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On the transverse symbol of vectorial distributions and some applications to harmonic analysis

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ABSTRACT

The transverse symbol of a vector-valued distribution supported on a submanifold is introduced and a micro-local vanishing theorem for spaces of such distributions invariant under a Lie group is proved. We give transparent proofs of results of Bruhat and Harish-Chandra on the irreducibility of parabolically or normally induced representations, and of Harish-Chandra in Whittaker theory.

1. INTRODUCTION

Let X be a manifold and E a Fréchet space; then the E-distributions on X are the elements of the dual space $C_c^\infty(X : E)'$. More generally, if F is a vector bundle on X, the $F \otimes E$-distributions are elements of the dual of $C_c^\infty(X : F \otimes E)$. Motivated by applications to harmonic analysis on Lie groups we consider the following problem. Given a closed submanifold O of X, describe the space of E-distributions on X with support contained in O. In particular, what can be said under the assumption of invariance of the distributions with respect to a Lie group of diffeomorphisms of X.

Initially we do not require invariance. To any such distribution T that is of transverse order $\leq r$ on all of O, we associate a globally defined $F \otimes E$-distribution $\sigma^{(r)}(T)$ that lives on O (in contrast to T which is only supported by O), F being the vector bundle which is the dual of the r-th graded transverse jet bundle $M^{(r)}$ of O embedded in X. Then $\sigma^{(r)}(T) = 0$ iff T has transverse order $< r$ on O. If O is a point, then T is a differential expression at that point, and $\sigma^{(r)}(T)$ can be identified canonically with the symbol of the differential expression. For this (and other) reasons we like to think of $\sigma^{(r)}(T)$ as the transverse symbol of T. We have not developed a general theory of this transverse symbol.

Suppose now that O is invariant under a Lie group H' of diffeomorphisms of X, and that H' also acts on E through a differentiable linear representation β. Now the description of the space of H'-invariant E-distributions on X supported by O is an important issue. If T is invariant under H', then so is $\sigma^{(r)}(T)$; in this manner the study of H'-invariant E-distributions supported by O is reduced to the study of H'-invariant $F \otimes E$-distributions that *live* on O. With a view to applications to harmonic analysis we are especially concerned with obtaining a sufficient condition for ensuring that there are no nonzero E-distributions supported by O invariant under H'. For any H'-homogeneous vector bundle F over O of finite rank and under some extra assumption, we are then able to use our method of transverse symbols to give the following criterion: in order that 0 is the only H'-invariant E-distribution supported by O, it is sufficient that there be some $y \in O$ satisfying

$$(1.1) \qquad (F'_y \otimes \Lambda'_y \otimes E')^{H'_y} = 0.$$

Here Λ is the line bundle of densities on the H'-orbit of y, while $'$ denotes the dual space, and $^{H'_y}$ the subspace of invariants under the stabilizer in H' of y. The point is that the transverse symbol $\sigma^{(r)}(T)$ $(r \in \mathbf{Z}_{\geq 0})$ of an H'-invariant E-distribution T on X of transverse order r supported by O, is an H'-invariant $F \otimes E$-distribution living on O; and its structure can be essentially reduced to its structure on a manifold $M \subset O$ transverse to $H' \cdot y$ at y. It is this that leads to the above powerful microlocal criterion for the vanishing of T.

The problem of determining vector-valued distributions invariant under a Lie group was first encountered by Bruhat [B], in his study of the irreducibility of induced representations of Lie groups. He was able to treat the case when the group in question had at most countably many orbits; by an induction on dimension this problem was reduced to the case of a single orbit, the distributions being supported by the orbit. Bruhat's method was later taken up by Harish-Chandra [Ha5] in the 1970's in his work on the irreducibility of representations induced from an arbitrary parabolic subgroup P of a semisimple Lie group G (unlike Bruhat who was concerned only with minimal such P). Later, in the years 1971–82 in his work on the spectral theory of Whittaker functions, Harish-Chandra [Ha4] developed a variant of Bruhat's method to prove the basic vanishing theorem that served as a foundation for his work. We say variant because the space of orbits in this case has positive dimension. The calculations of Bruhat and Harish-Chandra are quite involved, and in the Whittaker context Harish-Chandra's computations are even less transparent. This is because of two reasons: (a) neither Bruhat nor Harish-Chandra introduce an invariant object like the transverse symbol and so their calculations are always local in a problem which is best treated from a global point of view; (b) the Whittaker situation is complicated further by the appearance of moduli for the orbits.

