution of (5.1), so is for \(-x\).

(ii) When \( m = n = 1 \), the problem (5.1) was considered by Kaplan and Yorke in [15] with \( V'(t, x) = -f(x) \). Hence \( \mathcal{M} = \mathcal{J} = J_1 \). In this case, for a constant matrix function \( B(t) = aId_2 \), the index \( i^J_{\mathcal{M}}(\bar{B}) = 2j \) if
\[
\frac{(4j - 1)\pi}{2} < a \leq \frac{(4j + 3)\pi}{2},
\]

\( v^J_{\mathcal{M}}(\bar{B}) = 2 \) if \( a = \frac{(4j - 1)\pi}{2} \) and \( v^J_{\mathcal{M}}(\bar{B}) = 0 \) otherwise. Since \( V'(t, x) = -f(x) \), we should choose \( B_\infty = -\alpha ld \) and \( B_0 = -\beta ld \) to compute the index. If \( \alpha < \frac{\pi}{2} < \beta \) or in general \( \min(\alpha, \beta) < \frac{(1-4)i\pi}{2} \leq \max(\alpha, \beta) \), then \( |i_\infty - i_0| \geq 2 \). Furthermore, if \( v_\infty \neq 0 \) or \( v_0 \neq 0 \), then \( |i_\infty - i_0| \geq 4 \). This means that \( i_\infty \notin [i_0, i_0 + v_0] \). So when we replace the condition \( xf(x) > 0 \) for \( x \neq 0 \) with \( f'(x) \) bounded and with all other conditions in Theorem 1.1 of [15], the result is still true, i.e., there exists a pair nontrivial periodic solution \( x, -x \) of Eq. (1.4) with \( x(t + 2\tau) = -x(t) \). If \( \alpha < \frac{(1-4)i\pi}{2} \) and \( \beta > \frac{(3-4i)\pi}{2} \), then it is easy to see that (5.7) holds, so there exist at least two pair nontrivial periodic solutions. We note that the function \( H(t, z) \) is even in \( z \) since \( V(t, x) \) is even in \( x \), so the solutions appear in pairs. Moreover, if \( v_0 = 0 \), Eq. (5.1) may possess at least \( |i_\infty - i_0| - 1 \) pair nontrivial solutions under some conditions on the Hessian \( H''(t, z) \). We will prove this multiple result in a sequel paper.

Similarly, as consequences of Theorem 4.2, we have the following two results.

**Theorem 5.3.** Suppose \( V \) satisfies the condition \( (V_\infty) \) and the following conditions:

(V2) There exist constants \( a > 0 \) and \( p > 1 \) such that
\[
|V''(t, x)| \leq a(1 + |x|^p), \quad \forall (t, x) \in [0, 1] \times \mathbb{R}^n.
\]

(V3) The matrix function \( C_0(t) := V''(t, 0) \) satisfies
\[
V'(t, x) = C_0(t)x + o(|x|) \quad \text{as } |x| \to 0 \text{ uniformly in } t \in [0, 1].
\]

(V4) For \( v(t, x) = V(t, x) - \frac{1}{2}(C_\infty(t)x, x) \) with \( (t, x) \in \mathbb{R} \times \mathbb{R}^n \), either
\[
v(t, x) \to 0, \quad |v'(t, x)| \to 0 \quad \text{as } |x| \to +\infty \text{ uniformly in } t \quad (5.8)
\]
or
\[ v(t, x) \to \pm \infty, \quad |v(t, x)| = 0 \quad \text{as} \quad |x| \to \infty \quad \text{uniformly in} \quad t. \] (5.9)

Then the delay differential system (5.1) possesses a nontrivial 2m\tau-periodic solution \( x \) with \( x(t - m\tau) = -x(t) \), provided that
\[
\left[i^\mathcal{J}_M(B_0), i^\mathcal{J}_M(B_0) + v^\mathcal{J}_M(B_0)\right] \cap \left[i^\mathcal{J}_M(B_\infty), i^\mathcal{J}_M(B_\infty) + v^\mathcal{J}_M(B_\infty)\right] = \emptyset, \tag{5.10}
\]
where in (5.10) when \( m \in 2\mathbb{N}, \mathcal{J} = A_m, \mathcal{M} = T_m^{-1}, B_0(t) \) and \( B_\infty(t) \) are defined as in (5.2), (5.3), when \( m \in 2\mathbb{N} + 1, \mathcal{J} = A_{m-1}, \mathcal{M} = B_m^{-1}, B_0(t) \) and \( B_\infty(t) \) are defined as in (5.4), (5.5) with \( C_0(t) \) and \( C_\infty(t) \) defined in (V3) and (V\infty) respectively.

**Proof.** Take \( H(t, z) = V(t, x_1) + \cdots + V(t, x_m) \) for even \( m \) and \( H(t, z) = V(t, x_1) + \cdots + V(t, x_{m-1}) + V(t, -x_1 + x_2 - \cdots + x_{m-1}) \), \( z = (x_1, \ldots, x_m) \) for odd \( m \) in Theorem 4.2. It is easy to see that \( V \) satisfying the condition (V\infty) implies \( H \) satisfying the condition (H\infty). Also (V2) implies (H1), (V3) implies (H3) and (H4) follows from (V4). \( \square \)

5.2. First order delay Hamiltonian systems

For a function \( G \in C^2(\mathbb{R} \times \mathbb{R}^{2n}, \mathbb{R}) \) with \( G(t + \tau, x) = G(t, x) \), we consider the periodic solutions of the following first order delay Hamiltonian system
\[
\dot{x}(t) = J_n(\nabla G(t, x(t - \tau)) + \nabla G(t, x(t - 2\tau)) + \cdots + \nabla G(t, x(t - (m - 1)\tau))). \tag{5.11}
\]

Let \( z = (x_1, \ldots, x_m) \) and \( H(t, z) = G(t, x_1) + \cdots + G(t, x_m) \).

