Overgroups of symplectic group in linear group over commutative rings

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Abstract

For a commutative ring with identity 1, we give a complete description of all overgroups of symplectic group $Sp_{2n} R$ where either $n \geq 3$ or $n = 2$ and $R$ has no factor ring of two elements in general linear group $GL_{2n} R$.

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0. Introduction

Let $R$ be a commutative ring with identity 1 and $J$ an ideal in $R$ ($J = R$ is possible). $GL_n R$ denotes the group of all invertible $n$ by $n$ matrices over $R$. For any ideal $J$ of $R$, let $E_n J$ denote the subgroup of $GL_n R$ generated by all elementary matrices $e_{ij}(a) = I_n + aE_{ij}$ with $a \in J$, $i \neq j$, where $E_{ij}$ denotes the matrix with 1 at the position $(i, j)$ and zeros elsewhere. The normal subgroup of $E_n R$ generated by $E_n J$ is denoted by $E_n(R, J)$.

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Let

\[
Sp_{2n} \mathbb{R} = \left\{ A \in GL_{2n} \mathbb{R} : \ A^t H A = H \right\}
\]

where

\[
H = \begin{pmatrix}
0 & I_n \\
-I_n & 0
\end{pmatrix}
\]

and

\[
GSp_{2n} \mathbb{R} = \left\{ A \in GL_{2n} \mathbb{R} : \ A^t H A = u H, \right. \text{where} \ u \in \mathbb{R}^* \}
\]

Define

\[
CSp_{2n} J = Sp_{2n} \mathbb{R} \cdot E_{2n}(R, J),
\]

\[
CGSp_{2n} J = \left\{ A \in GL_{2n} \mathbb{R} : \ A(\text{mod} \ J) \in GSp_{2n}(R/J) \right\}. \quad (1)
\]

The main result of this paper is stated as follows.

**Theorem.** Let \( R \) be a commutative ring with \( 1 \). Assume that either \( n \geq 3 \) or \( n = 2 \) and \( R \) has no factor ring of two elements. Let \( X \) be an overgroup of \( Sp_{2n} \mathbb{R} \) in \( GL_{2n} \mathbb{R} \). Then there is a unique ideal \( J \) of \( R \) such that

\[
CSp_{2n} J \subseteq X \subseteq CGSp_{2n} J.
\]

Kantor [1] determined all overgroups of \( GL_n(F_{qr}) \) in \( GL_{nr}(F_q) \). For any pair of fields \( K \) and \( F \) with \( K \subset F \) and \( \dim_F K = r < \infty \), Shangzhi Li [4] described the overgroups of \( Sp_{2n} K \) in \( GL_{2n} F \). When \( r = 1 \), i.e., \( K = F \), the overgroups \( X \) of \( Sp_{2n} F \) in \( GL_{2n} F \) satisfy either

(i) \( Sp_{2n} F \subseteq X \subseteq GSp_{2n} F \), or
(ii) \( SL_{2n} F \subseteq X \subseteq GL_{2n} F \).

Before this, Mclaughlin [8] obtained a partial result. Some results on overgroups of \( G(K) \) in \( GL_n F \), where \( G(K) \) is a certain classical group over \( K \), can be found in [5–7]. Recently the author and Baodong Zheng [14] studied the overgroups \( X \) of \( Sp_{2n} R \) in \( GL_{2n} R \) where \( R \) is a local ring. In this paper, the author extends the result in [14] to a commutative ring \( R \) except for the case of \( n = 2 \) and \( R \) has factor ring of two elements.

It is obvious that our result covers the two cases stated above when \( R = F \) is a field, because \( R = F \) only has two ideals \( \{0\} \) and \( R \), and \( E_{2n} F = SL_{2n} F \) over any field \( F \). For \( n = 1 \), the result is trivial since \( Sp_2 R = SL_2 R \) and \( GSp_2 R = GL_2 R \).

1. Basic lemmas

With \( n \) fixed for any \( 1 \leq k \leq 2n \), set \( \sigma k = k + n \) if \( k \leq n \) and \( \sigma k = k - n \) if \( k > n \). For \( a \in R \) and \( 1 \leq i \neq j \leq 2n \) we define the elementary symplectic matrices \( \rho_{i,j}(a) \) and
\(\rho_{ij}(a)\) with \(j \neq \sigma i\) as follows: \(\rho_{i,\sigma i}(a) = I_{2n} + aE_{i,\sigma i}\), \(\rho_{ij}(a) = \rho_{\sigma j,\sigma i}(-a') = I_{2n} + aE_{ij}\).

Note that \(\rho_i,\sigma_i = I\).

