JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 151, 462–487 (1990)

Exponential Stability of the Solutions of Singularly Perturbed Systems with Impulse Effect

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Submitted by V. Lakshmikantham

Received November 18, 1988

In the present paper the exponential stability of the solutions of singularly perturbed systems with impulse effect is investigated. In order to obtain the main results the comparison method and piecewise continuous auxiliary functions which are analogues of Lyapunov's functions are used.

1. INTRODUCTION

In a number of papers [1–11] related to applications in various fields of science and technology, systems with impulses are considered.

In some of these papers [5–9] systems with impulse effect of the form

$$\frac{dx}{dt} = F(t, x), \qquad t \neq \tau_k, \\ \Delta x|_{t=\tau_k} = I_k(x) \tag{1}$$

are studied, where $\{\tau_k\}$ is an unbounded increasing sequence and x is an n-dimensional column matrix.

The system with impulse effect (1) is characterized by the fact that at the moments $\{\tau_k\}$ the mapping point (t, x) from the extended phase space "instantly" goes from the position $(\tau_k, x(\tau_k))$ into the position $(\tau_k, x(\tau_k) + I_k(x(\tau_k)))$. Assume that the solutions of system (1) are left continuous; i.e., at the moments τ_k the following relations hold

$$x(\tau_k - 0) = x(\tau_k), \qquad x(\tau_k + 0) = x(\tau_k) + \Delta x(\tau_k) = x(\tau_k) + I_k(x(\tau_k)).$$

The questions about the stability of the solutions of various classes of systems with impulse effect have been studied in [5–11].

A problem of great interest is that of finding sufficient conditions for stability of the solutions of singularly perturbed systems of the form (1).

0022-247X/90 $3.00

Concerning singularly perturbed systems of differential equations without impulses, there are some initial results published on this subject [12–15]. It is a characteristic of these papers that the basic mathematical apparatus used is the second method of Lyapunov.

In the present paper the exponential stability of the solutions of singularly perturbed systems with impulse effect is studied. In order to obtain the main results, the comparison method and piecewise continuous auxiliary functions which are analogues of Lyapunov's functions are used. Moreover, inverse theorems are proved (Lemmas 3 and 6) which guarantee the existence of piecewise continuous Lyapunov's functions with certain properties provided that the solution $x=0$ of system (1) is exponentially stable.

2. Preliminary Notes

We use the following notations:

R^m is an m-dimensional real space with a norm $|x|=|x_1|+\cdots+|x_m|$ of the vector $x=\mathrm{col}(x_1, ..., x_m)$; $R_+=[0, \infty)$; $B_\rho^m=\{x\in R^m; |x|<\rho\}$, $0<\rho\leqslant\infty$; $\|A\|=\max_{1\leqslant j\leqslant n}\sum_{i=1}^m |a_{ij}|$ is the norm of the matrix $A=(a_{ij})_{mn}$; E_m is the unit $(m\times m)$-matrix; C_0 is the class of continuous functions $W: R_+\times B_\rho^n\to R_+$ such that $W(t, 0)=0$ for $t\in R_+$; and C_1 is the class of differentiable functions contained in C_0.

For $(t_0, x_0)\in R_+\times B_\rho^n$ denote by $x(t)=x(t; t_0, x_0)$ the solution of system (1) for which $x(t_0+0)=x_0$ and by $J^+=J^+(t_0, x_0)$ the maximal interval of the form (t_0, ω) in which this solution is defined.

Henceforth $\{\tau_k\}_1^\infty$ is a fixed sequence of numbers:

$$0=\tau_0<\tau_1<\tau_2<\cdots, \qquad \lim_{k\to\infty}\tau_k=\infty.$$

Introduce the sets

$$G_k=(\tau_{k-1}, \tau_k)\times B_\rho^n \qquad (k=1, 2, ...)$$

and the classes $\mathscr{V}_0$ and $\mathscr{V}_1$ of piecewise continuous auxiliary functions [11]:

We say that the function $V: R_+\times B_\rho^n\to R_+$ belongs to the class $\mathscr{V}_0$ if

1. $V(t, x)$ is continuous in any of the sets G_k $(k=1, 2, ...)$ and $V(t, 0)=0$ for all $t\in R_+$.

2. For any $k=1, 2, ...$ and $x\in B_\rho^n$ there exist the finite limits

$$V(\tau_k-0, x)=\lim_{\substack{(t,y)\to(\tau_k,x)\\ t<\tau_k}} V(t, y), \qquad V(\tau_k+0, x)=\lim_{\substack{(t,y)\to(\tau_k,x)\\ t>\tau_k}} V(t, y)$$

and the equality $V(\tau_k-0, x)=V(\tau_k, x)$ holds.

Remark 1. If $t \neq \tau_k$, then $V(t+0, x)$ denotes $V(t, x)$.

We say that the function $V \in \mathscr{V}_0$ belongs to the class $\mathscr{V}_1$ if it is continuously differentiable in the sets G_k $(k = 1, 2, ...)$.

For $(t, x) \in G_k$ $(k = 1, 2, ...)$ we define

$$\dot{V}_{(1)}(t, x) = \frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x) F(t, x),$$

the derivative of the function $V \in \mathscr{V}_1$ with respect to system (1) and

$$D^+_{(1)} V(t, x) = \limsup_{\tau \to 0_+} \frac{1}{\tau} [V(t+\tau, x(t+\tau; t, x)) - V(t, x)],$$

the upper right derivative of the function $V \in \mathscr{V}_0$ with respect to the solutions of system (1).

Remark 2. If the function $V \in \mathscr{V}_0$ is locally Lipschitz continuous with respect to x in G_k, then for $(t, x) \in G_k$ we have [16]

$$D^+_{(1)} V(t, x) = \limsup_{\tau \to 0_+} \frac{1}{\tau} [V(t+\tau, x+\tau F(t, x)) - V(t, x)].$$

We consider the space R^m partially ordered in the following sense: we write $v \leqslant u$ $(v < u)$ if $v_i \leqslant u_i$ $(v_i < u_i)$ for $i = 1, ..., m$. Let $G \subset R^m$.

The function $F: R_+ \times G \to R^m$ is called quasimonotonely increasing in $R_+ \times G$ if for any pair $(t, u), (t, v)$ from $R_+ \times G$ for $i = 1, ..., m$ we have $F_i(t, v) \leqslant F_i(t, u)$ whenever $v_i = u_i$ and $v \leqslant u$.

The function $\psi: G \to R^m$ is called non-decreasing in G if $\psi(v) \leqslant \psi(u)$ for $v \leqslant u$ and $v, u \in G$.

In the proof of the main results we use the following lemmas:

LEMMA 1 [11]. *Let the following conditions be fulfilled*:

1. *The function* $F: R_+ \times G \to R^m$ *is continuous and quasimonotonely increasing in* $R_+ \times G$.

2. *The functions* $\psi_k: G \to R^m$ $(k = 1, 2, ...)$ *are non-decreasing in* G.

3. *The function* $u: (t_0, \omega) \to R^m$ *is the maximal (minimal) solution of the system*

$$\frac{du}{dt} = F(t, u) \qquad (t \in R_+, t \neq \tau_k),$$

$$u(\tau_k + 0) = \psi_k(u(\tau_k)) \qquad (\tau_k \in R_+)$$

such that $u(t_0 + 0) = u_0$, $(t_0, u_0) \in R_+ \times G$, $u(\tau_k + 0) \in G$ *if* $\tau_k \in (t_0, \omega)$.

4. *The function* $v\colon (t_0, \tilde{\omega}) \to R^m$ *is continuous for* $t \in (t_0, \tilde{\omega})$, $t \neq \tau_k$, *at the points* τ_k *it is left continuous and such that*:

4.1. $v(t) \in G$ *and* $v(\tau_k + 0) \in G$ *for* $t \in (t_0, \tilde{\omega})$ *and* $\tau_k \in (t_0, \tilde{\omega})$,

4.2. $v(t_0 + 0) \leqslant u_0$ $(u_0 \leqslant v(t_0 + 0))$,

4.3. $Dv(t) \leqslant F(t, v(t))$ $(F(t, v(t)) \leqslant Dv(t))$ *for* $t \in (t_0, \tilde{\omega})$, $t \neq \tau_k$,

4.4. $v(\tau_k + 0) \leqslant \psi_k(v(\tau_k))$ $(\psi_k(v(\tau_k)) \leqslant v(\tau_k + 0))$ *for* $\tau_k \in (t_0, \tilde{\omega})$,

where $Dv(t)$ *is some of the Dini derivatives of* $v(t)$.

LEMMA 2 [6]. *Let for* $\alpha \leqslant t < \beta \leqslant \infty$,

$$v(t) \leqslant c + \int_\alpha^t p(s)\, v(s)\, ds + \sum_{\alpha < \tau_k < t} \beta_k v(t_k),$$

where $c \geqslant 0, \beta_k \geqslant 0$ *are constants and the functions* $v\colon [\alpha, \beta) \to R_+$ *and* $p\colon [\alpha, \beta) \to R_+$ *are piecewise continuous in* $[\alpha, \beta)$. *Then*

$$v(t) \leqslant c \prod_{\alpha < \tau_k < t} (1 + \beta_k) \exp\left(\int_\alpha^t p(s)\, ds\right) \qquad \textit{for} \quad t \in [\alpha, \beta).$$

3. MAIN RESULTS

3.1. *Exponential Stability of the Zero Solution of Singularly Perturbed Systems with Impulse Effect*

Consider the system

$$\begin{aligned} \frac{dx}{dt} &= f(t, x, y), \\ \mu \frac{dy}{dt} &= g(t, x, y), \qquad t \neq \tau_k \\ \Delta x|_{t=\tau_k} &= I_k(x, y), \\ \Delta y|_{t=\tau_k} &= J_k(x, y), \qquad k = 1, 2, ..., \end{aligned} \tag{2}$$

where $x \in R^n$, $y \in R^m$; $f\colon R_+ \times B_H^n \times B_H^m \to R^n$; $g\colon R_+ \times B_H^n \times B_H^m \to R^m$; $I_k\colon B_H^n \times B_H^m \to R^n$; $J_k\colon B_H^n \times B_H^m \to R^m$; $0 < H \leqslant \infty$; and $\mu \in (0, \mu_0]$ is a small parameter.

