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Relatively terminal coalgebras

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ABSTRACT

Dana Scott's model of λ-calculus was based on a limit construction which started from an algebra of a suitable endofunctor F and continued by iterating F. We demonstrate that this is a special case of the concept we call coalgebra relatively terminal w.r.t. the given algebra A. This means a coalgebra together with a universal coalgebra-to-algebra morphism into A.

We prove that by iterating F countably many times we obtain the relatively terminal coalgebras whenever F preserves limits of ω^{op}-chains. If F is finitary, we need in general $\omega + \omega$ steps. And for arbitrary accessible (=bounded) set functors we need an ordinal number of steps in general. Scott's result is captured by the fact that in a CPO-enriched category, assuming that F is locally continuous, ω steps are sufficient for algebras given by projections.

1. Introduction

Terminal coalgebras of endofunctors F play an important role in the theory of systems expressed by F-coalgebras. Jan Rutten demonstrated in [9] that the terminal coalgebra is the coalgebra of behaviors of states in such systems. The classical construction (dualizing that of initial algebras in [2]) is to form the limit of the ω^{op}-chain

$$1 \xleftarrow{\alpha} F1 \xleftarrow{F\alpha} FF1 \xleftarrow{FF\alpha} \dots$$

where $\alpha : F1 \longrightarrow 1$ is the (trivial) terminal algebra. Another source of interest in terminal coalgebras stems from the model of untyped λ-calculus presented by Dana Scott [10]. However, Scott did not use a terminal coalgebra: rather, he used, for a "suitable" algebra $\alpha : FA \longrightarrow A$, the limit of the analogous ω^{op}-chain

$$A \xleftarrow{\alpha} FA \xleftarrow{F\alpha} FFA \xleftarrow{FF\alpha} \dots$$

The properties of the endofunctor F he used made it clear that F preserves this limit. Whenever this happens, we are going to prove that the limit carries the structure of a coalgebra for which the first projection (into A) is a universal coalgebra-to-algebra morphism. This is called a *coalgebra relatively terminal* to the given algebra. But in general, this limit $F^{\omega}A = \lim_{i<\omega} F^i A$ carries itself an obvious structure of an algebra $\overline{\alpha} : F(F^{\omega}A) \longrightarrow F^{\omega}A$. We prove that this algebra has always the same relatively terminal coalgebra as the original one.

For finitary set functors relatively terminal coalgebras are always obtained in $\omega + \omega$ steps: we first form the algebra $F^{\omega}A$ in ω steps, and then we perform the same construction on it—in the next ω steps we get a limit preserved by F, thus, yielding a relatively terminal coalgebra for $F^{\omega}A$, consequently, also for A. This generalizes the result of James Worell [14] that terminal coalgebras of finitary functors take $\omega + \omega$ steps. Surprisingly, finitary endofunctors of many-sorted sets can require an arbitrarily large number of steps for the terminal algebra.

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All accessible (=bounded) set functors F have relatively terminal coalgebras: if F preserves λ-filtered colimits, we need $\lambda + \lambda$ steps of iteration. More generally, every monomorphism preserving, accessible endofunctor of a locally presentable category has relatively terminal coalgebras obtained by the iterative construction. This is a new result even for (absolutely) terminal coalgebras: the proof that a terminal coalgebra exists, presented by Michael Barr [6], was not constructive.

2. Terminal coalgebras of accessible functors

Before coming to relatively terminal coalgebras, we formulate a result concerning terminal coalgebras: if an accessible functor preserves monomorphisms, then it has a terminal coalgebra obtained by the terminal chain. Recall that a functor is called *finitary* if it preserves filtered colimits, and λ-*accessible* if it preserves λ-filtered colimits for an infinite cardinal λ.

Notation 2.1. Let F be an endofunctor of a complete category. We denote by

$$F^i 1 \quad (i \in \mathbf{Ord})$$

the *terminal chain* with connecting morphisms $w_{i,j}$ $(i \geq j)$ defined by

$$F^0 1 = 1$$

$$F^{i+1} = F(F^i 1) \quad \text{and} \quad w_{i+1,j+1} = F w_{i,j} \quad (i \geq j)$$

and for limit ordinals i

$$F^i = \lim_{0<i} F^j 1 \text{ with the limit cone } w_{i,j}(i > j).$$

This determines an ordinal chain, unique up to natural isomorphism.

If this chain *converges* at i, i.e. the connecting morphism $F^{i+1}1 \longrightarrow F^i 1$ is invertible, then this inverse makes $F^i 1$ a coalgebra. This coalgebra is then terminal. See [2] where this was first proved in the dual form (initial algebra) and [6] where, independently, the present formulation was used.

Theorem 2.2 (*Worrell [14]*)**.** *Every λ-accessible endofunctor of* Set *has a terminal coalgebra obtained in $\lambda+\lambda$ steps. In particular, $\omega + \omega$ steps are sufficient for finitary functors.*

Example 2.3. Worrell's result does not generalize to many-sorted sets: for every cardinal k we can find a finitary functor requiring k steps of the terminal chain.