The following identities hold for elementary symplectic matrices (1 \(\leq i \neq j \leq 2n\)):

1. \(\rho_{ij}(a + b) = \rho_{ij}(a)\rho_{ij}(b)\).
2. \(\rho_{ij}(a)\rho_{jk}(b) = \rho_{jk}(ab)\) when \(i, j, k, \sigma i, \sigma j, \sigma k\) are all distinct.
3. \(\rho_{ij}(a)\rho_{ij}(b) = \rho_{ij}(2ab)\) when \(j \neq \sigma i\).
4. \(\rho_{ij}(a)\rho_{jk}(b)\rho_{ij}(c)\) when \(j \neq \sigma i\), where \(c = a^2b\) when \(i, j \leq n\) or \(n + 1 \leq i, j\) and \(c = -a^2b\) when \(j \leq n < i\) or \(i \leq n < j\).

**Lemma 1.2.** The following identities hold (1 \(\leq i \neq j \leq 2n\)):

1. \([\xi_{ij}(a), \rho_{jk}(b)] = \xi_{ik}(ab)\) when \(i, j, k\) are distinct and \(j \neq \sigma i\).
2. \([\xi_{ij}(a), \rho_{k,\sigma j}(b)] = \xi_{i,\sigma k}(c)\) when \(i, j, k\) are distinct, where \(c = ab\) when \(j, k \leq n\) or \(n + 1 \leq j, k\) and \(c = -ab\) when \(j \leq n < k\) or \(k \leq n < j\).

Note that the following matrices are in \(Sp_{2n}R\):

1. \(\pi_{k,\sigma k} = I_{2n} - E_{k,k} - E_{\sigma k,\sigma k} + E_{k,\sigma k} - E_{\sigma k,k}\) (1 \(\leq k \leq n\)),
2. \(w_{ij} = \left(\begin{array}{c} P_{ij} \\ (P_{ij})^{-1} \end{array}\right)\) where \(P_{ij}\) is the \((i, j)\)-permutation matrix (\(i \neq j \leq n\)).

Set \(v = v' H\) for \(v \in R^{2n}\). Let \(\{e_1, \ldots, e_{2n}\}\) denote the standard basis of \(R^{2n}\), i.e., \(\{e_1, \ldots, e_{2n}\} = I_{2n}\).

Let \(A \in GL_{2n}R\). By the definition of symplectic group, we have the following lemma.

**Lemma 1.3.** \(A \in Sp_{2n}R\) if and only if \(u_i = -v_{\sigma i}\) when 1 \(\leq i \leq n\) and \(u_i = v_{\sigma i}\) when \(n + 1 \leq i \leq 2n\), where \(u_i\) is the \(i\)th row of \(A^{-1}\) and \(v_{\sigma i}\) is the \(\sigma i\)th column of \(A\).

**Proof.** We need only to point out that \(A^{-1} = H^{-1} A' H\) if and only if \(A \in Sp_{2n}R\). \(\Box\)

Similarly, for the generalized symplectic group \(GSp_{2n}R\), we have the following lemma.

**Lemma 1.4.** \(A \in GSp_{2n}R\) if and only if there is an \(\alpha \in R^+\) such that \(u_i = -\alpha v_{\sigma i}\) when 1 \(\leq i \leq n\) and \(u_i = \alpha v_{\sigma i}\) when \(n + 1 \leq i \leq 2n\), where \(u_i\) is the \(i\)th row of \(A^{-1}\) and \(v_{\sigma i}\) is the \(\sigma i\)th column of \(A\).

For an invertible matrix \(g = (g_{ij})\), define the order \(o(g)\) of \(g\) to be the ideal generated by all \(g_{ii} - g_{jj}\) and \(g_{ij}\) (\(i \neq j\)). The order of a subgroup \(X\) in \(GL_{2n}R\) is \(o(X) = \sum_{g \in X} o(g)\).
2. Preliminary results

Lemma 2.1. Let $X$ be an overgroup of $\text{Sp}_{2n} R$ in $\text{GL}_{2n} R$ which contains an elementary matrix $\xi_{ij}(a)$ with $j \neq \sigma i$ and $a \in R$. Then $X$ contains $E_{2n}(aR)$.

Proof. Without loss of generality we assume that $X$ contains $\xi_{12}(a)$ and $n \geq 2$.

(i) $n = 2$. $X$ contains $\xi_{12}(ab) = [\xi_{12}(a), \rho_{24}(b)]$ for any $b \in R$ (note that $\rho_{24}(b) \in \text{Sp}_4 R \subset X$), i.e., $\xi_{12}(aR) \subset X$. Since $\rho_{14}(aR) \subset X$, we have $\xi_{12}(aR) \subset X$. By $\xi_{12}(ab) = [\xi_{14}(a), \rho_{24}(b)] \subset X$ and $\rho_{12}(aR) \subset X$, we have $\xi_{12}(aR) \subset X$. Since $H, w_{ij}$ (see Section 1) lie in $X$, we get all $\xi_{ij}(aR)$ in $X$. Then $E_{2n}(aR) \subset X$.

