Journal of Algebra **221,** 385–438 (1999)
Article ID jabr.1999.7984, available online at http://www.idealibrary.com on IDEAL®

Weak Hopf Algebras

I. Integral Theory and C^*-Structure

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Communicated by Eva Bayer-Fluckiger

Received June 3, 1998

We give an introduction to the theory of weak Hopf algebras proposed as a *coassociative* alternative of weak quasi-Hopf algebras. We follow an axiomatic approach keeping as close as possible to the "classical" theory of Hopf algebras. The emphasis is put on the new structure related to the presence of canonical subalgebras A^L and A^R in any weak Hopf algebra A that play the role of non-commutative numbers in many respects. A theory of integrals is developed in which we show how the algebraic properties of A, such as the Frobenius property, or semisimplicity, or innerness of the square of the antipode, are related to the existence of non-degenerate, normalized, or Haar integrals. In case of C^*-weak Hopf algebras we prove the existence of a unique Haar measure $h \in A$ and of a

[*] Supported by the Hungarian Scientific Research Fund, OTKA-T 016 233.
[†] Supported by the DFG, SFB 288 "Differentialgeometrie und Quantenphysik."
[‡] Supported by the Hungarian Scientific Research Fund, OTKA-T 020 285.

0021-8693/99 $30.00

canonical grouplike element $g \in A$ implementing the square of the antipode and factorizing into left and right elements $g = g_L g_R^{-1}$, $g_L \in A^L$, $g_R \in A^R$. Further discussion of the C^*-case will be presented in Part II.

CONTENTS

1. INTRODUCTION

Weak Hopf algebras have been proposed [2, 14, 20] as a new generalization of ordinary Hopf algebras that replaces Ocneanu's paragroup [16], in the depth 2 case, with a concrete "Hopf algebraic" object. The earlier proposals of *face algebras* [8] or *quantum groupoids* [17] are actually weak Hopf algebras even if not the most general ones. Also, the (finite-dimensional) *generalized Kac algebras* of Yamanouchi [25] are weak Hopf algebras in our sense [14], albeit with an involutive antipode.

In contrast to other Hopf algebraic constructions such as the quasi-Hopf algebras [6] or the weak quasi-Hopf algebras and rational Hopf algebras [7, 11, 22] weak Hopf algebras are *coassociative*. This allows one to define actions, coactions, and crossed products as easily as in the Hopf algebra case. On the other hand weak Hopf algebras have "weaker" axioms related to the unit and counit: The comultiplication is non-unital, $\Delta(1) \neq 1 \otimes 1$ (like in weak quasi-Hopf algebras) and the counit is only "weakly" multiplicative, $\varepsilon(xy) = \varepsilon(x1_{(1)})\varepsilon(1_{(2)}y)$. This kind of "weakness" is the "strength" of weak Hopf algebras because it allows (even in the finite-dimensional and semisimple case) the weak Hopf algebra to possess non-integral (quantum) dimensions.

Thus weak Hopf algebras are not special cases of weak quasi-Hopf algebras and also not more general than them. Nevertheless, in situations where only the representation category of the quantum group matters, these two concepts are equivalent. This is, of course, not surprising in view of MacLane's theorem on the equivalence of relaxed and strict monoidal categories [12]. In fact not all of the potential of this theorem is utilized by weak Hopf algebras because their representation category is not quite strict: Only the associator is trivial but not the left and right isomorphisms of the monoidal unit. Although a general analysis clarifying the role of representation categories of weak Hopf algebras within the set of monoidal categories is still missing the examples constructed in [2] using Ocneanu's cocycle suggest that they play a rather fundamental role, as long as they can accommodate to arbitrary $6j$-symbols.

So far weak Hopf algebras have been considered only under the additional assumption of finite dimensionality. Although a good deal of the results can be generalized to the infinite-dimensional case, finite dimension is particularly attractive because it implies self-duality. Just like finite Abelian groups or finite-dimensional Hopf algebras, the finite-dimensional weak Hopf algebras (WHA) are *self-dual* in the following sense. If A is a WHA then its dual space $\hat{A}$ is canonically equipped with a weak Hopf algebra structure. Furthermore this duality is reflexive, $(\hat{A})\hat{} \cong A$. This is a feature which makes WHAs more natural objects of study than either finite (non-Abelian) groups or finite-dimensional (weak) quasi-Hopf algebras.

The main motivation for studying WHAs comes from quantum field theory and operator algebras and consists roughly of the following two symmetry problems.

I. If $N \subset M$ is an inclusion of algebras satisfying certain conditions then find a (unique) "quantum group" G and an action of G on M such that $N = M^G$, the invariant subalgebra.

II. The dual problem is to find a quantum group $\hat{G}$ acting on N such that M is isomorphic to the crossed product $N \rtimes \hat{G}$.

Of course, determining the appropriate notion of quantum group, as well as its action, is part of the problem. If $N \subset M$ is a finite index irreducible depth 2 inclusion of von Neumann factors then the answer is known by [10] to be a finite-dimensional C^*-Hopf algebra. In [15] we showed that if we allowed the inclusion to be reducible and N and M to have arbitrary finite-dimensional centers then the appropriate quantum group was a C^*-weak Hopf algebra. Even in case of inclusions of certain associative (non-*) algebras the notion of a WHA over an arbitrary field K, introduced in this paper, may provide a useful invariant.

In Section 2 we introduce the axioms of weak bialgebras and weak Hopf algebras over a field K and discuss their consequences. If $K = \mathbb{C}$, the complex field, then these axioms are equivalent to those of [20]. The present axioms have the advantage of being manifestly self-dual and almost each of them having an ancestor among the Hopf algebra axioms which it generalizes. In discussing the consequences particular attention is paid to the canonical subalgebras A^L and A^R present in any WHA both of which reducing to the scalars $K1$ if A is a Hopf algebra. From many points of view these subalgebras behave like non-commutative generalizations of numbers. Just to mention some: 1. A^L and A^R are separable K-algebras. 2. The trivial left A-module is a representation on the K-space A^L (or on A^R). 3. The dual weak Hopf algebra $\hat{A}$ have left and right subalgebras $\hat{A}^L$ and $\hat{A}^R$ that are isomorphic to A^R and A^L, respectively. Of course, to realize the idea of A^L and A^R being "non-commutative numbers" one should completely get rid of the field K from the outset. As yet we have no concrete proposal for this scenario.

Section 3 is devoted to the study of integrals in weak Hopf algebras. Using the notion of *weak Hopf modules* which is a generalization of the Hopf modules [1, 19] we show that non-zero integrals exist. A weak Hopf version of Maschke's theorem characterizes semisimple WHAs as those possessing normalized integrals. Other important classes of WHAs are those which are Frobenius algebras. They are characterized by possessing non-degenerate left integrals. This class is a self-dual class by the duality theorem of non-degenerate integrals. We conclude with giving necessary and sufficient criteria for the existence of Haar integrals, i.e., normalized non-degenerate two-sided integrals in a WHA.

Section 5 contains the basic properties of weak C^*-Hopf algebras such as the existence of a Haar integral h and a canonical grouplike element $g \geq 0$ implementing S^2 and the modular automorphism of the Haar measure. As a consequence of the existence of Haar measures the dual of a C^*-weak Hopf algebra is a C^*-weak Hopf algebra again. Further analysis of C^*-WHAs will be given in Part II where we discuss the representation category and a notion of dimension which turns out to be non-commutative in case of solitonic representations [3].

2. THE WEAK HOPF CALCULUS

2.1. *The Axioms*

DEFINITION 2.1. A weak bialgebra (WBA) *is a quintuple* $(A, \mu, u, \Delta, \varepsilon)$ *satisfying Axioms* 1–3 *below*. If $(A, \mu, u, \Delta, \varepsilon, S)$ *satisfies Axioms* 1–4 *below it is called a* weak Hopf algebra (WHA).

Axiom 1. A is a finite-dimensional associative algebra over a field K with multiplication $\mu\colon A \otimes A \to A$ and unit $u\colon K \to A$. I.e., μ and u are K-linear and satisfy

$$\text{Associativity: } \mu \circ (\mu \otimes \mathrm{id}) = \mu \circ (\mathrm{id} \otimes \mu), \tag{1.1}$$

$$\text{Unit property: } \mu \circ (u \otimes \mathrm{id}) = \mathrm{id} = \mu \circ (\mathrm{id} \otimes u). \tag{1.2}$$

(Later on we suppress μ and u, just write xy for $\mu(x, y)$, and use the *unit element* $1 := u(1)$ instead of u.)

Axiom 2. A is a coalgebra over K with *comultiplication* $\Delta\colon A \to A \otimes A$ and *counit* $\varepsilon\colon A \to K$. I.e., Δ and ε are K-linear and satisfy

$$\text{Coassociativity: } (\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta, \tag{1.3}$$

$$\text{Counit property: } (\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon) \circ \Delta. \tag{1.4}$$

Axiom 3. For compatibility of the algebra and coalgebra structures we assume

Multiplicativity of the coproduct: For all $x, y \in A$,

$$\Delta(xy) = \Delta(x)\Delta(y). \tag{1.5}$$

Weak multiplicativity of the counit: For all $x, y, z \in A$,

$$\varepsilon(xyz) = \varepsilon(xy_{(1)})\varepsilon(y_{(2)}z), \tag{1.6a}$$

$$\varepsilon(xyz) = \varepsilon(xy_{(2)})\varepsilon(y_{(1)}z). \tag{1.6b}$$

Weak comultiplicativity of the unit,

$$\Delta^2(1) = (\Delta(1) \otimes 1)(1 \otimes \Delta(1)), \tag{1.7a}$$

$$\Delta^2(1) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1). \tag{1.7b}$$

Axiom 4. There exists a K-linear map $S\colon A \to A$, called the *antipode*, satisfying the following

Antipode axioms: For all $x \in A$,

$$x_{(1)}S(x_{(2)}) = \varepsilon(1_{(1)}x)1_{(2)}, \tag{1.8a}$$

$$S(x_{(1)})x_{(2)} = 1_{(1)}\varepsilon(x1_{(2)}), \tag{1.8b}$$

$$S(x_{(1)})x_{(2)}S(x_{(3)}) = S(x). \tag{1.9}$$

In Eqs. (1.6)–(1.9) we used a standard suffix notation for (iterated) coproducts, omitting as usual summation indices and a summation symbol.

In the terminology of [14] $(A, \mu, u, \Delta, \varepsilon)$ is called a *weak bialgebra* if it satisfies the axioms (1.1)–(1.5). There a weak bialgebra is called *monoidal* if it satisfies (1.6) and it is called *comonoidal* if it satisfies (1.7). As has been explored in detail in [14], these (co)monoidality axioms are precisely designed to render the category of A-modules (the category of A-comodules, respectively) monoidal.

The *dual* of a weak bialgebra (weak Hopf algebra) A is the dual space $\hat{A} := \mathrm{Hom}_K(A, K)$ equipped with structure maps $\hat{\mu}, \hat{u}, \hat{\Delta}, \hat{\varepsilon}$ ($, \hat{S}$) defined by transposing the structure maps of A by means of the canonical pairing $\langle\ ,\ \rangle\colon \hat{A} \times A \to K$,

$$\langle \varphi\psi, x\rangle := \langle \varphi \otimes \psi, \Delta(x)\rangle,$$

$$\langle \hat{1}, x\rangle := \varepsilon(x),$$

$$\langle \hat{\Delta}(\varphi), x \otimes y\rangle := \langle \varphi, xy\rangle,$$

$$\hat{\varepsilon}(\varphi) := \langle \varphi, 1\rangle,$$

$$\langle \hat{S}(\varphi), x\rangle := \langle \varphi, S(x)\rangle,$$

where $\varphi, \psi \in \hat{A}$ and $x, y \in A$.

Let f and g be maps from the m-fold tensor product $A^{\otimes m}$ to the n-fold tensor product $A^{\otimes n}$ such that they are composites of tensor products of the structure maps $\mu, u, \Delta, \varepsilon, S$ and of the twist maps τ_{ij} interchanging the ith and the jth A factors. Then the equality $f = g$ is called an *A-statement*. Similarly one defines the *$\hat{A}$-statements*. Now every A-statement $Q :: f = g$ determines an equivalent $\hat{A}$-statement $Q^T :: f^T = g^T$ obtained by reversing the order of composition and replacing μ with $\hat{\Delta}$, u with $\hat{\varepsilon}$, Δ with $\hat{\mu}$, ε with $\hat{u}$, and S with $\hat{S}$. The statement Q^T is called the *transpose* of Q. If we now substitute $\mu, u, \Delta, \varepsilon, S$, respectively, in place of $\hat{\mu}, \hat{u}, \hat{\Delta}, \hat{\varepsilon}, \hat{S}$ in the statement Q^T we obtain a new A-statement $Q^{\sim} :: f^{\sim} = g^{\sim}$ which is not equivalent to Q in general. This $Q^{\sim}$ will be called the *dual* of Q. For example, one can easily verify that the WBA axioms satisfy $(1.1)^{\sim} = (1.3)$, $(1.2)^{\sim} = (1.4)$, $(1.5)^{\sim} = (1.5)$, $(1.6\text{a})^{\sim} = (1.7\text{a})$, and $(1.6\text{b})^{\sim} = (1.7\text{b})$. Thus the weak bialgebra axioms form a self-dual set of statements. This implies that the dual of a WBA is a WBA, too. The same holds for weak Hopf algebras, since each one of the antipode axioms is a self-dual statement. As a consequence of self-duality if Q is a true statement in a WBA or in a WHA then $Q^{\sim}$ is also true there. This principle extends also to statements involving both A and $\hat{A}$ structure maps and canonical pairing(s).

As has been proven in [14], the above self-dual set of WHA axioms are equivalent to the non-self-dual set of axioms given in [20]. In this work we gradually reproduce all axioms of [20] as a consequence of the present ones.

For a weak Hopf algebra $(A, 1, \Delta, \varepsilon, S)$ the following conditions are equivalent

- A is a Hopf algebra;
- $\Delta(1) = 1 \otimes 1$;
- $\varepsilon(xy) = \varepsilon(x)\varepsilon(y)$;
- $S(x_{(1)})x_{(2)} = 1\varepsilon(x)$;
- $x_{(1)}S(x_{(2)}) = 1\varepsilon(x)$.

The proof of these assertions are either trivial or will become trivial after acquainting the weak Hopf calculus developed in the next subsections, see also [14].

2.2. *Weak Bialgebras*

In a WBA define the maps $\sqcap^L, \sqcap^R: A \to A$ by the formulae

$$\sqcap^L(x) := \varepsilon\big(1_{(1)}x\big)1_{(2)}, \qquad \sqcap^R(x) := 1_{(1)}\varepsilon\big(x1_{(2)}\big), \tag{2.1}$$

and introduce the notation $A^L := \sqcap^L(A)$, $A^R := \sqcap^R(A)$ for their images. The analogue objects in the dual bialgebra $\hat{A}$ will be denoted by $\hat{\sqcap}^L$, $\hat{\sqcap}^R$, $\hat{A}^L$, and $\hat{A}^R$, respectively.

Substituting $y = 1$ in Axiom (1.6b) one obtains immediately the identities

$$\varepsilon\big(x \sqcap^L(y)\big) = \varepsilon(xy), \tag{2.2a}$$

$$\varepsilon\big(\sqcap^R(x)y\big) = \varepsilon(xy), \tag{2.2b}$$

$$\sqcap^L \circ \sqcap^L = \sqcap^L, \tag{2.3a}$$

$$\sqcap^R \circ \sqcap^R = \sqcap^R. \tag{2.3b}$$

As a first application of the duality principle take[1] the duals of Eqs. (2.2a) and (2.2b),

$$1_{(1)} \otimes \sqcap^L\big(1_{(2)}\big) = 1_{(1)} \otimes 1_{(2)} = \sqcap^R\big(1_{(1)}\big) \otimes 1_{(2)}.$$

[1] In taking the transpose of a statement with $\sqcap^{L/R}$ use the fact that in a WBA $\langle \varphi, \sqcap^L(x)\rangle = \langle \hat{1}_{(1)} \otimes \hat{1}_{(2)}, 1_{(1)} \otimes x\rangle\langle \varphi, 1_{(2)}\rangle = \langle \hat{\sqcap}^L(\varphi), x\rangle$ and similarly $\langle \varphi, \sqcap^R(x)\rangle = \langle \hat{\sqcap}^R(\varphi), x\rangle$.

Then these are identities in any WBA. It follows that

$$\Delta(1) \in A^R \otimes A^L. \tag{2.4}$$

LEMMA 2.2. *The counit defines a non-degenerate bilinear form*

$$x^L \in A^L, \qquad y^R \in A^R \mapsto \varepsilon(y^R x^L) \in K.$$

Hence $A^L \cong A^R$ as K-spaces.

Proof.

$$\varepsilon(y^R x^L) = 0 \qquad \forall y^R \in A^R \quad \Rightarrow \quad x^L = \varepsilon\left(1_{(1)} x^L\right) 1_{(2)} = 0,$$

$$\varepsilon(y^R x^L) = 0 \qquad \forall x^L \in A^L \quad \Rightarrow \quad y^R = 1_{(1)} \varepsilon\left(y^R 1_{(2)}\right) = 0,$$

where we used (2.4). Q.E.D.

Returning to Eqs. (2.2a) and (2.2b) and substituting them into the definitions (2.1) one obtains

$$\sqcap^L\left(x \sqcap^L (y)\right) = \sqcap^L(xy), \tag{2.5a}$$

$$\sqcap^R\left(\sqcap^R(x) y\right) = \sqcap^R(xy). \tag{2.5b}$$

The duals of (2.5a) and (2.5b),

$$\Delta(A^L) \subset A \otimes A^L, \tag{2.6a}$$

$$\Delta(A^R) \subset A^R \otimes A \tag{2.6b}$$

tell us that A^L and A^R are left, respectively, right coideals in the coalgebra A. Using Axiom (1.7b) we can obtain explicit expressions for these coproducts

$$\Delta(x^L) = \varepsilon\left(1_{(1)} x^L\right) 1_{(2)} \otimes 1_{(3)} = \varepsilon\left(1_{(1')} x^L\right) 1_{(1)} 1_{(2')} \otimes 1_{(2)} = 1_{(1)} x^L \otimes 1_{(2)}, \tag{2.7a}$$

$$\Delta(x^R) = 1_{(1)} \otimes 1_{(2)} \varepsilon\left(x^R 1_{(3)}\right) = 1_{(1)} \otimes 1_{(1')} 1_{(2)} \varepsilon\left(x^R 1_{(2')}\right) = 1_{(1)} \otimes x^R 1_{(2)}, \tag{2.7b}$$

where x^L and x^R are meant to denote arbitrary elements of A^L, resp., A^R.

LEMMA 2.3. *For all $x \in A$ we have the identities*

$$x_{(1)} \otimes \sqcap^L(x_{(2)}) = 1_{(1)}x \otimes 1_{(2)}, \tag{2.8a}$$

$$\sqcap^R(x_{(1)}) \otimes x_{(2)} = 1_{(1)} \otimes x1_{(2)}. \tag{2.8b}$$

Proof. Using Axiom (1.7b) one obtains

$$\begin{aligned} x_{(1)} \otimes \varepsilon(1_{(1)}x_{(2)})1_{(2)} &= 1_{(1')}x_{(1)}\varepsilon(1_{(1)}1_{(2')}x_{(2)}) \otimes 1_{(2)} \\ &= 1_{(1)}x_{(1)}\varepsilon(1_{(2)}x_{(2)}) \otimes 1_{(3)} \\ &= 1_{(1)}x \otimes 1_{(2)}, \\ 1_{(1)}\varepsilon(x_{(1)}1_{(2)}) \otimes x_{(2)} &= 1_{(1)} \otimes \varepsilon(x_{(1)}1_{(1')}1_{(2)})x_{(2)}1_{(2')} \\ &= 1_{(1)} \otimes \varepsilon(x_{(1)}1_{(2)})x_{(2)}1_{(3)} \\ &= 1_{(1)} \otimes x1_{(2)}. \end{aligned}$$

Q.E.D.

