Journal of Computational and Applied Mathematics 224 (2009) 373–382

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Nontrivial solutions of m-point boundary value problems for singular second-order differential equations with a sign-changing nonlinear term

Lishan Liu[a,b,*], Bingmei Liu[a], Yonghong Wu[b]

[a] School of Mathematical Sciences, Qufu Normal University, Qufu 273165, Shandong, People's Republic of China

[b] Department of Mathematics and Statistics, Curtin University of Technology, Perth, WA 6845, Australia

ARTICLE INFO

Article history:
Received 1 March 2008
Received in revised form 9 May 2008

MSC:
45J05
34k30

Keywords:
Singular second-order differential equations
m-point BVP
Sign-changing
Eigenvalue
Topological degree

ABSTRACT

This paper concerns the existence of nontrivial solutions for the following singular m-point boundary value problem with a sign-changing nonlinear term

$$\begin{cases} (Lu)(t) + h(t)f(t,u) = 0, & 0 < t < 1, \\ u(0) = 0, & u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i), \end{cases}$$

where $(Lu)(t) = \left(\tilde{p}(t)u'(t)\right)' + q(t)u(t)$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $a_i \in [0, +\infty)$, $h(t)$ is allowed to be singular at $t = 0, 1$, and $f : [0, 1] \times (-\infty, +\infty) \to (-\infty, +\infty)$ is a sign-changing continuous function and may be unbounded from below. By applying the topological degree of a completely continuous field and the first eigenvalue and its corresponding eigenfunction of a special linear operator, some new results on the existence of nontrivial solutions for the above singular m-point boundary value problem are obtained. An example is then given to demonstrate the application of the main results. The work improves and generalizes the main results of [G. Han, Y. Wu, Nontrivial solutions of singular two-point boundary value problems with sign-changing nonlinear terms, J. Math. Anal. Appl. 325 (2007) 1327–1338; J. Sun, G. Zhang, Nontrivial solutions of singular superlinear Sturm-Liouville problem, J. Math. Anal. Appl. 313 (2006) 518–536].

1. Introduction

The purpose of this paper is to establish the existence of nontrivial solutions for the following singular m-point boundary value problem (BVP, for short) with a sign-changing nonlinear term

$$\begin{cases} (Lu)(t) + h(t)f(t,u) = 0, & 0 < t < 1, \\ u(0) = 0, & u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i), \end{cases} \tag{1.1}$$

where $(Lu)(t) = \left(\tilde{p}(t)u'(t)\right)' + q(t)u(t)$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $a_i \in [0, +\infty)$ with $\sum_{i=1}^{m-2} a_i\phi_1(\xi_i) < 1$ (ϕ_1 will be given in Lemma 2.1), $f : [0, 1] \times (-\infty, +\infty) \to (-\infty, +\infty)$ is a continuous sign-changing function and f may be unbounded from below, $h : (0, 1) \to [0, +\infty)$ is continuous and is allowed to be singular at $t = 0, 1$.

* Corresponding author at: School of Mathematical Sciences, Qufu Normal University, Qufu 273165, Shandong, People's Republic of China. Tel.: +86 537 4456234; fax: +86 537 4455076.

E-mail addresses: lls@mail.qfnu.edu.cn (L. Liu), liubingmei0315@163.com (B. Liu), yhwu@maths.curtin.edu.au (Y. Wu).

0377-0427/$ – see front matter © 2008 Elsevier B.V. All rights reserved.
doi:10.1016/j.cam.2008.05.007

Over the last thirty years, boundary value problems have attracted extensive attention due to their wide range of applications in applied mathematics, physics, biology and engineering (see, for example, [1,4,7,8,11–14] and references therein for more details). To our knowledge, most papers in the literature concern mainly the existence of positive solutions for the cases in which the nonlinear term is nonnegative. Results for the existence of solutions when the nonlinear term is sign-changing are rarely seen except for a few special cases [2,3,6,10].

Sun and Zhang [10] studied the following Sturm–Liouville problem

$$\begin{cases} -(Lu)(t) = h(t)f(u(t)), \quad 0 < t < 1, \\ R_1(u) = \alpha_1 u(0) + \beta_1 u'(0) = 0, \\ R_2(u) = \alpha_2 u(1) + \beta_2 u'(1) = 0, \end{cases} \tag{1.2}$$

in which f is bounded from below and is not necessarily nonnegative. By means of the topological degree theory, the authors established the existence of nontrivial solutions and positive solutions of the problem (1.2).

In [6], Han and Wu improved the results in [10] and obtained a new result on the existence of nontrivial solutions for the following BVP

$$\begin{cases} -u''(t) = h(t)f(u(t)), \quad 0 < t < 1, \\ u(0) = 0 = u(1), \end{cases} \tag{1.3}$$

where f is allowed to be unbounded from below. However, Han and Wu's results are limited to the cases in which the Green function of the boundary value problem is symmetric and the following condition (A_1^*) holds.

(A_1^*) There exist three constants $b > 0$, $c > 0$ and $\alpha \in (0, 1)$ such that

$$f(u) \geq -b - c|u|^{\alpha}, \quad \text{for any } u \in \mathbb{R}.$$

Inspired by the above work, the aim of this paper is to establish some simple criteria for the existence of nontrivial solutions to BVP (1.1) under some weaker conditions. The new features of this paper mainly include the following aspects. Firstly the nonlinear term f of BVP (1.1) is allowed to be sign-changing and unbounded from below and the above condition (A_1^*) is weakened to condition (A_1) given in Section 3. Secondly, the Green function is not necessarily symmetric, and thus our work improves the results in [6] and can be applied to more general problems. Thirdly, h has singularity at $t = 0$ and/or $t = 1$ and BVP (1.1) possesses the first-order derivative. Obviously, the problem in question is different from those in [1–4,6–13]. Finally, the main technique used here is the topological degree theory. To cope with the difficulties caused by the nonsymmetry of Green function, a special linear operator is sought and based on its first eigenvalue and positive eigenfunction, a linear continuous functional and a special cone are constructed for the study of the existence of nontrivial solutions.

