Chromatic equivalence classes of complete tripartite graphs

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Abstract

Some necessary conditions on a graph which has the same chromatic polynomial as the complete tripartite graph $K_{m,n,r}$ are developed. Using these, we obtain the chromatic equivalence classes for $K_{m,n,n}$ (where $1 \leq m \leq n$) and $K_{m_1,m_2,m_3}$ (where $|m_i - m_j| \leq 3$). In particular, it is shown that (i) $K_{m,n,n}$ (where $2 \leq m \leq n$) and (ii) $K_{m_1,m_2,m_3}$ (where $|m_i - m_j| \leq 3$, $2 \leq m_i$, $i = 1, 2, 3$) are uniquely determined by their chromatic polynomials. The result (i), proved earlier by Liu et al. [R.Y. Liu, H.X. Zhao, C.Y. Ye, A complete solution to a conjecture on chromatic uniqueness of complete tripartite graphs, Discrete Math. 289 (2004) 175–179], answers a conjecture (raised in [G.L. Chia, B.H. Goh, K.M. Koh, The chromaticity of some families of complete tripartite graphs (In Honour of Prof. Roberto W. Frucht), Sci. Ser. A (1988) 27–37 (special issue)]) in the affirmative, while result (ii) extends a result of Zou [H.W. Zou, On the chromatic uniqueness of complete tripartite graphs $K_{n_1,n_2,n_3}$ J. Systems Sci. Math. Sci. 20 (2000) 181–186].

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1. Introduction

We shall be concerned with finite, undirected graphs having neither loops nor multiple edges. Let $G$ be a graph and let $P(G; \lambda)$ denote its chromatic polynomial. Then the chromatic equivalence class of $G$, denoted $\mathcal{C}(G)$, is defined to be the set of all graphs which have the same chromatic polynomial as $G$. In the event that $\mathcal{C}(G) = \{G\}$, then $G$ is said to be chromatically unique.

Let $K_{m_1,m_2,\ldots,m_k}$ denote the complete $k$-partite graph whose $k$ ($\geq 2$) partite sets $V_1, V_2, \ldots, V_k$ are such that $|V_i| = m_i$, $i = 1, 2, \ldots, k$.

The first result concerning the question of whether or not $K_{m_1,m_2,\ldots,m_k}$ is chromatically unique seems to be attributed to Loerinc and Whitehead Jr. [8] who proved that $K_{1,\ldots,1,2,\ldots,2}$ is chromatically unique. Shortly afterwards, Chao and Novacky Jr. [1] generalized this result by proving that $K_{m_1,m_2,\ldots,m_k}$ is chromatically unique if $|m_i - m_j| \leq 1$ for all $i, j = 1, 2, \ldots, k$.

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If \( m_1 = 1 \), it is known that \( K_{1,m_2,\ldots,m_k} \) is chromatically unique if and only if \( \max\{m_2, \ldots, m_k\} \leq 2 \) (see [6]). If \( m_i \geq 2 \) for all \( i = 1, 2, \ldots, k \), it is not known in general whether or not \( K_{m_1,m_2,\ldots,m_k} \) is chromatically unique even when restricted to the case \( |m_i - m_j| \leq t \) for all \( i, j = 1, 2, \ldots, k \) where \( t \geq 2 \).

For restriction to the case \( k = 2 \), Teo and Koh [11] have shown that \( K_{m_1,m_2} \) is chromatically unique if \( 2 \leq m_1, m_2 \). However, as for the complete tripartite case, not much progress has been made. The first paper addressing this problem seems to be the paper [2] (see also [5]).

In the present paper, we determine the chromatic equivalence classes for the complete tripartite graphs \( K_{m,n,n} \), where \( 1 \leq m \leq n \) (Theorems 1 and 2) and \( K_{m_1,m_2,m_3} \) where \( |m_i - m_j| \leq 3 \) for all \( i, j = 1, 2, 3 \) (Theorem 3 and Proposition 1). In particular, it is shown that \( K_{m,n,n} \) is chromatically unique if \( 2 \leq m \leq n \), a result established recently by Liu et al. [7]. This answers a conjecture raised in [2] in the affirmative.

The main technique used in [2] to demonstrate the chromatic uniqueness of some complete tripartite graphs \( G = K_{m,n,r} \) was to compare the numbers of triangles and chordless 4-cycles in \( G \) with those in the graph \( Y \) for any \( Y \in \mathcal{C}(G) \). In the present situation, such a technique is no longer sufficient for drawing many conclusions. In the next section, we develop some necessary conditions on \( Y \) where \( Y \in \mathcal{C}(G) \) and \( G \) is the complete tripartite graph \( K_{m,n,r} \).

2. Machinery

Let \( K_n \) denote a complete graph on \( n \) vertices. If \( m \geq 3 \), let \( C^*_m \) denote a chordless cycle on \( m \) vertices.

Let \( G \) be a graph with \( p \) vertices and \( q \) edges. Let \( P(G; \lambda) = \sum_{i=1}^{p} a_i(G)\lambda^i \). Let \( n(A, G) \) denote the number of subgraphs in \( G \) that are isomorphic to \( A \). It is well known that \( a_p(G) = 1 \), \( a_{p-1}(G) = -q \) and \( a_{p-2}(G) = \binom{q}{2} - n(K_3, G) \) (see [9]).

Suppose \( Y \in \mathcal{C}(G) \). Then clearly, \( a_i(Y) = a_i(G) \) for each \( i = 1, 2, \ldots, p \). Thus, it follows that \( Y \) and \( G \) have the same numbers of vertices and edges, and \( n(K_3, Y) = n(K_3, G) \). Furthermore, in the event that \( G \) contains no \( K_4 \), it follows from Theorem 1 of [3] that \( n(C^*_4, Y) = n(C^*_4, G) \).