As a consequence we obtain the dual statements

$$x \sqcap^L(y) = \varepsilon(x_{(1)}y)x_{(2)}, \tag{2.9a}$$

$$\sqcap^R(x)y = y_{(1)}\varepsilon(xy_{(2)}). \tag{2.9b}$$

PROPOSITION 2.4. *Let A be a WBA. Then A^L and A^R are subalgebras of A containing* 1 *and*

$$x^L y^R = y^R x^L \quad \textit{for all } x^L \in A^L \quad \textit{and} \quad y^R \in A^R. \tag{2.10}$$

Proof. Eqs. (2.8a) and (2.8b) imply the relations

$$1_{(1)}1_{(1')} \otimes 1_{(2)} \otimes 1_{(2')} = 1_{(1)} \otimes \sqcap^L(1_{(2)}) \otimes 1_{(3)}, \tag{2.11a}$$

$$1_{(1)} \otimes 1_{(1')} \otimes 1_{(2)}1_{(2')} = 1_{(1)} \otimes \sqcap^R(1_{(2)}) \otimes 1_{(3)}. \tag{2.11b}$$

Now either Axiom (1.7a) or Axiom (1.7b) show that on the RHS of (2.11a) the first tensor factor belongs to A^R and on the RHS of (2.11b) the last factor belongs to A^L. This is sufficient for A^R, respectively A^L to be closed under multiplication. Hence they are algebras. Obviously $1 \in A^L \cap A^R$ since $\sqcap^L(1) = 1 = \sqcap^R(1)$. To see commutativity of left and right elements just compare Axioms (1.7a) and (1.7b). Q.E.D.

As the duals of the statements that A^L and A^R are subalgebras we obtain that Ker $\sqcap^L$ and Ker $\sqcap^R$ are coideals of the coalgebra A, i.e.,

$$\begin{aligned} &\Delta(\mathrm{Ker}\,\sqcap^C) \subset A \otimes \mathrm{Ker}\,\sqcap^C + \mathrm{Ker}\,\sqcap^C \otimes A, \\ &\varepsilon(\mathrm{Ker}\,\sqcap^C) = 0, \qquad C = L, R. \end{aligned} \tag{2.12}$$

On the other hand, being the annihilator of the left coideal $\hat{A}^L$, Ker $\sqcap^L$ is a left ideal of the algebra A and similarly, Ker $\sqcap^R$ is a right ideal.

LEMMA 2.5. *Consider A^L and A as left A^L-modules by left multiplication. Then $\sqcap^L\colon A \to A^L$ is a left A^L-module map. Analogously, $\sqcap^R\colon A \to A^R$ is a right A^R-module map. That is to say*

$$\sqcap^L\big(\sqcap^L(x)y\big) = \sqcap^L(x)\,\sqcap^L(y), \tag{2.13a}$$

$$\sqcap^R\big(x\,\sqcap^R(y)\big) = \sqcap^R(x)\,\sqcap^R(y) \tag{2.13b}$$

hold true for all $x, y \in A$.

Proof. At first use the definition of $\sqcap^{L/R}$, then Eqs. (2.2a) and (2.2b), and finally Eqs. (2.7a) and (2.7b),

$$\begin{aligned} \sqcap^L\big(\sqcap^L(x)y\big) &= \varepsilon\big(1_{(1)}\,\sqcap^L(x)y\big)1_{(2)} \\ &= \varepsilon\big(1_{(1)}\,\sqcap^L(x)\,\sqcap^L(y)\big)1_{(2)} = \sqcap^L(x)\,\sqcap^L(y), \\ \sqcap^R\big(x\,\sqcap^R(y)\big) &= 1_{(1)}\varepsilon\big(x\,\sqcap^R(y)1_{(2)}\big) \\ &= 1_{(1)}\varepsilon\big(\sqcap^R(x)\,\sqcap^R(y)1_{(2)}\big) = \sqcap^R(x)\,\sqcap^R(y). \end{aligned}$$

Q.E.D.

Our next assertion about WBAs establishes a canonical isomorphism between the left (right) subalgebra of A and the right (left) subalgebra of $\hat{A}$. Since the existence of a common non-trivial subalgebra of A and $\hat{A}$ for Hopf algebras is by far not typical, this result is the first hint toward the fundamental role A^L and A^R play in the theory of WHAs.

To formulate the statement we introduce the Sweedler arrow notation

$$x \rightharpoonup \varphi := \varphi_{(1)}\langle \varphi_{(2)}, x\rangle, \qquad \varphi \leftharpoonup x := \langle \varphi_{(1)}, x\rangle \varphi_{(2)}. \tag{2.14}$$

Since A is the dual WBA of $\hat{A}$, the Sweedler arrows $\varphi \rightharpoonup x$ and $x \leftharpoonup \varphi$ are also defined.

LEMMA 2.6. *The map* κ_A^L: $x^L \mapsto (x^L \rightharpoonup \hat{1})$ *is an algebra isomorphism from* A^L *onto* $\hat{A}^R$. *The map* κ_A^R: $x^R \mapsto (\hat{1} \leftharpoonup x^R)$ *is an algebra isomorphism from* A^R *onto* $\hat{A}^L$. *Furthermore, the restriction of the canonical pairing to* $\hat{A}^L \times A^L$, $\hat{A}^R \times A^R$, $\hat{A}^L \times A^R$, *or to* $\hat{A}^R \times A^L$ *is non-degenerate*.

Proof. Using Eqs. (2.11a) and (2.11b) and the defining properties $\langle \varphi \leftharpoonup x, y \rangle = \langle \varphi, xy \rangle, \ldots$ etc. of the Sweedler arrows one can easily verify that

$$\begin{aligned}(x^L \rightharpoonup \hat{1})(y^L \rightharpoonup \hat{1}) &= \hat{1}_{(1)}\hat{1}_{(1')}\langle \hat{1}_{(2)}, x^L \rangle \langle \hat{1}_{(2')}, y^L \rangle \\ &= \hat{1}_{(1)}\langle \hat{1}_{(2)}, x^L \rangle \langle \hat{1}_{(3)}, y^L \rangle \\ &= x^L y^L \rightharpoonup \hat{1},\end{aligned} \tag{2.15}$$

$$\begin{aligned}(\hat{1} \leftharpoonup x^R)(\hat{1} \leftharpoonup y^R) &= \langle \hat{1}_{(1)}, x^R \rangle \langle \hat{1}_{(1')}, y^R \rangle \hat{1}_{(2)}\hat{1}_{(2')} \\ &= \langle \hat{1}_{(1)}, x^R \rangle \langle \hat{1}_{(2)}, y^R \rangle \hat{1}_{(3)} \\ &= \hat{1} \leftharpoonup x^R y^R,\end{aligned} \tag{2.16}$$

$$(\hat{1} \leftharpoonup x^R) \rightharpoonup 1 = 1_{(1)}\langle \hat{1} \leftharpoonup x^R, 1_{(2)} \rangle = 1_{(1)}\varepsilon\big(x^R 1_{(2)}\big) = x^R, \tag{2.17}$$

$$1 \leftharpoonup (x^L \rightharpoonup \hat{1}) = \langle 1_{(1)}, x^L \rightharpoonup \hat{1} \rangle 1_{(2)} = \varepsilon\big(1_{(1)} x^L\big) 1_{(2)} = x^L. \tag{2.18}$$

Thus κ_A^L (κ_A^R) is an algebra map with inverse $\kappa_{\hat{A}}^R(\kappa_{\hat{A}}^L)$. As for the non-degeneracy

$$\begin{aligned}&\langle \varphi^R, x^L \rangle = 0 \qquad \forall \varphi^R \\ &\qquad \Rightarrow \quad x^L = \langle \hat{1}, x^L 1_{(1)} \rangle 1_{(2)} = \langle 1_{(1)} \rightharpoonup \hat{1}, x^L \rangle 1_{(2)} = 0, \\ &\langle \varphi^L, x^L \rangle = 0 \qquad \forall \varphi^L \\ &\qquad \Rightarrow \quad x^L = \langle \hat{1}, 1_{(1)} x^L \rangle 1_{(2)} = \langle \hat{1} \leftharpoonup 1_{(1)}, x^L \rangle 1_{(2)} = 0,\end{aligned}$$

and the transpose of these prove the claim. Q.E.D.

If $\{b_i\}$ is a K-basis of A and $\{\beta^i\} \subset \hat{A}$ is its dual basis, $\langle \beta^i, b_j \rangle = \delta_{ij}$, then

$$\sum_i \sqcap^L(b_i) \otimes \beta^i = \sum_i b_i \otimes \hat{\sqcap}^L(\beta^i) = 1 \leftharpoonup \hat{1}_{(1)} \otimes \hat{1}_{(2)}, \tag{2.19a}$$

$$\sum_i \sqcap^R(b_i) \otimes \beta^i = \sum_i b_i \otimes \hat{\sqcap}^R(\beta^i) = 1_{(1)} \otimes 1_{(2)} \rightharpoonup \hat{1}. \tag{2.19b}$$

This can be easily seen by pairing both sides of any of these equations with $\varphi \otimes x$ and applying the definitions (2.1).

The four arrow identities of the next remark are frequently used in later computations.

Remark 2.7. Let A be a WBA. Then for all $\varphi \in \hat{A}$, $x^L \in A^L$, and $x^R \in A^R$,

$$x^L \rightharpoonup \varphi = (x^L \rightharpoonup \hat{1})\varphi, \tag{2.20a}$$

$$\varphi \leftharpoonup x^R = \varphi(\hat{1} \leftharpoonup x^R), \tag{2.20b}$$

$$\varphi \leftharpoonup x^L = (\hat{1} \leftharpoonup x^L)\varphi, \tag{2.21a}$$

$$x^R \rightharpoonup \varphi = \varphi(x^R \rightharpoonup \hat{1}). \tag{2.21b}$$

2.3. *Weak Hopf Algebras*

In this subsection we show how the existence of an antipode relates $\sqcap^L$, A^L with $\sqcap^R$, A^R and derive the expected properties of S that have been axioms in earlier formulations. The two most important results will be invertibility of the antipode and separability of the algebras A^L and A^R. Let us start with the question of uniqueness of the antipode.

LEMMA 2.8. *The unit, the counit, and the antipode, if they exist, are unique. I.e., if* $(A, \mu, u, \Delta, \varepsilon, S)$ *and* $(A, \mu, u', \Delta, \varepsilon', S')$ *are both weak Hopf algebras then* $u' = u$, $\varepsilon' = \varepsilon$, *and* $S' = S$.

Proof. The uniqueness of the unit and the counit are obvious. Therefore $\sqcap^L$ and $\sqcap^R$ are common in these two WHAs. To prove $S' = S$ introduce the convolution product

$$(f \diamond g)(x) := f(x_{(1)})g(x_{(2)}), \qquad x \in A, \tag{2.22}$$

on functions $f, g \in \mathrm{Hom}_K(A, A)$. This is an associative operation in terms of which the antipode axioms take the form

$$\mathrm{id} \diamond S = \sqcap^L, \qquad S \diamond \mathrm{id} \diamond S = S, \qquad S \diamond \mathrm{id} = \sqcap^R.$$

Now S' satisfies the same equations with the same $\sqcap^L$, $\sqcap^R$, therefore

$$\begin{aligned} S' &= S' \diamond \mathrm{id} \diamond S' = S' \diamond \sqcap^L = S' \diamond \mathrm{id} \diamond S \\ &= \sqcap^R \diamond S = S \diamond \mathrm{id} \diamond S = S. \end{aligned}$$

Q.E.D.

As a preparation for the theorem below notice that the definitions (2.1) have counterparts involving the antipode,

$$\sqcap^L(x) = \varepsilon\big(S(x)1_{(1)}\big)1_{(2)}, \tag{2.23a}$$

$$\sqcap^R(x) = 1_{(1)}\varepsilon\big(1_{(2)}S(x)\big). \tag{2.23b}$$

As a matter of fact

$$\begin{aligned}\sqcap^L(x) &= \varepsilon\Big(1_{(1)}\sqcap^L(x)\Big)1_{(2)} = \varepsilon\Big(\sqcap^L(x)1_{(1)}\Big)1_{(2)}\\ &= \varepsilon\big(x_{(1)}S(x_{(2)})1_{(1)}\big)1_{(2)}\\ &= \varepsilon\Big(\sqcap^R(x_{(1)})S(x_{(2)})1_{(1)}\Big)1_{(2)} = \varepsilon\big(S(x_{(1)})x_{(2)}S(x_{(3)})1_{(1)}\big)1_{(2)}\\ &= \varepsilon\big(S(x)1_{(1)}\big)1_{(2)},\end{aligned}$$

where in the subsequent equations (2.1), (2.10), (1.8a), (2.2b), (1.8b), and finally (1.9) have been used. Equation (2.23b) can be proven analogously. As the duals of (2.23a) and (2.23b) we have automatically the identities

$$\sqcap^L(x) = S\big(1_{(1)}\big)\varepsilon\big(1_{(2)}x\big), \tag{2.24a}$$

$$\sqcap^R(x) = \varepsilon\big(x1_{(1)}\big)S\big(1_{(2)}\big). \tag{2.24b}$$

LEMMA 2.9. *In a WHA A the following identities hold*

$$\sqcap^L\circ S = \sqcap^L\circ\sqcap^R = S\circ\sqcap^R, \tag{2.25a}$$

$$\sqcap^R\circ S = \sqcap^R\circ\sqcap^L = S\circ\sqcap^L. \tag{2.25b}$$

Proof. It is sufficient to prove the first equalities in (2.25a) and (2.25b) because the second ones then follow by duality. So

$$\begin{aligned}\sqcap^L\circ S(x) &= \varepsilon\big(1_{(1)}S(x)\big)1_{(2)} = \varepsilon\big(1_{(1)}S(x_{(1)})x_{(2)}S(x_{(3)})\big)1_{(2)}\\ &= \varepsilon\Big(1_{(1)}S(x_{(1)})\sqcap^L(x_{(2)})\Big)1_{(2)} = \varepsilon\big(1_{(1)}S(x_{(1)})x_{(2)}\big)1_{(2)}\\ &= \sqcap^L\circ\sqcap^R(x).\end{aligned}$$

In a similar way one can verify $\sqcap^R\circ S = \sqcap^R\circ\sqcap^L$. Q.E.D.

The above lemma implies that $S(A^R)\subset A^L$ and $S(A^L)\subset A^R$. On the other hand Eqs. (2.24a) and (2.24b) say that $A^L\subset S(A^R)$ and $A^R\subset S(A^L)$. Therefore the antipode maps A^L onto A^R bijectively and maps A^R onto A^L bijectively.

THEOREM 2.10. *Let A be a WHA. Then the antipode is antimultiplicative and anticomultiplicative,*

$$S(xy) = S(y)S(x) \qquad x, y \in A, \tag{2.26}$$

$$S(x)_{(1)} \otimes S(x)_{(2)} = S(x_{(2)}) \otimes S(x_{(1)}) \qquad x \in A, \tag{2.27}$$

and the restrictions $S|_{A^L}$ and $S|_{A^R}$ are bijections such that

$$S(A^L) = A^R, \qquad S(A^R) = A^L. \tag{2.28}$$

The unit and the counit are S-invariant,

$$S(1) = 1, \tag{2.29a}$$

$$\varepsilon \circ S = \varepsilon. \tag{2.29b}$$

Furthermore $S\colon A \to A$ is invertible.

Proof. We have already shown (2.28). Equation (2.27) is the dual of (2.26) and (2.29a) is the dual of (2.29b). Equation (2.26) follows from

$$\begin{aligned} S(xy) &= S(x_{(1)}y_{(1)})x_{(2)}y_{(2)}S(x_{(3)}y_{(3)}) = S(x_{(1)}y_{(1)}) \sqcap^L \left(x_{(2)} \sqcap^L (y_{(2)})\right) \\ &= S(x_{(1)}y_{(1)})x_{(2)} \sqcap^L (y_{(2)})S(x_{(3)}) \\ &= \sqcap^R\left(\sqcap^R(x_{(1)})y_{(1)}\right)S(y_{(2)})S(x_{(2)}) \\ &= S(y_{(1)}) \sqcap^R (x_{(1)})y_{(2)}S(y_{(3)})S(x_{(3)}) \\ &= S(y_{(1)})y_{(2)}S(y_{(3)})S(x_{(1)})x_{(2)}S(x_{(3)}) \\ &= S(y)S(x). \end{aligned}$$

Next we prove (2.29b). As a matter of fact

$$\begin{aligned} \varepsilon(S(x)) &= \varepsilon\left(S(x_{(1)})x_{(2)}S(x_{(3)})\right) = \varepsilon\left(S(x_{(1)}) \sqcap^L (x_{(2)})\right) \\ &= \varepsilon\left(S(x_{(1)})x_{(2)}\right) = \varepsilon\left(\sqcap^R(x)\right) \\ &= \varepsilon(x). \end{aligned}$$

To prove invertibility of S notice that the descending chain $A \supset S(A) \supset S^2(A) \supset \cdots$ of WHAs all contain 1 by (2.29a). This implies the existence of $n \in \mathbb{N}$ such that

$$1 \in S^{n+1}(A) = S^n(A) \subset S^{n-1}(A).$$

We want to show that this implies $S^n(A) = S^{n-1}(A)$. Replacing A by $S^{n-1}(A)$ it is therefore enough to prove invertibility of S under the additional assumption $S^2(A) = S(A)$, implying

$$\operatorname{Ker} S \cap S(A) = 0.$$

In this case let $\bar{S} := S|_{S(A)}$, then $\bar{S}\colon S(A) \to S(A)$ is bijective and

$$P_S := \bar{S}^{-1} \circ S\colon A \to S(A)$$

is a multiplicative idempotent satisfying

$$P_S(xS(y)) = P_S(x)S(y), \qquad x, y \in A.$$

By (2.28) $A^{L,R} \subset S(A)$. Now taking into account the identity $x = x_{(1)}S(x_{(2)})x_{(3)}$, which follows directly from Axioms (1.8a) and (1.4), then using also $P_S(1) = 1$ we have

$$\begin{aligned} P_S(x) &= P_S\big(x_{(1)}S(x_{(2)})x_{(3)}\big) = P_S(x_{(1)})S(x_{(2)})x_{(3)} = P_S\big(x_{(1)}S(x_{(2)})\big)x_{(3)} \\ &= P_S(1)x_{(1)}S(x_{(2)})x_{(3)} = x, \end{aligned}$$

so $\operatorname{Ker} P_S = \operatorname{Ker} S = 0$. Q.E.D.

We are now able to derive (versions of) the original antipode axioms of [2, 20],

$$x_{(1)} \otimes x_{(2)}S(x_{(3)}) = 1_{(1)}x \otimes 1_{(2)}, \tag{2.30a}$$

$$S(x_{(1)})x_{(2)} \otimes x_{(3)} = 1_{(1)} \otimes x1_{(2)}, \tag{2.30b}$$

$$x_{(1)} \otimes S(x_{(2)})x_{(3)} = x1_{(1)} \otimes S\big(1_{(2)}\big), \tag{2.30c}$$

$$x_{(1)}S(x_{(2)}) \otimes x_{(3)} = S\big(1_{(1)}\big) \otimes 1_{(2)}x. \tag{2.30d}$$

The first two are just rewritings of the bialgebra identities (2.8a) and (2.8b). The second two are more delicate. Namely

$$\begin{aligned} &x_{(1)} \otimes S(x_{(2)})x_{(3)} \\ &\quad = x_{(1)} \otimes \varepsilon\big(x_{(2)}1_{(1)}\big)S\big(1_{(2)}\big) = x_{(1)}1_{(1')} \otimes \varepsilon\big(x_{(2)}1_{(2')}1_{(1)}\big)S\big(1_{(2)}\big) \\ &\quad = x_{(1)}1_{(1)}\varepsilon\big(x_{(2)}1_{(2)}\big) \otimes S\big(1_{(3)}\big) = x1_{(1)} \otimes S\big(1_{(2)}\big), \\ &x_{(1)}S(x_{(2)}) \otimes x_{(3)} \\ &\quad = S\big(1_{(1)}\big)\varepsilon\big(1_{(2)}x_{(1)}\big) \otimes x_{(2)} = S\big(1_{(1)}\big)\varepsilon\big(1_{(2)}1_{(1')}x_{(1)}\big) \otimes 1_{(2')}x_{(2)} \\ &\quad = S\big(1_{(1)}\big) \otimes \varepsilon\big(1_{(2)}x_{(1)}\big)1_{(3)}x_{(2)} = S\big(1_{(1)}\big) \otimes 1_{(2)}x. \end{aligned}$$

The following proposition also holds, if A is just a WBA, see [14].