The rest of the paper is organized as follows. Some preliminaries and a number of lemmas useful to the derivation of the main results are given in Section 2, then the proofs of the theorems are given in Section 3, followed by an example, in Section 4, to demonstrate the validity of our main results.

2. Preliminaries and lemmas

In this section, we present some preliminaries and lemmas that are useful to the proof of our main results.

Let $E = C[0, 1]$ be a Banach space with the maximum norm $\|u\| = \max_{0 \leq t \leq 1} |u(t)|$ for $u \in C[0, 1]$, E^* be the dual space of E. Define $P = \{u \in C[0, 1] \mid u(t) \geq 0, t \in [0, 1]\}$ and $B_r = \{u \in C[0, 1] \mid \|u\| < r\}$. Then P is a total cone in E, that is, $E = \overline{P - P}$. Let P^* be the dual cone of P, namely, $P^* = \{g \in E^* \mid g(u) \geq 0, \text{ for all } u \in P\}$ (see [5]).

For the sake of convenience, we first give the following assumptions:

(H_1) $\tilde{p}(t) \in C^1[0, 1]$, $\tilde{p}(t) > 0$, $q(t) \in C[0, 1]$, $q(t) < 0$, $t \in [0, 1]$.

(H_2) $h : (0, 1) \to [0, +\infty)$ is continuous, $h(t) \not\equiv 0$ and

$$\int_0^1 p(s)h(s)\mathrm{d}s < +\infty,$$

where $p(t) = \frac{1}{\tilde{p}(t)} \exp\left(\int_0^t \frac{\tilde{p}'(s)}{\tilde{p}(s)}\mathrm{d}s\right)$.

(H_3) $\sum_{i=1}^{m-2} a_i \phi_1(\xi_i) < 1$, where $\phi_1(t)$ is the unique solution of BVP (2.1) given below.

Lemma 2.1. *Assume that* (H_1) *is satisfied. Then*

$$\begin{cases} (L\phi_1)(t) = 0, \quad 0 < t < 1, \\ \phi_1(0) = 0, \quad \phi_1(1) = 1, \end{cases} \tag{2.1}$$

and

$$\begin{cases} (L\phi_2)(t) = 0, \quad 0 < t < 1, \\ \phi_2(0) = 1, \quad \phi_2(1) = 0, \end{cases}$$

have solutions ϕ_1 *and* ϕ_2 *respectively, and*

(i) ϕ_1 *is strictly increasing on* $[0, 1]$*;*

(ii) ϕ_2 *is strictly decreasing on* $[0, 1]$.

Proof. We will give a proof for (i). The proof for (ii) follows in a similar manner.

First we claim that $\phi_1(t) \geq 0$ on [0,1]. If not, there is some $\tau_0 \in (0, 1)$ such that $\phi_1(\tau_0) < 0$. Then there exists $\tau_1 \in (0, 1)$ such that

$$\phi_1(\tau_1) = \min_{t\in[0,1]} \phi_1(t) < 0.$$

As a result, $\phi_1'(\tau_1) = 0$ and $\phi_1''(\tau_1) \geq 0$. On the other hand, from (H_1) and $(L\phi_1)(t) = 0$ it follows that

$$\phi_1''(\tau_1) = -\frac{q(\tau_1)\phi_1(\tau_1)}{\tilde{p}(\tau_1)} < 0.$$

This is a contradiction.

Next we show $\phi_1(t) \neq 0$ for $t \in (0, 1]$. On the contrary, τ_2 is the first point in $(0, 1]$ such that $\phi_1(\tau_2) = 0$. Then there exists a point $\tau_3 \in (0, \tau_2)$ such that $\phi_1(\tau_3) = \max_{t\in[0,\tau_2]} \phi_1(t) > 0$. Thus it follows that $\phi_1'(\tau_3) = 0$ and $\phi_1''(\tau_3) \leq 0$. On the other hand,

$$\phi_1''(\tau_3) = -\frac{q(\tau_3)\phi_1(\tau_3)}{\tilde{p}(\tau_3)} > 0.$$

This is a contradiction.

Since $\phi_1(0) = 0$ and $\phi_1(t) > 0\ \forall t \in (0, 1]$, $\phi_1'(0) > 0$.

Finally we show that $\phi_1'(t) \neq 0$ on (0,1]. If not, we suppose that t_0 is the first point in (0,1] such that $\phi_1'(t_0) = 0$. Then, by integrating the differential equation in (2.1) and using $\phi_1(t) > 0$ and $q(t) < 0$ on (0,1], we have

$$-\tilde{p}(0)\phi_1'(0) = -\int_0^{t_0} q(t)\phi_1(t)\mathrm{d}t > 0.$$

On the other hand, $-\tilde{p}(t)\phi_1'(0) < 0$. This is a contradiction. Hence, $\phi_1'(t) > 0$ on [0,1) holds, and so $\phi_1(t)$ is strictly increasing on [0,1]. □

Lemma 2.2. *Assume that* (H_1) *and* (H_3) *are satisfied. Then, for any* $y \in C[0, 1]$, *the boundary value problem*

$$\begin{cases} (Lu)(t) + y(t) = 0, & 0 < t < 1, \\ u(0) = 0, & u(1) = \displaystyle\sum_{i=1}^{m-2} a_i u(\xi_i), \end{cases}$$

is equivalent to the integral equation

$$u(t) = \int_0^1 G(t, s)p(s)y(s)\mathrm{d}s + D^{-1}\phi_1(t) \int_0^1 \sum_{i=1}^{m-2} a_i G(\xi_i, s)p(s)y(s)\mathrm{d}s,$$

where

$$D = 1 - \sum_{i=1}^{m-2} a_i\phi_1(\xi_i), \qquad G(t, s) = \frac{1}{\rho} \begin{cases} \phi_1(t)\phi_2(s), & 0 \leq t \leq s \leq 1, \\ \phi_1(s)\phi_2(t), & 0 \leq s \leq t \leq 1, \end{cases}$$

and

$$\rho = \phi_1'(0) > 0. \quad \square$$

Define an operator $A : E \to E$ as follows:

$$(Au)(t) = \int_0^1 G(t, s)p(s)h(s)f(s, u(s))\mathrm{d}s + D^{-1}\phi_1(t) \int_0^1 \sum_{i=1}^{m-2} a_i G(\xi_i, s)p(s)h(s)f(s, u(s))\mathrm{d}s. \tag{2.2}$$

If u is a fixed point of A, then u is a solution of BVP (1.1) by means of Lemma 2.2.