Indeed, use k sorts and define

$$F : \mathsf{Set}^k \longrightarrow \mathsf{Set}^k$$

as follows: for every object $X = (X_i)_{i<k}$ put

$$(FX)_i = \begin{cases} X_i & \text{if } X_j \neq \varnothing \text{ for some } j < i \\ \varnothing & \text{else} \end{cases}$$

and define F on morphisms as expected. The terminal chain

$$(1, 1, 1, 1, \ldots) \longleftarrow (\varnothing, 1, 1, 1, \ldots) \longleftarrow (\varnothing, \varnothing, 1, 1, \ldots) \longleftarrow \ldots$$

needs k steps to converge to the initial object, the only (therefore terminal) coalgebra for F. However, F is finitary: let $X = \operatorname*{colim}_{t\in T} X^t$ be a filtered colimit. We need to prove

$$(FX)_i = \operatorname*{colim}_{t\in T} (FX^t)_i \quad \text{for all } i < k.$$

This is clear if $(FX)_i = \varnothing$. If not, we have $j < k$ with $\varnothing \neq X_j = \operatorname*{colim}_{t\in T} X_j^t$. Therefore, there exists t_0 with $X_j^{t_0} \neq \varnothing$. Since T is filtered, we can assume without loss of generality that t_0 is initial in T, thus, $x_j^t \neq \varnothing$ for all t. Then $(FX^t)_i = X_i^t$ for all i and the desired equality follows.

Remark 2.4. Recall that a category $\mathcal{A}$ is called *locally presentable*, see [7] or [3], if there exists an infinite cardinal λ such that

(a) $\mathcal{A}$ is complete and cocomplete

and

(b) $\mathcal{A}$ has a set of λ-presentable objects (i.e., such that their hom-functors are λ-accessible) whose closure under λ-filtered colimits is all of $\mathcal{A}$.

Theorem 2.5. *Every accessible endofunctor of a locally presentable category has a terminal coalgebra.*

In [8] a stronger result is proved: if F is accessible, then **Coalg**F is locally finitely presentable. Thus, it has a terminal object. This was explicitly used by Barr [6].

In case F preserve monomorphisms, more can be proved:

Theorem 2.6. *Let $\mathcal{A}$ be locally presentable and let F be an accessible endofunctor preserving monomorphisms. Then F has the terminal coalgebra $F^i 1$ for some ordinal i.*

Proof. (1) Choose a cardinal λ such that $\mathcal{A}$ is a locally λ-presentable category and F preserves λ-filtered colimits. Recall from [3] 1.19 and 1.58, that there exists a representative set A_λ of all λ-presentable objects, and that $\mathcal{A}$ is cowellpowered. Thus, we have a set $\overline{\mathcal{A}}_\lambda$ of representatives of all quotients of objects in $\mathcal{A}_\lambda$. Further, recall that every locally presentable category has a full embedding into $\mathsf{Set}^{\mathcal{C}}$ (for some small category $\mathcal{C}$) preserving limits and μ-filtered colimits for some infinite cardinal μ. From now on we consider $\mathcal{A}$ to be a full subcategory of $\mathsf{Set}^{\mathcal{C}}$ closed under limits and μ-filtered colimits. Moreover, the cardinal μ can be substituted by an arbitrary larger one. We thus assume without loss of generality that
(i) $\mu \geq \lambda$
(ii) $\mathcal{C}$ has less than μ morphisms
and
(iii) all objects of $\overline{A}_\lambda$ are μ-presentable in $\mathsf{Set}^{\mathcal{C}}$.
Condition (ii) implies that an object $X : \mathcal{C} \longrightarrow \mathsf{Set}$ of $\mathsf{Set}^{\mathcal{C}}$ is μ-presentable iff the sum of all cardinalities of the sets in the image of X is less than μ.

(2) Let $l_i : L \longrightarrow L_i$ $(i < \mu)$ be a limit of a μ^{op}-chain in $\mathsf{Set}^{\mathcal{C}}$. Then for every monomorphism

$$m : M \hookrightarrow L \qquad M\ \mu\text{-presentable}$$

there exists $i < \mu$ such that $l_i \cdot m$ is a monomorphism. In fact, this property clearly holds for limits of μ^{op}-chains in Set. Since M is μ-presentable, for every object C of $\mathcal{C}$ we have card $M(C) < \mu$, thus, for the limit $L(M)$ of $L_i(M)$ in Set there exists an ordinal i such that $(l_i)_C \cdot m_C$ is a monomorphism. Due to (ii) above our choice of i can be made independent of $C \in \mathcal{C}$. Since $(l_i \cdot m)_C$ is a monomorphism for every C, we conclude that $l_i \cdot m$ is a monomorphism.

(3) We prove that the connecting morphism

$$w_{\mu+1,\mu} : F^{\mu+1}1 \longrightarrow F^{\mu}1$$

is a monomorphism. Let $u_1, u_2 : X \longrightarrow F^{\mu+1}1$ be a pair of morphisms of $\mathcal{A}$ that $w_{\mu+1,\mu}$ merges, then we prove $u_1 = u_2$. Without loss of generality assume $X \in \mathcal{A}_\lambda$ (since $\mathcal{A}_\lambda$ is a generator of $\mathcal{A}$).