PROPOSITION 2.11. *Let A be a WHA over K. Then A^L and A^R are separable K-algebras, in particular, they are semisimple.*

Proof. Recall that an algebra A is separable if and only if there exists a $q \in A \otimes A$ such that $(x \otimes 1)q = q(1 \otimes x)$ holds for all $x \in A$ and furthermore $\mu(q) = 1$, where μ denotes the multiplication map of A [18]. Such a q will be called a *separating idempotent*.[2] So, our proof consists of showing that $q^L = S(1_{(1)}) \otimes 1_{(2)} \in A^L \otimes A^L$ and $q^R = 1_{(1)} \otimes S(1_{(2)}) \in A^R \otimes A^R$ are separating idempotents of A^L and A^R, respectively. In fact we prove the somewhat more general identities

$$x_{(1)} y^R \otimes x_{(2)} = x_{(1)} \otimes x_{(2)} S(y^R), \tag{2.31a}$$

$$x_{(1)} \otimes y^L x_{(2)} = S(y^L) x_{(1)} \otimes x_{(2)} \tag{2.31b}$$

valid for all $x \in A$ and $y^L \in A^L$, $y^R \in A^R$. Pairing the LHS of (2.31a) with $\varphi \otimes \psi$, we obtain

$$\begin{aligned}
&\langle \varphi \otimes \psi, \mathrm{LHS} \rangle \\
&\quad = \langle \varphi(y^R \rightharpoonup \hat{1}), x_{(1)} \rangle \langle \psi, x_{(2)} \rangle = \langle \varphi, x_{(1)} \rangle \langle (S(y^R) \rightharpoonup \hat{1}) \psi, x_{(2)} \rangle \\
&\quad = \langle \varphi \otimes \psi, \mathrm{RHS} \rangle.
\end{aligned}$$

The proof of (2.31b) is simply the mirror image of the above argument. Q.E.D.

2.4. *The "Trivial" Representation*

Since the counit of a WHA is in general not an algebra map, weak Hopf algebras may be lacking of any one-dimensional representation. Nevertheless the axioms ensure that any WHA A has a distinguished representation providing a unit object for the (relaxed) monoidal category of left A-modules. We shall discuss this category in detail in [3]. Now we concentrate only on the properties of this representation. We note that the trivial representation exists already in WBAs [14] and therefore the use of the antipode in this subsection is not obligatory.

Since the algebras $A^{L/R}$ occur on the right-hand side of Axioms (1.8a) and (1.8b) where in ordinary Hopf algebras the trivial representation stands, one expects that the "trivial representation" of WHAs must be a non-trivial representation acting on either $A^{L/R}$ or $\hat{A}^{L/R}$.

[2] In fact q is an idempotent only if considered as an element of $A \otimes A^{\mathrm{op}}$.

LEMMA 2.12. *The following left A-modules are isomorphic*:

$${}_A\hat{A}^R :: \text{the vector space } \hat{A}^R \text{ with action } x \cdot \varphi^R := x \rightharpoonup \varphi^R,$$

$${}_A\hat{A}^L :: \text{the vector space } \hat{A}^L \text{ with action } x \cdot \varphi^L := \varphi^L \leftharpoonup S(x),$$

$${}_A A^L :: \text{the vector space } A^L \text{ with action } x \cdot y^L := \sqcap^L(xy^L),$$

$${}_A A^R :: \text{the vector space } A^R \text{ with action } x \cdot y^R := \sqcap^R(y^R S(x)).$$

Proof. $\hat{S}$: $\hat{A}^L \to \hat{A}^R$ is an isomorphism of vector spaces and $\hat{S}(\varphi \leftharpoonup S(x)) = x \rightharpoonup \hat{S}(\varphi)$ is a general WHA identity. This proves the isomorphism of the first two A-modules. Similarly, S: $A^L \to A^R$ is an isomorphism of vector spaces and $S(\sqcap^L(xy)) = \sqcap^R(S(y)S(x))$ is a WHA identity. This proves the isomorphism of the last two A-modules.

To show the isomorphism of ${}_A\hat{A}^R$ with ${}_A A^L$ consider the bijection B: $\hat{A}^R \to A^L$, $B(\varphi^R) := 1 \leftharpoonup \varphi^R$. Then

$$\begin{aligned} B(x \rightharpoonup \varphi^R) &= 1 \leftharpoonup (x \rightharpoonup \varphi^R) = \langle 1_{(1)}x, \varphi^R \rangle 1_{(2)} \\ &= \langle 1_{(1)}(x \leftharpoonup \varphi^R), \hat{1} \rangle 1_{(2)} = \sqcap^L(x \leftharpoonup \varphi^R) \\ &= \sqcap^L(x(1 \leftharpoonup \varphi^R)) \\ &= \sqcap^L(xB(\varphi^R)), \end{aligned}$$

hence B is a left A-module map. Here, in the last-but-one equality we have used one of the four arrow identities of Remark 2.7. Q.E.D.

DEFINITION 2.13. By the trivial representation of the WHA A we mean the cyclic left A-module $V_\varepsilon := {}_A\hat{A}^R$ with A-action D_ε: $A \to \mathrm{End}_K \hat{A}^R$, $D_\varepsilon(x)\varphi := x \rightharpoonup \varphi$.

The third and fourth A-modules of the above lemma demonstrate that the restriction of the trivial representation to A^L (A^R) is equivalent to its left regular representation, hence faithful. This is one of the instances where $A^{L/R}$ appears in the role of a ground "field."

Later we will need the following strengthening of Lemma 2.6.

LEMMA 2.14. *Let A be a WHA and introduce the notation $Z^L := A^L \cap \mathrm{Center}\, A$, $Z^R := A^R \cap \mathrm{Center}\, A$, and $Z := A^L \cap A^R$. Then the isomorphism (of algebras) κ_A^L: $A^L \to \hat{A}^R$ restricts to an isomorphism $Z^L \to \hat{Z}$ and the isomorphism κ_A^R: $A^R \to \hat{A}^L$ restricts to the isomorphism $Z^R \to \hat{Z}$. Therefore*

$$Z^L \rightharpoonup \hat{1} = \hat{Z} = \hat{1} \leftharpoonup Z^R,$$

$$Z \rightharpoonup \hat{1} = \hat{Z}^R, \qquad \hat{Z}^L = \hat{1} \leftharpoonup Z.$$

The two isomorphisms have a common restriction to the hypercenter *Hypercenter* $A := Z^L \cap Z^R$ *and yields an isomorphism Hypercenter* $A \to$ *Hypercenter* $\hat{A}$.

Proof. Notice that for $c \in$ Center A one has $\hat{1} \leftharpoonup c = c \rightharpoonup \hat{1}$. Therefore $x^L \in Z^L \Rightarrow x^L \rightharpoonup \hat{1} = \hat{1} \leftharpoonup x^L \in \hat{Z}$. This proves $\kappa_A^L(Z^L) \subset \hat{Z}$.

If $z \in Z$ then $(z \rightharpoonup \hat{1})\varphi = z \rightharpoonup \varphi$ by (2.20a) and $z \rightharpoonup \varphi = \varphi(z \rightharpoonup \hat{1})$ by (2.21b). Hence $z \rightharpoonup \hat{1}$ is central. This proves $\kappa_A^L(Z) \subset \hat{Z}^R$.

Since $(\kappa_A^L)^{-1} = \kappa_{\hat{A}}^R$, the analogue inclusions $\kappa_A^R(Z^R) \subset \hat{Z}$ and $\kappa_A^R(Z) \subset \hat{Z}^L$ complete the proof. Q.E.D.

The unusual feature of the trivial representation of WHAs is that it can be decomposable. But this can occur only if the left and right subalgebras of the dual have non-trivial intersection as the next proposition claims.

PROPOSITION 2.15. *Let A be a WHA, let $(V_\varepsilon, D_\varepsilon)$ be its trivial representation as in Definition* 2.13. *Then*

$$\operatorname{End} V_\varepsilon = D_\varepsilon(Z^L) = D_\varepsilon(Z^R), \tag{2.32}$$

where $\operatorname{End} V_\varepsilon$ *denotes the algebra of A-module endomorphisms of V_ε.*

Proof. Let $T \in \operatorname{End} V_\varepsilon$ then $T(x \rightharpoonup \hat{1}) = x \rightharpoonup T(\hat{1})$, for $x \in A$, in particular

$$T(x^L \rightharpoonup \hat{1}) = x^L \rightharpoonup T(\hat{1}) = (x^L \rightharpoonup \hat{1})T(\hat{1}),$$
$$T(x^L \rightharpoonup \hat{1}) = T\big(S^{-1}(x^L) \rightharpoonup \hat{1}\big) = S^{-1}(x^L) \rightharpoonup T(\hat{1}) = T(\hat{1})(x^L \rightharpoonup \hat{1}),$$

where we have made use of Eqs. (2.20a) and (2.21b). Since by Lemma 2.6 $A^L \rightharpoonup \hat{1} = \hat{A}^R$, $\zeta := T(\hat{1}) \in$ Center $\hat{A}^R$, and $T(\varphi^R) = \varphi^R\zeta$. Thus $x \rightharpoonup \zeta = T(x \rightharpoonup \hat{1}) = (x \rightharpoonup \hat{1})\zeta$ holds for all $x \in A$. It follows that

$$\begin{aligned}\langle \hat{\sqcap}^L(\zeta), x\rangle \\ &= \langle \zeta, 1 \leftharpoonup (x \rightharpoonup \hat{1})\rangle = \langle (x \rightharpoonup \hat{1})\zeta, 1\rangle = \langle x \rightharpoonup \zeta, 1\rangle = \langle \zeta, x\rangle,\end{aligned}$$

i.e., $\zeta \in \hat{A}^L \cap \hat{A}^R \equiv \hat{Z}$. Now by Lemma 2.14 there exists a $z^L \in Z^L$ such that $\zeta = z^L \rightharpoonup \hat{1}$. We can conclude that

$$T(\varphi^R) = \zeta\varphi^R = (z^L \rightharpoonup \hat{1})\varphi^R = z^L \rightharpoonup \varphi^R = D_\varepsilon(z^L)\varphi^R,$$

i.e., $T = D_\varepsilon(z^L)$. This proves End $V_\varepsilon \subset D_\varepsilon(Z^L)$. The opposite inclusion is trivial since $D_\varepsilon(Z^L) \subset \operatorname{Center}(D_\varepsilon(A))$. This finishes the proof of End $V_\varepsilon = D_\varepsilon(Z^L)$.

Showing the other statement End $V_\varepsilon = D_\varepsilon(Z^R)$ one proceeds as above but chooses a $z^R \in Z^R$ such that $\zeta = \hat{1} \leftharpoonup z^R$. Then

$$T(\varphi^R) = \varphi^R(\hat{1} \leftharpoonup z^R) = \varphi^R(z^R \rightharpoonup \hat{1}) = z^R \rightharpoonup \varphi^R$$
$$= D_\varepsilon(z^R)\varphi^R$$

completes the proof. Q.E.D.

Notice that the above proposition does not imply that the trivial A-module is semisimple. It does imply, however, that V_ε has a decomposition $V_\varepsilon \cong \oplus_\nu V_\nu$ into indecomposable A-modules in which the indecomposables are disjoint, i.e., Hom $(V_\mu, V_\nu) = 0$ for all $\mu \neq \nu$.

DEFINITION 2.16. If $Z^L = K1$, or equivalently, if the trivial representation is indecomposable then the WHA is called pure.

The name "pure" comes from the C^*-setting when the trivial representation arises from the positive linear functional ε by the GNS construction. Thus A is pure iff ε is pure.

Nota bene pureness is not a self-dual notion, duals of pure WHAs may not be pure. Clearly, A is pure iff $Z^L \cong Z^R$ is trivial but $\hat{A}$ is pure iff Z is trivial.

3. WEAK HOPF MODULES AND INTEGRAL THEORY

As in Hopf algebras so in weak Hopf algebras the integrals play a decisive role in the structure analysis of these algebras. Using integrals we can formulate conditions for the algebra to be Frobenius, symmetric, or semisimple, and study questions related to innerness of S^2 or S^4. Furthermore we will be able to characterize those WHAs that have Haar measures. In deriving the basic properties of integrals the weak generalization of the fundamental theorem of Hopf modules is very useful. Unfortunately, it seems to be less powerful than in Hopf algebra theory (cf. [13]) where it implies the existence of non-degenerate integrals. It is an open problem yet whether all WHAs are Frobenius algebras. We can prove, however, that all of them are quasi-Frobenius algebras.

3.1. *Integrals in Weak Hopf Algebras*

The following definition provides the weak Hopf generalization of the well-known notion of integrals in a Hopf algebra [19].

DEFINITION 3.1. A left (right) integral in a weak Hopf algebra A is an element $l \in A$ ($r \in A$) satisfying

$$xl = \sqcap^L(x)l \qquad \left(rx = r\sqcap^R(x)\right) \tag{3.1}$$

for all $x \in A$. The space of left (right) integrals in A is denoted by $\mathscr{I}^L(A)(\mathscr{I}^R(A))$. Elements of $\mathscr{I} := \mathscr{I}^L(A) \cap \mathscr{I}^R(A)$ are called two-sided integrals. A left or right integral in A is called non-degenerate if it defines a non-degenerate functional on $\hat{A}$. $l \in \mathscr{I}^L(A)$ is called normalized if $\sqcap^L(l) = 1$, $r \in \mathscr{I}^R(A)$ is called normalized if $\sqcap^R(r) = 1$.

Some equivalent formulations of left (right) integrals are gathered in

LEMMA 3.2. *Let A be a weak Hopf algebra. Then the following statements for an element $l \in A$ are equivalent*:

(a) $l \in \mathscr{I}^L(A)$

(b) $l_{(1)} \otimes xl_{(2)} = S(x)l_{(1)} \otimes l_{(2)}$ *for all* $x \in A$

(c) $l \rightharpoonup \hat{A} \subset \hat{A}^L$

(d) $(\varphi \leftharpoonup x) \rightharpoonup l = S(x)(\varphi \rightharpoonup l)$ *for all* $\varphi \in \hat{A}$ *and* $x \in A$

(e) $(\mathrm{Ker}\, \sqcap^L)l = 0$

(f) $S(l) \in \mathscr{I}^R(A)$

Proof. (a) $\Rightarrow$ (b): Using (2.30b) and (2.7a) we have $l_{(1)} \otimes xl_{(2)} = [S(x_{(1)}) \otimes 1]\Delta(x_{(2)}l) = [S(x_{(1)})x_{(2)}S(x_{(3)}) \otimes 1]\Delta(l) = S(x)l_{(1)} \otimes l_{(2)}$. (b) $\Rightarrow$ (a): $xl = x_{(1)}l_{(1)}\varepsilon(x_{(2)}l_{(2)}) = x_{(1)}S(x_{(2)})l_{(1)}\varepsilon(l_{(2)}) = \sqcap^L(x)l$. (a) $\Leftrightarrow$ (c): For an $l \in A$ the equation $\langle l \rightharpoonup \varphi, x\rangle = \langle \hat{\sqcap}^L(l \rightharpoonup \varphi), x\rangle$ is clearly equivalent to the equation $\langle \varphi, xl\rangle = \langle \varphi, \sqcap^L(x)l\rangle$. (b) $\Leftrightarrow$ (d): By pairing the second tensor factor of (b) with an arbitrary $\varphi \in \hat{A}$. (a) $\Rightarrow$ (e): is obvious. (e) $\Rightarrow$ (a): $xl = [x - \sqcap^L(x)]l + \sqcap^L(x)l = \sqcap^L(x)l$. (f) $\Leftrightarrow$ (a): This follows by applying S to (3.1). Q.E.D.

Definition 3.1 as well as Lemma 3.2 provide rather technical characterizations of integrals. The next argument sheds some light on their real nature. Consider the left A-module map ε_R from the left regular A-module to the trivial A-module given by acting with the trivial representation on the cyclic vector $\hat{1}$,

$$\begin{aligned} \varepsilon_R : {}_A A &\to {}_A\hat{A}^R, \\ x &\mapsto (x \rightharpoonup \hat{1}). \end{aligned} \tag{3.2}$$

The existence of this (non-zero) map shows that $\mathrm{Hom}({}_A A, {}_A\hat{A}^R)$ is non-zero. However, there is in general no guarantee that $\mathrm{Hom}({}_A\hat{A}^R, {}_A A)$ is non-zero.

Left integrals are precisely the objects that label the possible homomorphisms of the latter type.

LEMMA 3.3. *Left integrals l in A are in one-to-one correspondence with left A-module homomorphisms $f\colon {}_A\hat{A}^R \to {}_A A$. The correspondence is given by $f \mapsto f(\hat{1}) \in \mathscr{I}^L$. What is more the above map provides an isomorphism $\mathscr{I}^L_A \cong \mathrm{Hom}({}_A\hat{A}^R, {}_A A)$ of right A-modules. In other words $\mathscr{I}^L_A$ is isomorphic to the A-dual of the trivial left A-module.*

Proof. If $f \in \mathrm{Hom}({}_A\hat{A}^R, {}_A A)$ then $xf(\hat{1}) = f(x \rightharpoonup \hat{1}) = f(\sqcap^L(x) \rightharpoonup \hat{1}) = \sqcap^L(x) f(\hat{1})$, hence $f(\hat{1}) \in \mathscr{I}^L$. This is obviously a right A-module map. It is invertible since for $l \in \mathscr{I}^L$ the map $f_l\colon \hat{A}^R \to A$, $f_l(\varphi^R) := (1 \leftharpoonup \varphi^R)l$ is a left A-module map and satisfies $f_l(\hat{1}) = l$. Q.E.D.

The identification of $\mathscr{I}^L$ with $\mathrm{Hom}({}_A\hat{A}^R, {}_A A)$ yields an A-valued bilinear form ${}_A\hat{A}^R \times \mathscr{I}^L_A \to A$ given by evaluation, $(\varphi^R, l) \mapsto f_l(\varphi^R)$. Replacing ${}_A\hat{A}^R$ with ${}_A A^L$ using the isomorphism of Lemma 2.12 we obtain that this bilinear form is nothing but multiplication in A,

$$ {}_A A^L \times \mathscr{I}^L_A \to {}_A A_A, \qquad (x^L, l) \mapsto x^L l, \tag{3.3} $$

and it is an A–A bimodule map. We claim that (3.3) is a non-degenerate bilinear form. From one side, $x^L l = 0$, $\forall x^L \in A^L \;\Rightarrow\; l = 0$, this is trivial. From the other side we will be able to prove this after having established that WHAs are quasi-Frobenius algebras in Theorem 3.11. As a matter of fact by Theorem 61.2 of [4] the left annihilator of the right annihilator of the left ideal $\mathrm{Ker}\,\sqcap^L$ is $\mathrm{Ker}\,\sqcap^L$ itself. Now by Lemma 3.2(e) the right annihilator of $\mathrm{Ker}\,\sqcap^L$ is just $\mathscr{I}^L$. Thus $x^L l = 0$, $\forall l \in \mathscr{I}^L \;\Rightarrow\; x^L = 0$ follows.

Now we turn to another characterization of left integrals that is related to conditional expectations. Notice at first that if $\lambda \in \mathscr{I}^L(\hat{A})$ then the map $E_\lambda\colon x \mapsto \lambda \rightharpoonup x$ is an A^L–A^L-bimodule map from A into A^L commuting with the right $\hat{A}$-action on A. In fact, all such maps arise from a left integral, as the following lemma shows.

LEMMA 3.4. *The left integrals $\lambda \in \mathscr{I}^L(\hat{A})$ are in one-to-one correspondence with right $\hat{A}$-module maps $E \in \mathrm{Hom}(A_{\hat{A}}, A^L_{\hat{A}})$ via*

$$ \lambda \mapsto E_\lambda, $$
$$ E \mapsto \varepsilon \circ E. $$

Proof. If λ is a left integral then E_λ is a right $\hat{A}$-module map and maps into A^L by Lemma 3.2(c).