For any $u \in E$, $t \in [0, 1]$, let us define two linear operators $K, T : E \to E$ by

$$(Ku)(t) = \int_0^1 G(t, s)p(s)h(s)u(s)\mathrm{d}s + D^{-1}\phi_1(t) \int_0^1 \sum_{i=1}^{m-2} a_i G(\xi_i, s)p(s)h(s)u(s)\mathrm{d}s \tag{2.3}$$

and

$$(Tu)(t)=\int_0^1 G(\tau,t)p(\tau)h(\tau)u(\tau)\mathrm{d}\tau+D^{-1}\sum_{i=1}^{m-2}a_iG(\xi_i,t)\int_0^1\phi_1(\tau)p(\tau)h(\tau)u(\tau)\mathrm{d}\tau. \tag{2.4}$$

Lemma 2.3. *Assume that* (H_1)–(H_3) *hold. Then linear operators* $K,T:E\to E$, *defined by* (2.3) *and* (2.4) *respectively, are completely continuous positive linear operators.*

Proof. By Lemma 2.1, we know that $0\le\phi_1(t)\le 1$, $t\in[0,1]$ and by Lemma 2.2, we have

$$G(s,s)\ge G(t,s),\qquad G(t,t)\ge G(t,s),\quad \text{for } t,s\in[0,1].$$

From (H_2) and Lemma 2.2 we know $\int_0^1 G(s,s)p(s)h(s)\mathrm{d}s\le\frac{1}{\rho}\int_0^1 p(s)h(s)\mathrm{d}s<+\infty$. Then by (H_3), for any $r>0$ and $u\in B_r$, $t\in[0,1]$

$$|(Ku)(t)|\le\left(1+D^{-1}\sum_{i=1}^{m-2}a_i\right)\|u\|\int_0^1 G(s,s)p(s)h(s)\mathrm{d}s<+\infty,$$

$$|(Tu)(t)|\le\|u\|\left(\int_0^1 G(\tau,\tau)p(\tau)h(\tau)\mathrm{d}\tau+D^{-1}\sum_{i=1}^{m-2}a_iG(\xi_i,\xi_i)\int_0^1 p(\tau)h(\tau)\mathrm{d}\tau\right)<+\infty.$$

Therefore $K,T:E\to E$ are well defined. As $G(t,s)$ are nonnegative for any $t,s\in[0,1]$ we have $K(P)\subset P$, $T(P)\subset P$. Thus, K and T are positive linear operators. Next we shall show that K and T are completely continuous. For any natural number $n\,(n\ge 2)$, we set

$$h_n(t)=\begin{cases}\inf_{t<s\le\frac{1}{n}}h(s), & 0\le t\le\dfrac{1}{n},\\ h(t), & \dfrac{1}{n}\le t\le\dfrac{n-1}{n},\\ \inf_{\frac{n-1}{n}\le s<t}h(s), & \dfrac{n-1}{n}\le t\le 1.\end{cases} \tag{2.5}$$

Then $h_n:[0,1]\to[0,+\infty)$ is continuous and $h_n(t)\le h(t)$, $t\in(0,1)$. For $t\in[0,1]$, let

$$(T_nu)(t)=\int_0^1 G(\tau,t)p(\tau)h_n(\tau)u(\tau)\mathrm{d}s+D^{-1}\sum_{i=1}^{m-2}a_iG(\xi_i,t)\int_0^1\phi_1(\tau)p(\tau)h_n(\tau)u(\tau)\mathrm{d}\tau. \tag{2.6}$$

It is obvious that $T_n:C[0,1]\to C[0,1]$ is completely continuous. For every $r>0$ and B_r, by (2.5), (2.6) and the absolute continuity of the integral, we have

$$\begin{aligned}\lim_{n\to\infty}\|T_nu-Tu\| &= \lim_{n\to\infty}\max_{t\in[0,1]}\left|\int_0^1 G(\tau,t)p(\tau)\left(h(\tau)-h_n(\tau)\right)u(\tau)\mathrm{d}\tau\right.\\ &\quad\left.+D^{-1}\sum_{i=1}^{m-2}a_iG(\xi_i,t)\int_0^1\phi_1(\tau)p(\tau)\left(h(\tau)-h_n(\tau)\right)u(\tau)\mathrm{d}\tau\right|\\ &\le \|u\|\lim_{n\to\infty}\left(\int_0^1 G(\tau,\tau)p(\tau)\left(h(\tau)-h_n(\tau)\right)\mathrm{d}\tau\right.\\ &\quad\left.+\ D^{-1}\sum_{i=1}^{m-2}a_iG(\xi_i,\xi_i)\int_0^1\phi_1(\tau)p(\tau)\left(h(\tau)-h_n(\tau)\right)\mathrm{d}\tau\right)\\ &\le r\lim_{n\to\infty}\left(\int_{e(n)} G(\tau,\tau)p(\tau)\left(h(\tau)-h_n(\tau)\right)\mathrm{d}\tau\right.\\ &\quad\left.+\ D^{-1}\sum_{i=1}^{m-2}a_iG(\xi_i,\xi_i)\int_{e(n)}\phi_1(\tau)p(\tau)\left(h(\tau)-h_n(\tau)\right)\mathrm{d}\tau\right)\\ &\le r\lim_{n\to\infty}\left(\int_{e(n)} G(\tau,\tau)p(\tau)h(\tau)\mathrm{d}\tau\right.\\ &\quad\left.+\ D^{-1}\sum_{i=1}^{m-2}a_iG(\xi_i,\xi_i)\int_{e(n)}\phi_1(\tau)p(\tau)h(\tau)\mathrm{d}\tau\right)\\ &=0,\end{aligned}$$

where $e(n) = [0, 1/n] \cup [(n-1)/n, 1]$. Then by the approximating theorem of completely continuous operators, $T : E \to E$ is completely continuous. In the same way, it is easy to prove that $K : E \to E$ is completely continuous. □