(ii) $n \geq 3$. By Lemma 1.2, all $\xi_{ij}(aR)$ for $1 \leq i \neq j \leq n$ and $n + 1 \leq i \neq j \leq 2n$ lie in $X$. Then $\xi_{ij}(ab) = [\xi_{i,j}(a), \rho_{j,j}(b)] \in X$ for all $1 \leq i \leq n$, $n + 1 \leq j \leq 2n$ and $n + 1 \leq i \leq 2n$, $1 \leq j \leq n$ ($j \neq \sigma i$). That means $X$ contains $E_{2n}(aR)$. □

For any ideal $J$ of $R$, define $SE_{2n}(R, J)$ to be the subgroup of $\text{GL}_{2n} R$ generated by the elements of the form $\rho_{ij}(r)\xi_{k,l}(a)\rho_{ij}(-r)$ with $a \in J$, $r \in R$ for all $k \neq l$, $i \neq j$. Note that $SE_{2n}(R, J)$ is different from $E\text{Sp}_{2n}(R, J)$, the normal subgroup of $E\text{Sp}_{2n} R$ generated by $E\text{Sp}_{2n} J$ (see [2,3]), and note that $E\text{Sp}_{2n}(R, J) \subseteq SE_{2n}(R, J)$.

Proposition 2.2. For any ideal $J$ of $R$ and $n \geq 2$, $SE_{2n}(R, J) = E_{2n}(R, J)$.

Proof. It is obvious that $SE_{2n}(R, J) \subseteq E_{2n}(R, J)$. Let us show that $h\xi_{k,l}(a)h^{-1} \in SE_{2n}(R, J)$ where $a \in J$ and $h \in \text{GL}_{2n} R$. Let $v$ stand for the $k$th column of $h$ and $w$ for the $l$th row of $h^{-1}$. Then $v \in U m_{2n}(R)$, the set of unimodular vectors over $R$, and $w \cdot v = 0$. We have $h\xi_{k,l}(a)h^{-1} = I_{2n} + v(au)$. By [9, Lemma 1.3], we can write $w$ in the form of $w = \sum_{i<j} a_{ij}(v_j e_i^j - v_i e_j^i)$ for some $a_{ij} \in R$. Thus

$$h\xi_{k,l}(a)h^{-1} = I_{2n} + v(au) = \prod_{i<j}(I_{2n} + v(aa_{ij}(v_j e_i^j - v_i e_j^i))).$$

Since $I_{2n} + v(aa_{ij}(v_j e_i^j - v_i e_j^i)) \equiv (I_{2n} + a'v_0 u_0')g$, where $g \in E_{2n} J$, $a' \in J$ and $v_0 = (0 \cdots 0, v_j, 0 \cdots 0, v_j, 0 \cdots 0, a'v_0 u_0')$, $u_0 = (0 \cdots 0, v_j, 0 \cdots 0, -v_i, 0 \cdots 0)'$, and $E_{2n} J \subseteq SE_{2n}(R, J)$, we need only to prove that $I_{2n} + a'v_0 u_0' \in SE_{2n}(R, J)$. We distinguish the following two cases:

(i) $j = \sigma i$. In this case, $u_0' = -v_0$. By the results in [2, Lemmas 1.2 and 1.5] we have $I_{2n} + a'v_0 u_0' = I_{2n} + (-a')v_0 u_0' \in E\text{Sp}_{2n}(R, J) \subseteq SE_{2n}(R, J)$.

(ii) $j \neq \sigma i$. We may assume that $1 \leq i, j \leq n$. In fact, if $1 \leq i \leq n$, $n + 1 \leq j \leq 2n$, using $\pi_{j, \sigma i}$ to conjugate $I_{2n} + a'v_0 u_0'$, we reduce to the case of $1 \leq i, j \leq n$.