Now let $E \in \mathrm{Hom}(A_{\hat{A}}, A^L_{\hat{A}})$. Then

$$E(x) = \varepsilon\big(1_{(1)}E(x)\big)1_{(2)} = \varepsilon \circ E\big(S^{-1}(1_{(1)})x\big)1_{(2)},$$

where we used the fact that a right $\hat{A}$-module map is an A^L–A^L-bimodule map by (2.20b) and (2.21a). Hence

$$E(x) = \varepsilon \circ E(x_{(3)})x_{(1)}S(x_{(2)}) = \sqcap^L(\lambda \rightharpoonup x),$$

where $\lambda := \varepsilon \circ E$. It remains to show that λ is a left integral. Then

$$\langle \varphi\lambda, x\rangle = \varepsilon(E(x \leftharpoonup \varphi)) = \varepsilon(E(x) \leftharpoonup \varphi) = \langle \hat{\sqcap}^L(\varphi)\lambda, x\rangle,$$

which proves the claim. Q.E.D.

The characterization of left integrals λ as "conditional expectations" E_λ provides a link to the theory of inclusions and "Jones extensions" [15].

The properties of the normalized and the non-degenerate left integrals are discussed in later subsections. Here we only note that λ is non-degenerate iff E_λ is non-degenerate and λ is normalized iff E_λ is unital.

There are two twisting operations $A \mapsto A^{\mathrm{op}}$ and $A \mapsto A_{\mathrm{cop}}$ that produce WHAs from WHAs. In the first one the multiplication μ is replaced with opposite multiplication $\mu^{\mathrm{op}}(x, y) = \mu(y, x)$ while in the second the coproduct is replaced by $\Delta^{\mathrm{op}}(x) = x_{(2)} \otimes x_{(1)}$. In both cases the antipode is replaced by S^{-1}. The left and right subalgebras−integrals and the dual WHAs of the resulting four twisted versions of a WHA A are related to those of A as in Table I. As an application of Table I we give here the twisted versions of the identity of Lemma 3.2(d),

$$(\varphi \leftharpoonup x) \rightharpoonup l = S(x)(\varphi \rightharpoonup l), \tag{3.4a}$$

$$(x \rightharpoonup \varphi) \rightharpoonup r = (\varphi \rightharpoonup r)S^{-1}(x), \tag{3.4b}$$

$$l \leftharpoonup (\varphi \leftharpoonup x) = S^{-1}(x)(l \leftharpoonup \varphi), \tag{3.4c}$$

$$r \leftharpoonup (x \rightharpoonup \varphi) = (r \leftharpoonup \varphi)S(x) \tag{3.4d}$$

for all $x \in A$, $\varphi \in \hat{A}$, $l \in \mathscr{I}^L$, and $r \in \mathscr{I}^R$.

TABLE I

	$\sqcap^L$	$\sqcap^R$	A^L	A^R	$\mathscr{I}^L$	$\mathscr{I}^R$	$\hat{A}$
$A = A(\mu, \Delta, S)$	$\sqcap^L$	$\sqcap^R$	A^L	A^R	$\mathscr{I}^L$	$\mathscr{I}^R$	$\hat{A}$
$A^{\mathrm{op}} = A(\mu^{\mathrm{op}}, \Delta, S^{-1})$	$S^{-1} \circ \sqcap^R$	$S^{-1} \circ \sqcap^L$	A^L	A^R	$\mathscr{I}^R$	$\mathscr{I}^L$	$\hat{A}_{\mathrm{cop}}$
$A_{\mathrm{cop}} = A(\mu, \Delta^{\mathrm{op}}, S^{-1})$	$S^{-1} \circ \sqcap^L$	$S^{-1} \circ \sqcap^R$	A^R	A^L	$\mathscr{I}^L$	$\mathscr{I}^R$	$\hat{A}^{\mathrm{op}}$
$A^{\mathrm{op}}_{\mathrm{cop}} = A(\mu^{\mathrm{op}}, \Delta^{\mathrm{op}}, S)$	$\sqcap^R$	$\sqcap^L$	A^R	A^L	$\mathscr{I}^R$	$\mathscr{I}^L$	$\hat{A}^{\mathrm{op}}_{\mathrm{cop}}$

3.2. *Weak Hopf Modules*

Let A be a WHA. Recall that a *left A-module* is a K-linear space M carrying a left action of the algebra A, denoted by $x \in A$, $m \in M \mapsto x \cdot m$. A *right A-module* is a left module M of the opposite algebra A^{op} with action denoted by $x \in A$, $m \in M \mapsto m \cdot x$. Since A is unital, all modules are assumed to be non-degenerate; i.e., 1 acts as the identity. The left A-module M is called *faithful* if $x \cdot m = 0$, $\forall m \in M$ implies $x = 0$.

The A-modules know nothing about the coalgebra structure of A. The *left A-comodules M* in turn are the comodules of the coalgebra A and carry no information about the algebra structure of A. The left coaction is denoted by $m \mapsto m_{-1} \otimes m_0 \in A \otimes M$. One defines the *right A-comodules* analogously and denotes the coaction as $m \mapsto m_0 \otimes m_1 \in M \otimes A$.

Because of the finite dimensionality of A there is a one-to-one correspondence between left (right) A-coactions on M and right (left) $\hat{A}$-actions on M given by

$$m \cdot \varphi = \langle \varphi, m_{-1} \rangle m_0, \qquad m_{-1} \otimes m_0 = \sum_i b_i \otimes m \cdot \beta^i, \tag{3.5}$$

$$\varphi \cdot m = m_0 \langle \varphi, m_1 \rangle, \qquad m_0 \otimes m_1 = \sum_i \beta^i \cdot m \otimes b_i. \tag{3.6}$$

Here $\{b_i\}$ denotes an arbitrary basis of A and $\{\beta^i\}$ is its dual basis: $\langle \beta^i, b_j \rangle = \delta_{ij}$. There are eight basic examples of A modules with the target space M being either A or its dual $\hat{A}$. These are

$$\begin{aligned} &{}_A A :: x \cdot y = xy, && A_A :: y \cdot x = yx, \\ &{}^A A :: x \cdot y = yS(x), && A^A :: y \cdot x = S(x)y, \\ &{}_A \hat{A} :: x \cdot \varphi = x \rightharpoonup \varphi, && \hat{A}_A :: \varphi \cdot x = \varphi \leftharpoonup x, \\ &{}^A \hat{A} :: x \cdot \varphi = \varphi \leftharpoonup S(x), && \hat{A}^A :: \varphi \cdot x = S(x) \rightharpoonup \varphi, \end{aligned}$$

where the Sweedler arrow notation (2.14) has been used. They all are faithful and non-degenerate due to the existence of a unit and a counit. To each of the A-modules in the above list there is a corresponding $\hat{A}$-comodule denoted by the same symbol. This identification is justified also by the fact that $N \subset M$ is an A-submodule if and only if it is an $\hat{A}$-subcomodule.

By analogy with our definition of left integrals, the space of *invariants* of a left A-module M is defined to be the subspace

$$\operatorname{Inv} M := \left\{ m \in M \,|\, x \cdot m = \sqcap^L(x) \cdot m, \forall x \in A \right\}. \tag{3.7}$$

By the same methods as in Lemma 3.3, $\operatorname{Inv} M$ is linearly isomorphic to $\operatorname{Hom}({}_A\hat{A}^R, {}_AM)$ via

$$\operatorname{Inv} M = \left\{ f(\hat{1}) \,|\, f \in \operatorname{Hom}\left({}_A\hat{A}^R, {}_AM\right) \right\}. \tag{3.8}$$

By duality, we define the *coinvariants* of a right A-comodule M as

$$\operatorname{Coinv} M := \left\{ m \in M \,|\, m_0 \otimes m_1 = m_0 \otimes \sqcap^L(m_1) \right\}. \tag{3.9}$$

Thus, $m \in \operatorname{Coinv} M \ \Leftrightarrow\ m_0 \otimes m_1 \in M \otimes A^L$ and for a left A-module M, the invariants $\operatorname{Inv} M \subset M$ coincide with $\operatorname{Coinv} M \subset M$ considered as an $\hat{A}$-comodule. Similarly, for a right A-module (left A-comodule) M the invariants (coinvariants) are

$$\operatorname{Inv} M = \left\{ m \in M \,|\, m \cdot x = m \cdot \sqcap^R(x), \forall x \in A \right\}, \tag{3.10}$$

$$\operatorname{Coinv} M = \left\{ m \in M \,|\, m_{-1} \otimes m_0 = \sqcap^R(m_{-1}) \otimes m_0 \right\}. \tag{3.11}$$

Notice that the (co)invariants do not form a sub(co)module, not even an $A^{L/R}$-submodule.

Remark 3.5. The invariants of the left (right) regular A-module are precisely the left (right) integrals of A:

$$\operatorname{Inv}{}_AA = \mathscr{I}^L(A), \qquad \operatorname{Inv} A_A = \mathscr{I}^R(A).$$

The invariants of ${}_A\hat{A}$ and $\hat{A}_A$, on the other hand, yield the left and right subalgebras, respectively:

$$\operatorname{Inv}{}_A\hat{A} = \hat{A}^L, \qquad \operatorname{Inv} \hat{A}_A = \hat{A}^R.$$

Investigating the structure of the mixed modules ${}_{\hat{A}}\hat{A}^A$ and ${}^A\hat{A}_{\hat{A}}$, that incorporates the whole bialgebra structure of A, one arrives at a weak generalization of the notion of Hopf modules [1, 19].

DEFINITION 3.6. A right weak Hopf module (right WHM) over A is a right A module M which is also a right A-comodule such that the compatibility relation

$$(m \cdot x)_0 \otimes (m \cdot x)_1 = m_0 \cdot x_{(1)} \otimes m_1 x_{(2)} \tag{3.12}$$

holds for $x \in A$, $m \in M$.

LEMMA 3.7. *Let M be a right WHM over A. Then for all $m \in M$*

(i) $m_0 \cdot \sqcap^R(m_1) = m$.

(ii) $\mathrm{Coinv}\, M = \{m \in M \,|\, m_0 \otimes m_1 = m \cdot 1_{(1)} \otimes 1_{(2)}\}$ *and* $\mathrm{Coinv}\, M$ *is a right* A^L *submodule*.

(iii) $E(m) := m_0 \cdot S(m_1)$ *provides a projection* $E\colon M \to \mathrm{Coinv}\, M$.

Proof. (i) Let M be a right WHM over A. Since

$$m_0 \varepsilon(m_1 x) = m_0 \cdot 1_{(1)} \varepsilon(m_1 1_{(2)} x) = m_0 \cdot 1_{(1)} \varepsilon(m_1 1_{(2)}) \varepsilon(1_{(3)} x)$$
$$= m \cdot 1_{(1)} \varepsilon(1_{(2)} x) \tag{3.13}$$

for all $x \in A$, we have $m_0 \otimes \hat{1} \leftharpoonup m_1 = m \cdot 1_{(1)} \otimes \hat{1} \leftharpoonup 1_{(2)}$, so that

$$m_0 \cdot \sqcap^R(m_1) = m_0 \cdot \left[\left(\hat{1} \leftharpoonup m_1\right) \rightharpoonup 1\right] = m \cdot 1_{(1)}\left[\left(\hat{1} \leftharpoonup 1_{(2)}\right) \rightharpoonup 1\right]$$
$$= m \cdot 1_{(1)} \sqcap^R \left(1_{(2)}\right) = m. \tag{3.14}$$

(ii) The inclusion $\supset$ follows from (2.4). Conversely, if $m \in \mathrm{Coinv}\, M$ then $m_0 \otimes m_1 \in M \otimes A^L$, implying by (2.3a), (2.10), and (3.13):

$$m_0 \otimes m_1$$
$$= m_0 \otimes \varepsilon(m_1 1_{(1)}) 1_{(2)} = m \cdot 1_{(1)} \otimes \varepsilon(1_{(2)} 1_{(1')}) 1_{(2')} = m \cdot 1_{(1)} \otimes 1_{(2)}.$$

Also $\Delta(A^L) \subset A \otimes A^L$ and therefore $\mathrm{Coinv}\, M$ is a right A^L-submodule.

(iii) To check that $n := m_0 \cdot S(m_1)$ is a coinvariant for all $m \in M$ we compute

$$n_0 \otimes n_1 = (m_0 \otimes m_1)\left(\cdot S(m_2)_{(1)} \otimes S(m_2)_{(2)}\right)$$
$$= m_0 \cdot S(m_3) \otimes m_1 S(m_2) = m_0 \cdot S\left(1_{(2)} m_1\right) \otimes S\left(1_{(1)}\right)$$
$$= n \cdot 1_{(1)} \otimes 1_{(2)}.$$

Since for $n \in \mathrm{Coinv}\, M$ we have $n_0 \cdot S(n_1) = n \cdot 1_{(1)} S(1_{(2)}) = n$, E is a projection onto $\mathrm{Coinv}\, M$. Q.E.D.

EXAMPLE 3.8. The right weak Hopf module ${}_{\hat{A}}\hat{A}^A$.

As a linear space the module is the dual WHA $\hat{A}$. The right action and coaction are

$$\varphi \cdot x := S(x) \rightharpoonup \varphi, \qquad \varphi_0 \otimes \varphi_1 := \sum_i \beta^i \varphi \otimes b_i. \tag{3.15}$$

Clearly, the right A-coaction is dual to the left $\hat{A}$-multiplication and therefore counital and right-coassociative. The compatibility condition (3.12) can be seen as

$$
\begin{aligned}
(\varphi \cdot x)_0 \otimes (\varphi \cdot x)_1 &= \sum_i \beta^i (S(x) \rightharpoonup \varphi) \otimes b_i \\
&= \sum_i \beta^i \big[S(x_{(1)}) \rightharpoonup x_{(2)} S(x_{(3)}) \rightharpoonup \varphi \big] \otimes b_i \\
&= \sum_i \beta^i \big[S(x_{(1)}) \rightharpoonup \big(x_{(2)} \rightharpoonup \hat{1} \big) \varphi \big] \otimes b_i \\
&= \sum_i \beta^i \big[S(x_{(2)}) x_{(3)} \rightharpoonup \hat{1} \big] \big[S(x_{(1)}) \rightharpoonup \varphi \big] \otimes b_i \\
&= \sum_i \big[S(x_{(2)}) x_{(3)} \rightharpoonup \beta^i \big] \big[S(x_{(1)}) \rightharpoonup \varphi \big] \otimes b_i \\
&= \sum_i S(x_{(1)}) \rightharpoonup \big[\big(x_{(2)} \rightharpoonup \beta^i \big) \varphi \big] \otimes b_i \\
&= \sum_i S(x_{(1)}) \rightharpoonup (\beta^i \varphi) \otimes b_i x_{(2)} \\
&= \varphi_0 \cdot x_{(1)} \otimes \varphi_1 x_{(2)}.
\end{aligned}
$$

The A-coinvariants of this WHM coincide with the $\hat{A}$-invariants of the dual left regular $\hat{A}$-module ${}_{\hat{A}}\hat{A}$ and therefore with the space of left integrals in $\hat{A}$ by Remark 3.5,

$$\operatorname{Coinv}\big({}_{\hat{A}}\hat{A}^A \big) = \mathscr{I}^L(\hat{A}). \tag{3.16}$$

The fundamental theorem of Hopf modules generalizes to the weak case as

THEOREM 3.9. *Let A be a WHA, M be a right WHM over A, and let $N =$ Coinv M denote the set of coinvariants of M. Since N is a right A^L submodule, one can form the A^L-module tensor product $N \otimes_{A^L} A$ and make it into a right WHM by the definitions*

$$(n \otimes a) \cdot x := n \otimes ax, \tag{3.17a}$$

$$(n \otimes a)_0 \otimes (n \otimes a)_1 := \big(n \otimes a_{(1)} \big) \otimes a_{(2)}, \tag{3.17b}$$

where $a, x \in A$, $n \in N$. Then the map

$$\alpha: N \otimes_{A^L} A \to M, \qquad n \otimes x \mapsto n \cdot x \tag{3.18}$$

is an isomorphism of right WHMs.

Recall that an *isomorphism of WHMs* is just a module isomorphism which is a comodule isomorphism at the same time.

Proof. That α is a module map and comodule map is easy to verify. To construct the inverse define

$$\begin{aligned} &\beta: M \to N \otimes_{A^L} A, \\ &\beta(m) = m_0 \cdot S(m_1) \otimes m_2 \equiv E(m_0) \otimes m_1. \end{aligned} \tag{3.19}$$

Then β is obviously a comodule map. We show that it is also a module map. Then

$$\begin{aligned} \beta(m \cdot x) &= m_0 \cdot x_{(1)} S(m_1 x_{(2)}) \otimes m_2 x_{(3)} \\ &= m_0 \cdot \sqcap^L(x_{(1)}) S(m_1) \otimes m_2 x_{(2)} \\ &= m_0 \cdot S(m_1 1_{(1)}) \otimes m_2 1_{(2)} x \\ &= \beta(m) \cdot x. \end{aligned}$$

We are left with showing that on the one hand

$$\begin{aligned} \beta \circ \alpha(n \otimes x) &= \beta(n \cdot x) = \beta(n) \cdot x = n \cdot 1_{(1)} S(1_{(2)}) \otimes 1_{(3)} x \\ &= n \otimes 1_{(1)} S(1_{(2)}) 1_{(3)} x \\ &= n \otimes x, \end{aligned}$$

and on the other hand

$$\alpha \circ \beta(m) = m_0 \cdot S(m_1) m_2 = m,$$

where in the last equality Lemma 3.7(i) has been used. Q.E.D.

Applying this theorem to the WHM of Example 3.8 we obtain the right WHM isomorphism

$${}_{\hat{A}}\hat{A}^A \cong \mathscr{I}^L(\hat{A}) \otimes_{A^L} {}_{\hat{A}}A_A. \tag{3.20}$$

COROLLARY 3.10. *In any WHA A the space of left integrals $\mathscr{I}^L(A) = S(\mathscr{I}^R(A))$ is non-zero and $\mathscr{I}^L(\hat{A})$ is the dual of $\mathscr{I}^R(A)$ with respect to the restriction of the canonical pairing. Moreover, choosing a basis $\{\lambda^a\}$ in $\mathscr{I}^L(\hat{A})$*

and taking its dual basis $\{r_a\}$ *in* $\mathscr{I}^R(A)$, *we have*

$$\hat{1} = \sum_a S(r_a) \rightharpoonup \lambda_a, \tag{3.21}$$

$$1 = \sum_a r_a \leftharpoonup \hat{S}(\lambda_a). \tag{3.22}$$

Proof. $\mathscr{I}^L(\hat{A}) \neq 0$ follows from (3.20). By inspecting the form of the projection $E: M \to N$ in Example 3.8 we get a projection $L: A \to \mathscr{I}^L(A)$ onto the left integrals,

$$L(x) = \sum_i \hat{S}^2(\beta^i) \rightharpoonup (b_i x). \tag{3.23}$$

Therefore the projection to the right integrals is

$$R(x) = S \circ L \circ S^{-1}(x) = \sum_i (xb_i) \leftharpoonup \hat{S}^2(\beta^i). \tag{3.24}$$

Similar expressions define the projections $\hat{L}$ and $\hat{R}$ to the dual integrals. Now it is easy to check that

$$\langle \hat{L}(\varphi), x\rangle = \langle \varphi, R(x)\rangle \tag{3.25}$$

proving the non-degeneracy of the restriction of the canonical pairing to $\mathscr{I}^L(\hat{A}) \times \mathscr{I}^R(A)$.

The dual bases satisfy $\langle \lambda^a, r_b\rangle = \delta_{ab}$ therefore

$$\begin{aligned} \langle \lambda^a, xS(r_a)\rangle &= \langle \hat{L}(\beta^i), xS(b_i)\rangle = \langle S^2(b_j) \rightharpoonup \beta^j\beta^i, xS(b_i)\rangle \\ &= \langle \beta^k, xS \circ \sqcap^R(b_k)\rangle \qquad (3.26) \\ &= \langle 1_{(2)} \rightharpoonup \hat{1}, xS(1_{(1)})\rangle, \qquad (3.27) \end{aligned}$$

where in the last step we used (2.19b). This proves (3.21). Equation (3.22) is the twisted version in $A^{\mathrm{op}}_{\mathrm{cop}}$. Q.E.D.

