Available online at www.sciencedirect.com
SCIENCE DIRECT
ELSEVIER
Stochastic Processes and their Applications 114 (2004) 251–263
stochastic processes and their applications
www.elsevier.com/locate/spa

A model of the term structure of interest rates based on Lévy fields

Sergio Albeverio[a,b,c,d,e], Eugene Lytvynov[f,*], Andrea Mahnig[a]

[a] *Institut für Angewandte Mathematik, Universität Bonn, Wegelerstr. 6, D-53115 Bonn, Germany*
[b] *SFB 611, Univ. Bonn, Germany*
[c] *CERFIM, Locarno, Switzerland*
[d] *Accademia di Architettura (USI), Mendrisio, Switzerland*
[e] *BiBoS, University of Bielefeld, Germany*
[f] *Department of Mathematics, University of Wales Swansea, Singleton Park, Swansea SA2 8PP, UK*

Received 30 January 2003; received in revised form 25 June 2004; accepted 28 June 2004
Available online 23 July 2004

Abstract

An extension of the Heath–Jarrow–Morton model for the development of instantaneous forward interest rates with deterministic coefficients and Gaussian as well as Lévy field noise terms is given. In the special case where the Lévy field is absent, one recovers a model discussed by D.P. Kennedy.

MSC: primary 91B28; 60J75; 60H15; 60G51

Keywords: Term structure of interest rates; Lévy fields; HJM model; Kennedy model

*Corresponding author. Tel./fax: +44-1792-602156.
E-mail addresses: albeverio@uni-bonn.de (S. Albeverio), e.lytvynov@swansea.ac.uk (E. Lytvynov), andrea.mahnig@gmx.de (A. Mahnig).

0304-4149/$ - see front matter
doi:10.1016/j.spa.2004.06.006

1. Introduction

Let us consider an interest rate market with $P_{s,t}$ being the price at time s of a bond paying one unit at time $t \geqslant s$. Then

$$P_{s,t} = \exp\left(-\int_s^t F_{s,u}\, \mathrm{d}u \right), \tag{1.1}$$

where $F_{s,t}$, $0 \leqslant s \leqslant t$, is called the *instantaneous forward rate*, or just the *forward rate*. Let R_t denote the *short rate* at time $t \geqslant 0$. The *discounted bond-price process* is then given by

$$Z_{s,t} := P_{s,t} \exp\left(-\int_0^s R_u\, \mathrm{d}u \right), \quad 0 \leqslant s \leqslant t. \tag{1.2}$$

Heath et al. [8] (see also [7]) proposed a model of interest rates and their associated bond prices in which the short rate and the forward rate are connected by

$$R_t = F_{t,t}, \quad t \geqslant 0 \tag{1.3}$$

and the forward rates are supposed to satisfy the stochastic differential equations

$$\mathrm{d}F_{s,t} = \alpha(s,t)\, \mathrm{d}s + \sum_{i=1}^m \beta_i(s,t)\, \mathrm{d}W_s^i. \tag{1.4}$$

Here, $W^1, \ldots, W^m$ are independent standard Brownian motions and $\alpha(s,t)$ and $\beta_i(s,t)$ are processes adapted to the natural filtration of the Brownian motions. This model was, in fact, an extension of the earlier work by Ho [10].

Kennedy [12] (see also [13]), while following the approach of modeling the forward rates, considered the case where $\{F_{s,t},\ 0 \leqslant s \leqslant t < \infty\}$ is a continuous Gaussian random field which has independent increments in the s-direction, that is, in the direction of evolution of 'real' time. This framework includes the Heath–Jarrow–Morton (HJM) model in the case where the coefficients $\alpha(s,t)$ and $\beta_i(s,t)$ in (1.4) are deterministic. An important example of application of the Kennedy model is the case where the forward rates are given by $F_{s,t} = \mu_{s,t} + X_{s,t}$ with $\mu_{s,t}$ being deterministic and $X_{s,t}$ a Brownian sheet (see, e.g., [1] for this concept). In the latter case, $F_{s,t}$ has independent increments also in the t direction. Furthermore, this may be intuitively thought of as the situation of (1.4) driven by an uncountably infinite number of Brownian motion. Kennedy [12] gave a simple characterization of the discounted bond-price process to be a martingale. In particular, he showed that the latter is true if and only if the expectation $\mu_{s,t}$ of $F_{s,t}$, $0 \leqslant s \leqslant t < \infty$, satisfies a simple relation.

In Björk et al. [4,5] (see also [3]), it was pointed out that, in many cases observed empirically, the interest rate trajectories do not look like diffusion processes, but rather as diffusions and jumps, or even like pure jump processes. Therefore, one needs to introduce a jump part in the description of interest rates. The authors of these papers considered the case where the forward rate process $\{F_{s,t},\ 0 \leqslant s \leqslant t\}$ is driven by a general marked point process as well as by a Wiener process [5], or by a rather general Lévy process [4], and the maturity time $t \geqslant 0$ is a continuous parameter of the model. In particular, an equivalence condition was given for a given probability measure to be a local martingale measure, i.e., for the discounted bond-price process $\{Z_{s,t},\ 0 \leqslant s \leqslant t\}$ to be a local martingale for each $t \geqslant 0$ [4, Propositions 5.3, 5.5], see also [8, Theorem 3.13]. This condition, formulated in terms of the coefficients for the forward rate dynamics, generalizes the result of Heath et al. [7] which was obtained for the diffusion case.

Other generalizations of the HJM model in which the forward rate process satisfies stochastic differential equations with an infinite number of independent standard Brownian motions (i.e., $m = \infty$ in (1.4)) were proposed in [14,15,18,19].

