Invariant Curves and Topological Invariants for Real Plane Analytic Vector Fields

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Let Z be a germ of a singular real analytic vector field at 0 ∈ ℝ². We give conditions on the multiplicity and the Milnor number of Z which imply that the foliation defined by Z has a characteristic orbit or an analytic invariant curve with the hypothesis that Z is a real generalized curve. Then it is proved that for a non-dicritical real generalized curve, the multiplicity mod 2 is invariant under bilipschitz homeomorphisms preserving foliations.

1. INTRODUCTION

In a famous paper [2], Camacho and Sad proved the existence of an invariant holomorphic germ (what they call a separatrix) for a singular holomorphic foliation at 0 ∈ ℂ². For a singular real analytic foliation at 0 ∈ ℝ², a reduced equation of this (complex) separatrix can be chosen with real coefficients. However, the zero set of this equation (the real part of the complex separatrix) may be reduced to the point 0, e.g., in the case of a center or a focus.

In this paper, we give sufficient algebraic conditions on the coefficients of the vector field defining the foliation, such that there exists a real analytic invariant curve (one dimensional over ℝ). The conditions are valid under the hypothesis that the foliation is a real generalized curve (RGC for short, inspired by the concept of generalized curve of [1]); a foliation is a RGC if there is no real saddle-node singularity at the end of the resolution process (see Section 2). For a real analytic plane vector field Z, set ν for its multiplicity and μ for its Milnor number. The main result of the paper (Propositions 3.1 and 3.9) is then that if Z is a RGC for which ν or μ is even, there exists a real separatrix (i.e., a real analytic invariant curve containing the origin). In the general case (i.e., without the RGC hypothesis),

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the algebraic conditions (namely \(r\) or \(u\) even) imply the existence of a characteristic orbit, i.e., a trajectory \(\gamma\) tending to \(O\) with a tangent (a characteristic orbit is not necessarily semi-analytic at \(O\)). Note that the main result of Camacho and Sad \([2]\), namely the existence of a (complex) separatrix, is not used in the proof. Similarly, in the paper \([1]\), the Camacho–Sad result is not used.

The last section addresses some problems of invariance of multiplicity and the Milnor number of two real analytic local singular foliations of the plane under some class of homeomorphisms sending leaf to leaf. In case both foliations are RGC and not dicritical, it is proved that the multiplicity mod 2 is invariant by a bilipchitz homeomorphism. The invariance of multiplicity seems more involved and is not even proved (as far as I know) for a homeomorphism in the complex case (see \([1]\) for some informations about this problem).

2. DEFINITIONS AND NOTATION

Throughout this paper, we use the same notation for a germ (of a function, a vector field, etc.), and a representative of this germ in a suitable open set.

Let \(Z = a(x, y) \frac{\partial}{\partial x} + b(x, y) \frac{\partial}{\partial y}\) be a germ at \(O \in \mathbb{R}^2\) of a real analytic vector field. Assume that it is defined in an open set \(U\), and that \(Z(O) = 0\).

We are interested in the foliation \(\mathcal{F}_Z\) induced in \(U\) by \(Z\), i.e., by the kernel of the differential form

\[
\omega = b(x, y) \, dx - a(x, y) \, dy
\]

assuming that \(a(x, y)\) and \(b(x, y)\) are relatively prime in the ring \(\mathbb{R}[x, y]\).

Set \(\nu_Z\) (or \(\nu\) if the context is clear) for the multiplicity of \(Z\) at \(O\): \(\nu_Z = \inf(e(a), e(b))\), where, for a series \(f\), \(e(f)\) stands for the order of \(f\) at \(O\).

Since we assume that \(a\) and \(b\) are relatively prime in \(\mathbb{R}[x, y]\), we can define

\[
\mu_Z^C = \dim_{\mathbb{R}} \mathbb{R}[x, y]_{(a, b)}
\]

(\(\mu_Z^C\) is the Milnor number of the vector field \(Z^C\), complexification of \(Z\)).

We set \(\mu_R^Z\) (or \(\mu_R\) for the index of the vector field \(Z\) at \(O\) (degree of the map induced by \((a, b)\) on a small circle \(S'_1\)).

If \(q\) is an isolated singular point of a complex foliation \(\mathcal{F}_C\) (on a complex manifold of dimension 2), we will set \(\mu^C(\mathcal{F}, q)\) for the Milnor number of a vector field inducing \(\mathcal{F}\) near \(q\).
It is well known (see, e.g., [1, 6]) that the foliation \( F \) of an open set \( U_C \subset \mathbb{C}^2 \) defined by the vector field \( Z \) has a desingularization \( \pi: S_C \to U_C \) defined by a sequence of blowing-up. The foliation \( \mathcal{F}_C \) on \( U_C \setminus \{ O \} \) lifts to a foliation on \( S_C \setminus \pi^{-1}(O) \), \( \pi^{-1}(O) \) being the exceptional divisor. This foliation can be extended to \( S_C \) into a foliation \( \mathcal{F}_C \) such that each component of \( \pi^{-1}(O) \) is either a union of leaves or a dicritical component. The foliation \( \mathcal{F}_C \) has the following properties:

1. The singular points of \( \mathcal{F}_C \) are simple, that is, the eigenvalues \( (\lambda_1, \lambda_2) \) of the linear part of the vector field defining \( \mathcal{F}_C \) near such a point satisfy one of the conditions:
   
   (a) \( \lambda_1, \lambda_2 \neq 0 \) and \( \lambda_1/\lambda_2 \notin \mathbb{Q} \), (node case).
   