Our approach to these results, based on the invariantly defined transverse symbol, makes the computation entirely transparent. This is the reason we have included these applications.

Bruhat's work is of course part of the literature. But Harish-Chandra's papers are still unpublished. Regarding the irreducibility of parabolically induced representations, Harish-Chandra's manuscript goes back to the 1970's and the results are apparently known to many. In any case, in his reply to an enquiry from G. van Dijk, Harish-Chandra gave a precise formulation of his theorem [Ha6]. As far as the Whittaker theory is concerned, Harish-Chandra's results are contained in the manuscript [Ha4] that was presented in the AMS conference at Toronto in 1982. He was actually unable to attend this meeting in person and his address was read for him by Varadarajan. It was during extended conversations with Harish-Chandra prior to this conference that V. had the opportunity to discuss the vanishing theorem and Harish-Chandra's proof of it. It should be clear from this that the results in the applications are clearly those of Harish-Chandra and that we are only presenting an approach that has some novel features.

Returning to our results, the simplest situation is when H' is transitive on O. However this is not adequate for all applications. If there are at most a countable number of H'-orbits in O, one can obtain a natural filtration on O based on the dimension of these orbits which induces a filtration on the space of invariant distributions, and for many purposes one can come down to the case of a single orbit. There are however situations when $H'\backslash O$ has positive dimension and this method will not apply; this is the case of Harish-Chandra's vanishing theorem in his Whittaker theory. To handle such cases we shall suppose that we are in one of the two following situations:

(i) H' is transitive on O; we shall generally write H for H'.

(ii) H' is not transitive on O but is a closed normal subgroup of a Lie group H of diffeomorphisms of X which leaves O invariant and is transitive on it. In this case we shall also suppose that the representation β of H' on E extends to a differentiable representation of H on E, also denoted by β.

In the case when $H'\backslash O$ is at most countable, when it becomes a question of treating the orbits one by one, the criterion (1.1) goes back to Bruhat [B]; here $X = G$ and $H' = P \times P$ with P a minimal parabolic subgroup of G, while $P \times P$ acts on G as usual from left and right. This approach was taken up later by Harish-Chandra to treat the case when P is no longer minimal. In the inductive arguments of Bruhat and Harish-Chandra the key step is to treat the case $H' = H$. Here the space of H-invariant $F \otimes E$-distributions living on O can be described completely. This allows us to give a more transparent treatment of the irreducibility of the representations of a semisimple Lie group that are induced from representations of parabolic subgroups. The irreducibility when the parabolic subgroup is cuspidal but not necessarily minimal has been treated by different methods by Harish-Chandra [Ha3, §41], and by Speh and Vogan [S–V].

When E is finite-dimensional we can prove (1.1) (now for all $y \in O$) without the extra assumption that the representation β of H' in E has an extension to H. In this case the analysis goes a little farther; and we have included it because we use it in giving an alternative treatment of Bruhat's work on representations induced from a normal subgroup and also because it may have further applications.

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2. TRANSVERSE SYMBOLS OF E-DISTRIBUTIONS

E-distributions. Let X be a C^∞ manifold of dimension n. We write $C^\infty(X)$, and $C_c^\infty(X)$, for the topological linear spaces of the C^∞ functions on X, and those with compact support, respectively, with their usual topologies. Further we denote by $\mathrm{Diff}^{(r)}(X)$ the $C^\infty(X)$-module of C^∞ differential operators on X of order $\leq r \in \boldsymbol{Z}_{\geq 0}$.

Let E be a Fréchet space over $\boldsymbol{C}$, and let E' be its dual space provided with the strong dual topology, that is, the topology of uniform convergence on bounded subsets of E. We write $C^\infty(X : E)$, and $C_c^\infty(X : E)$, for the topological linear spaces of the C^∞ mappings from X to E, and those with compact support, respectively. The topology on these spaces is defined in a similar fashion as for $C^\infty(X)$, and $C_c^\infty(X)$, respectively. If E_i $(i = 1, 2)$ are Fréchet spaces, the completions of the usual structures of topological tensor product on $E_1 \overline{\otimes} E_2$ all coincide; and we shall write $E_1 \overline{\otimes} E_2$ for the resulting complete topological linear space (cf. [G]).