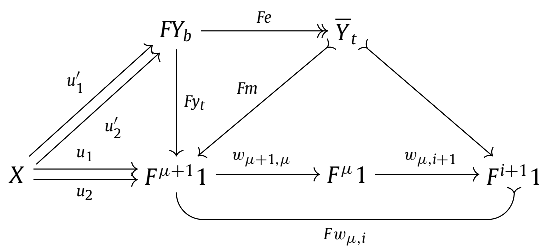
We express $F^\mu 1$ as a λ-filtered colimit of λ-presentable objects with a colimit cocone

$$y_t : Y_t \longrightarrow F^\mu 1\ (t \in T).$$

Since F preserves λ-filtered colimits, also

$$Fy_t : FY_t \longrightarrow F^{\mu+1}1\ (t \in T)$$

is a colimit cocone. This colimit is preserved by $\mathcal{A}(X, -)$ because X is λ-presentable. Consequently, u_1, u_2 factorize through Fy_t for some $t \in T$:



Factorize y_t as an epimorphism $e : Y_t \longrightarrow \overline{Y_t}$ followed by a strong monomorphism $m : \overline{Y}_t \longrightarrow F^\mu 1$, see [3], 1.61. By (iii), the object $\overline{Y}_t$ is μ-presentable in $\mathsf{Set}^{\mathcal{C}}$. By (2) there exists $i < \mu$ such that

$$w_{\mu,i} \cdot m : \overline{Y}_t \longrightarrow F^i 1 \text{ is monic}$$

and consequently, since F preserves monomorphisms,

$$Fw_{\mu,i} \cdot Fm \text{ is monic.}$$

This last monomorphism merges $Fe \cdot u'_1$ and $Fe \cdot u'_2$: see the diagram above and recall that $w_{\mu+1,\mu}$ merges $u_1 = Fy_t \cdot u'_1$ and $u_2 = Fy_t \cdot u'_2$. This proves

$$Fe \cdot u'_1 = Fe \cdot u'_2.$$

By composing with Fm we conclude

$$u_1 = Fy_t \cdot u'_1 = Fy_t \cdot u'_2 = u_2.$$

(4) All connecting morphisms $w_{i,\mu}$ with $i \geq \mu$ are monics. This follows from (3) by easy transfinite induction: recall that F preserves monics and $w_{i+1,\mu} = w_{\mu+1,\mu} \cdot Fw_{i,\mu}$, for limit steps use the fact that limits of chains of monics are formed by monics.

Every locally presentable category is wellpowered, see [3], 1.56. Thus, in the chain of subobjects $w_{i,\mu}$ of $F^{\mu}1$ there exists $i > j$ such that $w_{i,\mu}$ and $w_{j,\mu}$ represent the same subobject. From $w_{i,\mu} = w_{j,\mu} \cdot w_{i,j}$ we conclude that $w_{i,j}$ is invertible. Thus, so is $w_{j+1,j}$ (due to $w_{i,j} = w_{j+1,j} \cdot w_{i,j+1}$). □

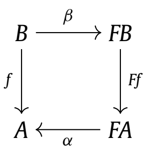
Open Problem 2.7. Can the assumption that F preserve monomorphisms be left out in the above theorem?

Remark 2.8 (*See [5]*)**.** If $\mathcal{A}$ is one of the categories sets, many-sorted sets, or vector spaces on a field then all (not necessarily accessible) functors having a terminal coalgebra have a convergent terminal chain. But this result is false e.g. for the category $\mathcal{A} = \mathsf{Set}^{\rightrightarrows}$ of graphs.

3. Relatively terminal coalgebras

Throughout this section an endofunctor F of a category $\mathcal{A}$ is assumed to be given. Recall the category **Alg**F of algebras $\alpha : FA \longrightarrow A$ for F: its morphisms, called *algebra homorphisms*, from (A, α) to (B, β) are morphisms $f : A \longrightarrow B$ in $\mathcal{A}$ with $f \cdot \alpha = \beta \cdot Ff$. Dually, **Coalg** F has objects $\alpha : A \longrightarrow FA$ and *coalgebra homomorphisms* from (A, α) to (B, β) are morphisms f with $\alpha \cdot f = Ff \cdot \beta$.

Given an algebra $\alpha : FA \longrightarrow A$ and a coalgebra $\beta : B \longrightarrow FB$, by a *coalgebra-to-algebra morphism* a morphism $f : B \longrightarrow A$ is meant such that the square



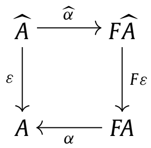
commutes. A precomposite of f with a coalgebra homomorphism yields another coalgebra-to-algebra homorphism. The same is true about post-composite f with an algebra homomorphism.

A *fixed point* is a (co)algebra $\alpha : FA \longrightarrow A$ with α invertible. Fixed points form full subcategories both in **Alg** F and **Coalg** F.