Lemma 2.4. *Assume that* (H_1)–(H_3) *hold. Then the special radii* $r(K) \neq 0$, $r(T) \neq 0$, *and* K *and* T *have positive eigenfunctions corresponding to their first eigenvalues* $\lambda_1 = (r(K))^{-1}$ *and* $\lambda_1' = (r(T))^{-1}$ *respectively.*

Proof. By (H_2), there is $t_1 \in (0, 1)$ such that $G(t_1, t_1)p(t_1)h(t_1) > 0$. The continuity of G, p and h tells us that there exists $[\alpha, \beta] \subset (0, 1)$ such that $t_1 \in (\alpha, \beta)$ and $G(t, s)p(s)h(s) > 0$, for $t, s \in [\alpha, \beta]$. Take $u \in C[0, 1]$ such that $u(t) \geq 0$, $t \in [0, 1]$, $u(t_1) > 0$ and $u(t) = 0$, $t \notin [\alpha, \beta]$. Then for any $t \in [\alpha, \beta]$, we have

$$\begin{aligned}
(Ku)(t) &= \int_0^1 G(t, s)p(s)h(s)u(s)\mathrm{d}s + D^{-1}\phi_1(t)\int_0^1 \sum_{i=1}^{m-2} a_i G(\xi_i, s)p(s)h(s)u(s)\mathrm{d}s \\
&\geq \int_0^1 G(t, s)p(s)h(s)u(s)\mathrm{d}s \geq \int_\alpha^\beta G(t, s)p(s)h(s)u(s)\mathrm{d}s > 0, \\
(Tu)(t) &= \int_0^1 G(\tau, t)p(\tau)h(\tau)u(\tau)\mathrm{d}\tau + D^{-1}\sum_{i=1}^{m-2} a_i G(\xi_i, t)\int_0^1 \phi_1(\tau)p(\tau)h(\tau)u(\tau)\mathrm{d}\tau \\
&\geq \int_0^1 G(\tau, t)p(\tau)h(\tau)u(\tau)\mathrm{d}\tau \geq \int_\alpha^\beta G(\tau, t)p(\tau)h(\tau)u(\tau)\mathrm{d}\tau > 0.
\end{aligned}$$

So there exists a constant $c > 0$ such that $c(Ku)(t) \geq u(t)$, $c(Tu)(t) \geq u(t)$, $t \in [0, 1]$. According to the Krein–Rutman theorem, we know that the special radii $r(K) \neq 0$, $r(T) \neq 0$ and K, T have positive eigenfunctions corresponding to their first eigenvalues $\lambda_1 = (r(K))^{-1}$ and $\lambda_1' = (r(T))^{-1}$ respectively. □

Let φ_1 and φ_2 denote the positive eigenfunctions of K and T respectively, i.e.

$$\lambda_1 K\varphi_1 = \varphi_1, \qquad \lambda_1' T\varphi_2 = \varphi_2. \tag{2.7}$$

Let K^* be the dual operator of K and T^* be the dual operator of T. If there exists $g \in P^* \setminus \{\theta\}$ satisfying that

$$\lambda_1' K^* g = g, \tag{2.8}$$

choose a number $\delta > 0$ such that

$$P(g, \delta) = \{u \in P \mid g(u) \geq \delta\|u\|\}, \tag{2.9}$$

then $P(g, \delta)$ is a cone in E and the following lemma (Lemma 2.5) holds.

Lemma 2.5. *Assume that* (H_1)–(H_3) *hold. Further, the following assumptions are satisfied:*

(C_1) *There exist* $\varphi_1, \varphi_2 \in P \setminus \{\theta\}$, $g \in P^* \setminus \{\theta\}$ *and* $\delta > 0$ *such that* (2.7), (2.8) *hold and* K *maps* P *into* $P(g, \delta)$.

(C_2) $H : E \to P$ *is a continuous operator and satisfies that*

$$\lim_{\|u\| \to +\infty} \frac{\|Hu\|}{\|u\|} = 0.$$

(C_3) $F : E \to E$ *is a bounded continuous operator and there exists* $u_0 \in E$ *such that* $Fu + u_0 + Hu \in P$ *for all* $u \in E$.

(C_4) *There exist* $v_0 \in E$ *and* $\zeta > 0$ *such that*

$$KFu \geq \lambda_1'(1 + \zeta)Ku - KHu - v_0, \quad \text{for all } u \in E.$$

Let $A = KF$, *then there exists* $R > 0$ *such that*

$$\deg(I - A, B_R, \theta) = 0,$$

where $B_R = \{u \in E \mid \|u\| < R\}$ *is an open ball with radius* R *in* E.

Proof. Choose a constant $L_0 = (\delta\lambda_1')^{-1}(1 + \zeta^{-1})\|g\| + \|K\| > 0$. From (C_2), for any $0 < \varepsilon < L_0^{-1}$, there exists $R_1 > 0$ such that $\|u\| > R_1$ implies that

$$\|Hu\| < \varepsilon\|u\|. \tag{2.10}$$

Now we shall show that

$$u \neq KFu + \mu\varphi_1, \quad \text{for any } u \in \partial B_R, \tag{2.11}$$

provided that R is sufficiently large.