Observe that (see [10, Lemma 1.1])

$$\begin{pmatrix} I_n + \mu v & 0 \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} I_n & \mu \\ 0 & I_n \end{pmatrix} \begin{pmatrix} I_n & 0 \\ v_0 & I_n \end{pmatrix} \begin{pmatrix} I_n & 0 \\ -\mu & I_n \end{pmatrix} \begin{pmatrix} I_n & 0 \\ v & I_n \end{pmatrix}^{-1}.$$
where $\mu = (\mu_1, 0)$ and $\mu_1 = (0 \cdots 0, a'v_1, \cdots, 0, a'v_j, 0 \cdots 0)$, $v = \left( \begin{smallmatrix} v_1 \\ \vdots \end{smallmatrix} \right)$ and $v_1 = (0 \cdots 0, v_j, 0 \cdots 0, -v_j, 0 \cdots 0)$, with $1 \leq i, j \leq n$. \(\omega = -\mu v (I_n + \mu v)^{-1} \in M_n(J)\). Since the first two matrices on the right side of the above equation lie in \(E_{2n}J\), we only need to show that

\[
\left( \begin{array}{cc} I_n & 0 \\ v & I_n \end{array} \right) \left( \begin{array}{cc} I_n & -\mu \\ 0 & I_n \end{array} \right)^{-1} \left( \begin{array}{cc} I_n & 0 \\ v & I_n \end{array} \right) \in SE_{2n}(R, J).
\]

Without loss of generality, we may assume $n = 2$. Then $\mu = \left( \begin{smallmatrix} a'v_1 \\ a'v_2 \end{smallmatrix} \right)$, $v = \left( \begin{smallmatrix} v_2 & -v_1 \\ v_1 & v_2 \end{smallmatrix} \right)$. Since $\left( \begin{array}{cc} I_2 & 0 \\ v & I_2 \end{array} \right) = \rho_{32}(v_2)\xi_{32}(-v_1)$ and $\rho_{31}(v_2) \in \text{Sp}_4(R)$, it is necessary only to prove that $\xi_{32}(v_1)\left( \begin{array}{cc} I_2 & \mu \\ 0 & I_2 \end{array} \right)\xi_{32}(-v_1) \in SE_4(R, J)$.

Note that

\[
\xi_{32}(v_1)\left( \begin{array}{cc} I_2 & \mu \\ 0 & I_2 \end{array} \right)\xi_{32}(-v_1) = \xi_{32}(v_1)\xi_{13}(a'v_1)\xi_{32}(-v_1)\xi_{32}(v_1)\xi_{23}(a'v_2)\xi_{32}(-v_1) = \xi_{12}(a'v_1)\xi_{13}(a'v_1)\xi_{23}(a'v_2)\xi_{32}(-v_1).
\]

Let us show that $\eta = \xi_{32}(v_1)\xi_{23}(a'v_2)\xi_{32}(-v_1) \in SE_4(R, J)$. Using $\pi_{13}$ to conjugate $\eta$, we have

\[
\pi_{13}\eta\pi_{13}^{-1} = \xi_{12}(v_1)\xi_{21}(a'v_2)\xi_{12}(-v_1) = \rho_{12}(v_1)\xi_{21}(a'v_2)\rho_{12}(-v_1) \in SE_4(R, J).
\]

Hence $\eta \in SE_4(R, J)$ as $\pi_{13} \in \text{ESp}_4(R)$. We complete the proof. \(\square\)

**Remark 2.3.** Since $E_{2n}(R, J)$ is normal in $GL_{2n}R$ when $n \geq 2$, $\text{Sp}_{2n}R \cdot E_{2n}(R, J)$ make sense. Moreover, if an overgroup $X$ of $\text{Sp}_{2n}R$ in $GL_{2n}R$ contains $E_{2n}J$ for some ideal $J$ of $R$, then $X$ contains $\text{CSp}_{2n}J = \text{Sp}_{2n}R \cdot E_{2n}(R, J)$ by Proposition 2.2.

For an ideal $J$ of $R$, let $\lambda_J$ denote the group homomorphism: $GL_{2n}R \to GL_{2n}(R/J)$ and $\text{Sp}_{2n}R \to \text{Sp}_{2n}(R/J)$ induced by the canonical ring homomorphism: $R \to R/J$. For a maximal ideal $M$ of $R$, the localization: $R \to R_M$ induces the group homomorphism $\varphi_M: GL_{2n}R \to GL_{2n}R_M$ and $\text{Sp}_{2n}R \to \text{Sp}_{2n}R_M$. Since $R_M$ is a local ring, $\text{ESp}_{2n}R_M = \text{Sp}_{2n}R_M$.

**Lemma 2.4.** Let $X$ be an overgroup of $\text{Sp}_{2n}R$ in $GL_{2n}R$. Assume that $X \not\subseteq \text{GSp}_{2n}R$. Then there exists a maximal ideal $M$ of $R$ such that $\varphi_M(X) \not\subseteq \varphi_M(\text{GSp}_{2n}R)$.