In the present paper, we follow the approach of Kennedy [12,13], but suppose that the forward rates $\{F_{s,t},\ 0 \leqslant s \leqslant t < \infty\}$ are driven by a Lévy field without a diffusion part. In particular, $\{F_{s,t}\}$ has independent increments in both the s and t directions. Analogously to Kennedy [12], we give, in this case, a characterization of the martingale measure. Furthermore, we do not assume that (1.3) a priori holds, but we derive it from a certain condition of independence and the martingale property of the discounted bond-price process.

We also show that, under a slight additional condition on the Lévy measure of the field, it is possible to choose the initial term structure $\{\mu_{0,t},\ t \geqslant 0\}$ in such a way that the forward interest rates are a.s. non-negative. This, of course, was impossible to reach in the framework of the Gaussian model, which caused problems in some situations (see [13, Section 1]). We then present two examples of application of our results: the cases where $F_{s,t}$ is a "Poisson sheet" (this case was discussed in [17], respectively a "gamma sheet." Finally, we mention the possibility of unification of the approaches of Kennedy and of the present paper, by considering $F_{s,t}$ as a sum of a Gaussian field and an independent Lévy field, and thus having a process with a diffusion part as well as a jump part.

2. The model based on Lévy fields

Let $\mathscr{D} := C_0^\infty(\mathbb{R}^2)$ denote the space of all real-valued infinitely differentiable functions on $\mathbb{R}^2$ with compact support. We equip $\mathscr{D}$ with the standard nuclear space topology, see, e.g., [2]. Then $\mathscr{D}$ is densely and continuously embedded into the real space $L^2(\mathbb{R}^2, \mathrm{d}x\,\mathrm{d}y)$. Let $\mathscr{D}'$ denote the dual space of $\mathscr{D}$ with respect to the "reference" space $L^2(\mathbb{R}^2, \mathrm{d}x\,\mathrm{d}y)$, i.e., the dual pairing between elements of $\mathscr{D}'$ and $\mathscr{D}$ is generated by the scalar product in $L^2(\mathbb{R}^2, \mathrm{d}x\,\mathrm{d}y)$. Thus, we get the standard (Gel'fand) triple

$$\mathscr{D}' \supset L^2(\mathbb{R}^2, \mathrm{d}x\,\mathrm{d}y) \supset \mathscr{D}.$$

We denote by $\langle \cdot, \cdot \rangle$ the dual pairing between elements of $\mathscr{D}'$ and $\mathscr{D}$. Let $\mathscr{C}(\mathscr{D}')$ denote the cylinder σ-algebra on $\mathscr{D}'$.

We define a centered Lévy noise measure as a probability measure v on $(\mathscr{D}', \mathscr{C}(\mathscr{D}'))$ whose Fourier transform is given by

$$\int_{\mathscr{D}'} \mathrm{e}^{\mathrm{i}\langle \omega, \varphi \rangle}\, v(\mathrm{d}\omega) = \exp\left(\int_{\mathbb{R}_+} \int_{\mathbb{R}^2} (\mathrm{e}^{\mathrm{i}\tau\varphi(x,y)} - 1 - \mathrm{i}\tau\varphi(x,y))\, \mathrm{d}x\,\mathrm{d}y\, \sigma(\mathrm{d}\tau) \right), \quad \varphi \in \mathscr{D} \tag{2.1}$$

(see, e.g., [6, Chapter III, Section 4]). Here, $\mathbb{R}_+ = (0, +\infty)$ and σ is a positive measure on $(\mathbb{R}_+, \mathscr{B}(\mathbb{R}_+))$, which is usually called the Lévy measure of the process. We suppose that σ satisfies the following condition:

$$\int_{\mathbb{R}_+} \tau^2 \, \sigma(\mathrm{d}\tau) < \infty. \tag{2.2}$$

The existence of the measure ν follows from the Minlos theorem.

For any $\varphi \in \mathscr{D}$, we easily have

$$\int_{\mathscr{D}'} \langle \omega, \varphi \rangle^2 \, \nu(\mathrm{d}\omega) = \int_{\mathbb{R}_+} \tau^2 \, \sigma(\mathrm{d}\tau) \int_{\mathbb{R}^2} \varphi(x,y)^2 \, \mathrm{d}x \, \mathrm{d}y. \tag{2.3}$$

Thus, the mapping $J : L^2(\mathbb{R}^2, \mathrm{d}x \, \mathrm{d}y) \to L^2(\mathscr{D}', \nu)$, $\mathrm{Dom}(J) = \mathscr{D}$, defined by

$$(J\varphi)(\omega) := \langle \omega, \varphi \rangle, \quad \varphi \in \mathscr{D}, \ \omega \in \mathscr{D}',$$

may be extended by continuity to the whole $L^2(\mathbb{R}^2, \mathrm{d}x \, \mathrm{d}y)$. For each $f \in L^2(\mathbb{R}^2, \mathrm{d}x \, \mathrm{d}y)$, we set $\langle \cdot, f \rangle := Jf$. Thus, the random variable (r.v.) $\langle \omega, f \rangle$ is well defined for ν-a.e. $\omega \in \mathscr{D}'$ and equality (2.3) holds for f replacing φ.

Let $\varkappa : [0, \infty)^2 \to \mathbb{R}_+$ be a continuous function. For each $s, t \geqslant 0$, we define the r.v. $X_{s,t}$ as follows:

$$X_{s,t}(\omega) := \langle \omega(x,y), \mathbf{1}_{[0,s]}(x) \mathbf{1}_{[0,t]}(y) \varkappa(x,y) \rangle, \quad \nu\text{-a.e. } \omega \in \mathscr{D}', \tag{2.4}$$

where x, y denote the variables in which the dualization is carried out. It follows from (2.1) that $X_{s,t}$ is centered and has independent increments in both the s and t directions.