In fact, if (2.11) is not true, then there exist $u_1 \in \partial B_R$ and $\mu_1 \geq 0$ satisfying

$$u_1 = KFu_1 + \mu_1\varphi_1. \tag{2.12}$$

By (C_4), (2.7) and the definition of conjugate operators, we get

$$\begin{aligned} g(u_1) &= g(KFu_1) + \mu_1 g(\varphi_1) \\ &\geq g(KFu_1) \\ &\geq \lambda_1'(1+\zeta)g(Ku_1) - g(KHu_1) - g(v_0) \\ &= \lambda_1'(1+\zeta)(K^*g)(u_1) - (K^*g)(Hu_1) - g(v_0) \\ &= (1+\zeta)g(u_1) - (\lambda_1')^{-1}g(Hu_1) - g(v_0). \end{aligned}$$

Thus,

$$g(u_1) \leq (\zeta\lambda_1')^{-1}g(Hu_1) + \zeta^{-1}g(v_0). \tag{2.13}$$

It follows from (2.7), (2.13) and (2.10) that

$$\begin{aligned} g(u_1 + KHu_1 + Ku_0) &= g(u_1) + (gK)(Hu_1) + (gK)(u_0) \\ &= g(u_1) + (\lambda_1')^{-1}g(Hu_1) + (\lambda_1')^{-1}g(u_0) \\ &\leq (\zeta\lambda_1')^{-1}g(Hu_1) + (\lambda_1')^{-1}g(Hu_1) + (\lambda_1')^{-1}g(u_0) + \zeta^{-1}g(v_0) \\ &\leq \varepsilon(1+\zeta^{-1})(\lambda_1')^{-1}\|g\|\|u_1\| + (\lambda_1')^{-1}g(u_0) + \zeta^{-1}g(v_0) \\ &= \varepsilon(1+\zeta^{-1})(\lambda_1')^{-1}\|g\|\|u_1\| + L_1, \end{aligned} \tag{2.14}$$

where $L_1 = (\lambda_1')^{-1}g(u_0) + \zeta^{-1}g(v_0)$ is a constant.

(C_3) shows $Fu_1 + u_0 + Hu_1 \in P$. Then (C_1) implies $\mu_1\varphi_1 = \mu_1\lambda_1 K\varphi_1 \in P(g, \delta)$. ($C_1$) and (2.12) tell us that

$$\begin{aligned} u_1 + KHu_1 + Ku_0 &= KFu_1 + \mu_1\varphi_1 + KHu_1 + Ku_0 \\ &= K(Fu_1 + Hu_1 + u_0) + \mu_1\varphi_1 \in P(g, \delta). \end{aligned}$$

By virtue of the definition of $P(g, \delta)$, we have

$$g(u_1 + KHu_1 + Ku_0) \geq \delta\|u_1 + KHu_1 + Ku_0\| \geq \delta\|u_1\| - \delta\|KHu_1\| - \delta\|Ku_0\|. \tag{2.15}$$

From (2.14) and (2.15), we know that

$$\begin{aligned} R = \|u_1\| &= \delta^{-1}g(u_1 + KHu_1 + Ku_0) + \|KHu_1\| + \|Ku_0\| \\ &\leq \varepsilon(\delta\lambda_1')^{-1}(1+\zeta^{-1})\|g\|\|u_1\| + L_1\delta^{-1} + \varepsilon\|K\|\|u_1\| + \|Ku_0\| \\ &= \varepsilon L_0\|u_1\| + L_2, \end{aligned}$$

where $L_2 = \|Ku_0\| + L_1\delta^{-1} > 0$ is a constant.

Since $\varepsilon L_0 < 1$, we obtain that (2.11) holds provided that R is sufficiently large. According to the property of omitting a direction for the Leray–Schauder degree, we have

$$\deg(I - A, B_R, \theta) = 0. \quad \square$$

Remark 2.1. It is worth mentioning that Lemma 2.5 is in essence different from Theorem 2.1 in paper [6]. Obviously g is closely related to the first eigenvalue of T. Secondly, in Lemma 2.5 the Green function of K is not necessarily symmetrical. In fact, if the Green function is symmetrical, then K, T have the same first eigenvalue and the same eigenfunction, that is, $\lambda_1' = \lambda_1$ and $\varphi_2 = \varphi_1$. In this special case, Lemma 2.5 turns into Theorem 2.1 in [6]. Finally, the operator H may include not only $|u|^\alpha$ but also many other unbounded functions, as shown in the example of Section 4. Hence, Lemma 2.5 improves the result in Theorem 2.1 in [6] and has a wider range of applications.

3. Main results

Theorem 3.1. *Assume that* (H_1)–(H_3) *hold and the following conditions are satisfied:*

(A_1) *There exist two nonnegative functions* $b(t), c(t) \in C[0, 1]$ *with* $c(t) \not\equiv 0$ *and one continuous even function* $B : (-\infty, +\infty) \to [0, +\infty)$ *such that*

$$f(t, u) \geq -b(t) - c(t)Bu, \quad \text{for all } t \in [0, 1], u \in \mathbb{R}.$$

Moreover, B *is nondecreasing on* $[0, +\infty)$ *and satisfies*

$$\lim_{u\to+\infty} \frac{Bu}{u} = 0.$$

(A_2) $f : [0, 1] \times (-\infty, +\infty) \to (-\infty, +\infty)$ *is continuous.*

(A_3)

$$\liminf_{u\to+\infty}\frac{f(t,u)}{u}>\lambda_1',\quad \textit{uniformly on } t\in[0,1].$$

(A_4)

$$\limsup_{u\to 0}\left|\frac{f(t,u)}{u}\right|<\lambda_1',\quad \textit{uniformly on } t\in[0,1].$$

Here λ_1' is the first eigenvalue of the operator T defined by (2.4). Then BVP (1.1) has at least one nontrivial solution.