**Proof.** By the assumption, there is an invertible matrix $g$ in $X$ such that $g \not\in \text{GSp}_{2n}R$. This means $g^tHg \not= uH$ for all $u \in R^*$, i.e., $H^{-1}g^tHg \not= uI_{2n}$. Note that $h = H^{-1}g^tHg \in X$ and $h$ is not a central element (i.e., $h \not= uI_{2n}$). If $\varphi_M(X) \subseteq \varphi_M(\text{GSp}_{2n}R)$ for each $M \in \text{max}(R)$, then $o(h)_M = 0$ for each $M \in \text{max}(R)$. This would imply $o(h) = 0$ since $o(h)$ is an $R$-module. This is a contradiction. \(\square\)

When no confusions can arise, $\varphi_M$ is simply denoted by $\varphi$. 
Lemma 2.5. Let \( M \) be a maximal ideal in \( R \) and let \( h \in \text{ESp}_{2n}R_M \). Assume that either \( n \geq 3 \) or \( n = 2 \) and \( R \) has no factor ring of two elements. Then there exists \( s \in R \setminus M \) such that

\[
\psi(\text{ESp}_{2n}(sR))h^{-1} \subseteq \psi(\text{ESp}_{2n}R).
\]  

(2)

Proof. For \( n \geq 3 \), the conclusion has been established in \([3]\) (see also \([11]\)). So we consider only the case of \( n = 2 \).

First we show that \( h\varphi(\text{ESp}_{2}n(Rc))h^{-1} \) lie in \( \text{ESp}_{4}(Rc) \) for any elementary symplectic matrix \( h \). Except for the following two cases, the consequence is easily derived from Lemma 1.1.

Case 1. \( h = \rho_{i,j}(b), \rho = \rho_{i,j}(c^3a) \).

By the assumption, 1 is the sum of elements of the form \( (r^2 - r)s \) with \( r, s \in R \). Then

\[
c^3a = \sum_k c^3a(r_k^2 - r_k)s_k \quad \text{and} \quad \rho = \prod_k \rho_{i,j}(c^3a(r_k^2 - r_k)s_k). \]

Note that

\[
\rho_{i,j}(\pm c^3a(r^2 - r)s) = \rho_{i,j}(\pm c^3a(r^2 - r)s) = \rho_{i,j}(\pm c^3a(r^2 - r)s) = \rho_{i,j}(c^3a(r^2 - r)s).
\]

where \( j \neq i \). Then

\[
h\rho_{i,j}(\pm c^3a(r^2 - r)s)h^{-1} = [h\rho_{i,j}(c)h^{-1}, h\rho_{i,j}(c)h^{-1}]^{-1}
\]

\[
\cdot [h\rho_{i,j}(c)h^{-1}, h\rho_{i,j}(c)h^{-1}]^{-1}
\]

\[
\cdot [h\rho_{i,j}(c)h^{-1}, h\rho_{i,j}(c)h^{-1}]^{-1}
\]

\[
\in \text{ESp}_{4}(Rc).
\]

i.e., \( hph^{-1} \in \text{ESp}_{4}(Rc) \).

Case 2. \( h = \rho_{j,i}(b), \rho = \rho_{j,i}(c^3a) \), where \( j \neq i \).

Then

\[
\rho_{j,i}(b)\rho_{j,i}(c^3a)\rho_{j,i}(b) = \rho_{j,i}(b)\left[\rho_{j,i}(c^3a)\rho_{j,i}(c^3a)\right]^{-1}\rho_{j,i}(c^3a)\rho_{j,i}(b)
\]

\[
\in \text{ESp}_{4}(Rc).
\]

Now let us consider the ring \( R' = R[x, y] \). The localization \( R' \to R_M[x, y] \) induces the group homomorphism: \( \text{Sp}_{2n}R' \to \text{Sp}_{2n}(R_M[x, y]) \), which is still denoted by \( \varphi \). Suppose that \( h \) can be written as a product of \( m \) elementary symplectic matrices. Let \( g(y) = h\rho_{j,i}(xy^3m)h^{-1} \). Then \( g(y) \in \text{ESp}_{2n}(R_M[x, y]) \) by the above results (replacing \( R \) with \( R_M[x, y] \)). Therefore \( h\rho_{j,i}(xy^3m)h^{-1} \) is a product of a finite number of \( \rho_{i,j}(x)h^{-1} \) where \( \omega_i \in R_M[x, y] \). Let \( s_1 \in R \setminus M \) be the common denominator of these \( \omega_i \). We
have $g(s_1 y) = h\rho_{ij}(x s_1 y) h^{-1} \in \varphi(E SP_{2n}(y R))$ for all $x = r \in R$. Hence $g(s_1 s_2) \in \varphi(E SP_{2n}(s_2 R)) \subset \varphi(E SP_{2n}(R))$. Now we take $s = (s_1 s_2)^{3w}$.