Introduce the following conditions (A):

(A1) The functions f, $\partial f/\partial x$, $\partial f/\partial y$, g, $\partial g/\partial x$, $\partial g/\partial y$, $\partial g/\partial t$ are continuous and bounded on $R_+ \times B_H^n \times B_H^m$.

(A2) The functions I_k, $\partial I_k/\partial x$, $\partial I_k/\partial y$, J_k, $\partial J_k/\partial x$, $\partial J_k/\partial y$ are continuous and uniformly bounded with respect to $k=1, 2, \ldots$ in $B_H^n \times B_H^m$.

(A3) $f(t, 0, 0)=0$, $g(t, 0, 0)=0$, $I_k(0, 0)=0$, $J_k(0, 0)=0$ for $t \in R_+$ and $k=1, 2, \ldots$.

(A4) There exists a continuously differentiable function $h: R_+ \times B_H^n \to B_H^m$, $(t, x) \to h(t, x)$ such that

$$|\psi(s; \alpha, \beta; \eta) - h(\alpha, \beta)| \leqslant A\, |\eta - h(\alpha, \beta)|\, e^{-\nu s} \qquad \text{for} \quad s \geqslant 0.$$

THEOREM 1. *Let the following conditions hold:*

1. *Conditions* (A) *are satisfied.*

2. *There exist constants* $\rho \in (0, H)$, $a>0$, $b>0$, $c>0$, $K>0$ *and functions* $V \in \mathscr{V}_0$ *and* $W \in C_1$

$$V: R_+ \times B_\rho^n \to R_+, \qquad (t, x) \to V(t, x),$$
$$W: R_+ \times B_\rho^n \times B_\rho^m \to R_+, \qquad (t, x, y) \to W(t, x, y)$$

such that for $t \in R_+$; $x, x_1 \in B_\rho^n$ *and* $y \in B_\rho^m$ *the following inequalities hold:*

$$a\,|x| \leqslant V(t, x) \leqslant b\,|x|, \tag{5}$$

$$D_{(3)}^+ V(t, x) \leqslant -c\,|x|, \qquad t \neq \tau_k, \tag{6}$$

$$V(\tau_k+0, x+I_k(x, h(\tau_k, x))) \leqslant V(\tau_k, x) \qquad (k=1, 2, \ldots), \tag{7}$$

$$|V(t, x) - V(t, x_1)| \leqslant K\,|x-x_1|, \tag{8}$$

$$a\,|y-h(t, x)|^2 \leqslant W(t, x, y) \leqslant b\,|y-h(t, x)|^2, \tag{9}$$

$$\dot{W}_{(4)}(t, x, y) \leqslant -c\,|y-h(t, x)|^2, \tag{10}$$

$$\left|\frac{\partial W}{\partial t}(t, x, y)\right| \leqslant K\,|y-h(t, x)|\,(|x| + |y-h(t, x)|), \tag{11}$$

$$\left|\frac{\partial W}{\partial x}(t, x, y)\right| \leqslant K\,|y-h(t, x)|, \tag{12}$$

$$\left|\frac{\partial W}{\partial y}(t, x, y)\right| \leqslant K\,|y-h(t, x)|. \tag{13}$$

Then, for μ *small enough, the solution* $x=0$, $y=0$ *of system* (2) *is exponentially stable.*

Proof. From conditions (A1–A4) it follows that there exist constants $L>0$, $r>0$, and $\rho_0 \in (0, \rho)$ such that for $t \in R_+$, $x, x_1 \in B_\rho^n$, $y, y_1 \in B_\rho^m$, and $k=1, 2, \ldots$ the following inequalities hold:

$$|f(t, x, y) - f(t, x_1, y_1)| \leqslant L(|x - x_1| + |y - y_1|), \tag{14}$$

$$h(t, 0) = 0 \quad \text{for} \quad t \in R_+,$$

$$g(t, x, h(t, x)) = 0 \quad \text{for} \quad (t, x) \in R_+ \times B_H^n,$$

$$g(t, x, y) \neq 0 \quad \text{for} \quad (t, x, y) \in R_+ \times B_H^n \times B_H^m \text{ and } y \neq h(t, x).$$

(A5) $\tau_k - \tau_{k-1} \geqslant \theta > 0$ for $k = 1, 2, \ldots$.

The investigation of the stability of the solution $x = 0$, $y = 0$ of system (2) can be reduced to the investigation of the stability of the solution $x = 0$ of the system

$$\begin{aligned} \frac{dx}{dt} &= f(t, x, h(t, x)), \qquad t \neq \tau_k, \\ \Delta x|_{t=\tau_k} &= I_k(x, h(\tau_k, x)), \qquad k = 1, 2, \ldots \end{aligned} \tag{3}$$

and the stability of the solution $y = h(\alpha, \beta)$ of the system

$$\frac{dy}{ds} = g(\alpha, \beta, y), \tag{4}$$

where $(\alpha, \beta) \in R_+ \times B_H^n$ are parameters.

DEFINITION 1. The solution $x = 0$, $y = 0$ of system (2) is called exponentially stable if there exist constants $\rho > 0$, $A \geqslant 1$, and $v > 0$ such that for any $t_0 \in R_+$ and $(x_0, y_0) \in B_\rho^n \times B_\rho^m$ the solution $x(t) = x(t; t_0, x_0, y_0)$, $y(t) = y(t; t_0, x_0, y_0)$ of system (2) satisfies the estimate

$$|x(t)| + |y(t)| \leqslant A(|x_0| + |y_0|)\, e^{-v(t - t_0)} \qquad \text{for} \quad t > t_0.$$

Analogously the exponential stability of the solution of system (3) is defined.

Let $\psi(s; \alpha, \beta; y_0)$ be the solution of system (4) for which $\psi(0; \alpha, \beta; y_0) = y_0$.

DEFINITION 2. The solution $y = h(\alpha, \beta)$ of system (4) is called exponentially stable uniformly with respect to $(\alpha, \beta) \in R_+ \times B_H^n$ if there exist constants $\rho > 0$, $A \geqslant 1$, and $\gamma > 0$ such that for any $(\alpha, \beta) \in R_+ \times B_H^n$ and $\eta \in B_H^m$, $|\eta - h(\alpha, \beta)| < \rho$ the solution $\psi(s; \alpha, \beta; \eta)$ of system (4) satisfies the estimate

$$|g(t, x, y) - g(t, x_1, y_1)| \leqslant L(|x - x_1| + |y - y_1|),$$

$$|h(t, x) - h(t, x_1)| \leqslant L|x - x_1|, \tag{15}$$

$$|I_k(x, y) - I_k(x_1, y_1)| \leqslant L(|x - x_1| + |y - y_1|), \tag{16}$$

$$|J_k(x, y) - J_k(x_1, y_1)| \leqslant L(|x - x_1| + |y - y_1|),$$

$$|h(t, x)| \leqslant L\,|x|, \tag{17}$$

$$|f(t, x, y)| \leqslant L(|x| + |y - h(t, x)|), \tag{18}$$

$$|I_k(x, y)| \leqslant L(|x| + |y - h(t, x)|), \tag{19}$$

$$|J_k(x, y)| \leqslant L(|x| + |y - h(t, x)|), \tag{20}$$

$$|x + I_k(x, y)| + |y + J_k(x, y)| \leqslant \rho_0 \qquad \text{for} \quad |x| + |y| \leqslant r. \tag{21}$$

From the relations

$$\begin{aligned} D^+_{(2)} V(t, x) \leqslant {} & \limsup_{\tau \to 0_+} \frac{1}{\tau} \\ & \times [V(t+\tau, x+\tau f(t, x, y)) - V(t+\tau, x+\tau f(t, x, h(t, x)))] \\ & + \limsup_{\tau \to 0_+} \frac{1}{\tau} [V(t+\tau, x+\tau f(t, x, h(t, x))) - V(t, x)], \end{aligned}$$

$$D^+_{(2)} W(t, x, y) = \dot{W}_{(2)}(t, x, y) = \frac{1}{\mu} \dot{W}_{(4)} + \frac{\partial W}{\partial t} + \frac{\partial W}{\partial x} f$$

in view of (6), (7), (14), (10), (11), (12), (18), (5), and (9), we obtain that the functions V and W satisfy inequalities of the form

$$\begin{aligned} & D^+_{(2)} V \leqslant -\kappa V + Q\sqrt{W}, \\ & D^+_{(2)} W \leqslant 2QV\sqrt{W} - 2\sigma W, \qquad t \neq \tau_k, \end{aligned} \tag{22}$$

where $\kappa > 0$ and $Q > 0$ are constants, $\sigma = \sigma(\mu) = c/2b\mu - K(L+1)/2a$, and $\lim_{\mu \to 0_+} \sigma(\mu) = +\infty$.