Another method of expressing the chromatic polynomial of \( G \) was introduced by Frucht [4]. A spanning subgraph is called \textit{special} if its connected components are complete graphs. Let \( s_i(G) \) denote the number of special spanning subgraphs of \( G \) with \( i \) components, \( i = 1, 2, \ldots, p \). Then

\[
P(G; \lambda) = \sum_{i=1}^{p} s_i(\overline{G})(\lambda)^i
\]

where \( (\lambda)_i = \lambda(\lambda - 1) \cdots (\lambda - i + 1) \) is the falling factorial and \( \overline{G} \) is the complement of \( G \). In this case, \( P(G; \lambda) \) is said to be expressed in a \textit{factorial basis}.

Clearly, \( s_p(\overline{G}) = 1 \) and \( s_{p-1}(\overline{G}) = q \) if \( \overline{G} \) has \( q \) edges. Note that if \( G \) has chromatic number \( \chi(G) = \chi \), then \( s_i(\overline{G}) = 0 \) for all \( i < \chi \).

Clearly if \( Y \in \mathcal{C}(G) \), then \( s_i(\overline{Y}) = s_i(\overline{G}) \) for all \( \chi(G) \leq i \leq p \).

The relationship between \( a_i(G) \) and \( s_i(\overline{G}) \) is given in the next lemma. Let \( S(n, k) \) denote the number of ways of partitioning a set of \( n \) elements into precisely \( k \) non-empty subsets. The number \( S(n, k) \) is known as the \textit{Stirling number of the second kind}. Note that \( \lambda^n = \sum_{k=1}^{n} S(n, k)(\lambda)_k \) and that \( S(n, k) = 0 \) for \( n < k \).

**Lemma 1.** Let \( P(G; \lambda) = \sum_{i=1}^{p} a_i(G)\lambda^i = \sum_{i=1}^{p} s_i(\overline{G})(\lambda)_i \). Then

\[
s_i(\overline{G}) = \sum_{r=1}^{p} a_r(G)S(r, i).
\]

**Corollary 1.** Let \( G \) and \( H \) be two graphs each on \( p \) vertices and having the same number of edges. Then

\[
s_{p-2}(\overline{G}) - s_{p-2}(\overline{H}) = n(K_3, H) - n(K_3, G).
\]

Let \( G \) be the complete tripartite graph \( K_{m_1,m_2,m_3} \). Suppose \( Y \in \mathcal{C}(G) \). We shall draw up some necessary conditions on \( Y \). Since \( G \) is connected and has \( p = m_1 + m_2 + m_3 \) vertices, \( q = \sum_{i<j} m_im_j \) edges and \( m_1m_2m_3 \) triangles, the
same is true for \( Y \). Moreover, \( Y \) is a tripartite graph obtained by deleting some \( e \) edges from the complete tripartite graph \( K_{s_1,s_2,s_3} \). Here \( s_1 + s_2 + s_3 = p \) and

\[
e = \sum_{i<j} s_is_j - \sum_{i<j} m_im_j.
\]

(1)

Equivalently, \( \overline{Y} \) is a union of three complete graphs \( K_{s_1}, K_{s_2} \) and \( K_{s_3} \) with \( e \) edges joining these subgraphs. Writing \( s_i = m_i + \alpha_i \) for \( i = 1, 2, 3 \), we have

\[
\alpha_1 + \alpha_2 + \alpha_3 = 0
\]

(2)

and it follows from Eq. (3) that

\[
e = \sum_{i<j} \alpha_i\alpha_j - \sum_{i=1}^3 m_i\alpha_i.
\]

(3)

If \( e = 0 \), then by noting that the numbers of vertices, edges and triangles in \( G \) and \( Y \) are each equal, which implies that the two polynomials \( \lambda^3 + \left( \sum_{i=1}^{s_i} m_i \right) \lambda^2 + \left( \sum_{i<j} m_im_j \right) \lambda + m_1m_2m_3 \) and \( \lambda^3 + \left( \sum_{i=1}^{s_i} \alpha_i \right) \lambda^2 + \left( \sum_{i<j} \alpha_i\alpha_j \right) \lambda + s_1s_2s_3 \) are the same, we have the following.

(\( \text{O1} \)) If \( e = 0 \), then \( \{s_1, s_2, s_3\} = \{m_1, m_2, m_3\} \), in which case \( Y \) is isomorphic to \( G \).

In what follows, we shall let \( K^e(s_1, s_2, s_3) \) denote the set of all connected tripartite graphs obtained by deleting \( e \) edges from the complete tripartite graph \( K_{s_1,s_2,s_3} \).

Note that, for any graph \( Y \in K^e(s_1, s_2, s_3) \), \( \overline{Y} \) is the union of three complete subgraphs \( K_{s_1}, K_{s_2} \) and \( K_{s_3} \) with \( e \) edges joining these subgraphs. Suppose, for any triplet \( (j, k, l) \) where \( \{j, k, l\} = \{1, 2, 3\} \), that there are \( a_j \) edges joining the subgraphs \( K_{s_j} \) and \( K_{s_l} \). Then

\[
e = a_1 + a_2 + a_3.
\]

(4)

\textbf{Definition.} Let \( E_i \) denote the set of all the \( a_i \) edges where \( i = 1, 2, 3 \). Two edges \( \beta \in E_r \) and \( \gamma \in E_s \), where \( r \neq s \), are said to be a \textit{coincidence pair} \( Y \) if they are incident with each other in \( \overline{Y} \).

The preceding discussions have lead to the following observation.