Before stating and proving the following lemma we want to point out that for any overgroup $X$ of $SP_{2n}R$ in $GL_{2n}R$ ($n \geq 2$) not contained in $GSP_{2n}R$, there is certainly an element $g$ in $X$ with at least one column $v_i$ such that $u_{\sigma i} \neq \alpha \tilde{v}_i$ (or $u_{\sigma i} \neq -\alpha \tilde{v}_i$), where $u_{\sigma i}$ is the $\sigma i$th row of $g^{-1}$.

In fact, if $g$ is almost generalized symplectic, i.e., there is a diagonal matrix $d = \text{diag}(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}, \ldots, \alpha_{2n})$ with $\alpha_i \neq \alpha_j$ for some $i \neq j$, such that $dg^{-1} = H^{-1}g'H$ for some $h \in GL_nR$, respectively), then we can find a symplectic matrix $\rho$ such that $\rho d \rho^{-1}$ is not diagonal for any $d = \text{diag}(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}, \ldots, \alpha_{2n})$ with $\alpha_i \neq \alpha_j$ for some $i \neq j$. Let $h = g \rho$. If $d h^{-1} = d_1 \rho^{-1}g^{-1} = H^{-1} \rho' g'H = H^{-1} h'H$ for a diagonal matrix $d_1 = \text{diag}(\alpha'_1, \ldots, \alpha'_n, \alpha'_{n+1}, \ldots, \alpha'_{2n})$ with $\alpha'_i \neq \alpha'_j$ for some $i \neq j$, we have $\rho d_1 \rho^{-1}g^{-1} = \rho H^{-1} \rho' g'H = H^{-1} g'H$. But $\rho d_1 \rho^{-1} \neq d$ ($\rho d_1 \rho^{-1}$ is not diagonal), this is contradictory to $dg^{-1} = H^{-1} g'H$.

**Lemma 2.6.** Let $X$ be an overgroup of $SP_{2n}R$ in $GL_{2n}R$ where either $n \geq 3$ or $n = 2$ and $R$ has no factor ring of two elements. Assume that $X \not\subset GSP_{2n}R$. Then there is $h \in E SP_{2n}RM$ for some maximal ideals $M$ of $R$ such that $\tilde{X} = h \varphi(X) h^{-1}$ contains an elementary matrix $\xi_{ij}(a)$ with $j \neq i$, $\sigma i$ and $a = c/s \in RM$ where $c \in R, s \in R \setminus M$.

**Proof.** By the assumption there is $g \in X$ which is not in $GSP_{2n}R$. By Lemma 2.4 there is maximal ideal $M$ of $R$ such that $\varphi(g) \notin \varphi(GSP_{2n}R)$. This means that there is a row $u_i$ in $\varphi(g^{-1})$ such that $u_i \neq \alpha \tilde{v}_i$, where $v_{\sigma i}$ is the $\sigma i$th column of $g$ and $\alpha \in R^*_M$. Without loss of generality, we can assume that $i = n + 1$, i.e., $v_{n+1} = v_1$ is the first column of $\varphi(g)$. Note that $E SP_{2n}RM = SP_{2n}RM$ and $E SP_{2n}RM$ acts transitively on the set of all unimodular vectors $v$ in $R^{2n}_M$. There exists $h \in E SP_{2n}RM$ such that $hv_1 = e_1$ and $u_{n+1} h^{-1} = (b_1, \ldots, b_n, b_{n+1}, \ldots, b_{2n})$. Since $hv_1 = v'_1 h'H = v'_1 h h^{-1}$ and $u_{n+1} \neq \alpha \tilde{v}_1$, there are some $b_i \neq 0 (b_i \notin M)$ in $u_{n+1} h^{-1}$ for $2 \leq i \neq n + 1 \leq 2n$ (note that $b_1 = 0$ since $u_{n+1} h^{-1}$ is the $(n+1)$th row of $(h \varphi(g))^{-1}$). Then $\tilde{X} = h \varphi(X) h^{-1} \supset \varphi(g) \rho_{1,n+1}(1) \varphi(g^{-1}) h^{-1}$ and $\eta = \varphi(g) \rho_{1,n+1}(1) \varphi(g^{-1}) h^{-1}$ has the following form:

$$I_{2n} + \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} (0, b_2 \cdots b_n, b_{n+1} \cdots b_{2n}) = \begin{pmatrix} 1 \\ w \end{pmatrix} I_{2n-1}$$

where $w = (b_2 \cdots b_n, b_{n+1} \cdots b_{2n})$ with some $b_i \neq 0$ for $2 \leq i \neq n + 1 \leq 2n$.