   (b) \( \lambda_1 = 0, \lambda_2 \neq 0 \), (saddle-node case).

2. To each projective line \( P \) appearing in the desingularization process (and in particular to each component \( E = \pi^{-1}(O) \)) is associated a weight \( w(P) \) defined as
   
   (a) \( 1 \), if the projective line \( P \) appears after exploding \( O \in U_C \),
   
   (b) the sum of the weights of the projective lines meeting at the singularity which is blown up to originate \( P \). (See [1], p. 159).

3. There is a canonical lifting \( \tilde{\sigma} \) on \( S_C \) of the complex conjugation \( \sigma \) (we always assume that \( U_C \) is invariant by \( \sigma \)). This is because a blowing-up is defined by real equations \( (x = \tilde{x}, y = i\tilde{x}) \) in a chart. The fixed part \( \tilde{S} \) of \( \tilde{\sigma} \) is a smooth real surface called the real desingularization of \( \mathcal{F} \). There is a foliation \( \mathcal{F} \) on \( \tilde{S} \) which lifts the foliation \( \mathcal{F} \) defined on \( U \setminus \{ O \} \) by \( Z \). The mapping \( \pi|: S \to U \) is a composition of blowing-up of real points.

**Definition 2.1.**

1. \( Z \) is a complex generalized curve (see [1]), CGC for short, if \( \mathcal{F}_C \) has no singularity with zero eigenvalue.

2. \( Z \) is a real generalized curve, RGC for short, if \( \mathcal{F} \) has no singularity with zero eigenvalue.

Note that, following [1], the desingularization of a CGC or of a RGC could have dicritical components.

**Definition 2.2.**

1. A characteristic orbit for \( Z \) is an orbit tending to \( O \) with a tangent (see [5]).

2. Following [1], we define a real (analytic) separatrix as a real analytic germ \( f \) invariant by the foliation defined by \( Z \), such that if \( f = \prod f_i \) is the decomposition of \( f \) into irreducible components, the topological dimension of the zero set \( Z(f_i) \) of each \( f_i \) is one (then, the ideals \( (f_i) \)'s are real in the sense of 2.3). Therefore, if \( \gamma \) is the zero set of \( f, \gamma \setminus \{ O \} \) is a union
of leaves (and therefore of characteristic orbits; note that we do not assume $\gamma$ irreducible, as in [1]).

3. The isolated separatrix of $\mathcal{F}_c$ is the (may be empty) union of all irreducible (complex) separatrices $C_i$ with reduced equation $f_i$ such that at the end of the resolution process, its strict transform $\tilde{f}_i$ meets transversally a nondicritical component of the exceptional divisor of the (complex) resolution of $Z^c$ (see [1, p. 170]). Set $f = \prod f_i$; note that we may assume that $f \in \mathbb{R}[x, y]$. If there are dicritical components in the resolution, the set of isolated separatrices may be empty (e.g., for the vector field $Z = x\partial / \partial x + y\partial / \partial y$). In case there is no isolated separatrix, we set $f = 1$.

We have the following properties of separatrices:

- The foliation $\mathcal{F}_c$ has always a (complex) analytic separatrix ([2]; see also [3]).

- If $\gamma$ is a characteristic orbit, its strict transform $\tilde{\gamma}$ by the resolution $\pi$ meets the exceptional divisor $\pi^{-1}(0)$ at a simple singularity (see [1] or [4]).

- If $\tilde{\gamma}$ is a separatrix (resp. a characteristic orbit) of the foliation $\mathcal{F}$ at some singular point $\tilde{P} \in \pi^{-1}(0)$, and if $\tilde{\gamma}$ is not contained in the exceptional divisor $\pi^{-1}(O)$, the image $\pi(\tilde{\gamma})$ is a separatrix (resp. a characteristic orbit); this is clear in the case of a characteristic orbit and results of the properness of $\pi$ in the case of a separatrix (see [2, p. 584]).

Let $f \in \mathbb{R}[x, y]$ be a germ of an analytic function, which we assume irreducible. We say that the ideal $(f)$ is real if the curve $C$ defined by $f = 0$ has topological dimension 1. We have the (easy) following result (see [8]).

**Proposition 2.3.** Let $f \in \mathbb{R}[x, y]$ be an irreducible germ; the following assertions are equivalent:

1. $(f)$ is real,
2. $f$ is irreducible in $\mathbb{C}[x, y]$,
3. $f$ takes locally positive and negative values.