The algebraic tensor product $C_c^\infty(X) \otimes E$ is canonically identified with a linear subspace of $C_c^\infty(X : E)$ via $f \otimes e \leftrightarrow (x \mapsto f(x)e)$, for $f \in C_c^\infty(X)$, $e \in E$ and $x \in X$. [G] has introduced a locally convex topology on $C_c^\infty(X) \otimes E$ such that this map extends to its completion, denoted by $C_c^\infty(X) \overline{\otimes} E$, defining an algebraic and topological isomorphism with $C_c^\infty(X : E)$

$$(2.1) \qquad C_c^\infty(X : E) \simeq C_c^\infty(X) \overline{\otimes} E.$$

An E-distribution on X is an element of the topological linear space

$$(2.2) \qquad \mathcal{D}'(X : E) := C_c^\infty(X : E)' \simeq (C_c^\infty(X) \overline{\otimes} E)',$$

consisting of the continuous linear forms on $C_c^\infty(X : E)$; and this dual space is provided with the strong dual topology.

We write $\mathrm{Lin}(C_c^\infty(X) : E')$ for the linear space of the E'-valued distributions on X, i.e., $T \in \mathrm{Lin}(C_c^\infty(X) : E')$ iff $T : C_c^\infty(X) \to E'$ is continuous linear. Also $\mathrm{Lin}(C_c^\infty(X) : E')$ is considered with the strong dual topology.

An E-distribution T defines canonically an E'-valued distribution $\tilde{T}$ on X, that is, we have a canonical isomorphism

$$T \leftrightarrow \tilde{T} : \mathcal{D}'(X : E) \leftrightarrow \mathrm{Lin}(C_c^\infty(X) : E').$$

In fact, if $T \in \mathcal{D}'(X : E)$ and $f \in C_c^\infty(X)$, then $\tilde{T}(f) \in E'$ is well-defined by $\langle \tilde{T}(f), e\rangle = \langle T, f \otimes e\rangle$, for all $e \in E$. Conversely, given $\tilde{T} \in \mathrm{Lin}(C_c^\infty(X) : E')$, we have the bilinear mapping

$$(f, e) \mapsto \langle \tilde{T}(f), e\rangle : C_c^\infty(X) \times E \to \boldsymbol{C}.$$

Obviously this map is separately continuous, and therefore (cf. (2.1)) it defines a continuous linear form: $T : C_c^\infty(X : E) \to \boldsymbol{C}$. That is, $T \in \mathcal{D}'(X : E)$.

Let $r \in \boldsymbol{Z}_{\geq 0}$. An E-distribution $T \in \mathcal{D}'(X : E)$ is said to be of order $\leq r$ if for every compact set $K \subset X$ there exist a constant $c > 0$, a finite number of elements, say $D_1, \dots, D_m$, in $\mathrm{Diff}^{(r)}(X)$, and a continuous seminorm ν on E such that, for all $f \in C_c^\infty(K : E)$

$$|\langle T, f\rangle| \leq c \sum_{1 \leq j \leq m} \sup_{x \in K} \nu(D_j f(x)).$$

We write $\mathcal{D}'^{(r)}(X : E)$ for the linear subspace of $\mathcal{D}'(X : E)$ consisting of elements of order $\leq r$. Similar to the scalar case (cf. [S, § III.6]) it can be shown that every E-distribution on X is of finite order over every relatively compact open subset of X. If $T \in \mathcal{D}'^{(r)}(X : E)$, then we have $\langle T, f\rangle = 0$, if $f \in C_c^\infty(X : E)$ satisfies

$$(2.3) \qquad Vf(x) = 0 \quad (V \in \mathrm{Diff}^{(r)}(X),\ x \in \mathrm{supp}(T)).$$

For a proof, see [Hö, Theorem 2.3.3].