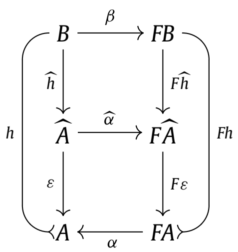
Definition 3.1. Let $\alpha : FA \longrightarrow A$ be an algebra. By a relatively terminal coalgebra is meant a coalgebra

$$\widehat{\alpha} : \widehat{A} \longrightarrow F\widehat{A}$$

together with a coalgebra-to-algebra homomorphism



is universal in the expected sense: for every coalgebra-to-algebra homomorphism $h : (B, \beta) \longrightarrow (A, \alpha)$ there exists a unique coalgebra homomorphism $\widehat{h} : (B, \beta) \longrightarrow (\widehat{A}, \widehat{\alpha})$ with $h = \varepsilon \cdot \widehat{h}$.



Lemma 3.2. *For every algebra $\alpha : FA \longrightarrow A$ the concept of relatively terminal colagebra for F is the same as the concept of terminal coalgebra for the endofunctor F_α of the slice category $\mathcal{A}/A$ defined by*

$$F_\alpha(X \xrightarrow{h} A) = (FX \xrightarrow{Fh} FA \xrightarrow{\alpha} A).$$

Proof. Let $\mathsf{Coalg}\ F/(A, \alpha)$ denote the category of coalgebras over (A, α), i.e., pairs $((B, \beta), h)$ consisting of a coalgebra (B, β) and a coalgebra-to-algebra homomorphism $h : B \longrightarrow A$. Morphisms from $((B, \beta), h)$ to $((B', \beta'), h')$ are precisely the coalgebra homomorphisms $u : B \longrightarrow B'$ with $h = h' \cdot u$. By definition the concept of a relatively terminal coalgebra is nothing else than a terminal object of Coalg $F/(A, \alpha)$. And this category is isomorphic to the category of coalgebras for F_α. Indeed, to give a coalgebra for F_α means to give an object $h : B \longrightarrow A$ of $\mathcal{A}/A$ and a morphism, say β, of $\mathcal{A}/A$:

$$\begin{array}{ccc} B & \xrightarrow{\beta} & FB \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle Fh} \\ A & \xleftarrow[\alpha]{} & FA \end{array}$$

Also morphisms of the two categories above obviously correspond. □

Corollary 3.3 (*Lambek's Lemma*). *All relatively terminal coalgebras are fixed points of* F.

Indeed, by Lambek's Lemma the terminal coalgebra of F_α is a fixed point of F_α, thus, a fixed point of F.

Examples 3.4. (1) Every fixed point $\alpha : FA \xrightarrow{\sim} A$ has the trivial relatively terminal coalgebra $\alpha^{-1} : A \longrightarrow FA$. This follows from the coincidence of coalgebra homomorphisms into this coalgebra and coalgebra-to-algebra homomorphisms into the given algebra.

(2) The relatively terminal colagebras for the trivial algebra $F1 \longrightarrow 1$ are precisely the terminal coalgebras for F. Thus in contrast to (1), these do not exist in general.

(3) For $F = \mathrm{Id}_{\mathsf{Set}}$ an algebra is a dynamic system given by a set A of states and a next-state function $\alpha : A \longrightarrow A$. The relatively terminal coalgebra can be described as the set $\widehat{A}$ of all runs $(x_n)_{n\in\mathbb{N}}$ in the dynamic system: here x_n are states such that the next state of x_{n+1} is x_n for every n. The coalgebra structure is given by $(x_n) \longmapsto (x_{n+1})$ and the universal map by $(x_n) \longmapsto x_0$. This follows from Corollary 3.9 below.

(4) For the power-set functor $\mathcal{P}$, coalgebras are the graphs G. Given an algebra $\alpha : \mathcal{P}A \longrightarrow A$, a coalgebra-to-algebra morphism is a labeling of the vertices of G in A,

$$f : G \longrightarrow A$$

with the property that the label of every vertex $x \in G$ is α applied to the set of labels of the neighbors of x:

$$f(x) = \alpha\{f(y);\ y \text{ a neighbor of } x\}.$$

Such a labeling is called an *A-decoration* of the graph. (Recall that a decoration, as introduced by Aczel [1], is a labeling of vertices by sets such that the label of every vertex is the set of all labels of all neighbors.)

It follows from Corollary 3.3 that no algebra has a relatively terminal coalgebra.

(5) For the finite power-set functor $\mathcal{P}_f$ coalgebras are the finitely branching graphs. For every algebra $\alpha : \mathcal{P}_f A \longrightarrow A$ we can describe the relatively terminal coalgebra $\widehat{A}$ analogously to the description of the (absolutely) terminal coalgebra due to Barr [6]. Let us call an A-labeled tree *extensional* if no node has two isomorphic maximal subtrees (w.r.t. isomorphisms respecting the labels). Every A-labeled tree has a unique extensional quotient, obtained by recursively identifying siblings defining isomorphic subtrees. We call two A-labeled trees t and s *Barr-equivalent* if for every $n \in \mathbb{N}$ the cuttings of t and s at level n have the same extensional quotient.