(\( \text{O2} \)) If \( G \) is the complete tripartite graph \( K_{m_1, m_2, m_3} \) and \( Y \in C(G) \), then \( Y \in K^e(s_1, s_2, s_3) \) where \( s_i, \alpha_i \) (for \( i \in \{1, 2, 3\} \) and \( e \) satisfy Eqs. (3)–(6).

\textbf{Lemma 2.} Let \( G \) and \( Y \) be as described in (\( \text{O2} \)). Suppose \( p = s_1 + s_2 + s_3 \) and \( q = s_1s_2 + s_2s_3 + s_3s_1 - e \). If \( e > 0 \), then \( |3s_i - p| < 2\sqrt{p^2 - 3q} \) for each \( i = 1, 2, 3 \).

\textbf{Proof.} Suppose the lemma is not true. Without loss of generality, suppose \( |3s_1 - p| \geq 2\sqrt{p^2 - 3q} \).

It is routine to check that this inequality simplifies to \( (p - s_1)(p + s_1) - 4q \leq 0 \) which implies that \( (s_2 + s_3)(4s_1 + s_2 + s_3) - 4q \leq 0 \). But then this further implies that \( (s_2 + s_3)^2 - 4s_2s_3 + 4e \leq 0 \) which yields \( (s_2 - s_3)^2 + 4e \leq 0 \), a contradiction because \( e > 0 \).

\textbf{Lemma 3.} Let \( G \) be the complete tripartite graph \( K_{m_1, m_2, m_3} \), \( p = m_1 + m_2 + m_3 \) and \( Y \in K^e(s_1, s_2, s_3) \) where \( s_i, \alpha_i \) (for \( i \in \{1, 2, 3\} \) and \( e \) satisfy Eqs. (3)–(6). Then for each \( j = 1, 2, 3 \),

\[
s_{p-2}(G) - s_{p-2}(\overline{Y}) \geq \prod_{i=1}^{3} (s_i - m_j) - \sum_{i=1}^{3} a_i(s_i - m_j).
\]

\textbf{Proof.} By Corollary 1, we have

\[
s_{p-2}(G) - s_{p-2}(\overline{Y}) = n(K_3, Y) - n(K_3, G).
\]

Since the number of triangles in \( Y \) is at least \( s_1s_2s_3 - (a_1s_1 + a_2s_2 + a_3s_3) \), it follows that

\[
s_{p-2}(G) - s_{p-2}(\overline{Y}) \geq s_1s_2s_3 - m_1m_2m_3 - (a_1s_1 + a_2s_2 + a_3s_3).
\]
Using the fact that \( s_1 + s_2 + s_3 = m_1 + m_2 + m_3 \) and Eq. (3), one can check that
\[
(s_1 - m_j)(s_2 - m_j)(s_3 - m_j) = s_1s_2s_3 - m_1m_2m_3 - em_j
\]
for each \( j = 1, 2, 3 \). Substituting \( e = a_1 + a_2 + a_3 \) into the above equation, the lemma follows. \( \square \)

**Lemma 4.** Let \( G \) and \( Y \) be as described in Lemma 3. Suppose further that \( Y \) contains no coincidence pair. Then for each \( j = 1, 2, 3 \),
\[
s_{p-2}(\overline{G}) - s_{p-2}(\overline{Y}) = \prod_{i=1}^{3} (s_i - m_j) - \sum_{i=1}^{3} a_i(s_i - m_j).
\]

**Proof.** By Corollary 1, we have
\[
s_{p-2}(\overline{G}) - s_{p-2}(\overline{Y}) = n(K_3, Y) - n(K_3, G).
\]
If \( Y \) contains no coincidence pair, then the number of triangles in \( Y \) is exactly \( s_1s_2s_3 - (a_1s_1 + a_2s_2 + a_3s_3) \). Applying an argument similar to that in the proof of Lemma 3, we get the conclusion of the lemma. \( \square \)

**Corollary 2.** Let \( G \) and \( Y \) be as described in Lemma 3. Suppose further that \( Y \) contains exactly one coincidence pair. Then for each \( j = 1, 2, 3 \),
\[
s_{p-2}(\overline{G}) - s_{p-2}(\overline{Y}) = \prod_{i=1}^{3} (s_i - m_j) - \sum_{i=1}^{3} a_i(s_i - m_j) + 1.
\]

Let \( J \) and \( H \) be two graphs whose chromatic polynomials are expressed in a factorial basis. Let \( J + H \) denote the join of \( J \) and \( H \). Then \( P(J + H; \lambda) = P(J; \lambda) \oplus P(H; \lambda) \), where the polynomial operator \( \oplus \) denotes the operation, known as umbral multiplication, in which factorials are multiplied as powers. (See [9,10].)

**Lemma 5.** Let \( G = K_{m_1,m_2,m_3} \) where \( 2 \leq m_1 \leq m_2 \leq m_3 \). Let \( Y \in C(G) \). Suppose further that \( Y \cong H + \overline{K}_t \) for some bipartite graph \( H \) and some \( t \in \{m_1, m_2, m_3\} \). Then \( Y \) is isomorphic to \( G \).

**Proof.** Assume without loss of generality that \( Y \cong H + \overline{K}_m \). We assert that \( H \cong K_{m_2,m_3} \). To see this, suppose on the contrary that \( H \) is not isomorphic to \( K_{m_2,m_3} \). The chromatic polynomials of \( H \) and \( K_{m_2,m_3} \) are respectively
\[
P(H; \lambda) = \sum_{i=1}^{m_3+m_3} s_i(\overline{H})(\lambda)_i
\]
and
\[
P(K_{m_2,m_3}; \lambda) = \sum_{i=1}^{m_2+m_3} s_i(\overline{K}_{m_2,m_3})(\lambda)_i.
\]
Now since the graph \( K_{m_2,m_3} \) is chromatically unique for \( 2 \leq m_2 \leq m_3 \) (see [11]), we must have
\[
P(H; \lambda) \neq P(K_{m_2,m_3}; \lambda).
\]
Note that \( s_1(\overline{H}) = s_1(\overline{K}_{m_2,m_3}) = 0 \) and \( s_2(\overline{H}) = s_2(\overline{K}_{m_2,m_3}) = 1 \). Let \( k \) be the smallest positive integer such that \( s_i(\overline{H}) = s_i(\overline{K}_{m_2,m_3}) \) for \( 2 \leq i < k \) but \( s_k(\overline{H}) \neq s_k(\overline{K}_{m_2,m_3}) \).