We note that, in the case where $\varkappa(x,y) \equiv 1$, $\{X_{s,t}, \ 0 \leqslant s \leqslant t\}$ is a Lévy process for each fixed $t > 0$. Indeed, it follows from (2.1) that the Fourier transform of $X_{s,t}$ is given by

$$\int_{\mathscr{D}'} \mathrm{e}^{\mathrm{i}\lambda X_{s,t}(\omega)} \, \nu(\mathrm{d}\omega) = \exp\left(st \int_{\mathbb{R}_+} (\mathrm{e}^{\mathrm{i}\tau\lambda} - 1 - \mathrm{i}\tau\lambda) \, \sigma(\mathrm{d}\tau) \right), \quad \lambda \in \mathbb{R}.$$

In particular, the Lévy measure of the process $\{X_{s,t}, \ 0 \leqslant s \leqslant t\}$ is equal to $t\sigma$.

Let $F_{s,t}$ be the forward rate for date t at time s, $0 \leqslant s \leqslant t < \infty$. We suppose that

$$F_{s,t} = \mu_{s,t} + X_{s,t}, \quad 0 \leqslant s \leqslant t < \infty, \tag{2.5}$$

where $\mu_{s,t}$ is deterministic and continuous in (s,t) on $\{0 \leqslant s \leqslant t < \infty\}$. The price at time s of a bond paying one unit at time $t \geqslant s$ is then given by (1.1). We note that the random variable $\int_s^t X_{s,u} \, \mathrm{d}u$ is ν-a.s. well defined and

$$\int_s^t X_{s,u}(\omega) \, \mathrm{d}u = \langle \omega(x,y), \mathbf{1}_{[0,s]}(x) \mathbf{1}_{[0,t]}(y) \varkappa(x,y) (t - (s \vee y)) \rangle, \quad \nu\text{-a.e. } \omega \in \mathscr{D}'. \tag{2.6}$$

Indeed, for each $f \in L^2(\mathbb{R}^2, \mathrm{d}x\,\mathrm{d}y)$, we have by (2.3)

$$\begin{aligned}
&\int_{\mathscr{D}'} \left(\int_s^t X_{s,u}(\omega)\,\mathrm{d}u \right) \langle \omega, f \rangle\, v(\mathrm{d}\omega) \\
&\quad = \int_s^t \int_{\mathscr{D}'} X_{s,u}(\omega) \langle \omega, f \rangle\, v(\mathrm{d}\omega)\,\mathrm{d}u \\
&\quad = \int_{\mathbb{R}_+} \tau^2\,\sigma(\mathrm{d}\tau) \int_s^t \int_{\mathbb{R}^2} \mathbf{1}_{[0,s]}(x)\mathbf{1}_{[0,u]}(y)\varkappa(x,y)f(x,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}u \\
&\quad = \int_{\mathbb{R}_+} \tau^2\,\sigma(\mathrm{d}\tau) \int_{\mathbb{R}^2} \mathbf{1}_{[0,s]}(x) \left(\int_s^t \mathbf{1}_{[0,u]}(y)\,\mathrm{d}u \right) \varkappa(x,y)f(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad = \int_{\mathbb{R}_+} \tau^2\,\sigma(\mathrm{d}\tau) \int_{\mathbb{R}^2} \mathbf{1}_{[0,s]}(x)\mathbf{1}_{[0,t]}(y)(t-(s \vee y))\varkappa(x,y)f(x,y)\,\mathrm{d}x\,\mathrm{d}y, \qquad (2.7)
\end{aligned}$$

which implies (2.6).

We denote by $\mathscr{F}_t$, $t \geqslant 0$, the σ-algebra generated by the r.v.'s $F_{u,v}$, $0 \leqslant u \leqslant t$, $u \leqslant v$, which describes the information available at time t.

For $t \geqslant 0$, let R_t be the short rate at time t. We suppose that $\{R_t, t \geqslant 0\}$ is a stochastic process defined on the probability space $(\mathscr{D}', \mathscr{C}(\mathscr{D}'), v)$ and adapted to the filtration $\{\mathscr{F}_t, t \geqslant 0\}$.

It is our aim now to find conditions under which the discounted bond-price process given by (1.2) is a martingale. Following Kennedy [12], we make the following assumption:

Assumption (A). For any $0 \leqslant s < t < \infty$, the r.v. $(R_t - F_{s,t})$ is independent of $\mathscr{F}_s$.

Lemma 2.1. *Suppose that* $R_t \in L^2(\mathscr{D}', v)$, $t > 0$. *Then Assumption (A) holds if and only if*

$$R_t = X_{t,t} + r_t, \quad v\text{-a.e.}, \qquad (2.8)$$

where

$$r_t = \int_{\mathscr{D}'} R_t\,\mathrm{d}v, \quad t > 0. \qquad (2.9)$$

Remark 2.1. In fact, in the above lemma, the assumption that $R_t \in L^2(\mathscr{D}', v)$ may be weakened.

Proof. The space $L^2(\mathscr{D}', v)$ is unitarily isomorphic to the symmetric Fock space over $L^2(\mathbb{R}_+ \times \mathbb{R}^2, \sigma \otimes \mathrm{d}x\,\mathrm{d}y)$, i.e., there exists a unitary operator

$$\begin{aligned}
&\mathscr{F}(L^2(\mathbb{R}_+ \times \mathbb{R}^2, \sigma \otimes \mathrm{d}x\,\mathrm{d}y)) \ni f \\
&\quad = (f^{(n)})_{n=0}^{\infty} \mapsto If = f^{(0)} + \sum_{n=1}^{\infty} I^{(n)}(f^{(n)}) \in L^2(\mathscr{D}', v),
\end{aligned}$$

where $I^{(n)}(f^{(n)})$ is a (certain image of a) multiple stochastic integral, cf. [16] (see also [11] for the case of a usual Lévy process).