Transverse order of an E-distribution. Suppose that O is a closed C^∞ submanifold that is regularly embedded in X and has dimension q; we set $n = p + q$. We write $\mathcal{D}'_O(X : E)$ for the distributions in $\mathcal{D}'(X : E)$ with support contained in O. If $x \in O$, we can select an open neighborhood U of x in X and local coordinates $(t : u) = (t^1, \dots, t^p : u^1, \dots, u^q)$ on U such that

$$(2.4) \qquad O \cap U = \{(t : u) \mid t^1 = \cdots = t^p = 0\}.$$

Mimicking (2.3) we say that $T \in \mathcal{D}'(X : E)$ has transverse order $\leq r \in \boldsymbol{Z}_{\geq 0}$ at $x \in O$, if the following holds. There exists an open neighborhood U of x in X such that

$$(2.5) \qquad \begin{cases} \langle T, f\rangle = 0, & \text{if } f \in C_c^\infty(U : E) \text{ satisfies } Vf|_{O\cap U} = 0, \\ & \text{for all } V \in \mathrm{Diff}^{(r)}(U). \end{cases}$$

If T satisfies this definition, then the transverse order of T is $\leq r$ at all points of O in a suitable neighborhood of x. Let $\mathcal{D}'^{(r)}_O(X : E)$ be the linear subspace of elements in $\mathcal{D}'(X : E)$ which have transverse order $\leq r$ at all points of O. Note that $\mathrm{supp}(T) \subset O$, for $T \in \mathcal{D}'^{(r)}_O(X : E)$; and this justifies the notation. We say that T lives on O if T belongs to $\mathcal{D}'(O : E) := \mathcal{D}'^{(0)}_O(X : E)$. This is the same as saying that $\langle T, f\rangle = 0$, if $f \in C_c^\infty(X : E)$ satisfies $f|_O = 0$. In this case there exists a unique distribution τ on O such that $\langle T, f\rangle = \langle \tau, f|_O\rangle$, for all $f \in C_c^\infty(X : E)$; and often T is identified with τ. Note that $T \in \mathcal{D}'^{(r)}_O(X : E)$ if $T \in \mathcal{D}'^{(r)}(X : E)$ and $\mathrm{supp}(T) \subset O$.

We shall now make a more refined study of $\mathcal{D}_O^{\prime(r)}(X:E)$. This space of E-distributions arises naturally if O is compact, or if we have a Lie group H consisting of diffeomorphisms of X all of which leave O invariant and act transitively on O and if we study distributions invariant with respect to the induced action of (a closed subgroup of) H.

Transverse jet bundle. For our purposes it is convenient to define $M^{(r)}$, the r-th graded subspace of the transverse jet bundle over O, for $r \in \boldsymbol{Z}_{\geq 0}$, as a locally free sheaf, or what amounts to the same, as a vector bundle over O.

For any $x \in O$, let $\boldsymbol{D}_x$ be the algebra of germs of C^∞ functions defined around x, and $\mathrm{Diff}_x^{(r)}$ the $\boldsymbol{D}_x$-module of germs of differential operators of order $\leq r$ defined around x. Let $V_x^{(r)}$ be the $\boldsymbol{D}_x$-submodule of $\mathrm{Diff}_x^{(r)}$ generated by germs of r-tuples $(v_1 \cdots v_r)$ of vector fields around x for which at least one of the v_i is tangent to O; let $I_x^{(r)} = \mathrm{Diff}_x^{(r-1)} + V_x^{(r)}$. Choosing local coordinates at x it is easy to see that $I_x^{(r)}$ actually is the stalk at x of a subsheaf $I^{(r)}$ of the sheaf $\mathrm{Diff}^{(r)}$ of differential operators of order $\leq r$ on X. Hence we have a well-defined quotient sheaf

$$(2.6) \qquad M^{(r)} = \mathrm{Diff}^{(r)}/I^{(r)},$$

with stalk at x equal to $M_x^{(r)} = \mathrm{Diff}_x^{(r)}/I_x^{(r)}$. We write $\partial \mapsto \bar{\partial}$ for the projection: $\mathrm{Diff}_x^{(r)} \to M_x^{(r)}$. Since $\mathrm{Diff}_x^{(r)}$ and $I_x^{(r)}$ both are $\boldsymbol{D}_x$-modules, $M_x^{(r)}$ also has the structure of a $\boldsymbol{D}_x$-module, and the projection is a mapping of $\boldsymbol{D}_x$-modules.