3.3. *Restrictions on the Algebraic Structure*

The existence of a weak Hopf structure on the K-algebra A involves certain restrictions on the algebra A, just like in case of Hopf algebras. In this subsection we show that any WHA A is quasi-Frobenius, i.e., self-injective. The notions of semisimple and separable algebras coincide within the class of WHAs. Moreover, we prove an analogue of Maschke's theorem which claims that A is semisimple if and only if it has normalized left integrals.

THEOREM 3.11. *Every weak Hopf algebra over a field K is a quasi-Frobenius algebra.*

Proof. By Theorem 61.2 of [4] it is sufficient to prove that the left regular A-module ${}_A A$ is injective. By the Nagao–Nakayama theorem injectivity of a left A-module is equivalent to that it is a direct sum of K-duals of principal indecomposable right A-modules. Since $\hat{A}^A$ is the K-dual of ${}_A A$, we need to show that $\hat{A}^A$ is a direct sum of principal indecomposable right A-modules, i.e., that $\hat{A}^A$ is projective. This in turn is a consequence of the fundamental theorem of WHMs.

As a matter of fact we have the right A-module isomorphisms

$$\hat{A}^A \cong \mathscr{I}^L(\hat{A}) \otimes_{A^L} A_A \cong P\big(\mathscr{I}^L(\hat{A}) \otimes_K A_A\big), \tag{3.28}$$

the first of which is the consequence of the fundamental theorem of the right WHM ${}_{\hat{A}}\hat{A}^A$, the second of which is a rather simple property of the amalgamated tensor product with respect to the separable algebra A^L. To explain the projection P here we make a digression.

LEMMA 3.12. *Define the map* $P\colon \mathscr{I}^L(\hat{A}) \otimes_K A \to \mathscr{I}^L(\hat{A}) \otimes_K A$ *by*

$$P(\lambda \otimes x) := S^2\big(1_{(1)}\big) \rightharpoonup \lambda \otimes 1_{(2)}x. \tag{3.29}$$

Then $P \circ P = P$ *and* $\operatorname{Ker} P$ *coincides with* $\operatorname{Ker} \pi$ *of the canonical projection* π *from the free right A module* $\mathscr{I}^L(\hat{A}) \otimes_K A_A$ *onto* $\mathscr{I}^L(\hat{A}) \otimes_{A^L} A_A$. *Therefore*

$$P\big(\mathscr{I}^L(\hat{A}) \otimes_K A_A\big) \overset{\pi|_{\operatorname{Im} P}}{\to} \mathscr{I}^L(\hat{A}) \otimes_{A^L} A_A \tag{3.30}$$

is an isomorphism of right A-modules.

Proof. The kernel of the canonical projection is

$$\operatorname{Ker} \pi = \operatorname{Span}_K\big\{\big(\lambda \otimes x^L y\big) - \big(S(x^L) \rightharpoonup \lambda \otimes x\big) \,\big|\, \lambda \otimes y \in \mathscr{I}^L(\hat{A}) \otimes A,\ x^L \in A^L\big\}. \tag{3.31}$$

If $\sum_i(\lambda_i \otimes x_i) \in \operatorname{Ker} \pi$ then obviously $\sum_i S^2(1_{(1)}) \rightharpoonup \lambda_i \otimes 1_{(2)}x_i = 0$, therefore $\operatorname{Ker} \pi \subset \operatorname{Ker} P$. Now assume $\sum \lambda_i \otimes x_i \in \operatorname{Ker} P$. Then

$$\begin{aligned}\sum_i \lambda_i \otimes x_i &= \sum_i \lambda_i \otimes S\big(1_{(1)}\big)1_{(2)}x_i \\ &= \sum_i \big[\lambda_i \otimes S\big(1_{(1)}\big)\big(1_{(2)}x_i\big) - S^2\big(1_{(1)}\big) \rightharpoonup \lambda_i \otimes 1_{(2)}x_i\big] \in \operatorname{Ker} \pi.\end{aligned}$$

This proves $\operatorname{Ker}\pi = \operatorname{Ker}P$. That P is a projection and a right A-module map is trivial to verify. Therefore $\pi|_{\operatorname{Im}P}$ is an A-module isomorphism. Q.E.D.

Back to the Proof of Theorem 3.11. In virtue of the above lemma the amalgamated tensor product $\mathscr{I}^L(\hat{A}) \otimes_{A^L} A_A$ is the direct summand of a free A-module, hence projective. By Eq. (3.28) this is isomorphic to $\hat{A}^A$. This proves projectivity of $\hat{A}^A$, hence injectivity of ${}_AA$. Q.E.D.

The equivalence of (c) and (d) of the next theorem provides a weak Hopf version of Maschke's theorem known for Hopf algebras as well [9]. Below we denote $\varepsilon_R(x) := x \rightharpoonup \hat{1}$.

THEOREM 3.13. *The following conditions on a WHA A over K are equivalent*:

(a) *A is semisimple*.

(b) *In the category of left A-modules the following exact sequence is split*

$$0 \to \operatorname{Ker}\varepsilon_R \to {}_AA \stackrel{\varepsilon_R}{\to} {}_A\hat{A}^R \to 0.$$

(c) *There exists a normalized left integral $l \in A$*.

(d) *A is a separable K-algebra*.

Proof. (a ⇒ c): If A is semisimple, then $\operatorname{Ker}\sqcap^L \equiv \operatorname{Ker}\varepsilon_R$ being a left ideal there exists $p = p^2 \in A$ such that $\operatorname{Ker}\sqcap^L = Ap$, whence $l = 1 - p$ is a normalized left integral by Lemma 3.2(e).

(b ⇔ c): Let $F \in \operatorname{Hom}({}_A\hat{A}^R, {}_AA)$ be such that $\varepsilon_R \circ F = \mathrm{id}$. Then $xF(\hat{1}) = F(x \rightharpoonup \hat{1}) = \sqcap^L(x)F(\hat{1})$, for $x \in A$, therefore $F(\hat{1}) \in \mathscr{I}^L(A)$. Moreover, $\hat{1} \in \varepsilon_R(F(\hat{1})) = F(\hat{1}) \rightharpoonup \hat{1}$ implying $\sqcap^L(F(\hat{1})) = 1$. Conversely, if $l \in \mathscr{I}^L$ is a normalized left integral then $F \in \operatorname{Hom}({}_A\hat{A}^R, {}_AA)$ given by $F(\varepsilon_R(x)) := xl$ satisfies $\varepsilon_R \circ F = \mathrm{id}$.

(c ⇒ d): Let l be a normalized left integral. Then $q = l_{(1)} \otimes S(l_{(2)})$ is a separating idempotent for A. As a matter of fact $\mu(q) = 1$ follows from the normalization $\sqcap^L(l) = 1$ while $(x \otimes 1)q = q(1 \otimes x)$ is precisely the left integral property of Lemma 3.2(b).

(d ⇒ a): This is a standard result [18]. Q.E.D.

3.4. *Non-degenerate Integrals*

Until now we have not been able to decide whether the WHM theorem of Subsection 3.2 implies the existence of non-degenerate integrals, as it does in the case of Hopf algebras. In the present subsection we show that the existence of non-degenerate integrals in the WHA $\hat{A}$ is equivalent to the existence of non-degenerate functionals on A, i.e., that A is a Frobenius algebra. As a byproduct we obtain that the class of Frobenius WHAs is self-dual.

The space $\mathscr{I}^R$ of right integrals can be viewed as a K-module, as a left A^L-module ${}_{A^L}\mathscr{I}^R$ by left multiplication, and as a left A-module ${}_A\mathscr{I}^R$ since it is a left ideal of A. From the latter point of view ${}_A\mathscr{I}^R$ is the dual of the trivial right A-module, ${}_A\mathscr{I}^R \cong \mathrm{Hom}(\hat{A}^L_A, A_A)$, by a twisted version of Lemma 3.3. As a K-module $\mathscr{I}^R$ has $\hat{\mathscr{I}}^L$ as its K-dual, $\hat{\mathscr{I}}^L \cong \mathrm{Hom}({}_K\mathscr{I}^R, {}_K K)$ the isomorphism being given by the restriction of the canonical pairing (see Corollary 3.10). The next lemma shows that $\hat{\mathscr{I}}^L$ is also the A^L-dual of $\mathscr{I}^R$ with right A^L-module structure precisely the one needed in Example 3.8, i.e., $\lambda \cdot x^L = S(x^L) \rightharpoonup \lambda$.

LEMMA 3.14. *The A^L-valued bilinear form*

$$(\ ,\ )_{A^L} : \mathscr{I}^R \times \hat{\mathscr{I}}^L \to A^L, \qquad (r, \lambda)_{A^L} = \lambda \rightharpoonup r \tag{3.32}$$

provides an isomorphism of right A^L-modules

$$(\hat{\mathscr{I}}^L)^{A^L} \to \mathrm{Hom}\big({}_{A^L}\mathscr{I}^R, {}_{A^L}A^L\big), \qquad \lambda \mapsto \big(r \mapsto (r, \lambda)_{A^L}\big), \tag{3.33}$$

i.e., $(\hat{\mathscr{I}}^L)^{A^L}$ is the A^L-dual of ${}_{A^L}\mathscr{I}^R$.

Proof. At first verify the following properties of the A^L-valued bilinear form. So

$$(x^L \cdot r, \lambda)_{A^L} = x^L (r, \lambda)_{A^L}, \tag{3.34a}$$

$$(r, \lambda \cdot x^L)_{A^L} = (r, \lambda)_{A^L} x^L, \tag{3.34b}$$

$$(r, \lambda)_{A^L} = 0 \qquad \forall r \in \mathscr{I}^R \quad \Rightarrow \quad \lambda = 0. \tag{3.34c}$$

The first two are simple WHA identities. The third one follows from the relation $\varepsilon((r, \lambda)_{A^L}) = \langle \lambda, r \rangle$ and from non-degeneracy of the canonical pairing $\langle\ ,\ \rangle$ on $\hat{\mathscr{I}}^L \times \mathscr{I}^R$ (Corollary 3.10). Now properties (a) and (b) tell us that $\lambda \mapsto (.,\lambda)_{A^L}$ is indeed the required A^L-module map and (c) ensures that it is injective. To show that it is surjective it is sufficient to find finite sets of elements $\{r_a\}$ in $\mathscr{I}^R$ and $\{\lambda^a\}$ in $\hat{\mathscr{I}}^L$ such that

$$\sum_a (r, \lambda^a)_{A^L} r_a = r \qquad \forall r \in \mathscr{I}^R. \tag{3.35}$$

For if such elements exist then any $f \in \mathrm{Hom}({}_{A^L}\mathscr{I}^R, {}_{A^L}A^L)$ can be written as $f = \sum_a \lambda^a \cdot f(r_a)$. As a matter of fact

$$f(r) = f\Big(\sum_a (r, \lambda^a)_{A^L} \cdot r_a\Big) = \sum_a (r, \lambda^a)_{A^L} f(r_a) = \Big(r, \sum_a \lambda^a \cdot f(r_a)\Big)_{A^L}$$

for all $r \in \mathscr{I}^R$. Now we claim that a pair of dual bases $\{r_a\}$ of $\mathscr{I}^R$ and $\{\lambda^a\}$ of $\hat{\mathscr{I}}^L$, in the sense of K-duality, i.e., $\langle \lambda^a, r_b \rangle = \delta_{ab}$, also satisfies (3.35). As a matter of fact for $\lambda \in \hat{\mathscr{I}}^L$ we have

$$\begin{aligned}
&\Big\langle \lambda, \sum_a (\lambda^a \rightharpoonup r) r_a \Big\rangle \\
&\quad = \varepsilon\Big(\sum_a (\lambda^a \rightharpoonup r)(\lambda \rightharpoonup r_a)\Big) = \sum_a \langle \lambda^a, r(\lambda \rightharpoonup r_a)\rangle \\
&\quad = \sum_a \langle \lambda^a, r\big(S(r_a) \leftharpoonup \hat{S}^{-1}(\lambda)\big)\rangle = \sum_a \langle \lambda^a_{(1)}, r\rangle\langle \hat{S}^{-1}(\lambda)\lambda^a_{(2)}, S(r_a)\rangle \\
&\quad = \sum_a \langle \lambda\lambda^a_{(1)}, r\rangle\langle \lambda^a_{(2)}, S(r_a)\rangle = \Big\langle \lambda\Big[\sum_a S(r_a) \rightharpoonup \lambda^a\Big], r\Big\rangle = \langle \lambda, r\rangle,
\end{aligned}$$

where in the last equality (3.21) has been used. Q.E.D.

Notice that Eq. (3.35) means that ${}_{A^L}\mathscr{I}^R$ is finitely generated projective.[3] Therefore by a general result (see, e.g., [5]):

$$\hat{\mathscr{I}}^L \otimes_{A^L} \mathscr{I}^R \cong \mathrm{End}\,{}_{A^L}\mathscr{I}^R. \tag{3.36}$$

On the other hand the isomorphism α of the WHM theorem, if restricted to $\hat{\mathscr{I}}^L \otimes_{A^L} \mathscr{I}^R$, yields an isomorphism onto $\hat{A}^L$. Thus we have the composition

$$\mathscr{E}\colon A^L \to \hat{A}^L \xrightarrow{\alpha^{-1}} \hat{\mathscr{I}}^L \otimes_{A^L} \mathscr{I}^R \to \mathrm{End}\,{}_{A^L}\mathscr{I}^R, \tag{3.37}$$

of isomorphisms. Evaluating it explicitly we obtain

$$\begin{aligned}
r \cdot \mathscr{E}(x^L) &= r \cdot \Big(\sum_{ij} S^2(b_i) \rightharpoonup \big(\beta^i\beta^j(\hat{1} \leftharpoonup x^L)\big) \otimes b_j\Big) \\
&= r \cdot \Big(\sum_j \hat{L}\big(\beta^j(\hat{1} \leftharpoonup x^L)\big) \otimes b_j\Big) \\
&= r \cdot \Big(\sum_a \lambda^a \otimes S(x^L) r_a\Big) = \sum_a (r, \lambda^a)_{A^L} S(x^L) r_a \\
&= S(x^L) r \qquad \forall r \in \mathscr{I}^R,\ x^L \in A^L.
\end{aligned} \tag{3.38}$$

This proves

[3] Although this is clear from the fact that A^L is semisimple, constructing the concrete bases λ^a, r_a is not in vain since it helps to compute the commutant in (3.38).

PROPOSITION 3.15. *The left modules ${}_{A^L}\mathscr{I}^R$ and ${}_{A^R}\mathscr{I}^R$ are faithful and the endomorphism algebra of ${}_{A^L}\mathscr{I}^R$ consists of left multiplications with elements of A^R. Therefore*

$$\mathrm{End}\,{}_{A^L}\mathscr{I}^R \cong A^L, \quad \textit{as algebras}. \tag{3.39}$$

The set $\mathscr{S}\mathrm{ec}\,A^L$ of equivalence classes of simple left A^L-modules will be called the *sectors* of A^L. For $a \in \mathscr{S}\mathrm{ec}\,A^L$ let V_a be a simple module from the class a and let $\mathscr{D}_a = \mathrm{End}\,V_a$ be the corresponding division algebra. Then by the Wedderburn structure theorem $A^L \cong \oplus_a M_{n_a}(\mathscr{D}_a)$. Let m_a denote the multiplicity of V_a in the semisimple module ${}_{A^L}\mathscr{I}^R$. Then $\mathrm{End}\,{}_{A^L}\mathscr{I}^R \cong \oplus_a M_{m_a}(\mathscr{D}_a)$ which is, by the proposition, isomorphic to A^L. This is possible only if there is a permutation

$$\tilde{}\;: \mathscr{S}\mathrm{ec}\,A^L \to \mathscr{S}\mathrm{ec}\,A^L, \quad \text{such that } n_{\tilde{a}} = m_a \quad \text{and} \quad \mathscr{D}_{\tilde{a}} = \mathscr{D}_a. \tag{3.40}$$

This means that $\mathscr{I}^R$, as an A^L-$\mathrm{End}\,{}_{A^L}\mathscr{I}^R$ bimodule, can be identified with a direct sum of matrices,

$${}_{A^L}\mathscr{I}^R \cong \oplus_a \mathrm{Mat}(n_a \times m_a, \mathscr{D}_a). \tag{3.41}$$

This allows us to compute its K-dimension and apply the Cauchy−Schwarz inequality to obtain the bound

$$\dim_K \mathscr{I}^R = \sum_a (\dim_K \mathscr{D}_a) n_a m_a \le \sum_a (\dim_K \mathscr{D}_a) n_a^2 = \dim_K A^L. \tag{3.42}$$

Equality holds here if $m_a = n_a$, $a \in \mathscr{S}\mathrm{ec}\,A^L$, i.e., iff ${}_{A^L}\mathscr{I}^R \cong {}_{A^L}A^L$. Now we are ready to prove

THEOREM 3.16. *Let A be a WHA over the field K. Then the following conditions are equivalent.*

(i) *A is a Frobenius algebra*;

(ii) $\dim_K \mathscr{I}^R = \dim_K A^L$;

(iii) *Non-degenerate integrals exist in A*;

(iv) *$\hat{A}$ is a Frobenius algebra.*

Proof. (i) ⇒ (ii) If ${}_A A \cong {}_A \hat{A}$ then their invariants $\mathscr{I}({}_A A) = \mathscr{I}^L$ and $\mathscr{I}({}_A \hat{A}) = \hat{A}^L$, respectively, (see Remark 3.5), are isomorphic as K-spaces. (ii) ⇒ (iii) As we have seen above the K-space isomorphism of $\mathscr{I}^R$ and A^L implies that ${}_{A^L}\mathscr{I}^R$ is isomorphic to the left regular module ${}_{A^L}A^L$. Since the latter is cyclic, there exists a cyclic vector $r \in {}_{A^L}\mathscr{I}^R$. Thus $l := S(r)$ is cyclic in $(\mathscr{I}^L)^{\hat{A}^L}$. As a matter of fact $\mathscr{I}^L = S(\mathscr{I}^R) = S(A^L r) = lA^R =$

$\hat{S}(\hat{A}^L) \rightharpoonup l$. Now interchanging the roles of A and $\hat{A}$ in the WHM theorem

$$A = \alpha\left(\mathscr{I}^L \otimes_{\hat{A}^L} \hat{A}\right) = \hat{S}(\hat{A}) \rightharpoonup \mathscr{I}^L = \hat{S}(\hat{A}) \rightharpoonup \left(\hat{S}(\hat{A}^L) \rightharpoonup l\right) = \hat{A} \rightharpoonup l,$$

hence l is a non-degenerate left integral in A. (iii) ⇒ (iv) is obvious since l is a non-degenerate functional on $\hat{A}$. (iv) ⇒ (i) Repeat the arguments above from (i) to (iv) with A replaced by $\hat{A}$. Q.E.D.

Weak Hopf algebras satisfying any one of the conditions of the above theorem will be called *Frobenius WHAs*. Note that since semisimple algebras are Frobenius, in a semisimple WHA there exist both normalized and non-degenerate integrals, although there may be no integral sharing both properties.[4]

As an immediate consequence of the above considerations we have

Remark 3.17. The following properties for $l \in \mathscr{I}^L$ ($r \in \mathscr{I}^R$) are equivalent:

(i) $l(r)$ is non-degenerate;

(ii) l is separating for $\mathscr{I}^L_{A^{L,R}}$ (r is separating for ${}_{A^{L,R}}\mathscr{I}^R$);

(iii) l is cyclic for $\mathscr{I}^L_{A^{L,R}}$ (r is cyclic for ${}_{A^{L,R}}\mathscr{I}^R$).

In a Frobenius WHA A the group of invertible elements $A^R_\times$ of A^R acts on the set $\mathscr{I}^L_*(A)$ of non-degenerate left integrals transitively and freely. A similar statement holds for the non-degenerate right integrals $\mathscr{I}^R_*$,

$$\mathscr{I}^L_* = lA^R_\times, \qquad \mathscr{I}^R_* = A^L_\times r \tag{3.43}$$

for any $l \in \mathscr{I}^L_*$ and $r \in \mathscr{I}^R_*$. A similar relation for the dual integrals shows that there are one-to-one correspondences between non-degenerate integrals of A and of $\hat{A}$. The theorem below selects a distinguished "natural" one-to-one correspondence.