Let $\widehat{A}$ be the coalgebra of all finitely branching A-decorated trees modulo Barr equivalence. This is a coalgebra of $\mathcal{P}_f$: to every tree assign the set of all children of the root. And $\widehat{A}$ has an A-decoration given by the label of the root. The proof that $\widehat{A}$ is relatively terminal is analogous to the proof for $A = 1$ (that all finitely branching trees modulo the Barr equivalence are terminal for $\mathcal{P}_f$) in [6].

Proposition 3.5. *Let $\mathcal{A}$ be a category in which subobjects of any object form a complete lattice. Given a monomorphisms-preserving endofunctor, then every "pre-fixed point", i.e., monomorphism $\alpha : FA \longrightarrow A$, has a relatively terminal coalgebra.*

Remark

We will prove that the algebra $\alpha : FA \longrightarrow A$ has the greatest subalgebra which is a fixed point, and the inverse yields the relatively terminal coalgebra. The universal arrow $\varepsilon : A \longrightarrow \widehat{A}$ is proved to be a monomorphism.

Proof. We have a function φ on the complete lattice of all subobjects of A assigning to a subobject $m : M \rightarrowtail A$ the subobject $\alpha \cdot Fm : FM \rightarrowtail A$. Since φ is monotone, by Knaster-Tarski fixed-point theorem [12] φ has the greatest fixed point. Now to be a fixed point of φ means precisely to be a subalgebra with an invertible structure morphism:

$$\begin{array}{ccc} FM & \overset{\sim}{\dashrightarrow} & M \\ {\scriptstyle Fm}\downarrow & \searrow^{\varphi(m)} & \downarrow{\scriptstyle m} \\ FA & \xrightarrow[\alpha]{} & A \end{array}$$

Let $\varepsilon : \widehat{A} \rightarrowtail A$ be the greatest fixed point of φ, and let $\widehat{\alpha} : \widehat{A} \longrightarrow F\widehat{A}$ be the corresponding isomorphism. Then ε is universal because every coalgebra-to-algebra homomorphism

$$\begin{array}{ccc} B & \xrightarrow{\beta} & FB \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle Fh} \\ A & \xleftarrow[\alpha]{} & FA \end{array}$$

factorizes through ε. (To verify this, it is only needed to prove that whenever h factorizes through a subobject $m : M \rightarrowtail A$, then it factorizes through $\varphi(m)$. Indeed, from $h = m\cdot k$ derive $h = \alpha\cdot F(m\cdot k)\cdot\beta = \varphi(m)\cdot Fk\cdot\beta$.) Thus we have a factorization $\widehat{h} : B \longrightarrow \widehat{A}$ with $\varepsilon\cdot\widehat{h} = h$, and then $\widehat{h}$ is a coalgebra homomorphism: to verify $\widehat{\alpha}\cdot\widehat{h} = F\widehat{h}\cdot\beta : B \longrightarrow F\widehat{A}$ we use that $\alpha\cdot F\varepsilon$ is a monomorphism with

$$\alpha\cdot F\varepsilon\cdot(\widehat{\alpha}\cdot\widehat{h}) = \varepsilon\cdot\widehat{h} = h = \alpha\cdot Fh\cdot\beta = \alpha\cdot F\varepsilon\cdot(F\widehat{h}\cdot\beta). \quad \square$$

Corollary 3.6. *Every relatively terminal coalgebra yields a coreflection of the given algebra in the full subcategory of* **Alg**F *formed by all fixed points. That is:*

(a) ε *is an algebra homomorphism*

$$\begin{array}{ccc} F\widehat{A} & \xrightarrow{\widehat{\alpha}^{-1}} & \widehat{A} \\ {\scriptstyle F\varepsilon}\downarrow & & \downarrow{\scriptstyle \varepsilon} \\ FA & \xrightarrow[\alpha]{} & A \end{array}$$

(b) *every fixed point* $\beta : FB \longrightarrow B$ *has the property that all algebra homomorphisms into* A *uniquely factorize through* ε.

Example 3.7. Unfortunately, we cannot *define* the relatively terminal coalgebras as coreflections in the subcategory of fixed points. Consider the modified power-set functor $\mathcal{P}'$ sending $\emptyset$ to $\emptyset$ and all nonempty sets X to $\mathcal{P}X$: it has almost no relatively terminal coalgebras, see Example 3.13. However, since $\emptyset$ is its only fixed point, this is the coreflection of every algebra in the subcategory of fixed points.

A limit construction 3.8. Recall from Notation 2.1 that a terminal coalgebra of F is obtained as a limit of the ω^{op}-chain

$$1 \longleftarrow F1 \longleftarrow FF1 \longleftarrow \ldots$$

whenever F preserves this limit. Applied to F_α of Lemma 3.2 this is the ω^{op}-chain obtained by iterating F on $A \xleftarrow{\alpha} FA$:

Corollary 3.9. *If F preserves limits of ω^{op}-chains, then every algebra $\alpha : FA \longrightarrow A$ has a relatively terminal coalgebra which is the limit $F^\omega A$ of the ω^{op}-chain*

$$A \xleftarrow{\alpha} FA \xleftarrow{F\alpha} FFA \xleftarrow{FF\alpha} \ldots \tag{1}$$

Example 3.10. (a) Let Σ be a (possibly infinitary) signature. Then Σ-algebras are algebras for the polynomial functor $H_\Sigma X = \coprod_{\sigma\in\Sigma} X^n$ where n is the arity of σ. This functor preserves limits of ω^{op}-chains.