If $f \in \mathbb{R}[x, y]$ (or $\mathbb{C}[x, y]$) is a reduced germ, $C$ the germ of curve defined by the equation $f = 0$, we will say simply that $C$ is a curve with equation $f$.

3. **EXISTENCE OF REAL SEPARATRIX**

In this section, we want to look at conditions for the existence of a real analytic separatrix for a germ of real plane analytic vector field $Z$. 
Throughout this section, $Z$ will be a real vector field defined on $U \subset \mathbb{R}^2$, for which $O$ is the only singular point.

**Proposition 3.1.** Assume that $Z$ is a RGC and that its multiplicity $\nu$ satisfies $\nu \equiv 0 \mod 2$. Then it has a real separatrix.

**Proof.** Let $\pi: \tilde{S}_c \rightarrow U_c$ be the desingularization of the complexification of $Z$.

If $D$ is a smooth analytic curve in $\mathbb{C}^2$, $q \in D$, and $f$ an analytic function, we define $e_D(f, D)$ as the order at $q$ of $f_D$ (i.e., if $q = O$ and $D$ is defined locally at $q$ by $y = 0$, $e_D(f, D)$ is the order of the series $f(x, 0)$).

Similarly [1], if $D$ is a smooth invariant curve and $q$ a singular point of a foliation $\mathcal{F}_C$, we set $e_{\mathcal{F}_C}(q, D)$ for the multiplicity of $\mathcal{F}_C$ at $q$ along $D$ (by definition, $e_{\mathcal{F}_C}(q, D) = 0$ if $q \notin D$; if $q \in D$, choosing convenient coordinates centered at $q$, we may assume that the foliation $\mathcal{F}_C$ is defined by

$$\dot{x} = x^n P(x) + y Q(x, y),$$
$$\dot{y} = y R(x, y),$$

where $y = 0$ stands for $D$ and $P(0) \neq 0$. Then $e_{\mathcal{F}_C}(q, D) = n$.

Let $E$ be an invariant line, component of the exceptional divisor of $\tilde{S}$, $q \in E$ a singularity of $\mathcal{F}_C$. As in [1], define:

$$\phi(q, E) = \begin{cases} e_{\mathcal{F}_C}(q, E), & \text{if } q \in E \text{ is not a corner} \\ e_{\mathcal{F}_C}(q, E) - 1, & \text{if } q \notin E \text{ is a corner}. \end{cases}$$

(1)

Recall that a corner is the intersection of two components of the exceptional divisor.

For simplicity, we will first prove Proposition 3.1 assuming that the desingularization of $Z^C$ has no dicritical component.

If $g$ is an analytic germ at $O \in U$, we will note $\tilde{g}_q$ the germ at $q \in \tilde{S}$ of the strict transform $\tilde{g}$ of $g$. Since we first assume to be in the nondicritical case, the set of irreducible (complex) separatrices is finite: they are all isolated. We will set $f$ for the product of the reduced equation of all the irreducible (complex) separatrices; as noted above, we may assume that $f \in \mathbb{R}\{x, y\}$, because the whole situation is invariant by complex conjugation $\sigma$.

If $q \in \tilde{S}$, we set $\tilde{f}_q$ for the germ at $q$ of the strict transform of $f$.

**Lemma 3.2.** Let $Z$ be a germ of analytic vector field at $O \in \mathbb{R}^2$. Assume that $O$ is an isolated singularity for $Z$ and that the (complex) desingularization of $Z$ has no dicritical component. We have the following formulas,

$$\nu + 1 = \sum v(E) \phi(q, E)$$

(2)
and, \( e \) being the order of the separatrix \( f \) at \( O \),

\[
e = \sum w(E) e_q(\tilde{f}_q, E),
\]

(3)

where both sums are taken on the set \( \text{Sing}(\tilde{F}_C) \) of singular points of the foliation \( \tilde{F}_C \).

The first relation is in [1, Theorem 1], and the second is easily proved following the same scheme. Note that in (3), we may have \( f_q = 1 \) for some singular point \( q \); this implies \( e_q(\tilde{f}_q, E) = 0 \); note also that the corners do not enter in (3) because the singularities of \( F_C \) are simple by hypothesis nor in (2) in the CGC hypothesis by definition of \( \phi(q, E) \). However, in the general case, there could exist a saddle-node singularity at a corner of the resolution. We now get immediately:

**Corollary 3.3.** Assume that there is no dicritical component in the desingularization of \( Z^C \); then

- if \( Z \) is a CGC, then \( e = v + 1 \).
- If \( Z \) is a RGC, then \( e \equiv v + 1 \mod 2 \)