Let $\mathscr{G}_t$ denote the subspace of $L^2(\mathscr{D}', \nu)$ consisting of $\mathscr{F}_t$-measurable functions, $t>0$. Denote

$$A_t := \mathbb{R}_+ \times \{(x,y) \in \mathbb{R}^2 : 0 \leqslant x \leqslant t,\ x \leqslant y < \infty\}.$$

Analogously to the proof of Lemma 3.3 in [16], one can show that a function $F \in L^2(\mathscr{D}', \nu)$ belongs to $\mathscr{G}_t$ if and only if

$$F = f^{(0)} + \sum_{n=1}^{\infty} I^{(n)}(f^{(n)}), \tag{2.10}$$

where for each $n \geqslant 1$ the function $f_n \in L^2(\mathbb{R}_+ \times \mathbb{R}^2, \sigma \otimes \mathrm{d}x\,\mathrm{d}y)^{\hat{\otimes} n}$ has support in A_t^n, i.e., $f^{(n)} = f^{(n)} \mathbf{1}_{A_t^n}$. Here, $\hat{\otimes}$ denotes symmetric tensor product.

Let us fix any s such that $0<s<t$ and take any $F \in \mathscr{G}_t$ which is independent of $\mathscr{F}_s$. Then, for any $G \in \mathscr{G}_s$, the covariance of F and G with respect to the measure ν is equal to zero. This implies that

$$F = f^{(0)} + \sum_{n=1}^{\infty} I^{(n)}(f^{(n)} \mathbf{1}_{A_t^n \setminus A_s^n}).$$

We also note that $f^{(0)} = \int_{\mathscr{D}'} F \,\mathrm{d}\nu$.

Next, by (2.4), for any $s, t \geqslant 0$, we have

$$X_{s,t} = I^{(1)}(\tau \mathbf{1}_{[0,s]}(x) \mathbf{1}_{[0,t]}(y) \varkappa(x,y)), \quad \nu\text{-a.e.} \tag{2.11}$$

Now, suppose that (A) holds. Since R_t is $\mathscr{F}_t$-measurable, we have a representation

$$R_t = r_t + \sum_{n=1}^{\infty} I^{(n)}(g^{(n)} \mathbf{1}_{A_t^n}), \tag{2.12}$$

where r_t is given by (2.9) and $g^{(n)} \in L^2(\mathbb{R}_+ \times \mathbb{R}^2, \sigma \otimes \mathrm{d}x\,\mathrm{d}y)^{\hat{\otimes} n}$.

Furthermore, by Assumption (A), for any $0<s<t$, the function $R_t - r_t - X_{s,t}$ is independent of $\mathscr{F}_s$. By (2.11) and (2.12)

$$R_t - r_t - X_{s,t} = I^{(1)}(g^{(1)}(\tau,x,y) \mathbf{1}_{A_t}(\tau,x,y) - \tau \mathbf{1}_{[0,s]}(x) \mathbf{1}_{[0,t]}(y)) + \sum_{n=2}^{\infty} I^{(n)}(g^{(n)} \mathbf{1}_{A_t^n})$$

and therefore

$$\begin{aligned} &g^{(1)}(\tau,x,y) = \tau \mathbf{1}_{[0,s]}(x) \mathbf{1}_{[0,t]}(y) \quad \text{on } A_s, \\ &g^{(n)} = 0 \quad \text{on } A_s^n, \quad n \geqslant 2. \end{aligned} \tag{2.13}$$

Letting $s \to t$, we get from (2.13)

$$\begin{aligned} &g^{(1)}(\tau,x,y) = \tau \mathbf{1}_{[0,t]}(x) \mathbf{1}_{[0,t]}(y) \quad \text{on } A_t, \\ &g^{(n)} = 0 \quad \text{on } A_t^n, \quad n \geqslant 2. \end{aligned} \tag{2.14}$$

By (2.11), (2.12) and (2.14), we deduce that (2.8) and (2.9) hold.

On the other hand, (2.8) and (2.9) evidently imply Assumption (A). $\square$

Taking Lemma 2.1 into account, in what follows we will assume that the short rate R_t is given by formulas (2.8) and (2.9). We will additionally assume that r_t is continuous in t on $[0, \infty)$.

Theorem 2.1. *The following statements are equivalent*:

(a) *For each* $t \geqslant 0$, *the discounted bond-price process* $\{Z_{s,t}, \mathscr{F}_s, 0 \leqslant s \leqslant t\}$ *is a martingale*.

(b) *We have, for all* $s, t \geqslant 0$, $s \leqslant t$,

$$\mu_{s,t} = \mu_{0,t} + \int_{\mathbb{R}_+} \int_0^t \int_0^s \tau\varkappa(x,y)(1 - \mathrm{e}^{-\tau\varkappa(x,y)(t-(x\vee y))})\,\mathrm{d}x\,\mathrm{d}y\,\sigma(\mathrm{d}\tau) \tag{2.15}$$

and for all $t \geqslant 0$

$$R_t = F_{t,t}, \quad \nu\text{-a.e.} \tag{2.16}$$

(c) $$P_{s,t} = \mathbb{E}(\mathrm{e}^{-\int_s^t R_u\,\mathrm{d}u} \mid \mathscr{F}_s) \quad \textit{for all } s,t \geqslant 0,\ s \leqslant t.$$