Making use of (7), (8), (16), (12), (13), (17), (19), (20), (5), and (9), we obtain that there exists a constant $T > 0$ such that

$$\begin{aligned} V(\tau_k + 0, x + I_k(x, y)) \leqslant V(\tau_k, x) + TW^{1/2}(\tau_k, x, y), \\ W(\tau_k + 0, x + I_k(x, y), y + J_k(x, y)) \leqslant T^2(V(\tau_k, x) + W^{1/2}(\tau_k, x, y))^2. \end{aligned} \tag{23}$$

Let $t_0 \in R_+$, $x_0 \in B^n_\rho$, $y_0 \in B^m_\rho$ and $x(t) = x(t; t_0, x_0, y_0)$, $y(t) = y(t; t_0, x_0, y_0)$ be a solution of system (2). From (22) and (23) it follows that the functions $v(t) = V(t, x(t))$, $w(t) = W(t, x(t), y(t))$ for $t \in J^+ = J^+(t_0, x_0, y_0)$, $\tau_k \in J^+$ satisfy

$$\begin{aligned} & D^+ v \leqslant -\kappa v + Q\sqrt{w}, \\ & D^+ w \leqslant 2Qv\sqrt{w} - 2\sigma w, \qquad t \neq \tau_k, \\ & v(\tau_k + 0) \leqslant v(\tau_k) + T\sqrt{w(\tau_k)}, \\ & w(\tau_k + 0) \leqslant T^2(v(\tau_k) + \sqrt{w(\tau_k)})^2. \end{aligned}$$

Then by Lemma 1 $v(t) \leqslant \xi(t)$, $w(t) \leqslant \eta(t)$, whee $\xi = \xi(t)$, $\eta = \eta(t)$ is the maximal solution of the initial value problem

$$\begin{aligned}
&\frac{d\xi}{dt} = -\kappa\xi + Q\sqrt{\eta},\\
&\frac{d\eta}{dt} = 2Q\xi\sqrt{\eta} - 2\sigma\eta, \qquad t \neq \tau_k,\\
&\xi(\tau_k + 0) = \xi(\tau_k) + T\sqrt{\eta(\tau_k)},\\
&\eta(\tau_k + 0) = T^2(\xi(\tau_k) + \sqrt{\eta(\tau_k)})^2,\\
&\xi(t_0 + 0) = v(t_0 + 0) \geqslant 0, \qquad \eta(t_0 + 0) = w(t_0 + 0) \geqslant 0.
\end{aligned} \tag{24}$$

It turns out that for sufficiently small μ the maximal solution of initial value problem (24) is equal to $\xi = \xi(t)$, $\eta = \psi^2(t)$, where $\xi(t), \psi(t)$ is the solution of the following initial value problem:

$$\begin{aligned}
&\frac{d\xi}{dt} = -\kappa\xi + Q\psi,\\
&\frac{d\psi}{dt} = Q\xi - \sigma\psi, \qquad t \neq \tau_k,\\
&\xi(\tau_k + 0) = \xi(\tau_k) + T\psi(\tau_k),\\
&\psi(\tau_k + 0) = T\xi(\tau_k) + T\psi(\tau_k),\\
&\xi(t_0 + 0) = v(t_0 + 0), \qquad \psi(t_0 + 0) = \sqrt{w(t_0 + 0)}.
\end{aligned} \tag{25}$$

Denote $\xi_k = \xi(\tau_k)$, $\psi_k = \psi(\tau_k)$, $\xi_k^+ = \xi(\tau_k + 0)$, $\psi_k^+ = \psi(\tau_k + 0)$, $z = \operatorname{col}(\xi, \psi)$. Then for $t \in (\tau_k, \tau_{k+1}]$ we have

$$z(t) = u(t - \tau_k)\, z_k^+, \tag{26}$$

where

$$u(t) = (u_{ij}(t))_1^2, \qquad u_{11}(t) = \frac{(\kappa + \lambda_1)\, e^{\lambda_2 t} - (\kappa + \lambda_2)\, e^{\lambda_1 t}}{\lambda_1 - \lambda_2},$$

$$u_{12}(t) = u_{21}(t) = Q\,\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}, \qquad u_{22}(t) = \frac{(\kappa + \lambda_1)\, e^{\lambda_1 t} - (\kappa + \lambda_2)\, e^{\lambda_2 t}}{\lambda_1 - \lambda_2},$$

$$\lambda_{1,2} = \frac{1}{2}\left(-\sigma - \kappa \pm \sqrt{(\sigma - \kappa)^2 + 4Q^2}\right),$$

Since $\kappa+\lambda_1>0$ and $\kappa+\lambda_2<0$, then

$$0\leqslant u_{12}(t)=u_{21}(t)\leqslant\frac{Qe^{\lambda_1 t}}{\lambda_1-\lambda_2}\leqslant ce^{\lambda_1 t}\qquad(t\geqslant 0),$$

$$0\leqslant u_{11}(t)\leqslant e^{\lambda_1 t},\qquad 0\leqslant u_{22}(t)\leqslant e^{\lambda_1 t}\qquad(t\geqslant 0),$$

$$u_{22}(t)\leqslant\frac{\kappa+\lambda_1-(\kappa+\lambda_2)\,e^{(\lambda_2-\lambda_1)\theta}}{\lambda_1-\lambda_2}\,e^{\lambda_1 t}\leqslant ce^{\lambda_1 t}\qquad(t\geqslant\theta),$$

$$\lim_{\mu\to 0_+} e^{\lambda_1\theta}=e^{-\kappa\theta}\in(0,1),\tag{27}$$

where

$$c=c(\mu)=\max\left\{\frac{Q}{\lambda_1-\lambda_2},\frac{\kappa+\lambda_1-(\kappa+\lambda_2)\,e^{(\lambda_2-\lambda_1)\theta}}{\lambda_1-\lambda_2}\right\}\to 0$$
$$\text{as}\quad \mu\to 0_+.\tag{28}$$

Hence,

$$u(t)\leqslant e^{\lambda_1 t}\begin{bmatrix}1 & c\\ c & 1\end{bmatrix}\qquad(t\geqslant 0)\tag{29}$$

and

$$u(t)\leqslant e^{\lambda_1 t}\begin{bmatrix}1 & c\\ c & c\end{bmatrix}\qquad(t\geqslant\theta).$$

In particular,

$$\begin{aligned}\xi_{k+1}&\leqslant(\xi_k^+ + c\psi_k^+)\,e^{\lambda_1(\tau_{k+1}-\tau_k)},\\ \psi_{k+1}&\leqslant(c\xi_k^+ + c\psi_k^+)\,e^{\lambda_1(\tau_{k+1}-\tau_k)}.\end{aligned}\tag{30}$$

Then (25) and (30) imply that

$$\xi_{k+1}^+\leqslant[(1+cT)\,e^{\lambda_1\theta}\xi_k^+ + (c+cT)\,e^{\lambda_1\theta}\psi_k^+]\,e^{\lambda_1(\tau_{k+1}-\tau_k-\theta)},$$
$$\psi_{k+1}^+\leqslant[(1+c)\,Te^{\lambda_1\theta}\xi_k^+ + 2cTe^{\lambda_1\theta}\psi_k^+]\,e^{\lambda_1(\tau_{k+1}-\tau_k-\theta)}.$$

In view of (27) and (28), we obtain that

$$z_{k+1}^+\leqslant e^{\lambda_1(\tau_{k+1}-\tau_k-\theta)}\,H(\mu)\,z_k^+,\tag{31}$$

where

$$H(\mu)=\begin{bmatrix}e^{-\kappa\theta}+\delta, & \delta\\ Te^{-\kappa\theta}+\delta, & \delta\end{bmatrix},\qquad \delta=\delta(\mu)\geqslant 0,\qquad \lim_{\mu\to 0_+}\delta(\mu)=0.\tag{32}$$

Let $q \in (0, 1)$. From (32) it follows that there exists $\mu_1 \in (0, \mu_0]$ such that if $\mu \in (0, \mu_1]$, then $c(\mu) \leqslant 1$ and the modules of the eigenvalues $\nu_1(\mu)$ and $\nu_2(\mu)$ of the matrix $H(\mu)$ are smaller than one: $|\nu_1| < 1$, $|\nu_2| < 1$. Hence there exists an integer $N > 0$ such that

$$\|H^N(\mu)\| \leqslant q \qquad \text{for} \quad \mu \in (0, \mu_1]. \tag{33}$$

Let $\tau_{j-1} \leqslant t_0 < \tau_j < \cdots < \tau_{j+i} < t \leqslant \tau_{j+i+1}$, $i = Np + r$, $0 \leqslant r < N$. Then from (26) and (31) we obtain the estimate

$$z(t) \leqslant u(t - \tau_{j+i}) \prod_{k=j+i}^{j+1} \left(e^{\lambda_1(\tau_k - \tau_{k-1} - \theta)} H(\mu)\right)$$
$$\times \begin{bmatrix} 1 & T \\ T & T \end{bmatrix} u(t_j - t_0)\, z(t_0 + 0)$$

and in view of (29), (33), and condition (A5) we find that

$$\xi(t) + \psi(t) \leqslant De^{-\nu(t - t_0)}(v(t_0 + 0) + \sqrt{w(t_0 + 0)}) \qquad (t > t_0), \tag{34}$$

where $D = \text{const}$, $\nu = \min(-(1/N\theta) \ln q,\ -\lambda_1(\mu_1)) > 0$.