Corollary 3.2. *Using the following condition (A_1^*) instead of (A_1), the conclusion of Theorem 3.1 remains true.*

(A_1^*) *There exist three constants $b>0$, $c>0$ and $\alpha\in(0,1)$ such that*

$$f(u)\ge -b-c|u|^{\alpha},\quad \textit{for any } u\in(-\infty,+\infty).$$

Proof of Theorem 3.1. We shall show that K satisfies all conditions in Lemma 2.5.

First we give some properties of $\varphi_2(t)$ which is the positive eigenfunction of T corresponding to its first eigenvalue λ_1'. By Lemma 2.2, $G(\tau,0)=G(\tau,1)=0$ for any $\tau\in[0,1]$. Then we obtain from (2.4) and $\lambda_1' T\varphi_2(t)=\varphi_2(t)$ that $\varphi_2(0)=\varphi_2(1)=0$ which implies that

$$\varphi_2'(0)>0,\qquad \varphi_2'(1)<0.$$

Thus

$$\lim_{s\to 0+}\frac{\varphi_2(s)}{s(1-s)}=\varphi_2'(0)>0,\qquad \lim_{s\to 1-}\frac{\varphi_2(s)}{s(1-s)}=-\varphi_2'(1)>0. \tag{3.1}$$

The maximum principle shows $\varphi_2(s)>0$, $s\in(0,1)$. This together with (3.1) yields that there are two numbers $\delta_1,\delta_2>0$ such that

$$\delta_1 s(1-s)\le\varphi_2(s)\le\delta_2 s(1-s),\quad \text{for any } s\in[0,1]. \tag{3.2}$$

It follows from Lemma 2.1 that

$$\lim_{s\to 0+}\frac{G(s,s)}{s(1-s)}=\frac{1}{\rho}\lim_{s\to 0+}\frac{\phi_1(s)\phi_2(s)}{s(1-s)}=\frac{1}{\rho}\phi_1'(0)>0,$$
$$\lim_{s\to 1-}\frac{G(s,s)}{s(1-s)}=\frac{1}{\rho}\lim_{s\to 1-}\frac{\phi_1(s)\phi_2(s)}{s(1-s)}=-\frac{1}{\rho}\phi_2'(1)>0.$$

On the other hand, $G(s,s)>0$ for any $s\in(0,1)$. Therefore, there exist two numbers σ_1 and $\sigma_2>0$ such that

$$\sigma_1 s(1-s)\le G(s,s)\le\sigma_2 s(1-s),\quad \text{for any } s\in[0,1]. \tag{3.3}$$

According to Lemmas 2.1 and 2.2, we have

$$G(t,s)+D^{-1}\phi_1(t)\sum_{i=1}^{m-2}a_iG(\xi_i,s)\le\left(1+D^{-1}\sum_{i=1}^{m-2}a_i\right)G(s,s),\quad t,s\in[0,1]. \tag{3.4}$$

By (3.2)–(3.4), we have

$$\varphi_2(s)\ge\delta_1\sigma_2^{-1}\left(1+D^{-1}\sum_{i=1}^{m-2}a_i\right)^{-1}\left(G(t,s)+D^{-1}\phi_1(t)\sum_{i=1}^{m-2}a_iG(\xi_i,s)\right),\quad \text{for } s\in[0,1] \tag{3.5}$$

and

$$\varphi_2(s)\le\sigma_1^{-1}\delta_2G(s,s),\quad s\in[0,1]. \tag{3.6}$$

Setting

$$g(u)=\int_0^1 p(t)h(t)\varphi_2(t)u(t)\mathrm{d}t,\quad \text{for any } u\in E, \tag{3.7}$$

from (H_2) and (3.6), we have that

$$\int_0^1 p(t)h(t)\varphi_2(t)u(t)\mathrm{d}t\le\sigma_1^{-1}\delta_2\|u\|\int_0^1 G(t,t)p(t)h(t)\mathrm{d}t<+\infty,$$

which shows that g is well defined on E. In the following we shall show that

$$\lambda_1' K^* g = g. \tag{3.8}$$

In fact, for any $u \in E$, $t, \tau \in [0, 1]$, by means of the definition of conjugate operators, we have

$$\begin{aligned}
(\lambda_1')^{-1} g(u) &= \int_0^1 p(t)h(t)\left((\lambda_1')^{-1}\varphi_2(t)\right)u(t)\mathrm{d}t \\
&= \int_0^1 p(t)h(t)\,(T\varphi_2)\,(t)u(t)\mathrm{d}t \\
&= \int_0^1 p(t)h(t)\left(\int_0^1 G(\tau,t)p(\tau)h(\tau)\varphi_2(\tau)\mathrm{d}\tau + D^{-1}\sum_{i=1}^{m-2} a_i G(\xi_i,t)\int_0^1 \phi_1(\tau)p(\tau)h(\tau)\varphi_2(\tau)\mathrm{d}\tau\right)u(t)\mathrm{d}t \\
&= \int_0^1 p(\tau)h(\tau)\varphi_2(\tau)\left(\int_0^1 G(\tau,t)p(t)h(t)u(t)\mathrm{d}t + D^{-1}\phi_1(\tau)\int_0^1\sum_{i=1}^{m-2} a_i G(\xi_i,t)p(t)h(t)u(t)\mathrm{d}t\right)\mathrm{d}\tau \\
&= \int_0^1 p(\tau)h(\tau)\varphi_2(\tau)\,(Ku)\,(\tau)\mathrm{d}\tau \\
&= g(Ku) = \left(K^* g\right)(u).
\end{aligned} \tag{3.9}$$

Let $\delta = (\lambda_1')^{-1}\delta_1\sigma_2^{-1}\left(1 + D^{-1}\sum_{i=1}^{m-2} a_i\right)^{-1} > 0$ and

$$P(g,\delta) = \{u \in P \mid g(u) \geq \delta\|u\|\}, \tag{3.10}$$

then it is easy to see that $P(g,\delta)$ is a cone in E.