THEOREM 3.18. *Let A be a WHA and let $l \in \mathscr{I}^L(A)$ be a left integral. If there exists a $\lambda \in \hat{A}$ such that $\lambda \rightharpoonup l = 1$ then it is unique, it is a left integral in $\hat{A}$, and both l and λ are non-degenerate. Moreover $l \rightharpoonup \lambda = \hat{1}$. Such a pair (l, λ) will be called a dual pair of left integrals.*

Similarly, elements $r \in \mathscr{I}^R_*$ and $\lambda \in \hat{\mathscr{I}}^L_*$ are in one-to-one correspondence by either one of the equivalent relations $\lambda \rightharpoonup r = 1$ or $\lambda \leftharpoonup r = \hat{1}$.

[4] As an example consider $M_2(\mathbb{Z}_2)$, the semisimple algebra of two by two matrices over the field of mod 2 residue classes. Fix a set of matrix units $\{e_{ij}\}$ and introduce the coproduct $\Delta(e_{ij}) := e_{ij} \otimes e_{ij}$. Then we have two normalized left integrals $l_j = \sum_i e_{ij}$ for $j = 1, 2$ neither of which is non-degenerate. The only non-degenerate left integral is $l = l_1 + l_2$ for which however $\sqcap^L(l) = 0$.

Proof. By Lemma 3.2(d) if l is a left integral such that $\lambda \rightharpoonup l = 1$ then[5] $l_R \circ \lambda_L = S$. Since S is invertible, both l_R and λ_L are invertible; i.e., l and λ are non-degenerate and λ is unique. To show that $\lambda \in \mathscr{I}^L(\hat{A})$,

$$\varphi\lambda \rightharpoonup l = \varphi \rightharpoonup 1 = \hat{\sqcap}^L(\varphi)\lambda \rightharpoonup l, \qquad \varphi \in \hat{A}$$

suffices since l_R is a bijection. It remains to show that $l \rightharpoonup \lambda = \hat{1}$ which eventually justifies the term "dual" left integral. For $l \in \mathscr{I}^L(A)$ and $\lambda \in \hat{A}$ we have

$$\begin{aligned} \lambda \rightharpoonup l = 1 \quad &\Leftrightarrow \quad xl_{(1)}\langle \lambda, l_{(2)}\rangle = x \qquad x \in A, \\ &\Leftrightarrow \quad \langle \lambda, xl_{(2)}\rangle S^{-1}(l_{(1)}) = x \qquad x \in A, \\ &\Rightarrow \quad \langle l \rightharpoonup \lambda, x\rangle = \varepsilon(x) \qquad x \in A, \\ &\Leftrightarrow \quad l \rightharpoonup \lambda = \hat{1}. \end{aligned}$$

The duality between $\hat{\mathscr{I}}^L$ and $\mathscr{I}^R$ follows from the above duality between $\hat{\mathscr{I}}^L$ and $\mathscr{I}^L$ by passing from A to A^{op}. The other two twisted versions of the theorem are not spelled out explicitly. They can also be obtained by applying the antipode to the above relations. Q.E.D.

Recall that the *quasi-basis* of a non-degenerate functional f on A is an element $\sum_i a_i \otimes b_i \in A \otimes A$ such that (cf. [23]):

$$\sum_i f(xa_i)b_i = x = \sum_i a_i f(b_i x), \qquad x \in A. \tag{3.44}$$

(If K is a field then this just means that $\{b_i\}$ is a K-basis of A and $\{a_i\}$ is its dual basis w.r.t. f.) In other words $\sum_i a_i \otimes b_i$ is simply the expression $\sum_i f_R^{-1}(\beta^i) \otimes b_i$ of the inverse of $f_R\colon A \to \hat{A}$ as an element of $A \otimes A$. The index of f is then defined by Index $f := \sum_i a_i b_i$ which belongs to Center A. Now let (l, λ) be a dual pair of left integrals. Then the quasi-basis of λ is $l_{(2)} \otimes S^{-1}(l_{(1)})$ and

$$\text{Index } \lambda = S^{-1} \circ \sqcap^L(l) \in Z^R. \tag{3.45}$$

In particular a non-degenerate left integral l is normalized if and only if its dual has index 1.

[5] Here we use the standard notations $f_L, f_R\colon A \to \hat{A}$ defined by $f_L(x) := f \leftharpoonup x$ and $f_R(x) := x \rightharpoonup f$ for any $f \in \hat{A}$.

3.5. *Two-sided Non-degenerate Integrals*

The space of two-sided integrals $\mathscr{I}(A) := \mathscr{I}^L(A) \cap \mathscr{I}^R(A)$ in a weak Hopf algebra A is a possibly zero subalgebra of A. The assumption $\mathscr{I}(A) \neq 0$ is independent of the assumption $\mathscr{I}^L_*(A) \neq \emptyset$ since Hopf algebras already provide examples [21] for $\mathscr{I}^L_*(A) \neq \emptyset$ and $\mathscr{I}(A) = 0$. In this subsection we make the stronger assumption $\mathscr{I}_*(A) := \mathscr{I}^L_*(A) \cap \mathscr{I}(A) \neq \emptyset$ and study some of the consequences. The main result will be finding a criterion for a WHA to be a symmetric algebra.

At first we observe that if a non-degenerate two-sided integral j exists then the subspace of two-sided integrals is obtained from j by the action of the central subalgebra $Z^R = A^R \cap \text{Center}\, A$,

$$\mathscr{I} = jZ^R, \qquad \mathscr{I}_* = jZ^R_\times \quad \text{for any } j \in \mathscr{I}_*. \tag{3.46}$$

As a matter of fact if $i \in \mathscr{I}$ then i is a left integral therefore there exists an $x^R \in \mathscr{A}^R$ such that $i = jx^R$. Thus for all $y \in A$ we have $jx^R \sqcap^R(y) = jx^R y = j \sqcap^R(x^R y)$. Since j is separating for the right A^R-action, $x^R \sqcap^R(y) = \sqcap^R(x^R y)$. Therefore

$$\begin{aligned} x^R S(y) &= x^R S(y_{(1)}) y_{(2)} S(y_{(3)}) = S(y_{(1)}) x^R y_{(2)} S(y_{(3)}) \\ &= S\big(1_{(1)} y\big) x^R 1_{(2)} = S(y) x^R, \end{aligned}$$

hence x^R is central.

Next we recall some facts about "modular automorphisms." Let A be a finite-dimensional Frobenius algebra over a field K and let $f\colon A \to K$ be a non-degenerate functional. Then the *modular automorphism* of f is defined to be the unique $\theta_f \in \text{Aut}\, A$ such that

$$f(xy) = f\big(y\theta_f(x)\big), \qquad x, y, \in A. \tag{3.47}$$

It is worth it to give two other equivalent definitions of θ,

$$f \leftharpoonup x = \theta_f(x) \rightharpoonup f, \qquad x \in A, \tag{3.48}$$

or simply

$$\theta_f = f_R^{-1} \circ f_L. \tag{3.49}$$

Since any two non-degenerate functionals f and g are related by $g = x \rightharpoonup f$, with $x \in A_\times$, the equivalence class $\theta_A := [\theta_f]$ of θ_f modulo inner automorphisms is independent of the choice of f. If A is a WHA which is Frobenius then one may ask the question whether $\theta_A = [S^2]$.

DEFINITION 3.19. A non-degenerate functional $f\colon A \to K$ over a WHA A is called a q-trace if $\theta_f = S^2$.

In the term "q-trace" the letter "q" has no individual meaning. One may as well read it as "skew trace" although we do not deny that our motivation came from the theory of q-deformed Hopf algebras.

LEMMA 3.20. *In a WHA A let l be a non-degenerate left integral. Then $S(l) = l$ if and only if its dual left integral λ is a q-trace.*

Proof. $\theta_\lambda = S^2$ is equivalent to that the quasi-basis of λ satisfies

$$l_{(2)} \otimes S^{-1}\big(l_{(1)}\big) = S\big(l_{(1)}\big) \otimes l_{(2)}. \tag{3.50}$$

Applying S to the second tensor factor we obtain $\Delta(l) = \Delta(S(l))$ which yields $l = S(l)$ by the existence of a counit. Q.E.D.

LEMMA 3.21. *If non-degenerate two-sided integrals exist then all two-sided integrals $i \in \mathscr{I}(A)$ are S-invariant, $S(i) = i$.*

Proof. If we can show only that the non-degenerate two-sided integrals are S-invariant then we are ready since $j = S(j) \in \mathscr{I}_*$ implies $S(jz^R) = S(z^R)j = z^Rj = jz^R$ for all $z^R \in Z^R$.

So let $j \in \mathscr{I}_*$. Then $S(j) \in \mathscr{I}_*$ thus there exists an invertible $z \in Z^R$ such that $S(j) = jz$. Let λ be the dual of j as a left integral. Then for arbitrary $x \in A$ and for $z^L = S^{-1}(z^{-1})$,

$$\begin{aligned} z^L S(x) &= z^L(\lambda \leftharpoonup x) \rightharpoonup j = (\lambda \leftharpoonup x) \rightharpoonup z^L j = (\lambda \leftharpoonup x) \rightharpoonup S^{-1}(j), \\ S^2(x) z^{-1} &= j \leftharpoonup \hat{S}^{-1}(\lambda \leftharpoonup x) = j_L \circ \hat{S}^{-1} \circ \lambda_L(x) = \lambda_R^{-1} \circ \lambda_L(x) \\ &= \theta_\lambda(x). \end{aligned}$$

Therefore $z^{-1} = \theta_\lambda(1) = 1$ and j is S-invariant. Q.E.D.

THEOREM 3.22. *The WHA A over K is a symmetric algebra if and only if it has non-degenerate two-sided integrals and the square of the antipode is an inner automorphism.*

Proof. Let A be a symmetric WHA, and $\tau \in \hat{A}$ be a non-degenerate trace. Then there exists a unique $i \in A$ such that $i \rightharpoonup \tau = \hat{1} = \tau \leftharpoonup i$. We claim that i is a two-sided integral. As a matter of fact

$$\begin{aligned} xi \rightharpoonup \tau = x \rightharpoonup \hat{1} = \sqcap^L(x) \rightharpoonup \hat{1} = \sqcap^L(x) i \rightharpoonup \tau, \\ \tau \leftharpoonup ix = \hat{1} \leftharpoonup x = \hat{1} \leftharpoonup \sqcap^R(x) = \tau \leftharpoonup i \sqcap^R(x), \end{aligned}$$

so by non-degeneracy of τ, $i \in \mathscr{I}$. This integral i is also non-degenerate. For any $x^R \in A^R$ one has $ix^R \rightharpoonup \tau = i \rightharpoonup \tau \leftharpoonup x^R = \hat{1} \leftharpoonup x^R$, hence i is separating for $\mathscr{I}^L_{A^R}$ so non-degenerate by Remark 3.17.

The innerness of S^2 in a symmetric algebra follows if we can construct a non-degenerate functional on A the modular automorphism of which is S^2. By Lemma 3.21 i is S-invariant so by Lemma 3.20 χ, the dual left integral to i, is such a non-degenerate q-trace.

Conversely, let $S^2 = \mathrm{Ad}_g$ with some $g \in A_\times$ and $i \in \mathscr{I}_*$. Denoting the dual left integral of i by χ again, $g^{-1} \rightharpoonup \chi$ is a non-degenerate trace. Q.E.D.

We close this subsection with a result arising from assuming the existence of non-degenerate two-sided integrals in both A and $\hat{A}$. Although the arising structure is reminiscent to that of the "distinguished grouplike element" in Hopf algebra theory it is not a generalization of that.

PROPOSITION 3.23. *Let A be a WHA and assume that both $\mathscr{I}_*(A)$ and $\mathscr{I}_*(\hat{A})$ are non-empty. Then S^4 is inner and the square of θ_A is the identity in* Out A. *Moreover and more explicitly, for $\hat{h} \in \mathscr{I}_*(\hat{A})$ there exist invertible elements $a_L \in A^L$ and $\alpha_L \in \hat{A}^L$ such that, with the notations $a_R = S(a_L)$ and $\alpha_R = \hat{S}(\alpha_L)$, we have*

$$\mathrm{Ad}_{a_L a_R^{-1}} = S^4, \tag{3.51}$$

$$\mathrm{Ad}_{a_L a_R} = \theta_{\hat{h}}^2, \tag{3.52}$$

$$a_L a_R^{-1} \rightharpoonup \psi \leftharpoonup a_L a_R^{-1} = \alpha_L \alpha_R \psi \alpha_R^{-1} \alpha_L^{-1}, \qquad \psi \in \hat{A}. \tag{3.53}$$

Proof. Choose $h \in \mathscr{I}_*(A)$ and $\hat{h} \in \mathscr{I}_*(\hat{A})$ and let λ be the dual of h and l be that of $\hat{h}$, as left integrals. Define

$$a_L = \hat{h} \rightharpoonup h, \qquad \alpha_L = h \rightharpoonup \hat{h}. \tag{3.54}$$

Then

$$\begin{aligned} \hat{1} \leftharpoonup a_L &= \langle \hat{1}_{(1)} \hat{h}, h \rangle \hat{1}_{(2)} = \langle \hat{h}_{(1)}, h \rangle \hat{h}_{(2)} \hat{S}\big(\hat{h}_{(3)}\big) = \hat{\sqcap}^L (\hat{h} \leftharpoonup h) \\ &= \hat{S}(\hat{h} \leftharpoonup h) = \alpha_L, \end{aligned} \tag{3.55}$$

and introducing a_R and α_R as above

$$la_L = l \leftharpoonup \alpha_R = l \leftharpoonup (\hat{h} \leftharpoonup h) = h(l \leftharpoonup \hat{h}) = h\big(\hat{S}^2(\hat{h}) \rightharpoonup l\big) = h, \tag{3.56}$$

where q-trace property of l and $\hat{S}$-invariance of $\hat{h}$ have been used. Similarly,

$$\alpha_R \rightharpoonup l = h = la_R, \tag{3.57}$$

$$a_R \rightharpoonup \lambda = \hat{h} = \lambda\alpha_R, \tag{3.58}$$

$$\lambda \leftharpoonup a_R = \hat{h} = \lambda\alpha_L. \tag{3.59}$$

Non-degeneracy of $\hat{h}$ and h now imply invertibility of a_L, a_R, α_L, and α_R. Hence Eq. (3.53) readily follows.

We can now compute the modular automorphism of $\hat{h}$ using the information $\theta_\lambda = S^2$. Thus

$$\hat{h} \leftharpoonup x = a_R \rightharpoonup \lambda \leftharpoonup x = a_R\theta_\lambda(x) \rightharpoonup \lambda \quad \Rightarrow \quad \theta_{\hat{h}} = \mathrm{Ad}_{a_R} \circ S^2. \tag{3.60}$$

Computing $\hat{S}(\hat{h} \leftharpoonup x)$ in two different ways

$$\begin{aligned}\hat{S}(\hat{h} \leftharpoonup x) &= S^{-1}(x) \rightharpoonup \hat{h} = \hat{h} \leftharpoonup \theta_{\hat{h}}^{-1}\big(S^{-1}(x)\big)\\ &= \hat{S}\big(\theta_{\hat{h}}(x) \rightharpoonup \hat{h}\big) = \hat{h} \leftharpoonup S^{-1}(\theta_{\hat{h}}(x))\end{aligned}$$

yields

$$S^{-1} \circ \theta_{\hat{h}} = \theta_{\hat{h}}^{-1} \circ S^{-1}, \tag{3.61}$$

and finally

$$\mathrm{Ad}_{a_L} \circ S^{-2} = \theta_{\hat{h}} = \mathrm{Ad}_{a_R} \circ S^2, \tag{3.62}$$

from which (3.51) and (3.52) follow immediately. Q.E.D.

3.6. *Haar Integrals*

Since finite-dimensional weak Hopf algebras do not go beyond the "compact" and "discrete" case, the following very conservative definition of Haar measure will suffice.

DEFINITION 3.24. An element h of a WHA A is called a Haar integral in A or Haar measure on $\hat{A}$ if h is a normalized two-sided integral, i.e., $h \in \mathscr{I}(A)$ and $\sqcap^L(h) = \sqcap^R(h) = 1$.

Obviously, if a Haar integral exists then it is a unique S-invariant idempotent. As a matter of fact let h and h' be Haar integrals. Then $h' = \sqcap^L(h)h' = hh' = h\sqcap^R(h') = h$. In particular $h^2 = h$. S-invariance follows from uniqueness since $S(h)$ is always a Haar integral if h is.

In finding criteria for the existence of a Haar measure in A an important role will be played by a special element $\chi \in \hat{A}$ the definition of which was inspired by similar computations in Hopf algebra theory [21],

$$\chi := \sum_i \beta^i \leftharpoonup S^{-2}(b_i) \equiv \hat{L}'(\hat{1}), \tag{3.63}$$

where $\{b_i\}$ and $\{\beta^i\}$ are dual bases of A and $\hat{A}$, respectively, and $\hat{L}'$: $\hat{A} \to \hat{A}$ is given by $\hat{L}'(\psi) := \sum_i \beta^i\psi \leftharpoonup S^{-2}(b_i)$. Note that $\hat{L}'$ is the "cop" version of the dual analogue $\hat{L}$ of the projection (3.23) onto the space of left integrals. Hence χ is a left integral in $(\hat{A})_{\mathrm{cop}}$ and therefore in $\hat{A}$.

As we see below if χ is non-degenerate and a q-trace then its dual left integral is automatically the Haar measure. To see that it is a q-trace let Tr_A be the standard trace on $\mathrm{End}_K\, A$ and introduce the notation $Q_-(x)y := yx$. Then for $x \in A$ we have

$$\begin{aligned}
\chi(x) &= \sum_i \langle \beta^i, S^{-2}(b_i)x\rangle = \mathrm{Tr}_A\, Q_-(x) \circ S^{-2},\\
\chi(xy) &= \mathrm{Tr}_A\, Q_-(y) \circ Q_-(x) \circ S^{-2} = \mathrm{Tr}_A\, Q_-(y) \circ S^{-2} \circ Q_-\big(S^2(x)\big)\\
&= \chi\big(yS^2(x)\big).
\end{aligned}$$

The next lemma is crucial in deciding whether χ is non-degenerate.

LEMMA 3.25. *Let l be a left integral in a WHA A and let $\chi \in \hat{A}$ be the q-trace left integral defined in Eq.* (3.63). *Then*

$$l \rightharpoonup \chi = \hat{S}^2(\hat{1} \leftharpoonup l). \tag{3.64}$$

Proof. Using the q-trace property of χ and then (2.19a),

$$\begin{aligned}
l \rightharpoonup \chi &= \sum_i \beta^i \leftharpoonup S^{-2}(b_i l) = \big(\hat{1} \leftharpoonup 1_{(1)}\big) \leftharpoonup S^{-2}(1_{(2)}l)\\
&= \langle \hat{1}_{(1)}, S^{-1}(1_{(2)})1_{(1)}\rangle \hat{1}_{(2)} \leftharpoonup S^{-2}(l) = \hat{1} \leftharpoonup S^{-2}(l) = \hat{S}^2(\hat{1} \leftharpoonup l).
\end{aligned}$$

Q.E.D.

PROPOSITION 3.26. *Let A be a weak Hopf algebra over a field K and let χ be given by* (3.63).

(i) *The Haar integral $h \in A$ exists if and only if χ is non-degenerate, in which case (h, χ) is a dual pair of left integrals. In particular Haar integrals are non-degenerate.*

(ii) *A left integral $l \in \mathscr{I}^L(A)$ is a Haar integral if and only if* $\sqcap^R(l) = 1$.

The characterization of Haar measures under (ii) is so simple that it could be well used as a definition of Haar measure. Notice that in that case the formal difference between the notions of normalized left integral and Haar measure were so tiny (change $\sqcap^L$ for $\sqcap^R$) that it would smear out the big conceptual difference: The existence of normalized left integrals is equivalent to semisimplicity while the existence of Haar measures is much stronger.