From (34), (5), (9), and (17) it follows that there exists a constant $A \geqslant 1$ such that for $t \in J^+$ the following estimate holds:

$$|x(t)| + |y(t)| \leqslant A(|x_0| + |y_0|)\, e^{-\nu(t - t_0)}. \tag{35}$$

Let $A\lambda < r$ and $|x_0| + |y_0| < \lambda$. Then from (35) it follows that $|x(t)| + |y(t)| < r$ for $t \in J^+$ and in view of (21) and conditions (A1) and (A2) we conclude that $J^+ = (t_0, \infty)$, i.e., inequality (35) holds for all $t > t_0$.

This completes the proof of Theorem 1.

LEMMA 3. *Let the following conditions hold*:

1. $0 < \tau_1 < \tau_2 < \cdots$, $\lim_{k \to \infty} \tau_k = \infty$.
2. *The function* $F(t, x)$ *is continuous in* $R_+ \times B^n_H$, *locally Lipschitz continuous with respect to* x *in* $R_+ \times B^n_H$, *and* $F(t, 0) = 0$ *for* $t \in R_+$.
3. *The functions* $I_k(x)$ $(k = 1, 2, \ldots)$ *are Lipschitz continuous with respect to* $x \in B^n_H$ *and* $I_k(0) = 0$ $(k = 1, 2, \ldots)$.
4. *The solution* $x = 0$ *of system* (1) *is exponentially stable.*

Then, if p *is a positive integer, there exist constants* $\rho \in (0, H)$, $a > 0$, $b > 0$, $c > 0$, *a function* $V: R_+ \times B^n_\rho \to R_+$, *and a positive function* $L: R_+ \to R_+$ *such that*:

1. $V \in \mathscr{V}_0$.
2. $a\,|x|^p \leqslant V(t, x) \leqslant b\,|x|^p$ *for* $(t, x) \in R_+ \times B^n_\rho$.
3. $|V(t, x) - V(t, y)| \leqslant L(t)\,|x - y|$ *for* $t \in R_+$; $x, y \in B^n_\rho$.
4. $D^+_{(1)} V(t, x) \leqslant -c\,|x|^p$ *for* $(t, x) \in R_+ \times B^n_\rho$, $t \neq \tau_k$.
5. $V(\tau_k + 0, x + I_k(x)) \leqslant V(\tau_k, x)$ *for* $x \in B^n_\rho$, $k = 1, 2, \ldots$.

Proof. Since the solution $x = 0$ is exponentially stable, then there exist constants $A \geqslant 1$, $v > 0$ and $\rho \in (0, H)$, $\rho A < H$ such that for $x_0 \in B^n_\rho$ and $t > t_0 \geqslant 0$ the following inequality holds:

$$|x(t; t_0, x_0)| \leqslant A\,|x_0|\,e^{-v(t - t_0)} \tag{36}$$

Let $0 < q < 1$. Then for $(t, x) \in R_+ \times B^n_\rho$ define

$$\begin{aligned} V(t, x) &= \sup_{\tau > 0} |x(t + \tau; t, x)|^p\,e^{vpq\tau} \qquad (t \neq \tau_k), \\ V(\tau_k, x) &= V(\tau_k - 0, x). \end{aligned} \tag{37}$$

From (36) and (37) it immediately follows that

$$|x|^p \leqslant V(t, x) \leqslant \sup_{\tau > 0} A^p\,|x|^p\,e^{vp(q-1)\tau} \leqslant A^p\,|x|^p.$$

Thus property 2 is proved for $t \neq \tau_k$.

Choose $T > 0$ so that $A^p e^{vp(1-q)T} \leqslant 1$. Then, if $\tau > T$, we have $A^p\,|x|^p\,e^{-vp(1-q)\tau} \leqslant |x|^p$ and

$$V(t, x) = \sup_{0 < \tau \leqslant T} |x(t + \tau; t, x)|^p\,e^{vpq\tau}, \tag{38}$$

From conditions 2 and 3 of Lemma 3 it follows that

$$|F(t, x) - F(t, y)| \leqslant M(t)\,|x - y| \qquad \text{for} \quad x, y \in B^n_{\rho A},\ t \in R_+,$$
$$|I_k(x) - I_k(y)| \leqslant M_k\,|x - y)| \qquad \text{for} \quad x, y \in B^n_{\rho A},\ k = 1, 2, \ldots,$$

where $M(t) \geqslant 0$, $M_k \geqslant 0$, and $M(t)$ is continuous on R_+.

Let $x, y \in B^n_\rho$ and $0 < \tau \leqslant T$. We apply Lemma 2 and obtain the estimate

$$\begin{aligned} &|x(t + \tau; t, x) - x(t + \tau; t, y)| \\ &\quad \leqslant |x - y| \prod_{t < \tau_k < t + \tau} (1 + M_k) \exp\left(\int_t^{t+\tau} M(s)\,ds\right). \end{aligned} \tag{39}$$

In view of (38), (39), and the inequality

$$|u^p - v^p| \leqslant p\,|u - v|\,(\max(u, v))^{p-1} \qquad (\text{for } u \geqslant 0, v \geqslant 0),$$

we obtain

$$|V(t,x)-V(t,y)| = \Big| \sup_{0<\tau\leqslant T} |x(t+\tau;t,x)|^p e^{\nu pq\tau} - \sup_{0<\tau\leqslant T} |x(t+\tau;t,y)|^p e^{\nu pq\tau}\Big|$$

$$\leqslant \sup_{0<\tau\leqslant T} \big||x(t+\tau;t,x)|^p - |x(t+\tau;t,y)|^p\big| e^{\nu pq\tau}$$

$$\leqslant L(t)\,|x-y|,$$

where $L(t)=p(A\rho)^{p-1}\prod_{t<\tau_k<t+T}(1+M_k)\exp(\int_t^{t+T} M(s)\,ds+\nu pqT)$.

Thus we have proved property 3 for $t\neq\tau_k$.

Let $x\in B_\rho^n$, $x_1\in B_\rho^n$, $\tau_{k-1}<t<\tau_k$ and let $\delta>0$ be such that $t+\delta<\tau_k$. Then

$$|V(t+\delta,x_1)-V(t,x)| \leqslant |V(t+\delta,x_1)-V(t+\delta,x)| + |V(t+\delta,x)-V(t+\delta,x(t+\delta;t,x))| + |V(t+\delta,x(t+\delta;t,x))-V(t,x)|. \tag{40}$$

From property 3 there follow the estimates

$$|V(t+\delta,x_1)-V(t+\delta,x)| \leqslant L(t+\delta)\,|x_1-x|,$$

$$|V(t+\delta,x)-V(t+\delta,x(t+\delta;t,x))| \leqslant L(t+\delta)|\,x-x(t+\delta;t,x)|.$$

Since for $t\neq\tau_k$, $\lim_{\delta\to 0}L(t+\delta)=L(t)$, and $\lim_{\delta\to 0}|x-x(t+\delta;t,x)|=0$, then the first two terms in the right-hand side of estimate (40) are small when $|x_1-x|$ and δ are small.

Denote $a(\delta)=\sup_{\tau>\delta}|x(t+\tau;t,x)|^p e^{\nu pq\tau}$.

The function $a(\delta)$ is non-increasing for $\delta\geqslant 0$ and $\lim_{\delta\to 0_+}a(\delta)=a(0)$ since $|x(t+\tau;t,x)|^p e^{\nu pq\tau}$ is a bounded and piecewise continuous function for $\tau\geqslant 0$ and continuous in some neighbourhood of $\tau=0$.

Then for the third term in (40) we obtain

$$0\leqslant |V(t+\delta,x(t+\delta;t,x))-V(t,x)|$$

$$=\Big|\sup_{s>0}|x(t+\delta+s;t+\delta,x(t+\delta;t,x))|^p e^{\nu pqs} - \sup_{\tau>0}|x(t+\tau;t,x)|^p e^{\nu pq\tau}\Big|$$

$$=\Big|\sup_{\tau>\delta}|x(t+\tau;t,x)|^p e^{\nu pq\tau}\cdot e^{-\nu pq\delta} - \sup_{\tau>0}|x(t+\tau;t,x)|^p e^{\nu pq\tau}\Big|$$

$$=|a(\delta)\,e^{-\nu pq\delta}-a(0)|\to 0 \qquad \text{as} \quad \delta\to 0_+.$$

Hence $V(t,x)$ is continuous for $x\in B_\rho^n$ and $t\neq\tau_k$.

Let $x \in B^n_\rho$, $t \in R_+$, $t \neq \tau_k$, $h > 0$, and $x_1 = x(t+h;\, t, x)$. Then

$$V(t+h, x_1) = \sup_{s>0} |x(t+h+s;\, t+h, x)|^p\, e^{vpqs}$$

$$= \sup_{\tau > h} |x(t+\tau;\, t, x)|^p\, e^{vpq\tau} \cdot e^{-vpqh} \leqslant V(t, x)\, e^{-vpqh}$$

or

$$\frac{1}{h}[V(t+h, x_1) - V(t, x)] \leqslant V(t, x)\frac{1}{h}[e^{-vpqh} - 1];$$

whence it follows that

$$D^+_{(1)} V(t, x) \leqslant -vpqV(t, x) \leqslant -\frac{vpq}{a}|x|^p,$$

which proves property 3.

