On ergodicity of systems with the asymptotic average shadowing property

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Abstract

In this note, we prove that every Lyapunov stable map from a compact metric space onto itself is topologically ergodic, provided it has the asymptotic average shadowing property.

Keywords: The asymptotic average shadowing property; Lyapunov stable; topologically ergodic

1. Introduction

It is known that numerous classes of real problems are modelled by a discrete dynamical system

$$x_{n+1} = f(x_n), \quad n = 0, 1, 2, \ldots$$

where $f$ is a continuous map from a metric space $X$ onto itself. The basic goal of the theory of discrete dynamical systems is to understand the nature of all orbits $x, f(x), f^2(x), \ldots, f^n(x), \ldots$. However, this is an impossible task, because we are often unable to compute the initial condition $x$ exactly in concrete situations. We just compute a value $x_0$ close to $x$. It may also be the case that we cannot compute $f(x_0)$ exactly, but just a value $x_1$ close to $f(x_0)$. Then we compute a value $x_2$ close to $f(x_1)$ and so on. In this way, we obtain a sequence $x_0, x_1, x_2, \ldots$ that can be thought of as the predicted behavior of the system $(X, f)$ at $x$. It is natural to ask whether or not this predicted behavior is close to the actual behavior of the system. This leads to the researches on shadowing properties.

The pseudo-orbit tracing property is one of the most important notions in dynamical systems (see [1]), which is closely related with stability and chaos of systems, see for instance [2–4]. The average shadowing property introduced by Blank [5] in studying chaotic dynamical systems is a good tool to characterize Anosov diffeomorphisms (see [6]). Yang [7] discussed the relation between the pseudo-orbit tracing property and topological ergodicity for maps, and showed that a chain transitive map with the pseudo-orbit tracing property is topologically ergodic. Gu and Guo [8] discussed the relation between the average shadowing property and topological ergodicity for flows, and showed that a Lyapunov stable flow with the average shadowing property is topologically ergodic. In a recent work, the author [9]...
introduced the notion of the asymptotic average shadowing property and found that the asymptotic average shadowing property is closely related with transitivity. Now a natural question arise: which system with the asymptotic average shadowing property is topologically ergodic. In this note, we try to discuss this problem and will show that a Lyapunov stable map with the asymptotic average shadowing property from a compact metric space onto itself is topologically ergodic. In addition, we also show that such a map is not topologically weakly mixing.

2. Some basic terminology

By a dynamical system, we mean a pair \((X, f)\), where \(X\) is a compact metric space with metric \(d\) and \(f : X \to X\) is a continuous map. If \(x \in X\), then the orbit of \(x\) is the sequence \(O(x, f) = \{ f^n(x) \}_{n=0}^{\infty}\).

For \(\delta > 0\), a sequence \(\{x_i\}_{i=0}^{\infty}\) of points in \(X\) is called a \(\delta\) pseudo-orbit of \(f\) if \(d(f(x_i), x_{i+1}) < \delta\) for every integer \(i \geq 0\). A sequence \(\{x_i\}_{i=0}^{\infty}\) is said to be \(\varepsilon\) traced by the point \(z\) in \(X\) if \(d(f^i(z), x_i) < \varepsilon\) for every integer \(i \geq 0\).

A map \(f\) is said to have the pseudo-orbit tracing property, if for any \(\varepsilon > 0\) there is a \(\delta > 0\) such that every \(\delta\) pseudo-orbit of \(f\) can be \(\varepsilon\) traced by some point in \(X\).