Proof. (ii) Assume $l \in \mathscr{I}^L(A)$ satisfies $\sqcap^R(l) = 1$. Then by Lemma 3.25 $l \rightharpoonup \chi = \hat{1}$. Therefore the duality Theorem (Theorem 3.18) implies that (l, χ) is a dual pair of non-degenerate left integrals. Since χ is a q-trace, Lemma 3.20 shows that l is an S invariant non-degenerate left integral. Furthermore $\sqcap^L(l) = \sqcap^L(S(l)) = S \circ \sqcap^R(l) = 1$. Thus l is a Haar integral. Now assume h is a Haar integral. Then obviously h is a left integral satisfying $\sqcap^R(h) = 1$.

(i) The "only if" part follows from the proof of (ii). Assume χ is non-degenerate and let h be its dual left integral. Then by Lemma 3.20 h is two-sided and by Lemma 3.25 it is normalized. Q.E.D.

However simple, the criteria of the above proposition are very difficult to verify in concrete situations. So it is worth looking for other criteria even if they are not applicable in full generality.

THEOREM 3.27. *Let A be a WHA over an algebraically closed field K. Then a necessary and sufficient condition for the existence of Haar measure $h \in A$ is that A is semisimple and there exists a $g \in A_\times$ such that $gxg^{-1} = S^2(x)$ for $x \in A$ and $\operatorname{tr} D_r(g^{-1}) \neq 0$ for all irreducible representation D_r of A.*

The assumption on K is used only to ensure that A is split semisimple, $A = \oplus_r M_{n_r}(K)$, once knowing that it is semisimple. In particular there will be a K-basis $\{e_r^{\alpha\beta}\}$ for A obeying matrix unit relations.

Proof. Sufficiency: Let $\tau: A \to K$ be the trace with trace vector $\tau_r = \operatorname{tr} D_r(g^{-1})$. Then τ is non-degenerate and has as quasi-basis the element

$$\sum_i x_i \otimes y_i := \sum_r \frac{1}{\tau_r} \sum_{\alpha, \beta=1}^{n_r} e_r^{\alpha\beta} \otimes e_r^{\beta\alpha}.$$

Notice that $\sum_i x_i g^{-1} y_i = 1$. Now we define $\chi' := g \rightharpoonup \tau$ and claim that χ' coincides with the χ of Eq. (3.63). As a matter of fact

$$\chi'(x) = \tau(gx) = \sum_i \tau\left(x_i g^{-1} y_i gx\right) = \sum_i \tau\left(x_i S^{-2}(y_i) x\right)$$
$$= \sum_i \langle \beta_i, S^{-2}(b_i) x \rangle = \chi(x),$$

where we used the fact that the dual of the basis $b_i = y_i$ is $\beta_i = \tau \leftharpoonup x_i$. Since χ' was non-degenerate by construction, we conclude that *the* χ of Eq. (3.63) is non-degenerate and therefore its dual left integral l has $\sqcap^R(l) = 1$ by Eq. (3.64). Therefore l is a Haar measure.

Necessity: If $h \in A$ is a Haar measure then A is semisimple by Theorem 3.13. Therefore A is a symmetric algebra and $\theta_A = \mathrm{id}$. This means that θ_ψ is inner for all non-degenerate functional ψ. In particular $\theta_\chi = S^2$ is inner where χ is the dual left integral of h. Choose a $g \in A_\times$ implementing S^2 and construct the non-degenerate trace $\tau := g^{-1} \rightharpoonup \chi$. Then

$$\tau(x) = \chi(xg^{-1}) = \mathrm{Tr}_A\, Q_-(xg^{-1}) \circ S^{-2} = \mathrm{Tr}_A\, Q_+(g^{-1}) \circ Q_-(x),$$

where $Q_+(x)y := xy$ is the left multiplication on A. Choosing a matrix unit basis to evaluate the trace we obtain

$$\tau(x) = \sum_r \mathrm{tr}\, D_r(g^{-1}) \mathrm{tr}\, D_r(x),$$

and by non-degeneracy of τ all components $\mathrm{tr}\, D_r(g^{-1})$ of the trace vector are non-vanishing. Q.E.D.

4. C^*-WEAK HOPF ALGEBRAS

In this section we introduce the C^*-structure in WHAs which is inevitable if WHAs are to be used as symmetries of inclusions of von Neumann algebras, in particular in quantum field theory. Utilizing the results of Sections 2 and 3 we establish the existence of two canonical elements in any C^*-WHA, the Haar measure h and the canonical grouplike element g. While the Haar measure is well known for C^*-Hopf algebras, the canonical grouplike element cannot be recognized in finite-dimensional Hopf algebras because it is always equal to 1. This is related to involutivity of the antipode in finite-dimensional C^*-Hopf algebras [24]. The very fact that C^*-WHAs can have non-involutive antipodes provides the sufficient flexibility for the emergence of non-integer dimensions.

4.1. *First Consequences of the C^*-structure*

DEFINITION 4.1. We define a *-WHA as a WHA $(A, 1, \Delta, \varepsilon, S)$ over the complex numbers $\mathbb{C}$ together with an antilinear involution * such that

(i) $(A, *)$ is a *-algebra,

(ii) Δ is a *-algebra map, i.e., $(x^*)_{(1)} \otimes (x^*)_{(2)} = (x_{(1)})^* \otimes (x_{(2)})^*$ for all $x \in A$.

By uniqueness of the unit, counit, and the antipode (see Lemma 2.8) we have the additional relations

$$1^* = 1, \qquad \varepsilon(x^*) = \overline{\varepsilon(x)}, \qquad S(x^*)^* = S^{-1}(x). \tag{4.1}$$

Now it is easy to check that the projections $\sqcap^L$ and $\sqcap^R$ satisfy

$$\sqcap^L(x)^* = \sqcap^L(S(x)^*), \qquad \sqcap^R(x)^* = \sqcap^R(S(x)^*), \tag{4.2}$$

therefore A^L and A^R are *-subalgebras of A. As an elementary exercise we obtain self-duality of the *-WHA:

Remark 4.2. Let A be a *-WHA and define a star operation on its dual as

$$\langle \varphi^*, x \rangle =: \overline{\langle \varphi, S(x)^* \rangle}. \tag{4.3}$$

Then $\hat{A}$ with this star operation becomes a *-WHA.

For a *-WHA A the canonical isomorphisms $\kappa_A^L\colon A^L \to \hat{A}^R$ and $\kappa_A^R\colon A^R \to \hat{A}^L$ of Lemma 2.6 become *-algebra isomorphisms.

We omit the discussion of further properties of *-WHAs and turn to the most important case of C^*-WHAs.

DEFINITION 4.3. A *-WHA A possessing a faithful *-representation is called a C^*-weak Hopf algebra, or C^*-WHA for short.

Being a finite-dimensional C^*-algebra any C^*-WHA can be uniquely characterized, as an algebra, by the dimensions $n_r \in \mathbb{N}$ of its blocks where r is running over the finite set $\mathscr{S}\mathrm{ec}\, A$ of equivalence classes of irreducible representations (i.e., the sectors) of A. So

$$A \cong \bigoplus_{r \in \mathscr{S}\mathrm{ec}\, A} M_{n_r}, \qquad M_{n_r} = \mathrm{Mat}(n_r, \mathbb{C}). \tag{4.4}$$

A^L and A^R are unital *-subalgebras therefore they are C^*-algebras as well and we have natural numbers n_a, $a \in \mathscr{S}\mathrm{ec}\, A^L$ and n_b, $b \in \mathscr{S}\mathrm{ec}\, A^R$ characterizing the type of A^L and A^R, respectively. Thus

$$A^c \cong \bigoplus_{a \in \mathscr{S}\mathrm{ec}\, A^c} M_{n_a}, \qquad c = L, R. \tag{4.5}$$

The antiisomorphism $S\colon A^L \to A^R$ establishes a bijection $a \mapsto \bar{a}$ of the blocks of A^L to the blocks of A^R such that $n_{\bar{a}} = n_a$. (We consider $\mathscr{S}\mathrm{ec}\, A^L$, $\mathscr{S}\mathrm{ec}\, A^R$, and $\mathscr{S}\mathrm{ec}\, A$ as disjoint sets which allows us to use one function n.)

The following elementary but important proposition will be the basic ingredient in proving both the existence of Haar measures and rigidity of the representation category of C^*-WHAs.

PROPOSITION 4.4. *Let A be a finite-dimensional C^*-algebra and $S\colon A \to A^{\mathrm{op}}$ an algebra isomorphism such that $(* \circ S)^2 = \mathrm{id}_A$. Then there exists $g \in A_\times$ such that*

(i) $g \geq 0$

(ii) $gxg^{-1} = S^2(x),\ x \in A$

(iii) $\mathrm{tr}_r(g) = \mathrm{tr}_r(g^{-1}),\ r \in \mathcal{S}\mathrm{ec}\ A$

(iv) $S(g) = g^{-1}$

where tr_r denotes trace in the irreducible representation D_r. An element $g \in A$ satisfying only the first three properties is already unique.

Proof. The restriction $S|_{\mathrm{Center}\,A}$ is an algebra automorphism therefore acts on the minimal central idempotents e_r as $S(e_r) = e_{\bar{r}}$ where $r \mapsto \bar{r}$ is a permutation of $\mathcal{S}\mathrm{ec}\ A$. Since $e_r^* = e_r$ and $* \circ S$ is an involution, $r \mapsto \bar{r}$ is an involution.

Choose matrix units $\{e_r^{\alpha\beta}\}$ for the C^*-algebra A and define the antiautomorphism $S_0\colon A \to A$ by $S_0(e_r^{\alpha\beta}) := e_{\bar{r}}^{\beta\alpha}$. Then $S_0^2 = \mathrm{id}_A$ and $* \circ S_0 = S_0 \circ *$. Since $S \circ S_0$ is an automorphism of A that acts as the identity on the center, there exists $C \in A$ invertible such that $S = \mathrm{Ad}_C \circ S_0$. It follows that

$$\begin{aligned} * \circ S(x) &= C^{-1*} S_0(x^*) C^*, \\ (* \circ S)^2(x) &= C^{-1*} S_0(C^{-1}) x S_0(C) C^* = x, \end{aligned} \tag{4.6}$$

therefore $S_0(C)C^*$ is central and so is its adjoint $K := CS_0(C^*) = S(C^*)C$. So

$$S^2(x) = CS_0\big(CS_0(x)C^{-1}\big)C^{-1} = CS_0(C^{-1})xS_0(C)C^{-1}, \qquad x \in A, \tag{4.7}$$

hence $T := CS_0(C^{-1}) = CC^*[S_0(C)C^*]^{-1} = CC^*K^{-1*}$ implements S^2 and its polar decomposition takes the form

$$T = ug', \qquad u = K^{-1*}(K^*K)^{1/2}, \qquad g' = C(K^*K)^{-1/2}C^*. \tag{4.8}$$

Using the centrality of the unitary part and the computations $S(T) = S(C)S^2(C^{-1}) = S(C)TC^{-1}T^{-1} = T^{-1}$ and $S(K) = S_0(K) = C^*S_0(C) = S_0(C)C^* = K^*$ we obtain that g' is positive invertible, implements S^2, and satisfies $S(g') = g'^{-1}$. These latter three properties, however, do not fix g

completely. If c is positive, central, and satisfies $S(c) = c^{-1}$ then $g = g'c$ will also satisfy the above three properties. Now defining

$$g := g'c \quad \text{where } c = \sum_r e_r \left(\frac{\mathrm{tr}_r(g'^{-1})}{\mathrm{tr}_r(g')} \right)^{1/2}, \tag{4.9}$$

it is easy to verify that g obeys (i)–(iv) of the proposition. If $f \in A$ satisfies only (i)–(iii) then $f = gc$ where c is positive invertible, central, and satisfies $D_r(c) = D_r(c)^{-1}$ for all irrep D_r. Hence $c = 1$, proving uniqueness of g. Q.E.D.

4.2. *The Haar Measure and Self-duality*

Recall that the Haar measure in a WHA A has been defined in Definition 3.24 as the unique element $h \in A$ making the integral $\int \varphi := \langle \varphi, h \rangle$ of a function $\varphi \colon A \to \mathbb{C}$ to be a non-degenerate functional invariant under left and right translations and normalized according to $\int \varphi^L = \hat{\varepsilon}(\varphi^L)$ for $\varphi^L \in \hat{A}^L$. The sufficient conditions for its existence given by Theorem 3.27 will be used here to prove the next theorem.

THEOREM 4.5. *In a C^*-WHA A Haar measure $h \in A$ exists. It is self-adjoint, $h^* = h$, and such that*

$$(\varphi, \psi) := \langle \varphi^* \psi, h \rangle, \qquad \varphi, \psi \in \hat{A} \tag{4.10}$$

is a scalar product on $\hat{A}$ making $\hat{A}$ a Hilbert space and making the left regular module ${}_{\hat{A}}\hat{A}$ a faithful $*$-representation of the $*$-WHA $\hat{A}$. Thus $\hat{A}$ is a C^*-WHA, too.

Proof. A being a finite-dimensional C^*-algebra is semisimple. By Proposition 4.4 there exists a g implementing S^2. This g was shown to be positive and invertible, hence $\mathrm{tr}\, D_r(g^{-1}) > 0$ for all $r \in \mathscr{S}\mathrm{ec}\, A$. Therefore all the conditions of Theorem 3.27 are satisfied and Haar measure h exists.

Since h is non-degenerate, $(\ ,\ )$ is a non-degenerate sesquilinear form on $\hat{A}$. So it remains to show positivity. By the equality

$$(\psi, \psi) = \langle \psi^* \psi, h \rangle = \overline{\langle \psi, S(h_{(1)})^* \rangle} \langle \psi, h_{(2)} \rangle, \tag{4.11}$$

positivity of $(\ ,\ )$ follows if we can show that $(S \otimes \mathrm{id}) \circ \Delta(h)$ belongs to the positive cone

$$\mathscr{P} = \left\{ \sum_k a_k^* \otimes a_k \,\middle|\, a_k \in A \right\} \subset A \otimes A. \tag{4.12}$$

Therefore the next lemma completes the proof. Q.E.D.

LEMMA 4.6. *Choose matrix units $\{e_q^{\alpha\beta}\}$ for A and let g denote the element determined in Proposition 4.4. If furthermore $\sum_i x_i \otimes y_i$ is the quasi-basis of the trace $\tau\colon A \to \mathbb{C}$ with trace vector $\tau_q = \mathrm{tr}_q(g^{-1})$ then*

$$S(h_{(1)}) \otimes h_{(2)} = \sum_i x_i \otimes g^{-1} y_i = \sum_{q \in \mathcal{S}\mathrm{ec}\, A} \frac{1}{\tau_q} \sum_{\alpha\beta} e_q^{\alpha\beta} g^{-1/2} \otimes g^{-1/2} e_q^{\beta\alpha}, \tag{4.13}$$

$$h_{(1)} \otimes S(h_{(2)}) = \sum_i x_i g \otimes y_i = \sum_{q \in \mathcal{S}\mathrm{ec}\, A} \frac{1}{\tau_q} \sum_{\alpha\beta} e_q^{\alpha\beta} g^{1/2} \otimes g^{1/2} e_q^{\beta\alpha}. \tag{4.14}$$

Proof. The quasi-basis of $\chi = g \rightharpoonup \tau$ is $\sum x_i g^{-1} \otimes y_i$ and since χ is the dual left integral of h, this quasi-basis is equal to $h_{(2)} \otimes S^{-1}(h_{(1)})$. This implies the first row. By property (iii) of Proposition 4.4 τ is an S-invariant trace, therefore its quasi-basis can also be written as $\sum_i y_i \otimes x_i = \sum_i S^{-1}(x_i) \otimes S^{-1}(y_i)$. Thus the second row follows from the first. Q.E.D.

From now on $h \in A$ will always denote the Haar measure of A and $\hat{h} \in \hat{A}$ that of $\hat{A}$.

LEMMA 4.7. *In a C^*-WHA A the counit is a positive linear functional, $\varepsilon(x^*x) \geq 0$, $x \in A$.*

Proof.

$$\begin{aligned} \varepsilon(x^*x) &= \varepsilon(x^*1_{(1)})\varepsilon(1_{(2)}1_{(2')})\varepsilon(1_{(1')}x) = \varepsilon\big(\sqcap^L(x)^* \sqcap^L(x)\big) \\ &= \langle \hat{h}, \sqcap^L(x)^* \sqcap^L(x)\rangle \geq 0, \end{aligned}$$

where we have used $\hat{h}|_{A^L} = \varepsilon|_{A^L}$, which follows from

$$\langle \hat{h}, x^L\rangle = \langle \hat{\sqcap}^L(\hat{h}), x^L\rangle = \langle \hat{1}, x^L\rangle$$

for all $x^L \in A^L$. Q.E.D.

A being semisimple the trivial representation V_ε decomposes into irreducibles V_q each of them with multiplicity 1 by Proposition 2.15. The sectors $q \in \mathcal{S}\mathrm{ec}\, A$ occurring in V_ε with non-zero multiplicity will be called *vacuum sectors*. So

$$V_\varepsilon \cong \bigoplus_{q \in \mathcal{V}\mathrm{ac}\, A} V_q. \tag{4.15}$$

By Proposition 2.15 there is a bijection $q \mapsto z_q^L$ from the set $\mathcal{V}\mathrm{ac}\, A$ of vacuum sectors to the set of minimal projections in Z^L such that, with

$z_q^R := S(z_q^L)$, we have

$$D_\varepsilon\big(z_q^L\big) = D_\varepsilon(e_q) = D_\varepsilon\big(z_q^R\big), \tag{4.16}$$

$$z_q^L = \sqcap^L(e_q) \qquad \sqcap^R(e_q) = z_q^R, \tag{4.17}$$

where e_q denotes the minimal central projection in A supporting the irreducible vacuum representation D_q.

LEMMA 4.8. *$D_r(h)$ is a one-dimensional projection for $r \in \mathscr{V}\mathrm{ac}\, A$ and $D_r(h) = 0$ if r is not a vacuum sector. The algebra of two-sided integrals is generated by minimal projections h_q:*

$$\mathscr{I}(A) = hAh = \mathrm{Span}\{h_q \,|\, q \in \mathscr{V}\mathrm{ac}\, A\}, \qquad h_q = he_q. \tag{4.18}$$

The non-degenerate two-sided integrals are precisely the invertible elements: $\mathscr{I}_*(A) = \mathscr{I}(A)_\times$.

Proof. If $D_r(h) \neq 0$ then pick up a non-zero vector v_r from the subspace $D_r(h)V_r$ of the irreducible A-module V_r and define

$$T\colon A^L \to V_r, \qquad Tx^L := D_r(x^L)v_r. \tag{4.19}$$

This map is a non-zero left A-module map if we equip A^L with the structure of the trivial A-module ${}_AA^L$ introduced in Lemma 2.12. Indeed,

$$D_r(x)Tx^L = D_r(xx^Lh)v_r = D_r\big(\sqcap^L(xx^L)h\big)v_r = T\sqcap^L(xx^L). \tag{4.20}$$

Therefore $r \in \mathscr{V}\mathrm{ac}\, A$. This proves that $D_r(h) = 0$ for $r \notin \mathscr{V}\mathrm{ac}\, A$.

Now let the Haar integral act on the trivial left A-module ${}_A\hat{A}^R$.

$$D_\varepsilon(h)\varphi^R = h \rightharpoonup \varphi^R \in \hat{A}^L \cap \hat{A}^R \equiv \hat{Z}. \tag{4.21}$$

Thus $D_\varepsilon(h)\colon \hat{A}^R \to \hat{Z}$ is a projection, onto. If z^L is a minimal projection in Z^L then $z^L \rightharpoonup \hat{1}$ is a minimal projection in $\hat{Z}$ by Lemma 2.14. Hence $D_\varepsilon(z^Lh)$ maps $\hat{A}^R$ onto $(z^L \rightharpoonup \hat{1})\hat{Z} \cong \mathbb{C}$. This proves that $D_\varepsilon(z^Lh)$, the restriction of which is precisely $D_q(h)$ for some $q \in \mathscr{V}\mathrm{ac}\, A$, is a one-dimensional projection. If $i \in \mathscr{I}$ then by the two-sided normalization of h one can write $i = hih$. Conversely, hxh is a two-sided integral for all $x \in A$. This proves the remaining assertions. Q.E.D.