Let $\tau_k \in R_+$ and $x \in B^n_\rho$ be fixed and $t_i \in (\tau_k, \tau_{k+1})$, $x_i \in B^n_\rho$, $u_i = x(t_i;\, \tau_k, x)$ for $i = 1, 2$. Then

$$|V(t_1, x_1) - V(t_2, x_2)| \leqslant |V(t_1, x_1) - V(t_1, u_1)| + |V(t_2, x_2) - V(t_2, u_2)| + |V(t_1, u_1) - V(t_2, u_2)|. \tag{41}$$

Taking into account that $V(t, x)$ and $F(t, x)$ are locally Lipschitz continuous, we obtain successively the estimates

$$|V(t_i, x_i) - V(t_i, u_i)| \leqslant L(t_i)\,|x_i - u_i| \qquad (i = 1, 2),$$

$$|x_i - u_i| \leqslant |x_i - x| + |u_i - x| \qquad (i = 1, 2),$$

$$|u_i - x| \leqslant \int_{\tau_k}^{t_i} M(s) \exp\left(\int_{\tau_k}^{s} M(\tau)\, d\tau\right) ds\, |x| \tag{42}$$

$$\equiv N(t_i)\,|x| \qquad (i = 1, 2),$$

$$|V(t_i, x_i) - V(t_i, u_i)| \leqslant L(t_i)\,|x_i - x| + L(t_i)\, N(t_i)\,|x| \qquad (i = 1, 2),$$

where

$$\lim_{t_i \to \tau_k + 0} L(t_i) = L(\tau_k + 0), \quad \lim_{t_i \to \tau_k + 0} N(t_i) = 0.$$

Since the function $a(\delta) = \sup_{\tau > \delta} |x(\tau_k + \tau;\, \tau_k, x)|^p\, e^{vpq\tau}$ is non-increasing for $\delta \geqslant 0$ and $\lim_{\delta \to 0_+} a(\delta) = a(0)$, then

$$|V(t_1, u_1) - V(t_2, u_2)|$$
$$= |\sup_{s>0} |x(t_1+s; t_1, u_1)|^p e^{\nu pqs} - \sup_{s>0} |x(t_2+s; t_2, u_2)|^p e^{\nu pqs}|$$
$$= |a(t_1 - \tau_k) e^{-\nu pq(t_1-\tau_k)} - a(t_2 - \tau_k) e^{-\nu pq(t_2-\tau_k)}| \to 0$$

as $t_i \to \tau_k + 0$, $i = 1, 2$ and in view of (41) and (42) we conclude that the limit $V(\tau_k + 0, x)$ exists. Analogously the existence of the limit $V(\tau_k - 0, x)$ is proved and since the equality $V(\tau_k - 0, x) = V(\tau_k, x)$ holds by definition, then $V \in \mathscr{V}_0$.

Let $\eta(t; t_0, x_0)$ be the solution of the initial value problem

$$\frac{d\eta}{dt} = F(t, \eta), \qquad \eta(t_0) = x_0.$$

Since for $\tau_{k-1} < \lambda < \tau_k < \mu < \tau_{k+1}$ and $s > \mu$ the relation

$$x(s; \mu, \eta(\mu; \tau_k, x + I_k(x))) = x(s; \lambda, \eta(\lambda; \tau_k, x))$$

holds, then

$$V(\mu, \eta(\mu; \tau_k, x + I_k(x))) \leqslant V(\lambda, \eta(\lambda; \tau_k, x))$$

and, passing to the limit for $\mu \to \tau_k + 0$ and $\lambda \to \tau_k - 0$, we obtain

$$V(\tau_k + 0, x + I_k(x)) \leqslant V(\tau_k - 0, x) = V(\tau_k, x).$$

This completes the proof of Lemma 3.

Remark 3. If for $t \in R_+$ and $k = 1, 2, \ldots$

$$M(t) = M_k = L, \qquad \tau_k - \tau_{k-1} \geqslant \theta > 0$$

then $L(t) \leqslant \rho(A\rho)^{p-1} \exp(\nu pqT + LT + \theta^{-1}T + 1) \equiv K$; i.e., the function $V(t, x)$ is Lipschitz continuous with respect to x in the domain $R_+ \times B^n_\rho$ with a constant K. Hence, the following corollary is valid.

COROLLARY 1. *If conditions* (A) *hold and the solution* $x = 0$ *of system* (3) *is exponentially stable, then there exist constants* $\rho \in (0, H)$, $a > 0$, $b > 0$, $c > 0$, $K > 0$, *and a function* $V \in \mathscr{V}_0$, $V\colon R_+ \times B^n_\rho \to R_+$ *which satisfies conditions* (5)–(8) *of Theorem* 1.

Introduce the following condition (B):

(B) There exists a constant $L > 0$ such that for $(t, x, y) \in R_+ \times B^n_H \times B^m_H$

$$\left|\frac{\partial h}{\partial t}(t, x)\right|, \left|\frac{\partial g}{\partial t}(t, x, y)\right| \leqslant L(|x| + |y - h(t, x)|).$$

LEMMA 4. *Let conditions* (A1), (A3), (A4), B *hold and let the solution* $y=h(\alpha, \beta)$ *of system* (4) *be exponentially stable, uniformly with respect to* $(\alpha, \beta) \in R_+ \times B^n_H$.

Then there exist constants $\rho \in (0, H)$, $a>0$, $b>0$, $c>0$, *and* $K>0$, *and a function* $W \in C_1$, $W: R_+ \times B^n_\rho \times B^m_\rho \to R_+$ *which satisfies conditions* (9)–(13) *of Theorem* 1.

Proof. Let $T \geqslant (2 \ln A + \ln 2)/2\nu$ and for $(t, x, y) \in R_+ \times B^n_\rho \times B^m_\rho$ define the function

$$W(t, x, y) = \int_0^T \sum_{i=1}^m (\psi_i(s; t, x; y) - h_i(t, x))^2 \, ds,$$

where the constants $A \geqslant 1$, $\nu > 0$, and $\rho \in (0, H)$ are chosen according to Definition 2.

Properties (9)–(12) were proved in [14], Lemma 2. We prove only property (13).

The matrix $Z(s)$ with entries $z_{ij}(s) = (\partial \psi_i / \partial y_j)(s; t, x; y)$ is a solution of the system

$$\frac{dZ}{ds} = \frac{\partial g}{\partial y}(t, x, \psi(s; t, x; y))Z, \qquad Z(0) = E_m$$

with bounded coefficients. Hence there exists a constant $M>0$ such that $|\partial \psi_i / \partial y_j| \leqslant M e^{Ms}$. Then

$$\left|\frac{\partial W}{\partial y_i}\right| = \left|2 \int_0^T \sum_{i=1}^m \frac{\partial \psi_i}{\partial y_j}(s; t, x; y)[\psi_i(s; t, x; y) - h_i(t, x)] \, ds\right|$$

$$\leqslant 2A\, |y - h(t, x)| \int_0^T M e^{Ms} e^{-\nu s} \, ds \equiv \frac{K}{m} |y - h(t, x)|.$$

This completes the proof of Lemma 4.

Introduce the following conditions (C):

(C1) The matrix-valued functions $D(t)$ and $(dD/dt)(t)$ of order m are continuous and bounded in R_+.

(C2) The real parts of all eigenvalues of $D(t)$ are bounded from above by a negative constant, uniformly with respect to $t \in R_+$.

(C3) The function $G: R_+ \times B^n_H \times B^m_H \to R^m$, $(t, x, y) \to G(t, x, y)$ is continuously differentiable and

$$G(t, 0, 0) = 0, \qquad \frac{\partial G}{\partial t} = O(|x| + |y|),$$

$$\frac{\partial G}{\partial x} = O(|x| + |y|), \qquad \frac{\partial G}{\partial y} = o(1)$$

as $|x| + |y| \to 0$, uniformly with respect to $t \in R_+$.

LEMMA 5 [14]. *Let the function g have the form*

$$g(t, x, y) = D(t)\, y + G(t, x, y),$$

where the functions $D(t)$ *and* $G(t, x, y)$ *satisfy conditions* (C).

Then there exists a function $W \in C_1$, $W: R_+ \times B_\rho^n \times B_\rho^m \to R_+$ *which satisfies conditions* (9)–(13) *of Theorem* 1.

An immediate consequence of Corollary 1 and Lemmas 3, 4, and 5 is the following theorem.

THEOREM 2. *Let the following conditions be fulfilled*:

1. *Conditions* (A) *hold.*
2. *The solution* $X = 0$ *of system* (3) *is exponentially stable.*
3. *One of the following conditions holds*:

3.1. *Condition* (B) *holds and the solution* $y = h(\alpha, \beta)$ *of system* (4) *is exponentially stable, uniformly in* $(\alpha, \beta) \in R_+ \times B_H^n$.

3.2. *Conditions* (C) *hold and the function g has the form* $g(t, x, y) = D(t)\, y + G(t, x, y)$.

Then, for μ *small enough, the solution* $x = 0$, $y = 0$ *of system* (2) *is exponentially stable.*

As a consequence of Theorem 2 we obtain an analogue of the theorem of Klimushev and Krasovskii [12] concerning the linear system with impulse effect,

$$\begin{aligned} \frac{dx}{dt} &= A(t)x + B(t)\, y, \\ \mu \frac{dy}{dt} &= C(t)x + D(t)\, y, \qquad t \neq \tau_k, \\ \Delta x|_{t=\tau_k} &= \alpha_k x + \beta_k\, y, \\ \Delta y|_{t=\tau_k} &= \gamma_k x + \delta_k\, y, \qquad k = 1, 2, ..., \end{aligned} \tag{43}$$

where the matrices A, B, C, D, α_k, β_k, γ_k, and δ_k are of the respective dimensions.