By taking the umbral multiplication and by equating the coefficients of \( (\lambda)_{k+1} \) in \( P(Y; \lambda) \) and \( P(G; \lambda) \), we have \( s_{k+1}(\overline{Y}) \neq s_{k+1}(\overline{G}) \), implying that \( P(Y; \lambda) \neq P(G; \lambda) \), which is a contradiction. Therefore we conclude that \( H \cong K_{m_2,m_3} \) and \( Y \) is isomorphic to \( G \). \( \square \)

**Lemma 6.** Let \( G \) and \( Y \) be as described in Lemma 3. Suppose further that \( Y \cong H + \overline{K}_n \) where \( H \) is a bipartite graph and \( n \) is a positive integer. If \( H \) is disconnected, then \( s_3(\overline{Y}) > s_3(\overline{G}) \).
Proof. Let $J_1, J_2, \ldots, J_t$ be the connected components of $H$ where $t \geq 2$. Note that each $J_i$ is a bipartite graph. For each $i = 1, 2, \ldots, t$, let $p_i$ denote the number of vertices in $J_i$. Then

$$P(J_i; \lambda) = \sum_{j=2}^{p_i} s_j(J_i)(\lambda)_j$$

since $s_1(J_i) = 0$ and $s_2(J_i) \geq 1$ for $i = 1, 2, \ldots, t$. As a result,\n
$$P(H; \lambda) = \prod_{i=1}^{t} P(J_i; \lambda)$$

$$= \prod_{i=1}^{t} (s_2(J_i)(\lambda)_2 + s_3(J_i)(\lambda)_3 + \ldots + (\lambda)_{p_i}).$$

Since $(\lambda)_2(\lambda)_k = (\lambda)_k + 2k(\lambda)_{k+1} + k(k-1)(\lambda)_k$, we have\n
$$s_2(H) = 2^{t-1} \prod_{i=1}^{t} s_2(J_i) \geq 2^{t-1}.$$

Now, since\n
$$P(Y; \lambda) = P(H; \lambda) \oplus P(\overline{K}_n; \lambda)$$

$$= (s_2(H)(\lambda)_2 + s_3(H)(\lambda)_3 + \ldots) \oplus \sum_{i \geq 1} s_i(K_n)(\lambda)_i$$

we see that $s_3(Y) = s_2(H)s_1(K_n) \geq 2^{t-1} > 1 = s_3(G)$ and this finishes the proof. \qed

3. $K_{m,n,n}$

In this section, we shall prove the chromatic uniqueness of the graph $K_{m,n,n}$ for $2 \leq m \leq n$ and obtain its chromatic equivalence class for $m = 1$. Incidentally, we note that the chromatic equivalence class for $K_{1,1,n}$ was obtained by Whitehead Jr. earlier in [12]. Note that $\mathcal{C}(K_{1,1,n})$ is the set of all 2-trees on $n + 2$ vertices because $K_{1,1,n}$ is itself a 2-tree on $n + 2$ vertices.

Lemma 7. Let $G$ and $Y$ be as described in Lemma 3 and let $Y \in \mathcal{C}(G)$. Suppose further that $1 \leq m_1 < m_2 = m_3$. Then $Y \cong H + \overline{K}_{m_2}$ for some bipartite graph $H$.

Proof. Note that by Lemma 2, for each $i = 1, 2, 3$, we have\n
$$|3s_i - p| < 2\sqrt{p^2 - 3q}$$

where $p = m_1 + m_2 + m_3$ and $q = m_1m_2 + m_2m_3 + m_3m_1$.

Since $1 \leq m_1 < m_2 = m_3$, the inequality on the right simplifies to $2(m_2 - m_1)$. This means that $m_1 < s_i$ which implies that\n
$$\alpha_1 > 0$$

and\n
$$\alpha_i > m_1 - m_2$$

for each $i = 2, 3$.

Using Lemma 3 with $j = 1$ and $j = 2$, we have, respectively,

$$s_{p-2}(G) - s_{p-2}(Y) \geq \alpha_1(m_2 - m_1 + \alpha_2)(m_2 - m_1 + \alpha_3) - a_1\alpha_1 - a_2(m_2 - m_1 + \alpha_2) - a_3(m_2 - m_1 + \alpha_3)$$

and\n
$$s_{p-2}(G) - s_{p-2}(Y) \geq (m_1 - m_2 + \alpha_1)\alpha_2\alpha_3 - a_1(m_1 - m_2 + \alpha_1) - a_2\alpha_2 - a_3\alpha_3.$$
Suppose \( \alpha_3 = 0 \). Then \( \alpha_1 + \alpha_2 = 0 \) (by Equation (4)) and this implies that \( \alpha_2 < 0 \) (because \( \alpha_1 > 0 \) by (7)) and that \( m_1 - m_2 + \alpha_1 < 0 \) (by using (8)). From (10), we have
\[
s_{p-2}(\overline{G}) - s_{p-2}(\overline{Y}) \geq -a_1(m_1 - m_2 + \alpha_1) - a_2\alpha_2 \geq 0. \tag{9}
\]

Since \( Y \in \mathcal{C}(G) \), equality holds in (11) and this implies that \( a_1 = a_2 = 0 \). Consequently, \( e = a_1 + a_2 + a_3 = a_3 \) and \( Y \cong J + \overline{K_{m_2}} \) for some bipartite graph \( J \).