**Proof.** The first assertion is in [1, Theorem 3]. To prove the second, set \( \text{Sing}(\tilde{F}) \) for the set of (real) singular points of the foliation \( \tilde{F} \); then we have clearly that

\[
\sum_{\text{Sing}(\tilde{F}_C)} w(E) \phi(q, E) \equiv \sum_{\text{Sing}(\tilde{F})} w(E) \phi(q, E) \mod 2
\]

because non-real singular points enter in pairs exchanged by complex conjugation \( \tilde{\sigma} \). For the same reason,

\[
\sum_{\text{Sing}(\tilde{F}_C)} w(E) e_q(\tilde{f}_q, E) \equiv \sum_{\text{Sing}(\tilde{F})} w(E) e_q(\tilde{f}_q, E) \mod 2.
\]

But the points of \( \text{Sing}(\tilde{F}) \) which enter in both sums are not corners (see above) or saddle-nodes (by RGC hypothesis). Then for such a point \( q \), we get \( e_q(\tilde{f}_q, E) = \phi(q, E) \) (\( = e_{\tilde{\sigma}}(q, E) \)) = 1, which gives \( e \equiv v + 1 \mod 2 \), using (2) and (3).

Proposition 3.1 follows immediately in the nondicritical case, because an analytic germ \( f \in R[x, y] \) with an odd multiplicity \( e \) has necessarily an irreducible component such that the ideal \( (f) \) is real (see 2.3: if \( (f) \) is not real, then \( f = gg \) in \( C[x, y] \) and its multiplicity is even).

Let us now look at the case when dicritical components exist. If a real dicritical component appears in the desingularization of \( Z \), then clearly there will exist an infinite number of real separatrices (see [2, p. 584]).
us therefore assume that there is no real dicritical component. Let \( f \) be the (reduced) germ with real coefficients such that equation \( f \) defines the isolated separatrix (see 2.2); if \( f = 1 \), \( e(f) = 0 \) by definition.

**Lemma 3.4.** With the above notation, we have

\[
v + 1 \equiv \sum w(E) \phi(q, E) \mod 2
\]

and, \( e \) being the order of the isolated separatrix \( f \) at \( O \),

\[
e \equiv \sum w(E) e(\mathcal{F}, E) \mod 2,
\]

where both sums are taken on the set \( \text{Sing}(\mathcal{F}) \) of real singular points of foliation \( \mathcal{F} \).

The proof is similar to the proof of (2) and (3) (see [1]): just replace “=” by “\( \equiv \mod 2 \)” at each stage of the proof. We then have again the congruence \( v + 1 \equiv e \mod 2 \) in case \( Z \) is a RGC, with the above notation. The proof of Proposition 3.1 is now complete.

**Corollary 3.5.** Let \( Z \) be a germ of an analytic vector field at \( O \in \mathbb{R}^2 \); then if \( v \equiv 0 \mod 2 \), there is always a characteristic orbit.

**Proof.** In view of Proposition 3.1, we may assume that a real saddle-node simple singular point \( q \) appears in the desingularization process of \( Z \).

We have the following result.

**Lemma 3.6.** Let \( \mathcal{F} \) be a foliation of a real analytic surface and \( q \) a saddle-node for \( \mathcal{F} \) with two real analytic smooth separatrices \( E \) and \( E_1 \) and no other characteristic orbit. Then, assuming that \( E \) is tangent to the eigenspace corresponding to the nonzero eigenvalue, we have that \( e(\mathcal{F}, q, E_1) \) is odd (and \( e(\mathcal{F}, q, E) = 1 \) by hypothesis).

**Proof.** If \( e(\mathcal{F}, q, E_1) \) were even, \( q \) would be an attractive (or repulsive) singular point in one of the half-spaces delimited by \( E \), but that is incompatible with the hypothesis that there are no characteristic orbits other than the components of \( E \setminus \{q\} \) and \( E_1 \setminus \{q\} \).

Now, if \( q \in \pi^{-1}(0) \) is not a corner, let \( E \) be the component of \( \pi^{-1}(O) \) passing by \( q \). Then there is a characteristic orbit for the foliation \( \mathcal{F} \), transversal to \( E \) at \( q \), which gives by blowing down a characteristic orbit. If \( q \) is a corner, say \( q = E \cap E_1 \), it might be that the only characteristic orbits of \( \mathcal{F} \) at \( q \) are components of \( E \setminus \{q\} \) and \( E_1 \setminus \{q\} \) as in Lemma 3.6 (if not, blowing down give a characteristic orbit). But if all simple saddle-nodes \( q \), appearing at the end of the desingularization process are at corners and
satisfy the hypotheses of Lemma 3.6, then \( \phi(q_i, E_i) \) is even for all such \( q_i \) by Lemma 3.6 and definition (1) of \( \phi(q_i, E_i) \). Then (4) is again valid for the foliation \( \tilde{F} \), the summation being taken on all the real simple singularities of \( \tilde{F} \) which are not at a corner, which implies by the same arguments as in the proof of Proposition 3.1 that in this case there is a real separatrix.

**Example 3.7.**

1. The vector field

\[
Z = x\partial_x - y^3 \partial_y
\]

satisfies the hypothesis of Lemma 3.6 at \( q = O \).

2. The vector field

\[
Z = y\partial_x - x\partial_y
\]

is a CGC, verifies \( v_Z = 1 \) (and \( \mu_Z = 1 \)), and has no real separatrix.