Next we shall show

$$K(P) \subset P(g,\delta). \tag{3.11}$$

For any $u \in P$, by (3.5), (3.9) and Lemma 2.2,

$$\begin{aligned}
g(Ku) &= (\lambda_1')^{-1} g(u) \\
&= \int_0^1 p(t)h(t)\left((\lambda_1')^{-1}\varphi_2(t)\right)u(t)\mathrm{d}t \\
&\geq (\lambda_1')^{-1}\delta_1\sigma_2^{-1}\left(1 + D^{-1}\sum_{i=1}^{m-2} a_i\right)^{-1}\left(\int_0^1 G(\tau,t)p(t)h(t)u(t)\mathrm{d}t\right. \\
&\quad \left. +\ D^{-1}\phi_1(\tau)\int_0^1\sum_{i=1}^{m-2} a_i G(\xi_i,t)p(t)h(t)u(t)\mathrm{d}t\right) \\
&= \delta\left(\int_0^1 G(\tau,t)p(t)h(t)u(t)\mathrm{d}t + D^{-1}\phi_1(\tau)\int_0^1\sum_{i=1}^{m-2} a_i G(\xi_i,t)p(t)h(t)u(t)\mathrm{d}t\right) \\
&= \delta(Ku)(\tau), \quad \text{for any } \tau \in [0,1].
\end{aligned}$$

Hence, $g(Ku) \geq \delta\|Ku\|$, i.e. $K(P) \subset P(g,\delta)$.

From the above proof we know that K satisfies condition (C_1) of Lemma 2.5.

Obviously $B : E \to P$ is a continuous operator. By (A_1), for any $\varepsilon > 0$, there is $L > 0$ such that

$$Bu < \varepsilon u, \quad \text{for } u > L.$$

Thus

$$B\|u\| < \varepsilon\|u\|, \quad \text{for } \|u\| > L.$$

On the other hand, B is nondecreasing on $[0, +\infty)$, which shows

$$Bu \leq B\|u\|, \quad \text{for } u > 0.$$

Since $B : (-\infty, +\infty) \to [0, +\infty)$ is an even function,

$$Bu \leq B\|u\|, \quad u \in E,$$

which implies that

$$\|Bu\| \le B\|u\|, \quad u \in E.$$

Therefore,

$$\|Bu\| \le B\|u\| < \varepsilon\|u\|, \quad \text{for any } \|u\| > L,$$

that is, $\lim_{\|u\|\to+\infty} \frac{\|Bu\|}{\|u\|} = 0$. Take $Hu = c_0 Bu$ for $u \in E$, where $c_0 = \max_{t\in[0,1]} c(t)$. Obviously, $\lim_{\|u\|\to+\infty} \frac{\|Hu\|}{\|u\|} = 0$ holds. Therefore H satisfies condition (C_2) in Lemma 2.5. Take $u_0(t) \equiv b$ and $(Fu)(t) = f(t, u(t))$ for $t \in [0, 1]$, $u \in E$, then it follows from (A_1) that

$$Fu + u_0 + Hu \in P, \quad \text{for all } u \in E,$$

which shows that condition (C_3) in Lemma 2.5 holds.

From (A_3) there exist $\varepsilon > 0$ and a sufficiently large number $l_1 > 0$ satisfying

$$F(u) \ge \lambda_1'(1+\varepsilon)u, \quad \text{for } u \ge l_1. \tag{3.12}$$

Combining (3.12) with (A_1), we have that there exists $b_1 \ge 0$ such that

$$F(u) \ge \lambda_1'(1+\varepsilon)u - b_1 - Hu, \quad \text{for all } u \in R. \tag{3.13}$$

Since K is a positive linear operator, from (3.13), we have

$$(KF)(u) \ge \lambda_1'(1+\varepsilon)(Ku)(t) - Kb_1 - (KHu)(t), \quad \text{for all } t \in [0, 1].$$

So condition (C_4) in Lemma 2.5 is satisfied.

Keeping in mind Lemma 2.5, we derive that there exists a sufficiently large number $R > 0$ such that

$$\deg(I - A, B_R, \theta) = 0. \tag{3.14}$$

From (A_4), it follows that there exist $0 < \varepsilon < 1$ and $0 < r < R$ such that

$$|f(t,u)| \le (1-\varepsilon)\lambda_1'|u(t)|, \quad t \in [0, 1],\ u \in E \text{ with } \|u\| \le r. \tag{3.15}$$

If there exist $u_1 \in \partial B_r$ and $\mu_1 \in [0, 1]$ such that $u_1 = \mu_1 A u_1$, then by (3.9) and (3.15), we have

$$\begin{aligned}
g(|u_1|) &= g(|\mu_1 A u_1|) = \mu_1 g(|Au_1|) \le g(|Au_1|)\\
&= g\left(\left|\int_0^1 G(t,s)p(s)h(s)f(s,u_1(s))\mathrm{d}s + D^{-1}\phi_1(t)\int_0^1 \sum_{i=1}^{m-2} a_i G(\xi_i,s)p(s)h(s)f(s,u_1(s))\mathrm{d}s\right|\right)\\
&\le (1-\varepsilon)\lambda_1' g\left(\int_0^1 G(t,s)p(s)h(s)|u_1(s)|\mathrm{d}s + D^{-1}\phi_1(t)\int_0^1 \sum_{i=1}^{m-2} a_i G(\xi_i,s)p(s)h(s)|u_1(s)|\mathrm{d}s\right)\\
&= (1-\varepsilon)\lambda_1' g(K|u_1|)\\
&= (1-\varepsilon)\lambda_1'(\lambda_1')^{-1} g(|u_1|)\\
&= (1-\varepsilon) g(|u_1|).
\end{aligned}$$

Therefore, $g(|u_1|) \le 0$.