Given a Σ-algebra A, then $\widehat{A} = \lim F^i A$ can be described as the set of all trees (up to isomorphism) labeled in $\Sigma \times A$ with the following property: given a node with label (σ, a) where σ is n-ary, this node has precisely n children, and their labels (σ_i, a_i) for $i < n$ satisfy

$$a = \sigma^A(a_i)_{i<n}$$

(b) CPO-enriched categories. As mentioned in the introduction, Dana Scott constructed in [S] a model of λ-calculus as a relatively terminal coalgebra for an endofunctor of the category of continuous lattices. Later Gordon Plotkin and Mike Smyth proved that the same procedure works in every category enriched over ωCPO, see [11], which means that the hom-sets $\mathcal{A}(X, Y)$ carry an ωCPO structure (i.e., a poset with a least element and joins of ω-chains) and composition is strict and continuous (i.e., preserves least element and ω-joins). We also assume that $\mathcal{A}$ has limits of ω^{op}-sequences which are ωCPO-enriched. An endofunctor F is called *locally continuous* provided that the induced maps from $\mathcal{A}(X, Y)$ to $\mathcal{A}(FX, FY)$ are continuous, i.e., $F(\sqcup f_n) = \sqcup Ff_n$ for every ω-chain (f_n) in $\mathcal{A}(X, Y)$.

For every algebra $\alpha : FA \longrightarrow A$ for which α is a projection, i.e., there exists $e : A \longrightarrow FA$ with $\alpha\cdot e = \mathrm{id}$ and $e\cdot\alpha \sqsubseteq \mathrm{id}$, the limit $F^\omega A$ is the relatively terminal coalgebra. This follows from the coincidence of the limit of the chain (1) and the colimit of the ω-chain $A \xrightarrow{e} FA \xrightarrow{Fe} FFA \xrightarrow{FFe} \ldots$ as established on [11].

Remark 3.11. We know from Proposition 3.5 that if $\alpha : FA \longrightarrow A$ is a monomorphism, then the relatively terminal coalgebra exists and $\varepsilon : \widehat{A} \longrightarrow A$ is a monomorphism. Now if $\alpha : FA \longrightarrow A$ is a split epimorphism, then so is $\varepsilon : \widehat{A} \longrightarrow A$:

Lemma 3.12. *Let $\alpha : FA \longrightarrow A$ be a split epimorphism. If a relatively terminal coalgebra exists, then $\varepsilon : \widehat{A} \longrightarrow A$ is also a split epimorphism.*

Proof. Given

$$\alpha \cdot m = \text{id for some } m : A \longrightarrow FA$$

we obtain a trivial coalgebra-to-algebra morphism

$$\begin{array}{ccc} A & \xrightarrow{m} & FA \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle F\mathrm{id}} \\ A & \xleftarrow[\alpha]{} & FA \end{array}$$

Then $\varepsilon \cdot \widehat{\mathrm{id}} = \mathrm{id}$ gives the desired splitting. $\square$

Example 3.13. For the functor $\mathcal{P}'$ of Example 3.7 no surjective algebra $\alpha : \mathcal{P}'A \longrightarrow A$ with $A \neq \emptyset$ has a relatively terminal coalgebra. Indeed, no fixed point $\widehat{A}$ of $\mathcal{P}'$ has a surjective map $\varepsilon : \widehat{A} \longrightarrow A$.

Notation 3.14. (a) The limit of the chain (1) is denoted by $F^\omega A$ with limit projections

$$\widehat{a}_i : F^\omega A \longrightarrow F^i A \quad (i < \omega).$$

Then $F^\omega A$ is an algebra: we have the unique algebra structure

$$\overline{\alpha} : F(F^\omega A) \longrightarrow F^\omega A$$

with

$$\widehat{a}_0 \cdot \overline{\alpha} = \alpha \cdot F\widehat{a}_0 \text{ and } \widehat{a}_{i+1} \cdot \overline{\alpha} = F\widehat{a}_i. \tag{2}$$

(b) For every coalgebra-to-algebra homomorphism

$$\begin{array}{ccc} B & \xrightarrow{\beta} & FB \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle Fh} \\ A & \xleftarrow[\alpha]{} & FA \end{array}$$

we obtain a cone $h_i : B \longrightarrow F^i A$ of the chain (1) by

$$h_0 = h \quad \text{and} \quad h_{i+1} = Fi \cdot \beta. \tag{3}$$

The unique factorization is denoted by

$$\overline{h} : B \longrightarrow F^\omega A.$$

Proposition 3.15. *The relatively terminal coalgebras for the three algebras*

$$FA \xrightarrow{\alpha} A, \qquad FFA \xrightarrow{F\alpha} FA \qquad \text{and} \qquad F(F^\omega A) \xrightarrow{\overline{\alpha}} F^\omega A$$

are the same.