3. (Kindly pointed to me by C. Camacho.) The vector field

\[
Z = (-y^2 + x^4) \partial_x + (xy + x^3 + x^3y) \partial_y
\]

is not a RGC. It has a characteristic orbit (conformally to Corollary 3.5 since \( v = 2 \) is even) but no real separatrix. In fact, in the chart of the blowing-up of \( O \) given by \( y = tx \), the foliation defined by \( Z \) lifts to a foliation \( \tilde{F}_C \) given by

\[
\begin{align*}
\dot{x} &= -t^2x + x^3 \\
\dot{t} &= t + x + t^3.
\end{align*}
\]

In this chart the exceptional divisor is defined by equation \( x = 0 \), and we have \( \text{Sing}(\tilde{F}_C) = \{(0, 0), (0, i), (0, -i)\} \) in the \((x, t)\) coordinates, with eigenvalues \((0, 1)\) at \((0, 0)\) (saddle node), and \((1, -2)\) at \((0, i)\) and \((0, -i)\) which by blowing-down give two complex conjugate separatrices.

Let us now look at a similar result as Proposition 3.1, but with the Milnor number \( \mu \) replacing the algebraic multiplicity \( v \). Let us first quote the following well-known result:

**Lemma 3.8.** Let \( Z = a(x, y) \partial_x + b(x, y) \partial_y \) be a germ at \( O \in \mathbb{R}^2 \) of a real analytic vector field. Then

\[
\mu_Z \equiv \mu_Z^C \mod 2.
\]

**Proof.** If \( c = (c_1, c_2) \) is a small regular value of \((a, b)\), the vector field

\[
\tilde{Z} = (a(x, y) - c_1) \partial_x + (b(x, y) - c_2) \partial_y
\]
has \( \mu_Z^C \) (complex) simple singularities in a small ball of center \( O \). Assume it has \( k \) real singularities; we have \( k \equiv \mu_Z^C \mod 2 \), and each real non-degenerate singularity has index \( \pm 1 \).

Since \( \mu_Z^R \equiv \mu_Z^C \mod 2 \), and we are interested only in their common value \( \mod 2 \), we delete the superscript \( R \) or \( C \). We have then:

**Proposition 3.9.** Assume that \( Z \) is a RGC, and that its (real or complex) Milnor number satisfies \( \mu_Z \equiv 0 \mod 2 \). Then it has a real separatrix.

**Proof.** As above, we may assume that there is no real dicritical component in the resolution of \( Z \) (otherwise, there exist an infinite number of real analytic separatrices). As above, we will set \( f = 0 \) for a reduced real equation of the isolated separatrix of \( Z^C \) (see Definition 2.2).

For a real (reduced) singular analytic germ \( g \in \mathbb{R}[x, y] \) at \( O \), set \( \mu(g) \) for its Milnor number, that is

\[
\mu(g) = \dim_{\mathbb{R}} \mathbb{R}[x, y] / (g_x, g_y).
\]

By convention, we set \( \mu(g) = -1 \) if \( g = 1 \) (or if \( g \) is invertible in the ring \( \mathbb{R}[x, y] \)).

**Lemma 3.10.** Let \( Z \) be a RGC. Assume that there is no real dicritical component in the resolution of \( Z \). Let \( f \in \mathbb{R}[x, y] \) be a reduced equation of the isolated separatrix. Then

\[
\mu_Z \equiv \mu(f) \mod 2.
\]

**Proof.** By induction on the number \( p \) of blowing-up needed to desingularize \( Z \), if \( p = 0 \), we have \( \mu_Z = \mu(f) = 1 \) since by the RGC hypothesis \( O \) is not a saddle-node.

In the general case, let us set \( \pi_1: \tilde{U} \to U \) for the blowing-up of the point \( O \). Let \( \mathcal{F}_C \) be the foliation induced on \( U_C \) by \( Z \), \( \mathcal{F}_C \) the corresponding foliation on \( \tilde{U}_C \) and let \( p_1, \ldots, p_s \) be the singularities of \( \mathcal{F}_C \), \( p_1, \ldots, p_s \) being real, and \( p_{s+1}, \ldots, p_k \) non-real. Then at each \( p_i, 1 \leq i \leq s \), the germ of \( \mathcal{F} \) is defined by a vector field \( \tilde{Z}_i \) which is again a RGC. From formulas (2.1) and (2.2) of [6, pp. 513, 514], called sometimes Noether's formulas, we have that

\[
\mu_Z \equiv \sum_{i=1}^{k} \mu(\mathcal{F}_C, p_i) + 1 \mod 2
\]
(the formulas in the dicritical and nondicritical cases are different, but the same mod 2). Then

\[ \mu_Z \equiv \sum_{i=1}^{s} \mu(F, p_i) + 1 \mod 2 \]  

since the non-real singularities enter by pairs.