A sequence \(\{x_i\}_{i=0}^{\infty}\) of points in \(X\) is called a \(\delta\) average pseudo-orbit of \(f\) if there is a positive integer \(N = N(\delta)\) such that for every integer \(n \geq N\) and every non-negative integer \(k\),

\[
\frac{1}{n} \sum_{i=0}^{n-1} d(f(x_{i+k}), x_{i+k+1}) < \delta.
\]

A map \(f\) is said to have the average shadowing property, if for any \(\varepsilon > 0\) there is a \(\delta > 0\) such that every \(\delta\) average pseudo-orbit \(\{x_i\}_{i=0}^{\infty}\) is \(\varepsilon\) shadowed in average by some point \(z \in X\), that is,

\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(z), x_i) < \varepsilon.
\]

In the following, we introduce the notion of the asymptotic average shadowing property.

**Definition 2.1.** A sequence \(\{x_i\}_{i=0}^{\infty}\) of points in \(X\) is called an asymptotic average pseudo-orbit of \(f\) if \(\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f(x_i), x_{i+1}) = 0\). A sequence \(\{x_i\}_{i=0}^{\infty}\) is said to be asymptotically shadowed in average by the point \(z\) in \(X\) if \(\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(z), x_i) = 0\).

A map \(f\) is said to have the asymptotic average shadowing property, if every asymptotic average pseudo-orbit of \(f\) can be asymptotically shadowed in average by some point in \(X\).

It is known from [9] that the pseudo-orbit tracing property does not imply the asymptotic average shadowing property and the asymptotic average shadowing property does not imply the pseudo-orbit tracing property.

A point \(x \in X\) is said to be stable point of \(f\) if for any \(\varepsilon > 0\) there is a \(\delta = \delta(x) > 0\) such that \(d(f^n(x), f^n(y)) < \varepsilon\) for every \(y \in X\) with \(d(x, y) < \delta\) and every positive integer \(n\). \(f\) is called Lyapunov stable, if every point of \(X\) is stable point of \(f\). \(f\) is said to have the sensitive dependence on initial conditions if every point of \(X\) is not stable point of \(f\).

If \(U\) and \(V\) are two nonempty subsets of \(X\), then we let

\[
N(U, V) = \{ n : f^n(U) \cap V \neq \emptyset, 0 \leq n < \infty \}.
\]

A map \(f\) is called topologically transitive if for any two nonempty open subsets \(U\) and \(V\) of \(X\), \(N(U, V) \neq \emptyset\). \(f\) is called topologically weakly mixing, if \(f \times f\) is topologically transitive. \(f\) is called topologically mixing, if for any two nonempty open subsets \(U\) and \(V\) of \(X\) there is a positive integer \(N\) such that \(N(U, V) \supset \{ N, N + 1, \ldots \}\).

A map \(f\) is called topologically ergodic if for any two nonempty open subsets \(U\) and \(V\) of \(X\), \(N(U, V)\) has positive upper density, that is,

\[
\tilde{D}(N(U, V)) = \limsup_{n \to \infty} \frac{\text{Card}[N(U, V) \cap \{0, 1, \ldots, n-1\}]}{n} > 0,
\]

where \(\text{Card}(E)\) denotes the number of members in the finite set \(E\).
It is well-known that

\[\text{mixing} \quad \Rightarrow \quad \text{weakly mixing} \quad \Rightarrow \quad \text{ergodic} \quad \Rightarrow \quad \text{transitive}.\]

3. Results and proofs

The main result of the paper is the following theorem.

**Theorem 3.1.** Let \( X \) be a compact metric space and \( f \) be a Lyapunov stable map from \( X \) onto itself. If \( f \) has the asymptotic average shadowing property, then \( f \) is topologically ergodic.

**Proof.** Suppose that \( U \) and \( V \) are two nonempty open subsets of \( X \). We choose \( x \in U \), \( y \in V \) and \( \varepsilon > 0 \) such that \( B(x, \varepsilon) \subset U \) and \( B(y, \varepsilon) \subset V \), where \( B(a, \varepsilon) = \{b \in X : d(a, b) < \varepsilon\} \).

Since \( f \) is Lyapunov stable and \( X \) is compact, there is a \( \delta > 0 \) such that for any \( u, v \in X \), \( d(u, v) < \delta \) implies \( d(f^n(u), f^n(v)) < \varepsilon \) for every integer \( n \geq 0 \).