Proof. (a) The category $\mathsf{Coalg}\,F/(A, \alpha)$ of coalgebras over (A, α), see Lemma 3.2, is isomorphic to the category of coalgebras over $(FA, F\alpha)$. Indeed, we have a functor

$$V : \mathsf{Coalg}\,F/(A, \alpha) \longrightarrow \mathsf{Coalg}\,F/(FA, F\alpha)$$

which is defined on objects by

$$\begin{array}{ccc} B & \xrightarrow{\beta} & FB \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle Fh} \\ A & \xleftarrow[\alpha]{} & FA \end{array} \quad \longmapsto \quad \begin{array}{ccc} B & \xrightarrow{\beta} & FB \\ {\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle F\beta} \\ FB & \overset{F\beta}{\dashrightarrow} & FFB \\ {\scriptstyle Fh}\downarrow & & \downarrow{\scriptstyle FFh} \\ FA & \xleftarrow[F\alpha]{} & FFA \end{array}$$

and on morphisms by $Vu = u$. This is inverse to the functor defined on objects by

$$\begin{array}{ccc} B & \xrightarrow{\beta} & FB \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle Fh} \\ FA & \xleftarrow[F\alpha]{} & FFA \end{array} \quad\longmapsto\quad \begin{array}{ccc} B & \xrightarrow{\beta} & FB \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle Fh} \\ FA & \overset{F\alpha}{\leftarrow - -} & FFA \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle F\alpha} \\ A & \xleftarrow[\alpha]{} & FA \end{array}$$

(b) The category of coalgebras over (A, α) is also isomorphic to the category of coalgebras over $(F^{\omega}A, \overline{\alpha})$. In the Notation 3.14 we have a functor

$$\overline{V} : \mathsf{Coalg}\, F/(A, \alpha) \longrightarrow CoalgF/(F^{\omega}A, \overline{\alpha})$$

defined on objects by

$$\begin{array}{ccc} B & \xrightarrow{\beta} & FB \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle Fh} \\ A & \xleftarrow[F\alpha]{} & FA \end{array} \quad\longmapsto\quad \begin{array}{ccc} B & \xrightarrow{\beta} & FB \\ {\scriptstyle \overline{h}}\downarrow & & \downarrow{\scriptstyle F\overline{h}} \\ F^{\omega}A & \xleftarrow[\overline{\alpha}]{} & F(F^{\omega}A) \end{array}$$

The right-hand square commutes since for every $i < \omega$ we have, due to (2) and (3),

$$\widehat{a}_{i+1} \cdot (\overline{\alpha} \cdot F\overline{h} \cdot \beta) = F(\widehat{a}_i \cdot \overline{h}) \cdot \beta = Fh_i \cdot \beta = \widehat{a}_{i+1} \cdot \overline{h}$$

The inverse functor is given on objects by

$$\begin{array}{ccc} B & \xrightarrow{\beta} & FB \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle Fh} \\ F^{\omega}A & \xleftarrow[\overline{\alpha}]{} & F(F^{\omega}A) \end{array} \quad\longmapsto\quad \begin{array}{ccc} B & \xrightarrow{\beta} & FB \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle Fh} \\ F^{\omega}A & \underset{\overline{\alpha}}{\leftarrow - -} & F(F^{\omega}A) \\ {\scriptstyle \widehat{a}_0}\downarrow & & \downarrow{\scriptstyle F\widehat{a}_0} \\ A & \xleftarrow[\alpha]{} & FA \end{array} \qquad \square$$

4. Finitary functors

Recall that an endofunctor F is called *finitary* if it preserves filtered colimits.

Remark 4.1. (a) For set functors this means that given an arbitrary element $x \in FX$, there exists a finite subset $u : U \hookrightarrow X$ such that x lies in the image of Fu.

(b) For example Id and $\mathscr{P}_f$ are finitary, and $\mathscr{P}$ is not. The functor H_{Σ} (Example 3.10) is finitary iff Σ is a finitary signature.

(c) James Worrell proved in [14] that finitary set functors F have terminal coalgebras obtained in $\omega+\omega$ steps of the iterative construction. We now prove that, analogously, every algebra $\alpha : FA \longrightarrow A$ has a relatively terminal coalgebra obtained in $\omega + \omega$ steps: the first ω steps yield the algebra $\overline{\alpha} : F(F^{\omega}A) \longrightarrow F^{\omega}A$ of Notation 3.14, the next ω steps:

$$F^{\omega}A \overset{\overline{\alpha}}{\longleftarrow} F(F^{\omega}A) \overset{F\overline{\alpha}}{\longleftarrow} FF(F^{\omega}A) \overset{FF\overline{\alpha}}{\longleftarrow} \dots \qquad (4)$$

are just the same construction applied to that new algebra. It turns out that $\overline{\alpha}$ is always a monomorphism so that the next limit is the intersection of the chain (4) of subobjects. And F preserves this intersection, consequently, $F^{\omega}(F^{\omega}A)$ is a relatively terminal coalgebra for $F^{\omega}A$ or, equivalently, for A: see Proposition 3.15.