Let \( E \) be the exceptional divisor of the blowing-up of \( O \). Fix \( i \) \((1 \leq i \leq s)\) and let \((x, t)\) be local coordinates at the point \( p_i \) such that \( x = 0 \) is a local equation for \( E \) near \( p_i \). In the nondicritical case (which is the case by hypothesis), \( E \) is a union of leaves of the foliation \( \mathcal{F} \). Then \( E \) is a component of the isolated separatrix of \( \mathcal{F} \) at \( p_i \), and we can set \( f_i = x f^i \) for an equation for this isolated separatrix. Let \( Z_i \) be an analytic vector field (with isolated singularity at \( p_i \)) defining the foliation \( \mathcal{F} \) near \( p_i \). Now, by induction hypothesis, we have

\[ \mu(Z_i, p_i) = \mu(f_i, p_i) \mod 2 \]

and therefore

\[ \mu_Z \equiv \sum_{i=1}^{s} \mu(f_i, p_i) + 1 \mod 2 \]

by (7).

Now, some of the \( f_i \)'s can a priori be 1, because we do not use the Camacho–Sad result or because a non-real \( p_i \) could be a saddle-node or have a dicritical component in its resolution. Assume that (reindexing the \( p_i \)'s), among the \( p_i \)'s such that \( f_i \neq 1, p_1, \ldots, p_t \) are real \((t \leq s)\), and \( p_{t+1}, \ldots, p_q \) non-real \((q \leq k)\).

Recall the following two formulas (see [9]): first, if a germ \( f \in \mathcal{R}[x, y] \) is of the form \( f = g_1 \cdots g_q \) with \( g_i \in \mathcal{R}[x, y] \) pairwise coprimes, then

\[ \mu(f) = \sum_{i=1}^{q} \mu(g_i) + 2 \sum_{i,j} (g_i, g_j) - q + 1, \]  

\((g_i, g_j)\) being the intersection multiplicity of the germs \( g_i \) and \( g_j \).

Second, let \( f \in \mathcal{R}[x, y] \) be a germ of analytic function with one tangent at \( O \) and \( \tilde{f} \) its strict transform by the blowing-up of \( O \). Then

\[ \mu(f) = \mu(\tilde{f}) \mod 2 \]  

(in fact, if \( v \) is the order of \( f \) at \( O \), we have \( \mu(f) = \mu(\tilde{f}) + v(v-1) \)).
Here, formula (8) applied to $f_i = x f_i$ gives

$$\mu(f_i) \equiv \mu(f_i) + 1 \mod 2.$$ 

Note that this last formula is again valid if $f_i = 1$ (with $\mu(1) = -1$ by definition). Therefore we have

$$\sum_{i=1}^s \mu(f_i, p_i) + 1 \equiv \sum_{i=1}^s \mu(f_i, p_i) + s + 1 \mod 2. \quad (10)$$

Now, we may write

$$f = \prod_{i=1}^q g_i,$$

each $g_i$ being a (complex) analytic germ having one tangent at $O$ corresponding to some point $p_i$ in the blowing-up; then the $g_i$'s are pairwise coprimes, and the strict transform of $g_i$ is $\tilde{g}_i = \tilde{f}_i$. Note that the RGC hypothesis, the fact that we assume that there is no real dicritical component in the resolution and the Camacho–Sad result imply that for $1 \leq i \leq s$, $\tilde{f}_i \neq 1$, but we do not need this fact in the proof.

Formula (9) gives

$$\mu(g_i) \equiv \mu(f_i, p_i) \mod 2$$

for $1 \leq i \leq t$, and formulas (8), (9) and (10) give:

$$\mu(f) \equiv \sum_{i=1}^q \mu(g_i) - q + 1 \equiv \sum_{i=1}^q \mu(\tilde{f}_i, p_i) - q + 1 \mod 2$$

$$\equiv \sum_{i=1}^t \mu(\tilde{f}_i, p_i) - t + 1 \equiv \sum_{i=1}^t \mu(f_i, p_i) + 1$$

$$\equiv \sum_{i=1}^t \mu(f_i, p_i) + 1 \equiv \mu_Z \mod 2. \quad (11)$$

Let us now finish the proof of 3.9. Let $f$ be an equation for the isolated separatrices and $r$ its number of (real or complex) branches. We have the following formula [7],

$$\mu(f) = 2\delta - r + 1,$$

where $\delta$ is an integer, analytic invariant of the germ $f$ (for instance, $\delta$ can be defined as the length of $\mathcal{E}/\mathcal{E}$, where $\mathcal{E}$ stands for the ring $R[x, y]/(f)$, and $\mathcal{E}$ for its normalization).
We have therefore
\[ \mu(f) \equiv r + 1 \mod 2. \]

Then the hypothesis and (11) imply that \( r \) is odd and therefore that \( f \) has a real branch. \( \square \)

As above, we have the following corollary:

**Corollary 3.11.** Let \( Z \) be a germ of an analytic vector field at \( O \in \mathbb{R}^2 \); then if \( \mu_Z \equiv 0 \mod 2 \), there is always a real characteristic orbit.