The Haar measure provides conditional expectations

$$E^L\colon A \to A^L, \qquad E^L(x) = \hat{h} \rightharpoonup x, \tag{4.22}$$

$$E^R\colon A \to A^R, \qquad E^R(x) = x \leftharpoonup \hat{h}. \tag{4.23}$$

As a matter of fact by Lemma 3.2(c) the image of E^L is in A^L since $\hat{h}$ is a left integral. E^L is unit preserving since $\hat{h}$ is normalized. Finally, E^L is positive since $\hat{h}$ is positive and Δ is a *-algebra map.

4.3. *The Canonical Grouplike Element*

In this subsection we investigate further properties of the element g of Proposition 4.4. We show that it is always a product of left and right elements, implying its grouplikeness immediately, and obtain expressions for the modular automorphisms of the Haar measures of A and $\hat{A}$.

PROPOSITION 4.9. *In a C*-WHA A there exists a unique $g \in A$ such that*

(i) $g \geq 0$ *and invertible,*

(ii) $gxg^{-1} = S^2(x)$ *for all* $x \in A$,

(iii) $h_{(2)} \otimes h_{(1)} = h_{(1)} \otimes gh_{(2)}g$.

Proof. *Existence*: Let g be the (unique) element defined by the conditions of Proposition 4.4. As in the proof of Lemma 4.6 let τ be the S-invariant trace with trace vector $\tau_q = \mathrm{tr}_q(g)$ and $\sum x_i \otimes y_i$ be its quasi-basis. Then

$$h_{(2)} \otimes h_{(1)} = \sum_i x_i \otimes S\left(g^{-1}y_i\right) = \sum_i S^{-1}(y_i) \otimes x_i g \tag{4.24}$$

$$= \sum_i S^{-1}\left(y_i g^{-1}\right) \otimes g x_i g = \sum_i g S^{-1}(y_i) \otimes g x_i g \tag{4.25}$$

$$= \sum_i S\left(g^{-1}y_i\right) \otimes g x_i g = h_{(1)} \otimes g h_{(2)} g. \tag{4.26}$$

Uniqueness: Let g and g' satisfy (i)–(iii). Then $g' = gc$ with c central, positive, and invertible. Furthermore, since (iii) is equivalent to

$$\langle \varphi\psi, h\rangle = \langle \psi(g \rightharpoonup \varphi \leftharpoonup g), h\rangle, \tag{4.27}$$

non-degeneracy of h implies

$$g' \rightharpoonup \varphi \leftharpoonup g' = g \rightharpoonup \varphi \leftharpoonup g, \qquad \varphi \in \hat{A}. \tag{4.28}$$

Therefore $c^2 \rightharpoonup \varphi \equiv c \rightharpoonup \varphi \leftharpoonup c = \varphi$ for all $\varphi \in \hat{A}$. Thus $c^2 = 1$ and, by positivity, $c = 1$. Q.E.D.

Notice that property (iii) of Proposition 4.9 is equivalent to the statement that the modular automorphism of the Haar functional $\varphi \mapsto \varphi(h)$ is expressible in the form

$$\theta_h(\psi) = g \rightharpoonup \psi \leftharpoonup g, \qquad \psi \in \hat{A}. \tag{4.29}$$

DEFINITION 4.10. Let A be a C^*-weak Hopf algebra. Then the unique element $g \in A$ determined either by the conditions of Proposition 4.4 or by the conditions of Proposition 4.9 is called the canonical grouplike element of A.

As one may suspect the canonical grouplike element is grouplike in the sense of

DEFINITION 4.11. An element x of a WHA A is called grouplike if

$$\Delta(x) = x1_{(1)} \otimes x1_{(2)} = 1_{(1)}x \otimes 1_{(2)}x, \tag{4.30}$$

$$S(x)x = 1. \tag{4.31}$$

We note that if (4.30) holds then condition (4.31) is equivalent to the assumption that x is invertible. One should emphasize that grouplike elements are not always like group elements if a *-operation is present. Namely, we allow for x not to be unitary. Thus there can be positive grouplike elements, for example, in a C^*-WHA.

If x is an invertible element factorizable as $x_L x_R^{-1}$ with $x_L \in A^L$ and $x_R = S(x_L) = S^{-1}(x_L)$ then x is automatically grouplike. As a matter of fact $\Delta(x) = x_L 1_{(1)} \otimes x_R^{-1} 1_{(2)} = x x_R 1_{(1)} \otimes x_R^{-1} 1_{(2)} = x1_{(1)} \otimes x1_{(2)}$. Now it follows from the next lemma that the canonical grouplike element g is grouplike.

LEMMA 4.12. *In a weak C^*-Hopf algebra A the elements $h \leftharpoonup \hat{h}$ and $\hat{h} \rightharpoonup h$ are positive and invertible. The canonical grouplike element of A can be factorized as*

$$g = g_L g_R^{-1}, \tag{4.32}$$

where

$$g_L := (\hat{h} \rightharpoonup h)^{1/2}, \qquad g_R = (h \leftharpoonup \hat{h})^{1/2}. \tag{4.33}$$

Proof. $\hat{h} \rightharpoonup h = E^L(h) = E^L(h^*h) \geq 0$ and similarly $h \leftharpoonup \hat{h} \geq 0$ by positivity of the conditional expectations (4.22). Invertibility follows from the existence of the dual left integral χ since $(h \leftharpoonup \hat{h}) \rightharpoonup \chi = \hat{S}(\hat{h})(h \rightharpoonup \chi) = \hat{h}$

can hold for the non-degenerate χ and $\hat{h}$ only if $h \leftharpoonup \hat{h}$ is invertible. Thus $\hat{h} \rightharpoonup h = S(h \leftharpoonup \hat{h})$ is invertible, too.

The next point is to observe that the three elements $\hat{h} \rightharpoonup h$, $h \leftharpoonup \hat{h}$, and g commute with each other. For g and any one of the others this follows from the fact that $\hat{h} \rightharpoonup h$ and $h \leftharpoonup \hat{h}$ are invariant under S^2. For the commutativity of the remaining two notice that one of them belongs to A^L the other to A^R. Now compare the following expressions,

$$\hat{h} = (h \leftharpoonup \hat{h}) \rightharpoonup \chi = (h \leftharpoonup \hat{h})g \rightharpoonup \tau, \tag{4.34}$$

$$\hat{h} = \hat{S}^{-1}(\hat{h}) = \tau \leftharpoonup g^{-1}(\hat{h} \rightharpoonup h) = g^{-1}(\hat{h} \rightharpoonup h) \rightharpoonup \tau. \tag{4.35}$$

By non-degeneracy of τ we obtain

$$(\hat{h} \rightharpoonup h)g^{-1} = (h \leftharpoonup \hat{h})g,$$

$$(\hat{h} \rightharpoonup h)(h \leftharpoonup \hat{h})^{-1} = g^2,$$

and taking the (positive) square root the lemma is proven. Q.E.D.

LEMMA 4.13. *The left–right components of the canonical grouplike element g of A and $\hat{g}$ of $\hat{A}$ obey*

$$\hat{g}_L = \hat{1} \leftharpoonup g_L = \hat{1} \leftharpoonup g_R \qquad g_L = 1 \leftharpoonup \hat{g}_L = 1 \leftharpoonup \hat{g}_R, \tag{4.36}$$

$$\hat{g}_R = g_R \rightharpoonup \hat{1} = g_L \rightharpoonup \hat{1} \qquad g_R = \hat{g}_R \rightharpoonup 1 = \hat{g}_L \rightharpoonup 1, \tag{4.37}$$

$$S(g_L) = g_R = S^{-1}(g_L) \qquad \hat{S}(\hat{g}_L) = \hat{g}_R = \hat{S}^{-1}(\hat{g}_L). \tag{4.38}$$

Proof. Since $g_L \in A^L$ and $g_R \in A^R$, they commute and both of them are invariant under $S^2 = \mathrm{Ad}_g$. So are the C^*-algebras generated by each of them, pointwise. Hence $S(g_L^{1/2})^* = S^{-1}(g_L^{1/2}) = S(g_L^{1/2})$ therefore $S(g_L) = S(g_L^{1/2})^2 \geq 0$. On the other hand $S(g_L)^2 = S(g_L^2) = g_R^2$, therefore $S(g_L)$ is the positive square root of g_R^2, i.e., $S(g_L) = g_R$.

Next we want to show that $\hat{1} \leftharpoonup (\hat{h} \rightharpoonup h) = h \rightharpoonup \hat{h}$. Since both sides belong to $\hat{A}^L$, the identity

$$\begin{aligned}\langle \hat{1} \leftharpoonup (\hat{h} \rightharpoonup h), x^R \rangle &= \varepsilon\big((\hat{h} \rightharpoonup h)S(x^R)\big) = \varepsilon\big(\hat{h} \rightharpoonup hS(x^R)\big) \\ &= \langle \hat{h} \leftharpoonup h, S(x^R) \rangle = \langle h \rightharpoonup \hat{h}, x^R \rangle\end{aligned}$$

valid for $x^R \in A^R$, suffices. Therefore $\hat{1} \leftharpoonup g_L^2 = \hat{g}_L^2$, or $\hat{1} \leftharpoonup g_R^2 = \hat{g}_L^2$. Now use the fact that $A^R \ni x^R \mapsto (\hat{1} \leftharpoonup x^R) \in \hat{A}^L$ is a $*$-algebra isomorphism. Hence passing to the square roots we obtain $1 \leftharpoonup g_R = \hat{g}_L$. All the remaining identities are simple consequences of this. Q.E.D.

PROPOSITION 4.14. *Let A be a C^*-WHA with dual $\hat{A}$ and let $h \in A$, $\hat{h} \in \hat{A}$ be the corresponding Haar measures. Then*

(i) *the modular automorphism of the Haar functional $\hat{h}$ is implemented by $g_L g_R$; i.e., for all $x \in A$ we have $\theta_{\hat{h}}(x) = g_L g_R x g_R^{-1} g_L^{-1}$;*

(ii) *the dual left integral of h can be expressed as $\chi = \hat{h}\hat{g}_R^{-2}$;*

(iii) *the S-invariant trace functional $\tau = g^{-1} \rightharpoonup \chi$ and the Haar functional $\hat{h}$ are related by*

$$\tau = \hat{g}_L^{-1}\hat{h}\hat{g}_R^{-1}, \tag{4.39}$$

$$\hat{h} = g_L g_R \rightharpoonup \tau. \tag{4.40}$$

Proof. (i) Using identities like $\hat{g}_L \rightharpoonup x = g_R x, \ldots,$ etc., which follow from Remark 2.7, one can easily verify $\hat{g} \rightharpoonup x \leftharpoonup \hat{g} = g_L g_R x g_R^{-1} g_L^{-1}$, for $x \in$ A.

(ii) The identity $\hat{h} \rightharpoonup h = g_L^2 = 1 \leftharpoonup \hat{g}_R^2$ implies $1 = \hat{h} \rightharpoonup h \leftharpoonup \hat{g}_R^{-2} = \hat{h} \rightharpoonup h g_L^{-2}$, hence $h g_L^{-2} = h g_R^{-2}$ is the dual left integral of $\hat{h}$. By duality, $\hat{h}\hat{g}_R^{-2}$ is the dual left integral χ of h.

(iii) $\tau = g^{-1} \rightharpoonup \hat{h}\hat{g}_R^{-2} = \hat{g}_R^{-1}(\hat{h}\hat{g}_R^{-2})\hat{g}_R = \hat{g}_L^{-1}\hat{h}\hat{g}_R^{-1}$ and $\tau = g^{-1} \rightharpoonup (g_R^{-2} \rightharpoonup \hat{h}) = g_L^{-1} g_R^{-1} \rightharpoonup \hat{h}$ completes the proof. Q.E.D.

Cyclicity and separability of the vector h in the right $A^{L,R}$-module $\mathscr{I}^L$ (cf. Remark 3.17) allows us to introduce $\hat{A}^R$-valued "Radon-Nikodym derivatives" of left integrals l with respect to the Haar measure. At first note that $l = \sqcap^L(h)l = hl = h \sqcap^R(l) = hS^{-1}(\sqcap^R(l))$ therefore using Remark 2.7 we have

$$\langle \varphi, l \rangle = \langle \varphi \rho_R, h \rangle = \langle \rho_L \varphi, h \rangle, \tag{4.41}$$

where $\rho_R = \sqcap^R(l) \rightharpoonup \hat{1}$ and $\rho_L = S^{-1}(\sqcap^R(l)) \rightharpoonup \hat{1} = \hat{S}^2(\rho_R)$.

PROPOSITION 4.15. *The bijections $\mathscr{I}^L(A) \to \hat{A}^R$ provided by the left and right Radon–Nikodym derivatives $l \mapsto \rho_L$ and $l \mapsto \rho_R$, respectively, obey the following properties.*

(i) *l is non-degenerate iff $\rho_{R,L}$ is invertible.*

(ii) *If l is non-degenerate then l is normalized iff $l^2 = l$.*

(iii) *l is of positive type; i.e., $\langle \varphi^*\varphi, l \rangle \geq 0$ for all $\varphi \in \hat{A}$, iff $\sqcap^R(l) \geq 0$ iff $\rho_R \in \hat{g}_R^{1/2}\hat{A}_+^R\hat{g}_R^{-1/2}$ where $\hat{A}_+^R$ is the cone of positive elements in $\hat{A}^R$. In this case $\rho_L = \rho_R^*$ and there exists a $\xi \in \hat{A}$ such that $\langle \varphi, l \rangle = \langle \xi^*\varphi\xi, h \rangle$ for $\varphi \in \hat{A}$.*

(iv) *Let λ be the dual left integral of l. Then the Radon–Nikodym derivatives of λ and l are related by $\sqcap^R(l)(\hat{\sqcap}^R(\lambda) \rightharpoonup 1) = g_R^{-2}$.*

Proof. (i) follows from cyclicity of h in $\mathcal{I}^L_{A^R}$. (ii) $l^2 = l$ implies $(\sqcap^L(l) - 1)l = 0$ and acting with $\lambda \rightharpoonup$, where λ is the dual left integral of l, one obtains $\sqcap^L(l) = 1$. The converse implication is trivial. (iii) As in the proof of Theorem 4.5 l is of positive type iff $S(l_{(1)}) \otimes l_{(2)}$ belongs to the positive cone (4.12). If it does then $\sqcap^R(l) = S(l_{(1)})l_{(2)} \geq 0$. Now assume $\sqcap^R(l) \geq 0$. Then introducing $\xi = \sqcap^R(l)^{1/2} \rightharpoonup \hat{1}$ we have $\sqcap^R(l)^{1/2} = \xi \rightharpoonup 1$, $S^{-1}(\sqcap^R(l)^{1/2}) = S(\xi \rightharpoonup 1)^* = (1 \leftharpoonup \hat{S}^{-1}(\xi))^* = 1 \leftharpoonup \xi^*$ therefore $l = hS^{-1}(\sqcap^R(l)^{1/2}) \sqcap^R(l)^{1/2} = \xi \rightharpoonup h \leftharpoonup \xi^*$ proving that l is of positive type. It remains to reformulate positivity of $\sqcap^R(l)$ in terms of ρ_R. Use the fact that the antimultiplicative map $x^R \mapsto (x^R \rightharpoonup \hat{1})$ from A^R to $\hat{A}^R$ sends the *-operation into a new involution, $x^{R*} \rightharpoonup \hat{1} = (S^{-1}(x^R) \rightharpoonup \hat{1})^* = (S^{-2}(x^R) \rightharpoonup \hat{1})^* = (g_R x^R g_R^{-1} \rightharpoonup \hat{1})^* = \hat{g}_R(x^R \rightharpoonup \hat{1})^* \hat{g}_R^{-1}$. Therefore the equality $\sqcap^R(l) = x^{R*}x^R$ for some $x^R \in A^R$ is equivalent to the equality $\rho_R = (x^R \rightharpoonup \hat{1})(x^{R*} \rightharpoonup \hat{1}) = \hat{g}_R^{1/2}\eta\eta^*\hat{g}_R^{-1/2}$ with $\eta = \hat{g}_R^{-1/2}(x^R \rightharpoonup \hat{1})\hat{g}_R^{1/2} \in \hat{A}^R$. (iv) follows by an elementary calculus starting from the identity $1 = \lambda \rightharpoonup l = \hat{h}\, \hat{\sqcap}^R(\lambda) \rightharpoonup hS^{-1}(\sqcap^R(l))$. Q.E.D.

APPENDIX: THE WEAK HOPF ALGEBRA $B \otimes B^{\mathrm{op}}$

Let B be a separable algebra over the field K and let $E\colon B \to K$ be a non-degenerate functional with index 1. These are the data needed for constructing a WHA structure on the algebra $B \otimes B^{\mathrm{op}}$. For a similar construction of a WBA see [14].

At first choose a basis $\{e_i\}$ of B over K and let $\{f_i\}$ be its dual basis w.r.t. E, i.e. $E(e_i f_j) = \delta_{ij}$. Then

(a) $\sum_i f_i \otimes e_i \in B \otimes B$ is independent of the choice of $\{e_i\}$;

(b) $\sum_i E(xf_i)e_i = x = \sum_i f_i E(e_i x)$, $x \in B$;

(c) $\sum_i f_i e_i = 1$;

(d) $\sum_i xf_i \otimes e_i = \sum_i f_i \otimes e_i x$, $x \in B$;

(e) if θ denotes the modular automorphism of E, i.e., $E(xy) = E(y\theta(x))$, $x, y \in B$, then

$$\sum_i f_i \otimes xe_i = \sum_i f_i \theta(x) \otimes e_i, \qquad x \in B;$$

(f) $\sum_i f_i \otimes e_i = \sum_i e_i \otimes \theta^{-1}(f_i) = \sum_i \theta(e_i) \otimes f_i$.

The algebra $B \otimes B^{\mathrm{op}}$ is the K-space $B \otimes B$ with multiplication $(a \otimes b)(x \otimes y) := (ax \otimes yb)$. Its WHA structure is defined by

$$\Delta(x \otimes y) = \sum_i (x \otimes f_i) \otimes (e_i \otimes y), \tag{A1}$$

$$\varepsilon(x \otimes y) = E(xy), \tag{A2}$$

$$S(x \otimes y) = y \otimes \theta(x). \tag{A3}$$

The verification of the WHA axioms is left to the reader. The left and right subalgebras of $B \otimes B^{\mathrm{op}}$ are $B \otimes 1$ and $1 \otimes B$, respectively, because we have

$$\sqcap^L(x \otimes y) = xy \otimes 1, \qquad \sqcap^R(x \otimes y) = 1 \otimes y\theta(x). \tag{A4}$$

Let A be an arbitrary WHA over K. Then $A^L A^R$ is a sub-WHA with hypercenter $A^L \cap A^R$. Thus $A^L A^R$ decomposes into a direct sum of WHAs each summand being isomorphic to a WHA of the type $B \otimes B^{\mathrm{op}}$.

Since $B \otimes B^{\mathrm{op}}$ is separable, by Theorem 3.13, it must contain a normalized left integral. Indeed,

$$l := \sum_i f_i \otimes e_i \equiv S^2(1_{(2)})1_{(1)} \tag{A5}$$

is such a left integral. What is more, it is non-degenerate.

Before looking for Haar integrals some remarks about innerness of θ are in order. The quantity $q = \sum_i e_i f_i$ always implements θ^{-1}; i.e., $xq = q\theta(x)$ for $x \in B$, but it is not necessarily invertible. (For example, for $B = M_2(\mathbb{Z}_2)$ and for any non-degenerate functional E the q is identically zero.) In fact q is invertible iff the left regular trace on B is non-degenerate (especially if K is of characteristic zero). Fortunately one can circumvent this nuisance by using the existence of a non-degenerate trace tr on any separable algebra B (see [5]). Then the Radon–Nykodim derivative γ of E w.r.t. tr provides an invertible element implementing θ,

$$E(x) = \mathrm{tr}(x\gamma), \qquad \theta(x) = \gamma x \gamma^{-1}, \qquad x \in B. \tag{A6}$$

This proves that θ is inner and therefore so is the square of the antipode, $S^2 = \theta \otimes \theta$.

Omitting the details we can now formulate the condition for the existence of the Haar measures h and $\hat{h}$ as follows. Haar measure in $B \otimes B^{\mathrm{op}}$ exists iff $\sum_i f_i \gamma^2 e_i$ is invertible and Haar measure in $\widehat{B \otimes B^{\mathrm{op}}}$ exists iff $E(1_B) \neq 0$.

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