By Lemma 2.5, $\tilde{X}$ is normalized by $\varphi(E SP_{2n}(s R))$ for some $s \in R \setminus M$. Since there is $b_i \neq 0$ for $2 \leq i \neq n + 1 \leq 2n$ in $w$, we have $\xi_{1,n+1}(-s b_i) = [\eta, \rho_{1,n}(s)] \in \tilde{X}$, where $s b_i \in R_M$, either for $2 \leq i \leq n$ or for $n + 2 \leq i \leq 2n$.  \[\square\]
Proposition 2.7. Let $X$ be an overgroup of $\text{Sp}_{2n} R$ in $\text{GL}_{2n} R$ where either $n \geq 3$ or $n = 2$ and $R$ has no factor ring of two elements. Assume that $X \not\subseteq \text{GSp}_{2n} R$. Then $X$ contains an elementary matrix $\xi_{ij}(a)$ with $j \neq i$, $\sigma_i$ and $a \in R$.

Proof. By Lemma 2.4, there is a maximal ideal $M$ in $R$ such that $\varphi(X) \not\subseteq \text{GSp}_{2n} R_M$. By Lemma 2.6, $\tilde{X} = hq(X)h^{-1}$ for some suitable $h \in \text{ESp}_{2n} R_M$ contains an elementary matrix $\xi_{ij}(b)$ with $j \neq \sigma_i$ and $b \in R_M$. Since $\tilde{X}$ is normalized by $\varphi(\text{ESp}_{2n}(sR))$ for some $s \in R \setminus M$ by Lemma 2.5, we can derive that $\varphi(X)$ contains an elementary matrix $\xi_{ij}(d)$ with $j \neq \sigma_i$ and $d \in R_M$. In fact, writing $h$ as $\prod_{k=1}^{l-1} \rho_{i_k,j_k}(r_k)$, we have

$$h_1 \varphi(X)h_1^{-1} = \prod_{k=1}^{l-1} \rho_{i_k,j_k}(r_k) \varphi(X) \left( \prod_{k=1}^{l-1} \rho_{i_k,j_k}(r_k) \right)^{-1} \ni \rho_{i_l,j_l}(-r_l) \xi_{ij}(b) \rho_{i_l,j_l}(r_l).$$

If $\rho_{i_l,j_l}(r_l)$ commutes with $\xi_{ij}(b)$, then $h_1 \varphi(X)h_1^{-1} \ni \xi_{ij}(d)$. Otherwise, we may replace $\xi_{ij}(b)$ by some $\xi_{pq}(b') \in \tilde{X}$, where $p \neq \sigma q$, $b' \in R_M$, such that $\xi_{pq}(b')$ commutes with $\rho_{i_l,j_l}(r_l)$ and then $h_1 \varphi(X)h_1^{-1} \ni \xi_{pq}(b')$. To explain that, we may take $\xi_{ij}(b) = \xi_{12}(b)$ and assume $n = 2$, which does not lose generality.

(1) $(i_l, j_l) = (2, 3)$ (or $(1,4), (2,4)$). Since

$$[\xi_{12}(b), \rho_{24}(s)] = \xi_{14}(sb) \in \tilde{X}$$

and $\xi_{14}(sb)$ commutes with $\rho_{i_l,j_l}(r_l)$ in these cases, we are done.

(2) $(i_l, j_l) = (2, 1)$. By the above results

$$[\rho_{31}(s), \xi_{14}(sb)] = \xi_{34}(s^2b) \in \tilde{X}$$

and $\xi_{34}(s^2b)$ commutes with $\rho_{23}(r_l)$. We have $h_1 \varphi(X)h_1^{-1} \ni \xi_{34}(s^2b)$.

(3) $(i_l, j_l) = (4, 1)$ (or $(3,2), (3,1)$). Since

$$[\rho_{31}(s), \xi_{12}(b)] = \xi_{32}(sb) \in \tilde{X}$$

and $\xi_{32}(sb)$ commutes with $\rho_{i_l,j_l}(r_l)$ in these cases, we are done.

Continuing the procedure as above, we get that $\varphi(X)$ contains an elementary matrix $\xi_{ij}(d)$ with $j \neq \sigma_i$ and $d = a/c \in R_M$, where $a \in R$, $c \in R \setminus M$. Since $\varphi(\text{ESp}_{2n} R) \subseteq \varphi(X)$,

$$[\rho_{\sigma_i,j}(c), \xi_{ij}(d)] = \xi_{\sigma_i,j}(a) \in \varphi(X),$$

i.e., there is $\eta \in X$ such that $\varphi(\eta) = \xi_{\sigma_i,j}(a)$. Set $g = \xi_{\sigma_i,j}(-a) \eta \in E_{2n} R$. Then $\varphi(g) = I_{2n}$, i.e., $ug = uI_{2n}$ for some $u \in R \setminus M$. Thus $g \varphi_{pq}(u) = \varphi_{pq}(u)g$ for all $p \neq q$ and $X$ contains

$$[\eta, \rho_{j,\sigma j}(u)] = [\xi_{\sigma i,j}(a)g, \rho_{j,\sigma j}(u)] = [\xi_{\sigma i,j}(a), \rho_{j,\sigma j}(u)] = \xi_{\sigma_i,j}(au).$$

Clearly $au \neq 0$. □
Corollary 2.8. Let R be a commutative ring with 1. Assume that either \( n \geq 3 \) or \( n = 2 \) and \( R \) has no factor ring of two elements. Then the normalizer of \( \text{Sp}_{2n} R \) in \( \text{GL}_{2n} R \) is \( \text{GSp}_{2n} R \).