(d) Our proof uses Worrell's idea, but is slightly simpler. Observe that we cannot apply Lemma 3.2 here, because F_{α} works on the category Set/A of A-sorted sets, and we know that Worrell's result does not extend to many-sorted sets (see Example 2.3).

Lemma 4.2. *For every finitary set functor F there exists a finitary set functor F' preserving intersections and agreeing with F on all nonempty sets (and functions).*

Proof. The existence of a functor F' such that F preserves finite intersections and agrees with F on all nonemtpy sets is established in [13], see also [4]. If F is finitary, so is F'. To prove that F' preserves all intersections, let $m = \bigcap_{i \in I} m_i$ be an intersection of subsets $m_i : M_i \longrightarrow X (i \in I)$. For every element $x \in FM$ lying in the image of Fm_i for all $i \in I$ we are to prove that x lies in the image of Fm. Choose u in Remark 4.1 with U of the minimal cardinality. Since F preserves the intersection $u \cap m_i$ for every $i \in I$, the minimality of U implies $u \subseteq m_i$. Thus, $u \subseteq m$, consequently, x lies in the image of Fm. □

Theorem 4.3. *For every algebra $\alpha : FA \longrightarrow A$ of a finitary set functor F the relatively terminal coalgebra is the limit of the ω^{op}-chain*

$$F^{\omega}A \xleftarrow{\overline{\alpha}} F(F^{\omega}A) \xleftarrow{F\overline{\alpha}} FF(F^{\omega}A) \xleftarrow{FF\overline{\alpha}} \dots$$

Remark

We will see that $\overline{\alpha}$ is monic, thus, the limit is an intersection.

Proof. (1) Let F preserve intersections. It is sufficient to prove that $\bar{\alpha}$ is monic: then F preserves the limit (=intersection) of the above chain. Given $x, y \in F(F^{\omega}A)$ there exists a finite subset $m : M \longrightarrow A^{\omega}$ with x, y in the image of Fm, see Remark 4.1. The limit cone $(\widehat{a_i})_{i<\omega}$ of Notation 3.14 fulfills, since M is finite: there exists j with $\widehat{a_j} \cdot m : M \longrightarrow F^jA$ monic. Thus $F\widehat{a_j} \cdot Fm$ is monic, hence, $x \neq y$ implies $F\widehat{a_j}(x) \neq F\widehat{a_j}(y)$. Since $\widehat{a}_{j+1} \cdot \bar{\alpha} = F\widehat{\alpha}_j$, see (2), we conclude $\bar{\alpha}(x) \neq \bar{\alpha}(y)$, as required.

(2) Let F be arbitrarily and let F' be the functor of Lemma 4.2. Every algebra $\alpha : FA \longrightarrow A$ with $A \neq \varnothing$ is also an algebra for F'. And the relatively terminal coalgebra for F' is clearly relatively terminal for F too. If $A = \varnothing$, then $FA = \varnothing$, thus, $\alpha = \mathrm{id}_{\varnothing}$ and the theorem holds triviality. □

5. Relatively terminal chain

Let $\alpha : FA \longrightarrow A$ be an algebra for an endofunctor of a complete category. The *relatively terminal chain* is the terminal chain of the endofunctor F_{α} of Lemma 3.2. Explicitly: it has objects F^iA ($i \in$ **Ord**) and connecting morphisms $\alpha_{i,j} : F^iA \longrightarrow F^jA$ $(i \geq j)$ defined by transfinite induction as follows:

$$F^0A = A, F^1A = FA \quad \text{and} \quad \alpha_{10} = \alpha$$

$$F^{i+1}A = F(F^iA) \quad \text{and} \quad \alpha_{i+1,j+1} = F\alpha_{i,j}$$

and for limit ordinals i

$$F^iA = \lim_{j<i} F^jA \text{ with the limit cone } \alpha_{i,j} (i > j).$$

Recall that the slice category $\mathcal{A}/A$ has all colimits and all connected limits computed as in $\mathcal{A}$. Therefore, every accessible endofunctor F yields an accessible endofunctors F_{α}. And if F preserves monomorphisms, so does F_{α}. Thus Theorems 2.5 and 2.6 yield

Corollary 5.1. *Every accessible endofunctor F of a locally presentable category has relatively terminal coalgebras for all algebras. If F preserves monomorphisms, then every algebra A has a relatively terminal coalgebra $\widehat{A} = F^iA$ for some ordinal i.*

Remark 5.2. In case of endofunctors of Set we can say more: whenever F (not necessarily accessible) has, for a given algebra $\alpha : FA \longrightarrow A$, a relatively terminal coalgebra, then $\widehat{A} = F^iA$ for some ordinal i. Indeed, use Lemma 3.2: since F_{α} is an endofunctor of the category Set$/A$ of A-sorted sets, we can apply the result of Remark 2.8: all terminal coalgebras in many-sorted sets are obtained by the terminal chain.

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