Proof. We first show the equivalence of (a) and (b). By our assumption, (A) holds. Hence, analogously to the proof of Theorem 1.1 in [12], we conclude that (a) is equivalent to the following condition to hold:

$$\int_{\mathscr{D}'} \exp\left(- \int_{s_1}^t (F_{s_1,u} - F_{s_2,u})\,\mathrm{d}u - \int_{s_2}^{s_1} (R_u - F_{s_2,u})\,\mathrm{d}u \right) \mathrm{d}\nu = 1 \tag{2.17}$$

for all $0 \leqslant s_2 \leqslant s_1 \leqslant t < \infty$. By (2.4), (2.5), (2.8), and (2.9), equality (2.17) is equivalent to

$$\begin{aligned} \int_{\mathscr{D}'} \exp\Bigg(&- \int_{s_1}^t \langle \omega(x,y), \mathbf{1}_{[s_2,s_1]}(x)\mathbf{1}_{[0,u]}(y)\varkappa(x,y)\rangle\,\mathrm{d}u \\ &- \int_{s_2}^{s_1} \langle \omega(x,y), \mathbf{1}_{[s_2,u]}(x)\mathbf{1}_{[0,u]}(y)\varkappa(x,y)\rangle\,\mathrm{d}u \Bigg) \nu(\mathrm{d}\omega) \\ = \exp&\left(\int_{s_1}^t (\mu_{s_1,u} - \mu_{s_2,u})\,\mathrm{d}u + \int_{s_2}^{s_1} (r_u - \mu_{s_2,u})\,\mathrm{d}u \right) \end{aligned} \tag{2.18}$$

for all $0 \leqslant s_2 \leqslant s_1 \leqslant t < \infty$. Analogously to (2.6) and (2.7), we have

$$\begin{aligned} \int_{s_1}^t \langle \omega(x,y), \mathbf{1}_{[s_2,s_1]}(x)\mathbf{1}_{[0,u]}(y)\varkappa(x,y)\rangle\,\mathrm{d}u + \int_{s_2}^{s_1} \langle \omega(x,y), \mathbf{1}_{[s_2,u]}(x)\mathbf{1}_{[0,u]}(y)\varkappa(x,y)\rangle\,\mathrm{d}u \\ = \langle \omega(x,y), \mathbf{1}_{[s_2,s_1]}(x)\mathbf{1}_{[0,t]}(y)\varkappa(x,y)(t - (x \vee y))\rangle, \quad \nu\text{-a.e. } \omega \in \mathscr{D}'. \end{aligned}$$

Hence, it follows from (2.1) that condition (2.18) is equivalent to

$$\begin{aligned} \int_{\mathbb{R}_+} \int_0^t \int_{s_2}^{s_1} (\mathrm{e}^{-\tau\varkappa(x,y)(t-(x\vee y))} - 1 + \tau\varkappa(x,y)(t - (x \vee y)))\,\mathrm{d}x\,\mathrm{d}y\,\sigma(\mathrm{d}\tau) \\ = \int_{s_1}^t (\mu_{s_1,u} - \mu_{s_2,u})\,\mathrm{d}u + \int_{s_2}^{s_1} (r_u - \mu_{s_2,u})\,\mathrm{d}u \end{aligned} \tag{2.19}$$

for all $0 \leqslant s_2 \leqslant s_1 \leqslant t < \infty$. We remark that the function under the sign of integral on the left-hand side of (2.19) is integrable. Indeed, let us set $C_t := \sup_{x,y \in [0,t]} \varkappa(x,y)$. Then, for all $\tau \in (0,1]$ and $x, y \in [0,t]$, we have

$$\begin{aligned} &|\mathrm{e}^{-\tau\varkappa(x,y)(t-(x\vee y))} - 1 + \tau\varkappa(x,y)(t-(x\vee y))| \\ &\quad \leqslant \sum_{n=2}^{\infty} \frac{(\tau C_t t)^n}{n!} \leqslant \tau^2 C_t^2 t^2 \exp(\tau C_t t) \leqslant \tau^2 C_t^2 t^2 \exp(C_t t). \end{aligned} \tag{2.20}$$

This, together with the fact that $\int_{(0,1]} \tau^2 \, \sigma(\mathrm{d}\tau) < \infty$, yields the integrability on $(0,1] \times [0,t] \times [s_1, s_2]$. Furthermore, for $\tau \in (1, +\infty)$ and $x, y \in [0,t]$, we have

$$|\mathrm{e}^{-\tau\varkappa(x,y)(t-(x\vee y))} - 1 + \tau\varkappa(x,y)(t-(x\vee y))| \leqslant 1 + \tau t C_t.$$

This, together with the fact that $\int_{(1,+\infty)} \tau \, \sigma(\mathrm{d}\tau) < \infty$, completes the proof of the integrability.

We now fix $t > 0$ and suppose, for a moment, that $\mu_{s,t}$ has the following form:

$$\mu_{s,t} = \mu_{0,t} + \int_0^t \int_0^s \Psi_t(x,y) \, \mathrm{d}x \, \mathrm{d}y, \tag{2.21}$$

where $\Psi_t(x,y)$ is an integrable function on $[0,t]^2$, and

$$r_t = \mu_{t,t}, \quad t \geqslant 0. \tag{2.22}$$

Then,

$$\begin{aligned} &\int_{s_1}^t (\mu_{s_1,u} - \mu_{s_2,u}) \, \mathrm{d}u + \int_{s_2}^{s_1} (r_u - \mu_{s_2,u}) \, \mathrm{d}u \\ &\quad = \int_0^t \int_{s_2}^{s_1} \Psi_t(x,y)(t-(x\vee y)) \, \mathrm{d}x \, \mathrm{d}y. \end{aligned} \tag{2.23}$$

Comparing (2.23) with (2.19), we see that condition (2.19) is, at least formally, satisfied if $\Psi_t(x,y)$ has the form

$$\begin{aligned} \Psi_t(x,y) = (t-(x\vee y))^{-1} \int_{\mathbb{R}_+} (&\mathrm{e}^{-\tau\varkappa(x,y)(t-(x\vee y))} - 1 \\ &+ \tau\varkappa(x,y)(t-(x\vee y))) \, \sigma(\mathrm{d}\tau). \end{aligned} \tag{2.24}$$