**Proof.** As above, we may assume that all simple saddle-nodes appearing at the end of the desingularization process of \( F \) are at corners and satisfy the hypotheses of Lemma 6. Then the foliation \( \mathcal{F} \) satisfies the following hypothesis:

(H): There is no real dicritical component in the resolution of \( Z \), and the real saddle-nodes appearing in the resolution process have a finite number of characteristic orbits.

For a saddle-node which has a finite number of characteristic orbits, the proof of Lemma 3.6 shows that its complex Milnor number \( \mu_q \) is odd (being equal to \( e_F(q, E_1) \), with the notation of 3.6); note that \( e_F(q, E_1) \) can be computed at the formal level, even if there is no analytic invariant curve tangent to the eigenspace corresponding to the 0 eigenvalue.

Now, Proposition 3.9 is valid for a vector field \( Z \) satisfying (H) and \( \mu_Z \equiv 0 \mod 2 \); just replace the equality \( \mu_Z = 1 \) by \( \mu_Z \equiv 1 \mod 2 \) in the first step of the proof of Lemma 3.10. \( \square \)

**Remark 3.12.** The hypotheses of Propositions 3.1 and 3.9 (namely \( \nu \equiv 0 \mod 2 \) and \( \mu \equiv 0 \mod 2 \)) for a real analytic germ of vector field are independent, as shown by the following examples.

1. In Example 3.7 (3), one has \( \nu(Z) = 2 \), \( \mu(Z) = 8 \). Modifying slightly \( Z \) to get
   \[ Z_1 = (-y^2 + x^4) \frac{\partial}{\partial x} + (xy + x^3 + x^4y) \frac{\partial}{\partial y} \]
   gives \( \nu(Z_1) = 2 \), \( \mu(Z_1) = 9 \).

2. Let \( Z \) be the field
   \[ Z = (py^{p-1}) \frac{\partial}{\partial x} + (qx^{q-1}) \frac{\partial}{\partial y} \]
   \( (Z \) is the Hamiltonian field corresponding to the germ \( y^p - x^q = 0 \), one has \( \nu(Z) = \min(p, q) \), \( \mu(Z) = (p - 1)(q - 1) \). Then \( \mu \) can be even (e.g., if \( f \) is irreducible, i.e., \( (p, q) = 1 \)), and simultaneously \( \nu \) can be odd.\]
4. TOPOLOGICAL AND DIFFERENTIAL INVARIANTS

In this section, we study some invariants of a germ of foliation at \((\mathbb{R}^2, O)\) defined by an analytic germ of vector field \(Z\) with isolated singularity at \(O\), under various types of homeomorphisms.

4.1. Topological Invariants

As in [1], Theorem A, we have that \(\mu^R_Z\) is a topological invariant.

Proposition 4.1. Let \(Z\) and \(\tilde{Z}\) be real analytic germs of vector fields at \(p\) and \(\tilde{p}\), locally topologically equivalent. Then

\[
\mu^R_Z = \mu^R_{\tilde{Z}}.
\]

The proof is exactly the same than the proof of [1] for the complex case.

Remark 4.2. 1. There is no invariance property (under topological equivalence) concerning the multiplicity \(v\). In fact, for any pair \((p, q)\) of relatively prime integers, the singular foliation defined by the vector field \(Z = qx^{q-1}\partial/\partial y + py^{p-1}\partial/\partial x\), which gives the leaves \(y^p - x^q = \lambda (\lambda \in \mathbb{R})\) is topologically equivalent to the trivial foliation \(x = \lambda\). Moreover, it is not known (at least to me) if the multiplicity \(v\) is a topological invariant for germs of complex analytic foliations (see [1]).

2. The foliations defined by \(Z = x\partial/\partial x + y\partial/\partial y\) and \(Z' = (x - y(1 - r^2))\partial/\partial y - (y + x(1 - r^2))\partial/\partial x\), \(r^2 = x^2 + y^2\), are topologically equivalent, but the first one has an infinite number of (analytic) separatrices (the lines \(y = \lambda x\)) and the second no real separatrix (it is a focus).

4.2. Bilipschitz Invariants

We say that two germs of foliations \(\mathcal{F}\) and \(\mathcal{F}'\) at \(p\) and \(p'\), respectively, are locally bilipschitz equivalent if there exists a germ of bilipschitz homeomorphism:

\[
\phi: (U, p) \rightarrow (U', p')
\]

sending a leaf of \(\mathcal{F}\) into a leaf of \(\mathcal{F}'\).

Note that the bilipschitz character of \(\phi\) is not sufficient to distinguish between a focus and a radial field (like in the above Remark 4.2(2)). We have the following fact, kindly pointed out to me by R. Moussu:

Lemma 4.3. let \(X_j, j = 1, 2\), be two germs of \(C^\infty\) germs of vector fields at \(O \in \mathbb{R}^2\) such that for \(j = 1, 2\), \(\text{Spec}DX_j(O) = (\lambda_j, \hat{\lambda}_j)\), with \(\text{Re}(\lambda_j) < 0\). Then the flows of \(X_1\) and \(\text{Re}(\lambda_1)/\text{Re}(\lambda_2) X_2\) are bilipschitz conjugated (and therefore the two foliations are bilipschitz equivalent).
Apply this lemma to the vector fields $X_1 = -(x\partial / \partial x + y\partial / \partial y)$, and $X_2 = X_1 + (x(x\partial / \partial y - y\partial / \partial x))$.