Next, we assume that \( \alpha_3 \neq 0 \) and there are two cases to consider. In each case we show that \( s_{p-2}(\overline{G}) - s_{p-2}(\overline{Y}) > 0 \), thereby establishing a contradiction because \( Y \in \mathcal{C}(G) \).

Case (1): \( \max\{s_1, s_2, s_3\} = s_3 \)

In this case, since \( s_3 \geq s_1 \) and \( s_3 \geq s_2 \), we have
\[
\alpha_1 \leq m_2 - m_1 + \alpha_3 \tag{10}
\]
and
\[
m_2 - m_1 + \alpha_2 \leq m_2 - m_1 + \alpha_3. \tag{11}
\]

Using (12) and (13) and the fact that \( e = a_1 + a_2 + a_3 \), inequality (9) reduces to
\[
s_{p-2}(\overline{G}) - s_{p-2}(\overline{Y}) \geq (m_2 - m_1 + \alpha_3)(\alpha_1(m_2 - m_1 + \alpha_2) - e). \tag{12}
\]

Note that, from (4) and (5), we have
\[
\alpha_1(m_2 - m_1 + \alpha_2) - e = \alpha_3^2 > 0 \tag{13}
\]
because \( \alpha_3 \neq 0 \). By (8), \( m_2 - m_1 + \alpha_3 > 0 \). Consequently, by (14), we have \( s_{p-2}(\overline{G}) - s_{p-2}(\overline{Y}) > 0 \).

Case (2): \( \max\{s_1, s_2, s_3\} \neq s_3 \)

Without loss of generality, we may assume that \( s_2 \leq s_3 \) (by interchanging \( \alpha_2 \) and \( \alpha_3 \) if necessary). In this case, since \( s_2 \leq s_3 \leq s_1 \), we have
\[
\alpha_2 \leq \alpha_3 \leq m_1 - m_2 + \alpha_1. \tag{14}
\]

Suppose \( m_1 - m_2 + \alpha_1 = 0 \). Then by (16), we have \( \alpha_2 \leq 0 \) and \( \alpha_3 \leq 0 \). Note that by (16), we have \( \alpha_2 \neq 0 \) because \( \alpha_3 \neq 0 \). Therefore \( \alpha_2 < 0 \) and \( \alpha_3 < 0 \). By (10), we have
\[
s_{p-2}(\overline{G}) - s_{p-2}(\overline{Y}) \geq -a_2\alpha_2 - a_3\alpha_3 \geq 0. \tag{15}
\]

Since \( Y \in \mathcal{C}(G) \), equality holds in (17) and this implies that \( a_2 = a_3 = 0 \). Consequently, \( e = a_1 + a_2 + a_3 = a_1 \) and \( Y \cong J + \overline{K_{m_1+a_1}} = J + \overline{K_{m_2}} \) for some bipartite graph \( J \).

Now suppose \( m_1 - m_2 + \alpha_1 \neq 0 \). Using (16) and the fact that \( e = a_1 + a_2 + a_3 \), inequality (10) reduces to
\[
s_{p-2}(\overline{G}) - s_{p-2}(\overline{Y}) \geq (m_1 - m_2 + \alpha_1)(a_2\alpha_3 - e). \tag{16}
\]

Note that, from (4) and (5), we have
\[
\alpha_2\alpha_3 - e = \alpha_1(m_1 - m_2 + \alpha_1). \tag{17}
\]

Consequently, by (18), we have \( s_{p-2}(\overline{G}) - s_{p-2}(\overline{Y}) > 0 \) because \( \alpha_1 > 0 \) by (7).

This completes the proof. \( \square \)

**Theorem 1.** The complete tripartite graph \( K_{m,n,n} \) is chromatically unique for all integers \( m \) and \( n \) such that \( 2 \leq m \leq n \).

**Proof.** Let \( G \) be the complete tripartite graph \( K_{m,n,n} \). If \( m = n \), then as was remarked earlier in the introduction, \( G \) is chromatically unique (see [1]). Hence we assume that \( m < n \).

Let \( Y \in \mathcal{C}(G) \). Then by Lemma 7, \( Y \cong H + \overline{K_n} \) for some bipartite graph \( H \). By Lemma 5, \( Y \) is isomorphic to \( G \) and \( K_{m,n,n} \) is chromatically unique. \( \square \)

Let \( T_m \) denote the set of all trees on \( m \) vertices.
Theorem 2. For any positive integer $n$, the chromatic equivalence class of $K_{1,n}$ is given by $C(K_{1,n}) = \{ T + \overline{K}_n \mid T \in \mathcal{T}_{n+1} \}.$

Proof. If $Y$ is a graph of the form $T + \overline{K}_n$ where $T$ is a tree on $n + 1$ vertices, then $Y \in C(K_{1,n})$ because
\[
P(Y; \lambda) = P(T; \lambda) \oplus P(\overline{K}_n; \lambda) = P(K_{1,n}; \lambda) \oplus P(\overline{K}_n; \lambda) = P(K_{1,n}; \lambda).
\]

On the other hand, suppose $Y \in C(K_{1,n})$. Then by Lemmas 6 and 7, we see that $Y \cong H + \overline{K}_n$ for some connected bipartite graph $H$. Here, the number of edges in $H$ is $n^2 + 2n - (1 + n)n = n$. That is, $H$ is a connected graph on $n + 1$ vertices and $n$ edges. Hence $H$ is a tree and this completes the proof. \qed

4. $K_{m_1,m_2,m_3}$ with $|m_i - m_j| \leq 3$

In this section, we prove the chromatic uniqueness of the graph $K_{m_1,m_2,m_3}$ for $2 \leq m_1 \leq m_2 \leq m_3$ and $|m_i - m_j| \leq 3$ for any $i, j \in \{1, 2, 3\}$. This extends a result of Zou [13]. Also, we obtain its chromatic equivalence class when $m_1 = 1$.