3. Proof of the theorem

Lemma 3.1. Under the conditions of Proposition 2.7, there exists a unique ideal \( J \) of \( R \) such that \( \text{Sp}_{2n} R \cdot E_{2n} (R, J) \leq X \).

Proof. If \( X \subseteq \text{GSp}_{2n} R \), then \( \text{Sp}_{2n} R \cdot E_{2n} (R, 0) = \text{Sp}_{2n} R \subseteq X \), and \( \text{Sp}_{2n} R \) is normal in \( X \) by Corollary 2.8.

Now suppose that \( X \not\subseteq \text{GSp}_{2n} R \). By Proposition 2.7 and Lemma 2.1, \( E_{2n} (a R) \subseteq X \) for some \( a \in R \). Let \( J = \{ x \in R \colon E_{2n} (x R) \subseteq X \} \). It is easy to show that \( J \) is an ideal of \( R \). Thus \( E_{2n} (R, J) \subseteq X \) by Proposition 2.2.

Denote \( \overline{R} = R/J \) and \( \overline{X} = \lambda_{J} (X) \). We have \( \text{Sp}_{2n} \overline{R} \subseteq \overline{X} \). If \( \overline{X} \not\subseteq \text{GSp}_{2n} \overline{R} \), then there exists some \( \xi_{ij} (\overline{\sigma}) \in \overline{X} \) with \( j \neq i \) and \( \overline{\sigma} \neq \overline{\xi} \in \overline{R} \) by Proposition 2.7. Note that \( a \not\in J \). Thus there is \( h \in X \) such that \( \lambda_{J} (h) = \lambda_{J} (\xi_{ij} (a)) \). Take \( g = \xi_{ij} (-a) h \in \ker \lambda_{J} \). Choose \( \rho = \rho_{j, \sigma j} (1) \). By [12, Theorem 3], \( [\rho, g] \in E_{2n} (R, J) \subseteq X \). Since \( \xi_{ij} (a) [\rho, g] \xi_{ij} (-a) \in E_{2n} (R, J) \subseteq X \), we have

\[
\xi_{i, \sigma j} (a) = [\xi_{ij} (a), \rho_{j, \sigma j} (1)] = \xi_{ij} (a) [\rho_{j, \sigma j} (1), g] \xi_{ij} (-a) [h, \rho_{j, \sigma j} (1)] \in X.
\]

This is contradictory to that \( a \not\in J \). Thus, \( \overline{X} \) must be in \( \text{GSp}_{2n} \overline{R} \). Hence \( J \) is maximal such that \( \text{Sp}_{2n} R \cdot E_{2n} (R, J) \subseteq X \), and is uniquely determined.

The normality of \( \text{GSp}_{2n} J = \text{Sp}_{2n} R \cdot E_{2n} (R, J) \) in \( X \) can be derived from Proposition 2.2, Remark 2.3, Proposition 2.7, and the above results directly. \( \square \)

Now let us complete the proof of the theorem.

By Lemma 3.1, we need only to show that \( X \subseteq \text{CGSp}_{2n} J \). Since \( \text{CGSp}_{2n} J \leq X \) and \( \lambda_{J} (\text{CGSp}_{2n} J) = \text{Sp}_{2n} (R/J) \), we have \( \text{Sp}_{2n} (R/J) \) is normal in \( \lambda_{J} (X) \). By Corollary 2.8, \( \lambda_{J} (X) \subseteq \text{GSp}_{2n} (R/J) \). Hence \( X \subseteq \lambda_{J}^{-1} (\lambda_{J} (X)) \subseteq \lambda_{J}^{-1} (\text{GSp}_{2n} (R/J)) \). Since \( X \subseteq \text{GL}_{2n} R \), so, \( X \subseteq \lambda_{J}^{-1} (\text{GSp}_{2n} (R/J)) \cap \text{GL}_{2n} R = \{ g \in \text{GL}_{2n} R \colon \lambda_{J} (g) \in \text{GSp}_{2n} (R/J) \} = \text{CGSp}_{2n} J \).

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References