On the other hand, $\varphi_2(t) > 0$ for all $t \in (0, 1)$ by the maximum principle, and $u_1(t)$ attains zero on isolated points by the Sturm theorem. Hence

$$g(|u_1|) = \int_0^1 p(t)h(t)\varphi_2(t)\,|u_1(t)|\,\mathrm{d}t > 0.$$

This is a contradiction. Thus

$$u \ne \mu A u, \quad \text{for all } u \in \partial B_r \text{ and } \mu \in [0, 1].$$

It follows from the homotopy invariance of the Leray–Schauder degree that

$$\deg(I - A, B_r, \theta) = 1. \tag{3.16}$$

By (3.14), (3.16) and the additivity of the Leray–Schauder degree, we obtain

$$\deg(I - A, B_R \setminus \overline{B}_r, \theta) = \deg(I - A, B_R, \theta) - \deg(I - A, B_r, \theta) = -1.$$

As a result, A has at least one fixed point on $B_R \setminus \overline{B}_r$, namely, BVP (1.1) has at least one nontrivial solution. □

Remark 3.1. Since the Green function of BVP (1.1) is not necessarily symmetrical, Theorem 2.1 in [6] and Theorem 1 in [10] are not applicable to the boundary value problem in general. In order to overcome the difficulties caused by the nonsymmetry, we seek one special linear operator T and use its first eigenvalue and its corresponding eigenfunction to construct a linear continuous functional g of P^*. Then we establish a cone to solve the problem. The method is new and the results obtained in this paper improve and extend those in [6,10].

4. An example

In this section, we construct an example to demonstrate the application of our main result obtained in Section 3.

Let $h(t) = \frac{1}{\sqrt{t(1-t)}}$ and

$$f(t,u) = \begin{cases} \sum_{i=1}^{n}(-1)^i a_i - |u|^\alpha \ln(|u|+1) + \ln 2, & u \in (-\infty, -1], \\ \sum_{i=1}^{n} a_i u^i, & u \in [-1, +\infty), \end{cases} \tag{4.1}$$

where $0 < a_1 < \lambda_1'$ and $a_n > 0$. Then h is singular at $t = 0, 1$ and f is unbounded from below. Then h is singular at $t = 0, 1$ and f is unbounded from below. Take $c(t) = 1$, $b(t) = \sum_{i=1}^{n} a_i + \ln 2$, $Bu = |u|^\alpha \ln(|u|+1)$. Then

$$f(t,u) \geq -b(t) - c(t)Bu.$$

It is easy to prove that all the conditions in Theorem 3.1 are satisfied. As a result, BVP (1.1) with the $h(t)$ and $f(t,u)$ given by (4.1) has at least one nontrivial solution.

Remark 4.1. Note that the Green function is not symmetrical in this example and $Hu = |u|^\alpha \ln(|u|+1)$ satisfies

$$\lim_{u\to+\infty} \frac{Hu}{|u|^\alpha} = \lim_{u\to+\infty} \ln(|u|+1) = +\infty.$$

Therefore Theorem 2.1 in [6] is not applicable to this example. On the other hand, Theorem 3.1 in our paper can solve the case in [6] because $Hu = |u|^\alpha$ satisfies all the conditions of Lemma 2.5 in this paper. Hence, our results are applicable to more general cases than those in previous work.

Acknowledgements

The first and second authors were supported financially by the National Natural Science Foundation of China (10771117, 10471075), the State Ministry of Education Doctoral Foundation of China (20060446001) and the Natural Science Foundation of Shandong Province of China (Y2007A23). The third author was supported financially by the Australia Research Council through an ARC Discovery Project Grant.

References

[1] D. Anderson, Multiple positive solutions for a three-point boundary value problems, Math. Comput. Modelling 27 (1998) 49–57.
[2] R.P. Agarwal, D. O'Regan, Existence criteria for singular boundary value problems with sign-changing nonlinearities, J. Differential Equations 183 (2002) 409–433.
[3] R.P. Agarwal, S. Staněk, Nonnegative solutions of singular boundary value problems with sign changing nonlinearities, Comput. Math. Appl. 46 (2003) 1827–1837.
[4] C.P. Gupta, Solvability of a three-point nonlinear boundary value problem for a second order ordinary differential equations, J. Math. Anal. Appl. 168 (1988) 540–551.
[5] D. Guo, Nonlinear Functional Analysis, Shangdong Science and Technology Press, Jinan, 1985 (in Chinese).
[6] G. Han, Y. Wu, Nontrivial solutions of singular two-point boundary value problems with sign-changing nonlinear terms, J. Math. Anal. Appl. 325 (2007) 1327–1338.
[7] B. Liu, Positive solutions of a nonlinear three-point boundary value problem, Comput. Math. Appl. 44 (2002) 201–211.
[8] B. Liu, L. Liu, Y. Wu, Positive solutions for singular second-order three-point boundary value problems, Nonlinear Anal. 66 (2007) 2756–2766.
[9] R. Ma, H. Wang, Positive solutions of nonlinear three-point boundary-value problems, J. Math. Anal. Appl. 279 (2003) 216–227.
[10] J. Sun, G. Zhang, Nontrivial solutions of singular superlinear Sturm–Liouville problem, J. Math. Anal. Appl. 313 (2006) 518–536.
[11] P. Singh, A second-order singular three-point boundary value problem, Appl. Math. Lett. 17 (2004) 969–976.
[12] Y. Sun, L. Liu, Solvability for nonlinear second-order three-point boundary value problem, J. Math. Anal. Appl. 296 (2004) 265–275.
[13] J.R.L. Webb, Positive solutions of some three-point boundary value problems via fixed point index theory, Nonlinear Anal. 47 (2001) 4319–4332.
[14] G. Zhang, J. Sun, Positive solutions of m-point boundary value problems, J. Math. Anal. Appl. 291 (2004) 406–418.