Lemma 8. Let $G$ and $Y$ be as described in Lemma 3. Suppose further that $Y \in C(G)$, $1 \leq m_1 \leq m_2 \leq m_3$ and that $Y \cong J + \overline{K}_{s_i}$ for some bipartite graph $J$, $i \in \{1, 2, 3\}$. Then $Y \cong H + \overline{K}_t$ for some bipartite graph $H$ and some $t \in \{m_1, m_2, m_3\}$.

Proof. The case $m_2 = m_3$ has been settled in Lemma 7. We therefore assume that $m_2 < m_3$.

Note that, since $Y \cong J + \overline{K}_{s_i}$ for some bipartite graph $J$, we have $e = a_i$ for some $i = 1, 2, 3$. Hence $Y$ contains no coincidence pair. By Lemma 4, for each $j = 1, 2, 3$, we have
\[
s_{p-2}(\overline{G}) - s_{p-2}(\overline{Y}) = (s_1 - m_j)(s_2 - m_j)(s_3 - m_j) - e(s_i - m_j).
\]

Note that, if $s_i = m_j$ for some $j$, then we are done; otherwise, since $Y \in C(G)$, we have $s_{p-2}(\overline{G}) - s_{p-2}(\overline{Y}) = 0$ and this leads to
\[
(s_{r_1} - m_j)(s_{r_2} - m_j) = e \tag{18}
\]
where $r_1 \neq r_2$ and $r_1, r_2 \neq i$.

Substituting $j = r_1$ and $j = r_2$ into (20), we have
\[
(s_{r_1} - m_{r_1})(s_{r_2} - m_{r_1}) = (s_{r_1} - m_{r_2})(s_{r_2} - m_{r_2})
\]
which simplifies to $s_{r_1} + s_{r_2} = m_{r_1} + m_{r_2}$. This leads to $s_i = m_j$.

This completes the proof. \qed

Lemma 9. Let $G$ and $Y$ be as described in Lemma 3. Suppose further that $1 \leq m_1 \leq m_2 \leq m_3$ where $|m_i - m_j| \leq 3$ and $m_3 - m_2 \leq 2$. Then $e \leq 2$. Moreover equality holds if and only if $Y \in K^2(m_1 + 1, m_1 + t, m_1 + 2)$ for some $t \in \{1, 2\}$.

Proof. Since $|m_i - m_j| \leq 3$, we have $G \cong K_{m,m+r,m+s}$ where $m_1 = m$ and $0 \leq r \leq s \leq 3$. Furthermore, since $m_3 - m_2 \leq 2$, we have $s - r \leq 2$.

Now if $r = s$, then $G$ is chromatically unique by Theorem 1 and this implies that $e = 0$ and the lemma follows.

Therefore we may assume that $0 \leq r < s \leq 3$ where $0 \leq r \leq 2$ and $1 \leq s \leq 3$ and $s - r \leq 2$.

From (4) and (5), we have
\[
e = -\alpha_2(\alpha_2 + r + \alpha_3) - \alpha_3(\alpha_3 + s). \tag{19}
\]

Now, by Lemma 2, for each $i = 1, 2, 3$ we have
\[
|3s_i - p| < 2\sqrt{p^2 - 3q}
\]
where \( p = 3m + r + s \) and \( q = 3m^2 + 2m(r + s) + rs \). The right-hand side of the above inequality simplifies to \( 2\sqrt{(r+s)^2 - 3rs} \). Note that \((r+s)^2 - 3rs \leq s^2 \) (because \((r+s)^2 - 3rs > s^2 \) leads to \( r^2 - rs = r(r - s) > 0 \), which is a contradiction). This means that \( \frac{r}{3} < s_i - m < \frac{3r+s}{3} \). Consequently, we have

\[
\frac{r - 4s}{3} < \alpha_3 < \frac{r}{3}.
\]  

(20)

From (22), we have \( \frac{4(r-s)}{3} < r + \alpha_3 < \frac{4s}{3} \) which implies that \( |r + \alpha_3| \leq 2 \) because \( 0 \leq r \leq 2 \) and \( s - r \leq 2 \). Since the maximum value of \( -\alpha_i(\alpha_i + a) \) is \( \frac{a^2}{4} \), it follows that

\[
-\alpha_2(\alpha_2 + r + \alpha_3) \leq 1
\]

(21)

and

\[
-\alpha_3(\alpha_3 + s) \leq \frac{s^2}{4} < 3.
\]

(22)


Hence, from (19), (21) and (22), we have \( e \leq 3 \).

Now, we assert that if \( -\alpha_2(\alpha_2 + r + \alpha_3) = 1 \) then \( -\alpha_3(\alpha_3 + s) \leq 0 \). To see this, we note that, if \( -\alpha_2(\alpha_2 + r + \alpha_3) = 1 \), then either \( \alpha_2 = -1 \) and \( r + \alpha_3 = 2 \) and hence \( -\alpha_3(\alpha_3 + s) = (r - 2)(s - r + 2) \leq 0 \) (since \( r - 2 \leq 0 \)), or else \( \alpha_2 = 1 \) and \( r + \alpha_3 = -2 \) and hence \( -\alpha_3(\alpha_3 + s) = (r + 2)(s - r - 2) \leq 0 \) (since \( s - r \leq 2 \)). This proves the assertion.