THEOREM 3. *Let the following conditions be fulfilled*:

1. *The matrix-valued functions* $A(t)$, $B(t)$, $C(t)$, *and* $D(t)$ *are continuous and bounded for* $t \in R_+$ *together with their first derivatives.*

2. *The real parts of all eigenvalues of $D(t)$ are bounded from above by a negative constant, uniformly with respect to $t \in R_+$.*

3. *There exist constants $\theta > 0$ and $c > 0$ such that for $k = 1, 2, \ldots$*

$$\tau_k - \tau_{k-1} \geqslant \theta; \qquad \|\alpha_k\| \leqslant c,\ \|\beta_k\| \leqslant c,\ \|\gamma_k\| \leqslant c,\ \|\delta_k\| \leqslant c.$$

4. *The solution $x = 0$ of the system*

$$\frac{dx}{dt} = [A(t) - B(t)\, D^{-1}(t)\, C(t)]\,x, \qquad t \neq \tau_k,$$

$$\Delta x|_{t=\tau_k} = [\alpha_k - \beta_k D^{-1}(\tau_k)\, C(\tau_k)]\,x, \qquad k = 1, 2, \ldots$$

is exponentially stable.

Then, for μ small enough, the solution $x = 0$, $y = 0$ of system (43) *is exponentially stable.*

EXAMPLE 1. Consider the singularly perturbed system

$$\frac{dx}{dt} = x - y, \qquad \mu \frac{dy}{dt} = 2x - 3y, \qquad t \neq k,$$
$$\Delta x|_{t=k} = x - 2y, \qquad k = 1, 2, \ldots \tag{44}$$

and the related equaltions

$$\frac{dx}{dt} = \frac{x}{3},\ t \neq k, \qquad \Delta x|_{t=k} = -\frac{x}{3},\ k = 1, 2, \ldots \tag{45}$$

and

$$\frac{dy}{ds} = 2x - 3y. \tag{46}$$

The solution $y = \frac{2}{3}x$ of Eq. (46) is exponentially stable uniformly with respect to $x \in R$.

Moreover, for the solution $\varphi(t; t_0, x_0)$ of (45) we have

$$\varphi(t; t_0, x_0) = e^{(1/3)(t-n)} q^{n-k}\, \tfrac{2}{3}\, e^{(1/3)(k-t_0)} x_0$$
$$< e^{2/3} q^{n-k} x_0,$$

where $k - 1 \leqslant t_0 < k$, $n < t \leqslant n + 1$, $q = \frac{2}{3} e^{1/3}$.

Since $0 < q < 1$, then the solution $x = 0$ of Eq. (45) is exponentially stable. Then, in virtue of Theorem 3 for μ small enough, the solution $x = 0$, $y = 0$ of system (44) is exponentially stable.

3.2. *Partial Exponential Stability of the Zero Solution of Singularly Perturbed Systems with Impulse Effect*

Before we formulate and prove the main results in this section, we make some preliminary considerations.

Consider the systems

$$\begin{aligned} \frac{dx}{dt} &= f(x, y), \\ \frac{dy}{dt} &= g(x, y), \qquad t \neq \tau_k, \\ \Delta x|_{t=\tau_k} &= I_k(x, y), \\ \Delta y|_{t=\tau_k} &= J_k(x, y), \qquad k = 1, 2, ..., \end{aligned} \tag{47}$$

where $f: B_H^n \times B_H^m \to R^n$, $g: B_H^n \times B_H^m \to R^m$, $I_k: B_H^n \times B_H^m \to R^n$, $J_k: B_H^n \times B_H^m \to R^m$, $H > 0$.

Let $x = 0$, $y = 0$ be a solution of system (47).

DEFINITION 3. The solution $x = 0, y = 0$ of system (47) is called y-exponentially stable if there exist constants $\rho > 0$, $A \geqslant 1$, and $\nu > 0$ such that for any $x_0 \in B_\rho^n$, $y_0 \in B_\rho^n$, $t_0 \in R_+$, and $t > t_0$ the following inequalities hold:

$$|x(t; t_0, x_0, y_0)| \leqslant A,$$

$$|y(t; t_0, x_0, y_0)| \leqslant A\,|y_0|\, e^{-\nu(t - t_0)}.$$

LEMMA 6. *Let the following conditions hold*:

1. *The functions* f, g, I_k, J_k $(k = 1, 2, ...)$ *are Lipschitz continuous with respect to* $(x, y) \in B_H^n \times B_H^m$ *with a constant* $L > 0$ *and* $f(0, 0) = I_k(0, 0) = 0$, $g(0, 0) = J_k(0, 0) = 0$.
2. $\tau_k - \tau_{k-1} \geqslant \theta > 0$ $(k = 1, 2, ...)$.
3. *The solution* $x = 0$, $y = 0$ *of system* (47) *is y-exponentially stable.*

Then there exist constants $a > 0$, $b > 0$, $c > 0$, $K > 0$, $\rho \in (0, H)$, *and a function* $V \in \mathscr{V}_0$, $V: R_+ \times B_\rho^n \times B_\rho^m \to R_+$ *such that for any* $t \in R_+$, $x, x_1 \in B_\rho^n$, $y, y_1 \in B_\rho^m$, *and* $k = 1, 2, ...$ *the following inequalities hold*:

$$a\,|y| \leqslant V(t, x, y) \leqslant b\,|y|, \tag{48}$$

$$D^+_{(47)} V(t, x, y) \leqslant -c\,|y|, \qquad t \neq \tau_k, \tag{49}$$

$$|V(t, x, y) - V(t, x_1, y_1)| \leqslant K(|x - x_1| + |y - y_1|), \tag{50}$$

$$V(\tau_k + 0, x + I_k(x, y), y + J_k(x, y)) \leqslant V(\tau_k, x, y). \tag{51}$$

Proof. Let the constants $A \geqslant 1$, $\nu > 0$, and $\rho \in (0, H)$ be chosen according to Definition 3. Let $0 < q < 1$ and for $(t, x, y) \in R_+ \times B_\rho^n \times B_\rho^m$ define

$$V(t, x, y) = \sup_{\tau > 0} |y(t+\tau; t, x, y)| \, e^{\nu q \tau} \qquad \text{for} \quad t \neq \tau_k,$$

$$V(\tau_k, x, y) = V(\tau_k - 0, x, y).$$

Further on, the proof of Lemma 6 repeats the arguments from the proof of Lemma 3 and we omit it.

Consider the system

$$\frac{dy}{ds} = g(x, y), \tag{52}$$

where $x \in B_H^n$ is a parameter. Assume that the equation $g(x, y) = 0$ has a unique solution $y = \bar{y}(x)$ for any $x \in B_H^n$.

LEMMA 7. *Let the function $g(x, y)$ be Lipschitz continuous with respect to $(x, y) \in B_H^n \times B_H^m$ and let the solution $y = \bar{y}(x)$ of system (52) be exponentially stable, uniformly with respect to $x \in B_H^n$.*

Then there exist constants $a > 0$, $b > 0$, $c > 0$, $K > 0$, $\rho \in (0, H)$, and a function $W \in C_0$, $W\colon B_\rho^n \times B_\rho^m \to R_+$ such that for any $x, x_1 \in B_\rho^n$ and $y, y_1 \in B_\rho^m$ the following inequalities hold:

$$a\,|y - \bar{y}(x)| \leqslant W(x, y) \leqslant b\,|y - \bar{y}(x)|, \tag{53}$$

$$D_{(52)}^{+} W(x, y) \leqslant -c\,|y - \bar{y}(x)|, \tag{54}$$

$$|W(x, y) - W(x_1, y_1)| \leqslant K(|x - x_1| + |y - y_1|). \tag{55}$$

The proof of Lemma 7 is carried out as the proof of Lemma 3, making use of the function

$$W(x, y) = \sup_{\tau \geqslant 0} |\psi(s; x; y) - \bar{y}(x)| \, e^{\nu q \tau} \qquad (0 < q < 1).$$

Consider the system

$$\begin{aligned} \varepsilon_\nu \frac{dy_\nu}{dt} &= f_\nu(y_0, ..., y_m), \qquad && t \neq \tau_k, \\ \Delta y_\nu|_{t=\tau_k} &= I_{\nu k}(y_0, ..., y_m), \qquad && k = 1, 2, ..., \nu = 0, 1, ..., m, \end{aligned} \tag{56}$$

where $y_\nu \in B_H^{n_\nu}$, $0 < H \leqslant \infty$, $f_\nu\colon B_H^{n_0} \times \cdots \times B_H^{n_m} \to R^{n_\nu}$, $I_{\nu k}\colon B_H^{n_0} \times \cdots \times B_H^{n_m} \to R^{n_\nu}$, and $\varepsilon_\nu = \varepsilon_\nu(\mu) > 0$ for $\mu \in (0, \mu_0]$.

Introduce the following conditions (D):

(D1) $\tau_k - \tau_{k-1} \geqslant \theta > 0$ $(k = 1, 2, \ldots)$.

(D2) $\varepsilon_0 \equiv \varepsilon_1 = 1$, the functions $\varepsilon_\nu = \varepsilon_\nu(\mu)$ are continuous in the interval $(0, \mu_0]$ and

$$\lim_{\mu \to 0_+} \frac{\varepsilon_{\nu+1}(\mu)}{\varepsilon_\nu(\mu)} = 0 \qquad (\nu = 1, \ldots, m-1).$$

(D3) Let $f_\nu^m \equiv f_\nu$ for $\nu = 0, \ldots, m$.