To show that this inserted into (2.21) indeed gives a solution of (2.19), we have to verify that the $\Psi_t(x,y)$ given by (2.24) is integrable on $[0,t]^2$. Analogously to (2.20), we get

$$\begin{aligned} &\int_{(0,1]} \int_0^t \int_0^t |(t-(x\vee y))^{-1}(\mathrm{e}^{-\tau\varkappa(x,y)(t-(x\vee y))} - 1 \\ &\qquad + \tau\varkappa(x,y)(t-(x\vee y)))| \, \mathrm{d}x \, \mathrm{d}y \, \sigma(\mathrm{d}\tau) \\ &\quad \leqslant \int_{(0,1]} \int_0^t \int_0^t \sum_{n=2}^{\infty} \frac{\tau^n \varkappa(x,y)^n (t-(x\vee y))^{n-1}}{n!} \, \mathrm{d}x \, \mathrm{d}y \, \sigma(\mathrm{d}\tau) \\ &\quad \leqslant t^3 C_t^2 \mathrm{e}^{tC_t} \int_{(0,1]} \tau^2 \, \sigma(\mathrm{d}\tau) < \infty. \end{aligned} \tag{2.25}$$

Next,

$$\int_{(1,+\infty)} \int_0^t \int_0^t |(t-(x\vee y))^{-1}(\mathrm{e}^{-\tau\varkappa(x,y)(t-(x\vee y))}-1 + \tau\varkappa(x,y)(t-(x\vee y)))|\,\mathrm{d}x\,\mathrm{d}y\,\sigma(\mathrm{d}\tau)$$

$$\leqslant \int_{(1,+\infty)} \int_0^t \int_0^t |(t-(x\vee y))^{-1}(\mathrm{e}^{-\tau\varkappa(x,y)(t-(x\vee y))}-1)|\,\mathrm{d}x\,\mathrm{d}y\,\sigma(\mathrm{d}\tau) + t^2 C_t \int_{(1,+\infty)} \tau\,\sigma(\mathrm{d}\tau)$$

$$\leqslant 2t^2 C_t \int_{(1,+\infty)} \tau\,\sigma(\mathrm{d}\tau), \tag{2.26}$$

where we used the estimate: $1-\mathrm{e}^{-\alpha}\leqslant\alpha$ for all $\alpha\geqslant 0$. Thus, by (2.21), (2.22), and (2.24)–(2.26), statement (a) holds for

$$\mu_{s,t} = \mu_{0,t} + \int_{\mathbb{R}_+} \int_0^t \int_0^s (t-(x\vee y))^{-1}(\mathrm{e}^{-\tau\varkappa(x,y)(t-(x\vee y))}-1 + \tau\varkappa(x,y)(t-(x\vee y)))\,\mathrm{d}x\,\mathrm{d}y\,\sigma(\mathrm{d}\tau) \tag{2.27}$$

and $R_t = F_{t,t}$ ν-a.e., $t\geqslant 0$. As will be seen from below, the right-hand side of (2.15) and (2.27) do indeed coincide, so that (b) implies (a).

Let us now suppose that (a), or equivalently (2.19), holds. Setting in (2.19) $s_2 = s$ and $s_1 = t$, we get

$$\int_{\mathbb{R}_+} \int_0^t \int_s^t (\mathrm{e}^{-\tau\varkappa(x,y)(t-(x\vee y))}-1+\tau\varkappa(x,y)(t-(x\vee y)))\,\mathrm{d}x\,\mathrm{d}y\,\sigma(\mathrm{d}\tau) = \int_s^t (r_u-\mu_{s,u})\,\mathrm{d}u, \quad 0\leqslant s\leqslant t<\infty. \tag{2.28}$$

Differentiating (2.28) in t yields

$$\int_{\mathbb{R}_+} \int_0^t \int_s^t \tau\varkappa(x,y)(1-\mathrm{e}^{-\tau\varkappa(x,y)(t-(x\vee y))})\,\mathrm{d}x\,\mathrm{d}y\,\sigma(\mathrm{d}\tau) = r_t-\mu_{s,t}, \quad 0\leqslant s\leqslant t<\infty. \tag{2.29}$$

Setting $s=t$ in (2.29) gives (2.21), or equivalently (2.16). Next, setting $s=0$ in (2.29) gives

$$\int_{\mathbb{R}_+} \int_0^t \int_0^t \tau\varkappa(x,y)(1-\mathrm{e}^{-\tau\varkappa(x,y)(t-(x\vee y))})\,\mathrm{d}x\,\mathrm{d}y\,\sigma(\mathrm{d}\tau) = r_t-\mu_{0,t}, \quad t\geqslant 0. \tag{2.30}$$

Subtracting (2.29) from (2.30) implies (2.15). Thus, (b) implies (a). Furthermore, this also yields that the right-hand side of (2.15) and (2.27) coincide, which finishes the proof of the equivalence of (a) and (b).