Suppose \( e = 3 \). Then we have \( -\alpha_2(\alpha_2 + r + \alpha_3) = 1 \) and \( -\alpha_3(\alpha_3 + s) = 2 \). However this is impossible by the preceding assertion. Therefore \( e \leq 2 \).

Suppose \( e = 2 \). If \( -\alpha_2(\alpha_2 + r + \alpha_3) = 1 \), then \( -\alpha_3(\alpha_3 + s) = 1 \). However this is impossible because, by the previous assertion, we have \( -\alpha_3(\alpha_3 + s) \leq 0 \). Therefore \( -\alpha_2(\alpha_2 + r + \alpha_3) = 0 \) (which implies that either \( \alpha_2 = 0 \) or else \( \alpha_2 + \alpha_3 = -r \)) and \( -\alpha_3(\alpha_3 + s) = 2 \) (which implies that \( s = 3 \) and \( \alpha_3 \in \{-2, -1\} \)).

Now if \( \alpha_2 = 0 \) then \( \alpha_1 = -\alpha_3 \) and we have \( Y \in K^2(m - \alpha_3, m + r, m + 3 + \alpha_3) \). If \( \alpha_2 + \alpha_3 = -r \) then \( \alpha_1 = r \) and \( r + \alpha_2 = -\alpha_3 \) and we have \( Y \in K^2(m + r, m - \alpha_3, m + 3 + \alpha_3) \). Since \( \alpha_3 \in \{-2, -1\} \), we have \( Y \in K^2(m + 1, m + r, m + 2) \).

Since \( s = 3 \) and \( s - r \leq 2 \), we have \( 1 \leq r \leq 2 \). \( \square \)

**Lemma 10.** Let \( G = K_{m,m+r,m+3} \) where \( 1 \leq m \) and \( 1 \leq r \leq 2 \) and let \( Y \) be as described in Lemma 3. If \( Y \not\cong J + \overline{K}_{s_i} \) for any bipartite graph \( J \) and for any \( i \in \{1, 2, 3\} \), then \( Y \not\in \mathcal{C}(G) \).

**Proof.** By Lemma 9, \( e \leq 2 \).

Since \( Y \not\cong J + \overline{K}_{s_i} \) for any bipartite graph \( J \) and for any \( i \in \{1, 2, 3\} \), it follows that \( e = 2 \).

By Lemma 9 again, we have \( Y \in K^2(m + 1, m + r, m + 2) \) for some \( r \in \{1, 2\} \). That is, \( \alpha_1 = 1 \), \( \alpha_2 = 0 \), and \( \alpha_3 = -1 \).

Suppose \( Y \in \mathcal{C}(G) \). We shall establish a contradiction by showing that \( n(C^*_4, Y) < n(C^*_4, G) \). The following identity is helpful. For each \( r \in \{1, 2\} \),

\[
\binom{m+r}{2} + 2\binom{m+1}{2}\binom{m+2}{2} + 2\binom{m+r}{2} = \binom{m}{2}\left(\binom{m+r}{2} + \binom{m+3}{2}\right) + m^2 + 2m.
\]

Let \( e_1, e_2 \in E_1 \cup E_2 \cup E_3 \).

Case (1): \( \{e_1, e_2\} \) is a coincidence pair of \( Y \).

By Corollary 2 with \( j = 2 \), we have

\[
s_{p-2}(\overline{G}) - s_{p-2}(\overline{Y}) = -a_1(1 - r) - a_3(2 - r) + 1.
\]

Since \( r \in \{1, 2\} \), we see that \( s_{p-2}(\overline{G}) - s_{p-2}(\overline{Y}) = 0 \) only if \( r = 1 \) and \( \alpha_3 = 1 \). That is, \( Y \in K^2(m + 1, m + 1, m + 2) \) where \( \overline{Y} \) is the union of three complete subgraphs \( K_{m+1}, K_{m+1} \), and \( K_{m+2} \) with one edge joining the two subgraphs \( K_{m+1} \) and \( K_{m+1} \) and another edge joining the two subgraphs \( K_{m+1} \) and \( K_{m+2} \). This is because \( Y \) contains a coincidence pair.

Now, the number of \( C^*_4 \) in \( Y \) is

\[
= \binom{m+1}{2}^2 + 2\binom{m+1}{2}\binom{m+2}{2} + \binom{m+1}{2} + \binom{m}{2} - m^2 - m(m + 1)
\]

\[
= \binom{m+1}{2}^2 + 2\binom{m+1}{2}\binom{m+2}{2} + 2\binom{m}{2} - 2m(m + 1)
\]
The chromatic equivalence class of $K_{3,3}$ (6)

Lemma 10

Let $G$ be the graph $K_{m,m+1,m+3}$ where $2 \leq m_1 \leq m_2 \leq m_3$. Then $G$ is chromatically unique if $|m_i - m_j| \leq 3$ for all $i, j = 1, 2, 3$.

Proof. Let $G = K_{m_1,m_2,m_3}$ and let $Y \in \mathcal{C}(G)$.

By (O2), $Y \in \mathcal{K}^c(s_1,s_2,s_3)$ where $s_i, a_i$ (for $i \in \{1, 2, 3\}$) and $e$ satisfy Eqs. (3)–(6).

Since the graph $K_{m,m,m+3}$ is chromatically unique for $2 \leq m$ (see [2]), we may assume that $|m_i - m_j| \leq 3$ and $m_1 - m_2 \leq 2$.

By Lemma 9, $e \leq 2$.

Now if $e = 0$, then by (O1), $Y$ is isomorphic to $G$.