Assume that for any $(y_0, \ldots, y_{m-1}) \in B_H^{n_0} \times \cdots \times B_H^{n_{m-1}}$ the equation

$$f_m^m(y_0, \ldots, y_{m-1}, y_m) = 0$$

has a unique solution $y_m = \bar{y}_m = \bar{y}_m(y_0, \ldots, y_{m-1})$ and define for $\nu = 0, 1, \ldots, m-1$ and $k = 1, 2, \ldots$

$$f_\nu^{m-1} \equiv \bar{f}_\nu^m = f_\nu^m(y_0, \ldots, y_{m-1}, \bar{y}_m(y_0, \ldots, y_{m-1})),$$
$$I_{\nu k}^{m-1} \equiv \bar{I}_{\nu k}^m = I_{\nu k}^m(y_0, \ldots, y_{m-1}, \bar{y}_m(y_0, \ldots, y_{m-1})).$$

Assume inductively that for $\nu = m-2, \ldots, 1$ and $(y_0, \ldots, y_\nu) \in B_H^{n_0} \times \cdots \times B_H^{n_\nu}$ the equation

$$f_{\nu+1}^{\nu+1}(y_0, \ldots, y_\nu, y_{\nu+1}) = 0$$

has a unique solution $y_{\nu+1} = \bar{y}_{\nu+1} = \bar{y}_{\nu+1}(y_0, \ldots, y_\nu)$ and define for $i = 0, \ldots, \nu$ and $k = 1, 2, \ldots$

$$f_i^\nu \equiv \bar{f}_i^{\nu+1} \equiv f_i^{\nu+1}(y_0, \ldots, y_\nu, \bar{y}_{\nu+1}(y_0, \ldots, y_\nu)),$$
$$I_{ik}^\nu \equiv \bar{I}_{ik}^{\nu+1} \equiv I_{ik}^{\nu+1}(y_0, \ldots, y_\nu, \bar{y}_{\nu+1}(y_0, \ldots, y_\nu)).$$

(D4) There exists a constant $L > 0$ such that for any $\nu = 0, 1, \ldots, m$; $k = 1, 2, \ldots$; $y_\nu, y_\nu^* \in B_H^{n_\nu}$; and $j = 2, \ldots, m$;

$$y_j, y_j^* \in B_H^{n_j}$$

$$|f_\nu(y_0, \ldots, y_m) - f_\nu(y_0^*, \ldots, y_m^*)| \leqslant L \sum_{i=0}^m |y_i - y_i^*|,$$

$$|f_\nu(y_0, \ldots, y_m)| \leqslant L \sum_{i=1}^m |y_i|,$$

$$|I_{\nu k}(y_0, \ldots, y_m) - I_{\nu k}(y_0^*, \ldots, y_m^*)| \leqslant L \sum_{i=0}^m |y_i - y_i^*|,$$

$$|I_{\nu k}(y_0, ..., y_m)| \leqslant L \sum_{i=1}^{m} |y_i|,$$

$$|\bar{y}_j(y_0, ..., y_{j-1}) - \bar{y}_j(y_0^*, ..., y_{j-1}^*)| \leqslant L \sum_{i=0}^{j} |y_i - y_i^*|,$$

$$|\bar{y}_j(y_0, ..., y_{j-1})| \leqslant L \sum_{i=1}^{j} |y_i|.$$

(D5) The solution $y_0 = 0$, $y_1 = 0$ of the system

$$\begin{aligned} \varepsilon_0 \frac{dy_0}{dt} &= f_0^1(y_0, y_1), \\ \varepsilon_1 \frac{dy_1}{dt} &= f_1^1(y_0, y_1), \qquad t \neq \tau_k, \\ \Delta y_0|_{t=\tau_k} &= I_{0k}^1(y_0, y_1), \\ \Delta y_1|_{t=\tau_k} &= I_{1k}^1(y_0, y_1), \qquad k = 1, 2, ... \end{aligned} \tag{57}$$

is y_1-exponentially stable.

(D6) For any $\nu = 2, ..., m$ the solution $y_\nu = \bar{y}_\nu = \bar{y}_\nu(y_0, ..., y_{\nu-1})$ of the system

$$\frac{dy_\nu}{ds} = f_\nu^\nu(y_0, ..., y_{\nu-1}, y_\nu) \tag{58}$$

is exponentially stable, uniformly with respect to $(y_0, ..., y_{\nu-1}) \in B_H^{n_0} \times \cdots \times B_H^{n_{\nu-1}}$.

THEOREM 4. *If conditions* (D) *hold, then the solution* $y = 0, ..., y_m = 0$ *of system* (56) *is* $(y_1, ..., y_m)$*-exponentially stable.*

Proof. From conditions (D1), (D3), (D4), and (D5) it follows that the conditions of Lemma 6 hold. Hence there exist constants $a > 0$, $b > 0$, $c > 0$, $K > 0$, $\rho \in (0, H)$, and a function $V \in \mathscr{V}_0$, $V: R_+ \times B_\rho^{n_0} \times B_\rho^{n_1} \to R_+$ such that for $t \in R_+$; $y_i, y_i^* \in B_\rho^{n_i}$, $i = 0, 1$, and $k = 1, 2, ...$ the following inequalities hold:

$$a\,|y_1| \leqslant V(t, y_0, y_1) \leqslant b\,|y_1|, \tag{59}$$

$$D_{(57)}^+ V(t, y_0, y_1) \leqslant -c\,|y_1|, \qquad t \neq \tau_k, \tag{60}$$

$$|V(t, y_0, y_1) - V(t, y_0^*, y_1^*)| \leqslant K(|y_0 - y_0^*| + |y_1 - y_1^*|), \tag{61}$$

$$V(\tau_k + 0, y_0 + I_{0k}^1(y_0, y_1), y_1 + I_{1k}^1(y_0, y_1)) \leqslant V(\tau_k, y_0, y_1). \tag{62}$$

Consider the system

$$\frac{dy_2}{ds} = f_2^2(y_0, y_1, y_2), \tag{63}$$

where $(y_0, y_1) \in B_H^{n_0} \times B_H^{n_1}$ are parameters.

From conditions (D3), (D4), and (D6) it follows that system (63) satisfies the conditions of Lemma 7. Hence there exist constants $a > 0$, $b > 0$, $c > 0$, $K > 0$, $\rho \in (0, H)$, and a function $W \in C_0$, $W: B_\rho^{n_0} \times B_\rho^{n_1} \times B_\rho^{n_2} \to R_+$ such that for any $t \in R_+$; $y_i, y_i^* \in B_\rho^{n_i}$, $i = 0, 1, 2$ the following inequalities hold:

$$a\,|y_2 - \bar{y}_2(y_0, y_1)| \leqslant W(y_0, y_1, y_2) \leqslant b\,|y_2 - \bar{y}_2(y_0, y_1)|, \tag{64}$$

$$D_{(63)}^+ W(y_0, y_1, y_2) \leqslant -c\,|y_2 - \bar{y}_2(y_0, y_1)|, \tag{65}$$

$$|W(y_0, y_1, y_2) - W(y_0^*, y_1^*, y_2^*)| \leqslant L \sum_{i=0}^{2} |y_i - y_i^*|. \tag{66}$$

Consider the system

$$\begin{aligned} \varepsilon_v \frac{dy_v}{dt} &= f_v^2(y_0, y_1, y_2), \qquad t \neq \tau_k, \\ \Delta y_v|_{t=\tau_k} &= I_{vk}^2(y_0, y_1, y_2), \qquad k = 1, 2, ..., v = 0, 1, 2. \end{aligned} \tag{67}$$

In view of the relations

$$\begin{aligned} D_{(67)}^+ V \leqslant \limsup_{\tau \to 0_+} \frac{1}{\tau} \Bigg[& V\left(t+\tau, y_0 + \frac{\tau}{\varepsilon_0} f_0^2, y_1 + \frac{\tau}{\varepsilon_1} f_1^2\right) \\ & - V\left(t+\tau, y_0 + \frac{\tau}{\varepsilon_0} \bar{f}_0^2, y_1 + \frac{\tau}{\varepsilon_1} \bar{f}_1^2\right)\Bigg] \\ & + \limsup_{\tau \to 0_+} \frac{1}{\tau} \left[V\left(t+\tau, y_0 + \frac{\tau}{\varepsilon_0} f_0^1, y_1 + \frac{\tau}{\varepsilon_1} f_1^1\right) - V(t, y_0, y_1)\right], \end{aligned}$$

$$\begin{aligned} D_{(67)}^+ W \leqslant \limsup_{\tau \to 0_+} \frac{1}{\tau} \Bigg[& W\left(y_0 + \frac{\tau}{\varepsilon_0} f_0^2, y_1 + \frac{\tau}{\varepsilon_1} f_1^2, + \frac{\tau}{\varepsilon_2} f_2^2\right) \\ & - W\left(y_0, y_1 + \frac{\tau}{\varepsilon_2} f_2^2\right)\Bigg] \\ & + \limsup_{\tau \to 0_+} \frac{1}{\tau} \left[W\left(y_0, y_1, y_2 + \frac{\tau}{\varepsilon_2} f_2^2\right) - W(y_0, y_1, y_2)\right], \end{aligned}$$