Let us now show the equivalence of (b) and (c). Using Assumption (A), analogously to Kennedy [12], we conclude that (c) is equivalent to

$$\int_{\mathscr{D}'} \exp\left(-\int_s^t (R_u - F_{s,u})\,\mathrm{d}u\right)\mathrm{d}\nu = 1, \quad 0\leqslant s\leqslant t<\infty. \tag{2.31}$$

But (2.31) is a special case of (2.17) with $s_2 = s$ and $s_1 = t$. Therefore, (b) implies (c). On the other hand, it follows from the proof of (a)⇒(b) that (2.31) implies (b). Thus, the proof is complete. □

Corollary 2.1. *Suppose that the Lévy measure σ additionally satisfies*

$$\langle\tau\rangle_\sigma := \int_{\mathbb{R}_+} \tau\,\sigma(\mathrm{d}\tau)<\infty. \tag{2.32}$$

Suppose that statement (a) *of Theorem* 2.1 *holds and suppose that the initial term structure $\{\mu_{0,t},\ t\geqslant 0\}$ satisfies*

$$\mu_{0,t}\geqslant \int_0^t\int_0^t \varkappa(x,y)\,\mathrm{d}x\,\mathrm{d}y\cdot\langle\tau\rangle_\sigma, \quad t\geqslant 0. \tag{2.33}$$

Then the forward rate process $\{F_{s,t},\ 0\leqslant s\leqslant t<\infty\}$ and the spot rate process $\{R_t,\ t\geqslant 0\}$ take on non-negative values ν-a.s.

Proof. By (2.5) and Theorem 2.1, we get

$$F_{s,t} = \mu_{0,t} - \int_{\mathbb{R}_+}\int_0^t\int_0^s \tau\varkappa(x,y)\mathrm{e}^{-\tau\varkappa(x,y)(t-(x\vee y))}\,\mathrm{d}x\,\mathrm{d}y\,\sigma(\mathrm{d}\tau) + \widetilde{X}_{s,t},$$
$$0\leqslant s\leqslant t<\infty,$$

where

$$\widetilde{X}_{s,t} := X_{s,t} + \int_0^t\int_0^s \varkappa(x,y)\,\mathrm{d}x\,\mathrm{d}y\cdot\langle\tau\rangle_\sigma.$$

Under condition (2.32), the measure ν is concentrated on the set of all signed measures of the form $\sum_{n=1}^\infty \tau_n\delta_{(x_n,y_n)}(\mathrm{d}x\,\mathrm{d}y) - \langle\tau\rangle_\sigma\,\mathrm{d}x\,\mathrm{d}y$, where δ_a denotes the Dirac measure with mass at a, $\tau_n\in\operatorname{supp}\sigma$, $n\in\mathbb{N}$, and $\{(\tau_n,x_n,y_n)\}_{n=1}^\infty$ is a locally finite set in $\mathbb{R}_+\times\mathbb{R}^2$, see, e.g., Lytvynov [16]. Therefore, by (2.4), $\widetilde{X}_{s,t}$ takes on non-negative values ν-a.s. Furthermore, it follows from (2.33) that

$$\mu_{0,t} - \int_{\mathbb{R}_+}\int_0^t\int_0^s \tau\varkappa(x,y)\mathrm{e}^{-\tau\varkappa(x,y)(t-(x\vee y))}\,\mathrm{d}x\,\mathrm{d}y\,\sigma(d\tau)\geqslant 0, \quad t\geqslant 0,\ \ 0\leqslant s\leqslant t,$$

from where the statement about $F_{s,t}$ follows. Finally, by Theorem 2.1, (2.16) holds, which implies the statement about R_t. □

Let us consider two examples of a measure ν satisfying the assumptions of Theorem 2.1 and Corollary 2.1.

Example 1 (*Poisson sheet*). We take as ν the centered Poisson measure π_z with intensity parameter $z>0$, see, e.g., [9]. The Lévy measure σ has now the form $z\delta_1$. Thus, the Fourier transform of π_z is given by

$$\int_{\mathscr{D}'} \mathrm{e}^{\mathrm{i}\langle\omega,\varphi\rangle}\,\pi_z(\mathrm{d}\omega) = \exp\left[\int_{\mathbb{R}^2} (\mathrm{e}^{\mathrm{i}\varphi(x,y)} - 1 - \mathrm{i}\varphi(x,y))\, z\,\mathrm{d}x\,\mathrm{d}y\right], \quad \varphi \in \mathscr{D}.$$

We set $\varkappa(x,y) \equiv 1$. Then, $X_{s,t}$ given by (2.4) with the underlying probability measure $\nu = \pi_z$ is, by definition, a Poisson sheet, and for each fixed $t>0$, $\{X_{s,t},\ 0 \leqslant s \leqslant t\}$ is a centered Poisson process with intensity parameter tz. Formula (2.15) now reads as follows:

$$\mu_{s,t} = \mu_{0,t} + z\big((2-s)\mathrm{e}^{s-t} - 2e^{-t} - s + st\big).$$

Condition (2.33) now means $\mu_{0,t} \geqslant zt^2$, $t \geqslant 0$.

Example 2 (*Gamma sheet*). We take as ν the centered gamma measure γ_z with intensity parameter $z>0$, see, e.g., [16]. The Lévy measure σ on $\mathbb{R}_+$ has the form

$$\sigma(\mathrm{d}\tau) = \frac{\mathrm{e}^{-\tau}}{\tau}\, z\,\mathrm{d}\tau.$$

The Fourier transform of γ_z may be written as follows:

$$\int_{\mathscr{D}'} \mathrm{e}^{\mathrm{i}\langle\omega,\varphi\rangle}\,\gamma_z(\mathrm{d}\omega) = \exp\left(-\int_{\mathbb{R}^2} (\log(1 - \mathrm{i}\varphi(x,y)) + \varphi(x,y))\, z\,\mathrm{d}x\,\mathrm{d}y\right),$$
$$\varphi \in \mathscr{D},\ |\varphi| < 1.$$

We set $\varkappa(x,y) \equiv 1$. Then, $X_{s,t}$ given by (2.4) with the underlying probability measure $\nu = \gamma_z$ is, by definition, a gamma sheet, and for each $t>0$, $\{X_{s,t},\ 0 \leqslant s \leqslant t\}$ is a centered gamma process with intensity parameter tz. Formula (2.15) now reads as follows:

$$\mu_{s,t} = \mu_{0,t} + z\left(st + 2s + 2(1+t)\log\left(\frac{1+t-s}{1+t}\right) - s\log(1+t-s)\right).$$

Condition (2.33) means $\mu_{0,t} \geqslant zt^2$, $t \geqslant 0$.