The following results are also consequences of Theorem 4.1 and Theorem 4.2.

**Theorem 5.4.** Suppose \( G \) satisfies the conditions (V1) and (V\infty) in Theorem 5.1. Then the system (5.11) possesses an \( m\tau \)-periodic solution \( x_0 \). Suppose \( z_0(t) \) is the solution of (1.17) corresponding to \( x_0 \). Let \( B_0(t) = H^\mathcal{J}(t, z_0(t)) \) and
\[
B_\infty(t) = \begin{pmatrix}
C_\infty(t) & 0 & \cdots & 0 \\
0 & C_\infty(t) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & C_\infty(t)
\end{pmatrix}.
\]

Set \((i_0, v_0) = (i^\mathcal{J}_M(B_0), v^\mathcal{J}_M(B_0)), (i_\infty, v_\infty) = (i^\mathcal{J}_M(B_\infty), v^\mathcal{J}_M(B_\infty))\) with \( \mathcal{J} = J_{n,m}, \mathcal{M} = P_m^{-1} \) defined in Section 1. If (4.2) holds, the system (5.11) possesses at least two \( m\tau \)-periodic solutions. Furthermore, if \( z_0 \) is not pseudo-degenerated, and (4.3) holds, then the system (5.11) possesses at least three \( m\tau \)-periodic solutions.

**Theorem 5.5.** Suppose \( G \) satisfies the conditions (V2), (V3), (V4) and (V\infty) in Theorem 5.3. Then the system (5.11) possesses a nontrivial solution, provided the twisted condition (5.10) holds with \( \mathcal{J} = J_{n,m}, \mathcal{M} = P_m^{-1} \) and the matrix functions \( B_0(t), B_\infty(t) \) are defined as in (5.2), (5.3) with \( C_0(t) \) and \( C_\infty(t) \) defined in (V3) and (V\infty) respectively.

The proofs of the above two results are the same as in that of Theorem 5.1 and Theorem 5.3, respectively. We omit the details.
5.3. Second order delay Hamiltonian systems

For a function \( V \in C^2(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}) \) with \( V(t + \tau, x) = V(t, x) \), we consider the \( m\tau \)-periodic solutions of the following second order delay Hamiltonian system

\[
\ddot{x}(t) = -\left[ \nabla V(t, x(t - \tau)) + \nabla V(t, x(t - 2\tau)) + \cdots + \nabla V(t, x(t - (m-1)\tau)) \right]. \tag{5.12}
\]

Let \( z = (x, y) = (x_1, \ldots, x_m, y_1, \ldots, y_m) \), and \( \mathcal{H}(t, z) = -\frac{1}{2}(A_m y, y) - V(t, x_1) - \cdots - V(t, x_m) \). From Theorem 4.1 and Theorem 4.2, we obtain the following results.

**Theorem 5.6.** Suppose \( V \) satisfies the conditions \((V1)\) and \((V\infty)\) in Theorem 5.1. Then the system (5.12) possesses an \( m\tau \)-periodic solution \( x_0 \). Suppose \( w_0(t) \) is the solution of (1.19) corresponding to \( x_0 \). Let \( B_0(t) = \mathcal{H}''(t, w_0(t)) \),

\[
B_\infty(t) = \begin{pmatrix} -R_\infty(t) & 0 \\ 0 & -A_m \end{pmatrix}, \quad R_\infty(t) = \begin{pmatrix} C_\infty(t) & 0 & \cdots & 0 \\ 0 & C_\infty(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_\infty(t) \end{pmatrix}_{mn \times mn}
\]

Set \((i_0, v_0) = (i_{A_m}(B_0), v_{A_m}(B_0)), (i_\infty, v_\infty) = (i_{\mathcal{H}'}(B_\infty), v_{\mathcal{H}'}(B_\infty))\) with \( \mathcal{J} = J_{mn}, \mathcal{M} = \mathcal{P}_{n,m}^{-1} \) defined in Section 1. If (4.2) holds, the system (5.12) possesses at least two solutions. Furthermore, if \( w_0 \) is not pseudo-degenerated, and (4.3) holds, then the system (5.12) possesses at least three solutions.

**Theorem 5.7.** Suppose \( V \) satisfies the conditions \((V2), (V3), (V4)\) and \((V\infty)\) in Theorem 5.3. Then the system (5.12) possesses a nontrivial solution, provided the twisted condition (5.10) holds with \( \mathcal{J} = J_{mn}, \mathcal{M} = \mathcal{P}_{n,m}^{-1} \).

\[
B_0(t) = \begin{pmatrix} -R_0(t) & 0 \\ 0 & -A_m \end{pmatrix}, \quad R_0(t) = \begin{pmatrix} C_0(t) & 0 & \cdots & 0 \\ 0 & C_0(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_0(t) \end{pmatrix}_{mn \times mn}
\]

and \( B_\infty(t) \) defined as in Theorem 5.6.

**References**