$$
\begin{aligned}
&V(\tau_k+0,\ y_0+I_{0k}^2,\ y_1+I_{1k}^2)\\
&\quad = V(\tau_k+0,\ y_0+I_{0k}^1,\ y_1+I_{1k}^1)+V(\tau_k+0,\ y_0+I_{0k}^2,\ y_1+I_{1k}^2)\\
&\qquad - V(\tau_k+0,\ y_0+\bar I_{0k}^2,\ y_1+\bar I_{1k}^2),\\
&W(y_0+I_{0k}^2,\ y_1+I_{1k}^2,\ y_2+I_{2k}^2)\\
&\quad = W(y_0,\ y_1,\ y_2)+W(y_0+I_{0k}^2,\ y_1+I_{1k}^2,\ y_2+I_{2k}^2)\\
&\qquad - W(y_0,\ y_1,\ y_2),
\end{aligned}
$$

taking into account (59)–(62), (63)–(66), and condition (D4), we conclude that there exist constants $\kappa>0$, $\beta>0$, and $T>0$ such that

$$
\begin{aligned}
&D^+_{(67)}V(t,\ y_0,\ y_1)\\
&\qquad \leqslant -\kappa V(t,\ y_0,\ y_1)+Q(\mu)\,W(y_0,\ y_1,\ y_2),\\
&D^+_{(67)}W(y_0,\ y_1,\ y_2)\\
&\qquad \leqslant Q(\mu)\,V(t,\ y_0,\ y_1)-\sigma(\mu)\,W(y_0,\ y_1,\ y_2), \qquad t\neq\tau_k,\\
&V(\tau_k+0,\ y_0+I_{0k}^2,\ y_1+I_{1k}^2)\\
&\qquad \leqslant V(\tau_k,\ y_0,\ y_1)+TW(y_0,\ y_1,\ y_2),\\
&W(y_0+I_{0k}^2,\ y_1+I_{1k}^2,\ y_2+I_{2k}^2)\\
&\qquad \leqslant TV(\tau_k,\ y_0,\ y_1)+TW(y_0,\ y_1,\ y_2),
\end{aligned}
\tag{68}
$$

where

$$
Q(\mu)=T\left(\frac{1}{\varepsilon_0(\mu)}+\frac{1}{\varepsilon_1(\mu)}\right), \qquad \sigma(\mu)=\frac{\beta}{\varepsilon_2(\mu)}-T\left(\frac{1}{\varepsilon_0(\mu)}+\frac{1}{\varepsilon_1(\mu)}\right).
$$

Let $y_0(t)$, $y_1(t)$, $y_2(t)$ be a solution of system (67) defined for $t\in J^+=(t_0,\omega)$ and $V(t)=V(t,\ y_0(t),\ y_1(t))$, $w(t)=W(y_0(t),\ y_1(t),\ y_2(t))$. From (68) it follows that for $t\in J^+$ the following estimates hold:

$$
\begin{aligned}
&D^+v(t)\leqslant -\kappa v(t)+Q(\mu)\,w(t),\\
&D^+w(t)\leqslant Q(\mu)\,v(t)-\sigma(\mu)\,w(t), \qquad t\neq\tau_k,\\
&v(\tau_k+0)\leqslant v(\tau_k)+Tw(\tau_k),\\
&w(\tau_k+0)\leqslant Tv(\tau_k)+Tw(\tau_k), \qquad \tau_k\in J^+.
\end{aligned}
\tag{69}
$$

Taking into account

$$
\lim_{\mu\to 0_+}\frac{Q(\mu)}{\sigma(\mu)}=0, \qquad \lim_{\mu\to 0_+}\sigma(\mu)=+\infty,
$$

and condition (D1), as in the proof of Theorem 1, we conclude that there exist constants $A_0 \geqslant 1$, $\nu > 0$ such that for $t \in J^+$ we have

$$v(t) \leqslant A_0(v(t_0+0) + w(t_0+0))\, e^{-\nu(t-t_0)}$$
$$w(t) \leqslant A_0(v(t_0+0) + w(t_0+0))\, e^{-\nu(t-t_0)}. \tag{70}$$

From (70), making use of (59), (63), and condition (D4), we get that there exists a constant $A_1 \geqslant 1$ such that for $t \in J^+$

$$|y_1(t)| + |y_2(t)| \leqslant A_1(|y_1(t_0+0)| + |y_2(t_0+0)|)\, e^{-\nu(t-t_0)}. \tag{71}$$

In view of conditions (D1), (D4), estimate (71), and the equality

$$y_0(t) = y_0(t_0+0) + \int_{t_0}^{t} f_0^2(y_0(s), y_1(s), y_2(s))\, ds$$
$$+ \sum_{t_0 < \tau_k < t} I_{0k}^2(y_0(\tau_k), y_1(\tau_k), y_2(\tau_k))$$

we obtain that there exists a constant $A_2 \geqslant 1$ such that for $t \in J^+$

$$|y_0(t)| \leqslant A_2(|y_0(t_0+0)| + |y_1(t_0+0)| + |y_2(t_0+0)|). \tag{72}$$

From estimates (71) and (72) it immediately follows that the solution $y_0 = 0$, $y_1 = 0$, $y_2 = 0$ of system (67) is (y_1, y_2)-exponentially stable. Repeating the above arguments, we obtain successively that for $\nu = 3, ..., m$ the solution $y_0 = 0$, $y_1 = 0, ..., y_m = 0$ of the system

$$\varepsilon_i \frac{dy_i}{dt} = f_i^\nu(y_0, ..., y_\nu), \qquad t \neq \tau_k,$$
$$\Delta y_i|_{t=\tau_k} = I_{ik}^\nu(y_0, ..., y_\nu), \qquad k = 1, 2, ..., i = 0, ..., \nu$$

is $(y_0, ..., y_\nu)$-exponentially stable. For $\nu = m$ the assertion of Theorem 4 is proved.

An important particular case of system (56) is the system

$$\frac{dy_0}{dt} = f_0(y_0, y_1, ..., y_m),$$
$$\varepsilon_\nu \frac{dy_\nu}{dt} = \sum_{i=1}^{m} A_{\nu i}(y_0)\, y_i, \qquad t \neq \tau_k,$$
$$\Delta y_0|_{t=\tau_k} = I_{0k}(y_0, y_1, ..., y_m),$$
$$\Delta y_\nu|_{t=\tau_k} = \sum_{i=1}^{m} B_{\nu k i}(y_0)\, y_i, \qquad k = 1, 2, ..., \nu = 1, ..., m, \tag{73}$$

where $A_{\nu i}$ and $B_{\nu k i}$ are matrices of respective dimensions.

Introduce the following conditions (E):

(E1) Let $A^m_{\nu i} \equiv A_{\nu i}$, $B^m_{\nu k i} = B_{\nu k i}$ for $\nu = 1, \ldots, m$, $i = 1, \ldots, m$, $k = 1, 2, \ldots$.

Assume that the matrix $A^m_{mm}(y_0)$ is invertible for any $y_0 \in B^{n_0}_H$ and define the matrices

$$A^{m-1}_{\nu i} = A^m_{\nu i} - A^m_{\nu m}(A^m_{mm})^{-1} A^m_{mi},$$
$$B^{m-1}_{\nu k i} = B^m_{\nu k i} - B^m_{\nu k m}(A^m_{mm})^{-1} A^m_{mi}$$

for $\nu, i = 1, \ldots, m-1$; $k = 1, 2, \ldots$.

Assume inductively that the matrix $A^l_{ll}(y_0)$ is invertible for $y_0 \in B^{n_0}_H$ and define the matrices

$$A^{l-1}_{\nu i} = A^l_{\nu i} - A^l_{\nu l}(A^l_{ll})^{-1} A^l_{li},$$
$$B^{l-1}_{\nu k i} = B^l_{\nu k i} - B^l_{\nu k l}(A^l_{ll})^{-1} A^l_{li},$$

$\nu, i = 1, 2, \ldots, l-1$; $k = 1, 2, \ldots$.

(E2) For any $\nu, i = 1, \ldots, m$ the matrices $A_{\nu i}(y_0)$ and $B_{\nu k i}(y_0)$ are differentiable and bounded in the domain $B^{n_0}_H$, uniformly with respect to $k = 1, 2, \ldots$.

(E3) For any $\nu = 2, \ldots, m$ the real parts of all eigenvalues of the matrix $A^\nu_{\nu\nu}(y_0)$ are bounded from above by a negative constant, uniformly with respect to $y_0 \in B^{n_0}_H$.

(E4) The functions $f_0(y_0, \ldots, y_m)$ and $I_{0k}(y_0, \ldots, y_m)$ satisfy condition (D4).

(E5) The solution $y_0 = 0$, $y_1 = 0$ of the system

$$\frac{dy_0}{dt} = f^1_0(y_0, y_1),$$

$$\frac{dy_1}{dt} = A^1_{11}(y_0)\, y_1, \qquad t \neq \tau_k,$$

$$\Delta y_0|_{t=\tau_k} = I^1_{0k}(y_0, y_1),$$
$$\Delta y_1|_{t=\tau_k} = B^1_{1k1}(y_0)\, y_1, \qquad k = 1, 2, \ldots$$

is y_1-exponentially stable.

As a consequence of Theorem 4 we obtain the following theorem.

THEOREM 5. *Let conditions* (D1), (D2), *and* (E) *hold. Then the solution* $y_0 = 0, \ldots, y_m = 0$ *of system* (73) *is* $(y_1, \ldots, y_m)$*-exponentially stable.*

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