It is possible to construct a model of forward interest rates which unifies the approach of Kennedy [12] to modeling the forward interest rate with our approach. Indeed, consider $F_{s,t}$ in the form

$$F_{s,t} = \mu_{s,t} + X_{s,t} + Y_{s,t}, \quad 0 \leqslant s \leqslant t < \infty, \tag{2.34}$$

where $\mu_{s,t}$ and $X_{s,t}$ are as in formula (2.5) (thus, as in our approach) and $Y_{s,t}$ is a centered continuous Gaussian random field that is independent of $X_{u,v}$, $0 \leqslant u \leqslant v < \infty$, and has covariance

$$\operatorname{Cov}(Y_{s_1,t_1}, Y_{s_2,t_2}) = c(s_1 \wedge s_2, t_1, t_2), \quad 0 \leqslant s_i \leqslant t_i,\ i = 1, 2,$$

with a function c satisfying $c(0, t_1, t_2) \equiv 0$ (as in Kennedy's approach). Furthermore, set

$$R_t = r_t + X_{t,t} + Y_{t,t}, \quad t \geqslant 0, \tag{2.35}$$

where r_t is deterministic and continuous in t.

The following theorem may be proved by combining the proof of Theorem 1.1 in Kennedy [12] and the proof of Theorem 2.1.

Theorem 2.2. *Theorem* 2.1 *remains valid for the forward rates* $\{F_{s,t},\ 0 \leqslant s \leqslant t < \infty\}$ *given by* (2.34) *and the short rates given by* (2.35) *if we set the deterministic term* $\mu_{s,t}$ *in statement* (b) *to be*

$$\mu_{s,t} = \mu_{0,t} + \int_0^t \int_0^s \int_{\mathbb{R}_+} \tau\varkappa(x, y)(1 - \mathrm{e}^{-\tau\varkappa(x,y)(t-(x\vee y))})\, \sigma(\mathrm{d}\tau)\, \mathrm{d}x\, \mathrm{d}y + \int_0^t c(s \wedge u, u, t)\, \mathrm{d}u$$

for all $0 \leqslant s \leqslant t < \infty$.

Acknowledgements

The financial support of the SFB 611 is gratefully acknowledged. We would also like to thank the referee for a careful reading of the manuscript and making very useful suggestions.

References

[1] R.J. Adler, The Geometry of Random Fields, Wiley, Chichester, 1981.
[2] Yu.M. Berezansky, Z.G. Sheftel, G.F. Us, Functional Analysis, vol. 2, Birkhäuser Verlag, Basel, 1996.
[3] T. Björk, B.J. Christensen, Interest rate dynamics and consistent forward rate curves, Math. Finance 9 (1999) 323–348.
[4] T. Björk, G. Di Masi, Yu. Kabanov, W. Runggaldier, Towards a general theory of bond markets, Finance Stochast. 1 (1997) 141–174.
[5] T. Björk, Yu. Kabanov, W. Runggaldier, Bond market structure in the presence of marked point processes, Math. Finance 7 (1997) 211–239.
[6] I.M. Gelfand, N.Ya. Vilenkin, Generalized Functions, Applications of Harmonic Analysis, vol. 4, Academic Press, New York, 1964.
[7] D.C. Heath, R.A. Jarrow, A. Morton, Bond pricing and the term structure of interest rates: a discrete time approximation, J. Financial Quantum Anal. 25 (1990) 419–440.
[8] D.C. Heath, R.A. Jarrow, A. Morton, Bond pricing and the term structure of interest rates: a new methodology for contingent claims valuation, Econometrica 60 (1992) 77–105.

[9] T. Hida, Stationary Stochastic Processes, Princeton University Press, Princeton, NJ, 1979.
[10] T. Ho, S. Lee, Term structure movements and pricing interest rate contingent claims, J. Finance 1 (1986) 1011–1029.
[11] K. Itô, Spectral type of the shift transformation of differential processes with stationary increments, Trans. Amer. Math. Soc. 81 (1956) 253–266.
[12] D.P. Kennedy, The term structure of interest rates as a Gaussian random field, Math. Finance 4 (1994) 247–258.
[13] D.P. Kennedy, Characterizing Gaussian models of the term structure of interest rates, Math. Finance 7 (1997) 107–118.
[14] S. Kusuoka, Term structure and SPDE, in: S. Kusuoka, T. Maruyama (Eds.), Advances in Mathematical Economics, vol. 2, Springer, Tokyo, 2000, pp. 67–85.
[15] E. Lütkebohmert, Endlich dimensionale Darstellungen für das erweiterte Zinsmodell von Heath, Jarrow und Morton, Diploma Thesis, Bonn University, Bonn, 2000.
[16] E. Lytvynov, Orthogonal decompositions for Lévy processes with an application to the gamma, Pascal, and Meixner processes, Infinite Dimensional Anal. Quantum Probab. Related Topics 6 (2003) 73–102.
[17] A. Mahnig, Modellierung der Zinsstrukturen durch ein Poisson Sheet, Diploma Thesis, Bonn University, Bonn, 2002.
[18] S.A. Popovici, Modellierung von Zinsstrukturkurven mit Hilfe von stochastischen partiellen Differentialgleichungen Diploma Thesis, Bonn University, Bonn, 2001.
[19] I. Yalovenko, Modellierung des Finanzmarktes und unendlich dimensionale stochastische Prozesse, Diploma Thesis, Bochum University, Bochum, 1998.