We have the following proposition:

**Proposition 4.4.** Let $Z$ and $Z'$ be real analytic germs of plane vector fields at $p$ and $p'$ inducing germs of foliations $\mathcal{F}$ and $\mathcal{F}'$ locally bilipschitz equivalent. Assume that in the resolution process of $Z$ (resp. $Z'$) there is no real dicritical component and that $Z$ and $Z'$ are RGC. Then if $v$ (resp. $v'$) is the order of $Z$ (resp. $Z'$) at $p$ ($p'$), we have

$$v \equiv v' \mod 2.$$ 

**Proof.** Let $e_C$ denotes the multiplicity of the (complex) separatrix of $Z$. We have $e_C \equiv v + 1 \mod 2$ since $Z$ is a RGC (see the remark after Lemma 3.4). Similarly, we get $e_C \equiv v' + 1 \mod 2$. Now, let $C$ (resp. $C'$) be the real separatrix of $Z$ (resp. $Z'$) (the real separatrix is the union of the real components of the complex separatrix), $e$ (resp. $e'$) the multiplicity of $C$ (resp. $C'$). We have clearly $e \equiv e_C \mod 2$ and $e' \equiv e'_C \mod 2$. We call a connected component of $C\setminus\{p\}$ a half-branch of $C$. Then, if $\phi$ is a local bilipschitz homeomorphism, it sends $C$ onto $C'$ (more precisely, it sends each half-branch of $C$ onto a half-branch of $C'$, since it must send a characteristic orbit onto a characteristic orbit). Proposition 4.4 is therefore in the considered case a consequence of:

**Lemma 4.5.** Let $C$ and $C'$ be two germs of real analytic curves bilipschitz equivalent, $e$ and $e'$ their multiplicities. Then,

$$e \equiv e' \mod 2.$$ 

**Proof.** (a) Let us take two half-branches $\gamma_1$ and $\gamma_2$ of $C$ with the same tangent $T$. Let $T'$ be the orthogonal line to $T$ by $p$. If $\gamma_1$ and $\gamma_2$ are separated by $T'$, $d(x, p)$ is equivalent to $d(x, \gamma_2)$ for $x \in \gamma_1$ (that means that there exist two constants $c_1$ and $c_2$ and a neighborhood $U$ of $p$ such that for $x \in \gamma_1 \cap U$ one has $c_1 d(x, p) \leq d(x, \gamma_2) \leq c_2 d(x, p)$). If not, i.e., if they are on the same side in relation to $T'$, $d(x, \gamma_2)$ is equivalent to $d(x, p)^k$ for some integer $k > 1$. This exponent $k$ is invariant by a bilipschitz homeomorphism. In particular, this remark proves the lemma in the case where $C$ is an irreducible germ of curve whose ideal is real (see 2.3).

(b) An irreducible component of $C$ is called even (resp. odd) if its multiplicity is even (resp. odd). The germ $C$ has a finite number of tangents $(T_i)_{1 \leq i \leq \ell}$ at $p$. For each tangent $T_i$, let $C_i$ be the union of the irreducible components of $C$ which are tangent to $T_i$, $e_i$ the multiplicity of $C_i$, and $e = \sum e_i$ the multiplicity of $C$. As above a line $T'_i$ orthogonal to $T_i$ at $p$ (with
some orientation) separates the half-branches of $C_i$ into right and left ones. Assume that we have $l$ left half-branches, and $r$ right ones. We have

$$l + r \equiv 0 \pmod{2} \quad \text{and} \quad e_i \equiv l \equiv r \pmod{2}$$

because $e_i$ is congruent mod 2 to the number of odd components, each odd component has one left half-branch and one right half-branch, and an even one has two half-branches of the same kind. Therefore, we get

$$e = \sum_{i=1}^{s} e_i \equiv \sum_{i=1}^{s} r_i \pmod{2}. \quad (12)$$

The image by $\phi$ of the right half-branches of $C_i$ are half-branches of $C'$ with the same tangent $T'_i$ at $p'$, and on the same side of an orthogonal line to that tangent, which we may assume to be the right one (see (a)) above: two right half-branches with the same tangent have a greater contact than two half-branches with different tangents or than one left and one right half-branch with the same tangent, and this fact is invariant by bilipschitz homeomorphism. Let $r'_i$ be the number of right half-branches of $C'$ with tangent $T'_i$; we have then $r_i = r'_i$, which proves the lemma, using (12).

We conjecture that Proposition 4.4 is true in general, i.e., without the hypothesis that $Z$ is a RGC and has no dicritical components in its resolution.

REFERENCES