We define a sequence \( \{w_i\}_{i=0}^{\infty} \) as follows. Let

\[
\begin{align*}
    w_0 &= x, \\
    w_1 &= y, \\
    w_2 &= x, \\
    w_3 &= y, \\
    w_4 &= x_-, y_-, y = w_7
\end{align*}
\]

\[
\vdots
\]

\[
\begin{align*}
    w_{2^k} &= x_{-2^k+1}, \ldots, x_{-2}, x, y_{-2^k+1}, \ldots, y_{-2}, y = w_{2^{k+1}-1}
\end{align*}
\]

where \( f(x_{-j}) = x_{-j+1} \) for every \( j > 0 \), \( x_0 = x \) and \( f(y_{-l}) = y_{-l+1} \) for every \( l > 0 \), \( y_0 = y \). It is easy to see that for \( 2^k \leq n < 2^{k+1} \),

\[
\frac{1}{n} \sum_{i=0}^{n-1} d(f(w_i), w_{i+1}) < \frac{2(k+1) \times D}{2^k},
\]

where \( D = \max\{d(x, y) : x, y \in X\} \) is the diameter of \( X \). Hence

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f(w_i), w_{i+1}) = 0.
\]

That is, \( \{w_i\}_{i=0}^{\infty} \) is an asymptotic average pseudo-orbit of \( f \). Since \( f \) has the asymptotic average shadowing property, there is a point \( \omega \in X \) such that

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(\omega), w_i) = 0. \tag{*}
\]

For \( z \in \{x, y\} \), let

\[
J_z = \{i : w_i \in \{z_{-2^i+1}, z_{-2^i-1}, \ldots, z_{-1}, z\} \text{ and } d(f^i(w), w_i) < \delta\}.
\]

We have the following claim:

**Claim.** \( J_z \) has positive upper density, that is, \( \tilde{D}(J_z) > 0 \).

**Proof of Claim.** Without loss of generality, we assume \( z = x \). Suppose on the contrary that \( \tilde{D}(J_x) = 0 \), then we have

\[
\lim_{n \to \infty} \frac{\text{Card}(J_x \cap \{0, 1, \ldots, n-1\})}{n} = 0.
\]

Let

\[
J_x' = \{i : w_i \in \{x_{-2^i+1}, x_{-2^i-1}, \ldots, x_{-1}, x\} \text{ and } d(f^i(w), w_i) \geq \delta\}.
\]
Then
\[ \lim_{n \to \infty} \frac{\text{Card}(J'_i \cap \{0, 1, \ldots, n-1\})}{n} = \frac{1}{2}. \]

Hence, for any \( \rho \in (0, 1/2) \) there is a positive integer \( N \) such that
\[ \frac{\text{Card}(J'_i \cap \{0, 1, \ldots, n-1\})}{n} > \frac{1}{2} - \rho, \]
for every \( n \geq N \). So,
\[ \limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(w), w_j) \geq \limsup_{n \to \infty} \frac{1}{n} \sum_{i \in J'_i \cap \{0, 1, \ldots, n-1\}} d(f^i(w), w_j) \]
\[ \geq \delta \limsup_{n \to \infty} \frac{\text{Card}(J'_i \cap \{0, 1, \ldots, n-1\})}{n} \]
\[ \geq \delta \left( \frac{1}{2} - \rho \right). \]

Since \( \rho \) is arbitrary, we have
\[ \limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(w), w_j) \geq \frac{1}{2} \delta. \]

This contradicts the formula (*). Therefore \( \tilde{D}(J_x) > 0 \). The proof of Claim is completed. \( \square \)

Now, let \( J_m(y) = \{ i \in J : w_i = y_{-m} \} \), for every \( m \geq 0 \). Then, by Claim, there is an integer \( m_0 \geq 0 \) such that \( \tilde{D}(J_{m_0}(y)) > 0 \).