If $e = 1$, then $Y \cong J + \overline{K}_{s_i}$ for some bipartite graph $J$ and some $i \in \{1, 2, 3\}$. By Lemmas 5 and 8, again we have that $Y$ is isomorphic to $G$. Therefore $e = 2$. By Lemma 9, $Y \in \mathcal{K}^2(m_1 + 1,m_1 + t,m_1 + 2)$ for some $t \in \{1, 2\}$.

Alternatively, we may write $G = K_{m,m+r,m+3}$ where $0 \leq r \leq s \leq 3$ and $2 \leq m$. By the conditions imposed on the $m_i$’s, we see that $1 \leq r \leq s \leq 3$ and $4 \leq r + s \leq 5$.

By Theorem 1, the graph $K_{m,m+r,m+3}$ is chromatically unique. Hence we need only consider the case where $G$ is the graph $K_{m,m+r,m+3}$ where $r \in \{1, 2\}$.

If $Y \not\cong J + \overline{K}_{s_i}$ for any bipartite graph $J$ and for any $i \in \{1, 2, 3\}$, then by Lemma 10, $Y \not\in \mathcal{C}(G)$. Therefore $Y \cong J + \overline{K}_{s_i}$ for some bipartite graph $J$ and some $i \in \{1, 2, 3\}$. By Lemmas 5 and 8, we have $Y$ is isomorphic to $G$. 

Proposition 1. The chromatic equivalence class of $K_{1,r,4}$, where $r \in \{2, 3\}$, is given by $\mathcal{C}(K_{1,r,4}) = \{T + \overline{K}_2 \mid T \in \mathcal{T}_5\}$ and $\mathcal{C}(K_{1,4}) = \{T + \overline{K}_3, S + \overline{K}_4 \mid T \in \mathcal{T}_5, S \in \mathcal{T}_4\}$.

Proof. Suppose $Y$ is isomorphic to $T + \overline{K}_r$ or $S + \overline{K}_4$ where $r \in \{2, 3\}$, $T \in \mathcal{T}_5$ and $S \in \mathcal{T}_4$. Then $Y \in \mathcal{C}(K_{1,r,4})$. This can be verified directly by computing the chromatic polynomials of these graphs.

On the other hand, suppose $Y \in \mathcal{C}(K_{1,r,4})$. Then, by (O2), $Y \in \mathcal{K}^c(s_1,s_2,s_3)$ where $s_i, a_i$ (for $i \in \{1, 2, 3\}$) and $e$ satisfy Eqs. (3)–(6).

Now, if $Y \not\cong J + \overline{K}_{s_i}$ for any bipartite graph $J$ and for any $i \in \{1, 2, 3\}$, then by Lemma 10, $Y \not\in \mathcal{C}(K_{1,r,4})$, a contradiction. Therefore $Y \cong J + \overline{K}_{s_i}$ for some bipartite graph $J$ and some $i \in \{1, 2, 3\}$. By Lemma 8, $Y \cong H + \overline{K}_t$ for some bipartite graph $H$ and some $t \in \{1, r, 4\}$.

Note that, by Lemma 9, $e \leq 2$.

Suppose $e = 2$. Then by Lemma 9, we have $Y \in \mathcal{K}^2(2,r,3)$. Since $Y \cong J + \overline{K}_{s_i}$, $Y$ contains no coincidence pair.
Since \( \alpha_2 = 0 \), by using Lemma 4 with \( j = 2 \), we have
\[
s_{p-2}(G) - s_{p-2}(\overline{Y}) = -a_1(2-r) - a_3(3-r).
\]

Since \( r \in \{2, 3\} \), we see that \( s_{p-2}(G) - s_{p-2}(\overline{Y}) = 0 \) only if either \( r = 2 \) and \( a_3 = 0 \) or else \( r = 3 \) and \( a_1 = 0 \).
This implies that, for each \( r \in \{2, 3\} \), \( Y \cong H + \overline{K}_r \) where \( H \) is a bipartite graph on five vertices and four edges. By Lemma 6, \( H \) is connected and hence is a tree.

Suppose \( e = 1 \).

Suppose \( Y \cong H + \overline{K}_1 \). Then \( H \) is a bipartite graph with \( r + 4 \) vertices and \( 4r \) edges. In fact, \( H \) is the complete bipartite graph \( K_{s,t} \) with an edge deleted. Here \( s + t = r + 4 \) and \( st = 4r + 1 \). Since \( r \in \{2, 3\} \), the only possible solution is \( s = 3 = t \) with \( r = 2 \). However, this implies that \( n(C^*_4, Y) = 5 < 6 = n(C^*_4, K_{1,2,4}) \), a contradiction because \( Y \in C(K_{1,2,4}) \). Hence \( Y \not\cong H + \overline{K}_1 \).

If \( Y \cong H + \overline{K}_r \) then \( H \) is a bipartite graph on five vertices and four edges. By Lemma 6, \( H \) is connected and hence is a tree.

If \( Y \cong H + \overline{K}_4 \) then \( H \) is a bipartite graph on \( r + 1 \) vertices and \( r \) edges. By Lemma 6, \( H \) is connected and hence is a tree.

Suppose \( e = 0 \). Then by (O1), \( Y \) is isomorphic to \( K_{1,r,4} \).
This completes the proof. \( \square \)

Suppose \( 2 \leq m < n \). Let \( \mathcal{J}(m, n) = \{T + \overline{K}_m, S + \overline{K}_n \mid T \in \mathcal{T}_{n+1}, S \in \mathcal{T}_{m+1}\} \). Then it is easy to see that \( \mathcal{J}(m, n) \subseteq C(K_{1,m,n}) \). However, we do not know whether or not equality holds. So, we end this paper by posing the following problem.

**Question.** What is the chromatic equivalence class for the graph \( K_{1,m,n} \) where \( 2 \leq m < n \)?

**References**