Choose \( i_0 > 0 \) and \( 0 \leq k_0 \leq 2^{i_0-1} - 1 \) such that \( f^{i_0}(w) \in B(x_{-k_0}, \delta) \). For any \( j \in J_{m_0}(y) \) with \( j \geq i_0 + k_0 \), we have \( f^j(w) \in B(y_{-m_0}, \delta) \). Since \( f \) is Lyapunov stable, we have
\[ f^{i_0+k_0}(w) \in B(x, \varepsilon) \quad \text{and} \quad f^{j+m_0}(w) \in B(y, \varepsilon). \]

Let \( n_j = (j + m_0) - (i_0 + k_0) \). We \( f^{n_j}(B(x, \varepsilon)) \cap B(y, \varepsilon) \neq \emptyset \). So, \( f^{n_j}(U) \cap V \neq \emptyset \). Thus
\[ \tilde{D}(K(U, V)) \geq \tilde{D}(J_{m_0}(y)) > 0. \]

This shows that \( f \) is topologically ergodic.

The proof of Theorem 3.1 is completed. \( \square \)

The following theorem is due to Yang [7]. For the sake of completeness, we give a proof of this theorem.

**Theorem 3.2.** Let \( X \) be a compact metric space containing at least two points and \( f : X \to X \) be continuous map. If \( f \) is topologically weakly mixing, then \( f \) has the sensitive dependence on initial conditions.

**Proof.** Suppose on the contrary that \( f \) has not the sensitive dependence on initial conditions. There is at least one stable point \( z \) of \( f \).

Suppose that \( a \) and \( b \) are two distinct points of \( X \). We choose two open subsets \( U \) and \( V \) of \( X \) such that \( a \in U \), \( b \in V \) and \( U \cap V = \phi \). Let \( d = d(U, V) \). Clearly, \( d > 0 \). For any \( \varepsilon \in (0, d) \), there is \( \delta > 0 \) such that if \( d(z, z') < \delta \) then we have \( d(f^n(z), f^n(z')) < \varepsilon/2 \) for all positive integer \( n \).

Since \( f \) is topologically weakly mixing, there is a positive integer \( N \) such that
\[ (f \times f)^N(B(z, \delta) \times B(z, \delta)) \cap (U \times V) \neq \phi. \]

It follows that \( f^N(B(z, \delta)) \cap U \neq \phi \) and \( f^N(B(z, \delta)) \cap V \neq \phi \). Hence, there are \( x \in B(z, \delta) \) such that \( f^N(x) \in U \) and \( y \in B(z, \delta) \) such that \( f^N(y) \in V \). Since \( z \) is stable point for \( f \), we have
\[ d(f^N(x), f^N(z)) < \frac{\varepsilon}{2} \quad \text{and} \quad d(f^N(y), f^N(z)) < \frac{\varepsilon}{2}. \]
Thus,
\[ d(f^N(x), f^N(y)) \leq d(f^N(x), f^N(z)) + d(f^N(z), f^N(y)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \]

Note that \( f^N(x) \in U \) and \( f^N(y) \in V \), we have \( d(U, V) < \varepsilon \). On the other hand, \( d(U, V) \geq d(\overline{U}, \overline{V}) = d > \varepsilon \). This is a contradiction. Therefore, \( f \) has sensitive dependence on initial conditions.

The proof of Theorem 3.2 is completed. \( \square \)

**Remark 3.3.** We have known from Xiong [10] that topological transitivity is strictly weaker than topological ergodicity. By Theorems 3.1 and 3.2, we know that a Lyapunov stable map with the asymptotic average shadowing property from a compact metric space onto itself is topologically ergodic but not topologically weakly mixing, further, it is not topologically mixing. Therefore, topological ergodicity is strictly weaker than topological mixing. Moreover, we also know that topological ergodicity is different from topologically weak mixing, although both properties lie between topological transitivity and topological mixing.

**References**