If we take local coordinates $(t:u)$ at x satisfying (2.4), it is easy to show that the elements $\overline{\partial_t^\alpha}$ $(|\alpha| = r)$ form a free basis for the sections of $M^{(r)}$ around x. Thus $M^{(r)}$ is a vector bundle over O of finite rank. This is the so-called r-th graded part of the transverse jet bundle on O. We have that $M^{(r)}$ is the r-th symmetric power of $M^{(1)}$. We write $M^{(r)'}$ for the bundle dual to $M^{(r)}$.

Transverse symbols. We now define the transverse symbol $\sigma(T)$ of an E-distribution T supported by O, as an $M^{(r)'} \otimes E$-distribution living on O. In view of the isomorphism $M^{(r)} \otimes (C_c^\infty(O) \mathbin{\overline{\otimes}} E)' \simeq (C_c^\infty(O) \mathbin{\overline{\otimes}} (M^{(r)'} \otimes E))'$ it follows from (2.2) that

$$(2.7) \qquad M^{(r)} \otimes \mathcal{D}'(O:E) \simeq \mathcal{D}'(O:M^{(r)'} \otimes E).$$

If $X = \boldsymbol{R}^n$ and $E = \boldsymbol{C}$, we have the following result for $T \in \mathcal{D}_O^{\prime(r)}(X:E)$ (cf. [S, Theorem XXXVII]). For each $x \in O$ there exist a neighborhood U of x as in (2.4) and uniquely determined distributions $\tau_\alpha \in \mathcal{D}'(O:E)$ with $|\alpha| \leq r$, such that on U we have

$$(2.8) \qquad T = \sum_{|\alpha| \leq r} (-1)^{|\alpha|} \tau_\alpha \partial_t^\alpha.$$

The proof can be carried over directly to the present situation, with $\boldsymbol{R}^n$ replaced by X and $\boldsymbol{C}$ by E.

We shall prove that the map $T \mapsto \sum_{|\alpha|=r} \partial_t^\alpha \otimes \tau_\alpha$, which assigns to T the top-order part of its representation in (2.8), in fact is coordinate-independent, although the τ_α in $\mathcal{D}'(O:E)$ as well as the ∂_t^α depend on the choice of the coordinates $(t:u)$.

Theorem 2.1. *For any $T \in \mathcal{D}_O'^{(r)}(X:E)$ there is a $\sigma(T) = \sigma^{(r)}(T) \in \mathcal{D}'(O:M^{(r)'} \otimes E)$ with the following property. If T is as in* (2.8), *then*

$$\sigma(T) = \sum_{|\alpha|=r} \overline{\partial_t^\alpha} \otimes \tau_\alpha.$$

The element $\sigma(T)$ is uniquely determined by this requirement. The map

$$\sigma : \mathcal{D}_O'^{(r)}(X:E) \to \mathcal{D}'(O:M^{(r)'} \otimes E)$$

is injective modulo the linear subspace $\mathcal{D}_O'^{(r-1)}(X:E)$ and is independent of the choice of local coordinate systems.

Proof. Let U be a neighborhood of x as in (2.4). It is a question of proving that a different representation of T and $\sigma(T)$ leads to the same result when tested against an arbitrary $s \in \Gamma_c^\infty(U:M^{(r)'} \otimes E)$. The proof is completely elementary and is omitted. □

3. INVARIANT DISTRIBUTIONS

H'-actions. Let H be a Lie group. As usual we denote the left and right action of H on itself by

$$l(h')h = h'h, \qquad r(h')h = hh'^{-1} \quad (h', h \in H).$$

Now suppose that H is a Lie group of C^∞ diffeomorphisms of X all of which leave O invariant. We assume

$$H \text{ acts transitively on } O,$$

and we write $(h, y) \mapsto h \cdot y$ for the action. For any $x \in O$, we denote by H_x the stabilizer in H of x. Since O is regularly embedded in X, we have the diffeomorphism $O \simeq H/H_x$. Let $H' \subset H$ be a closed subgroup.

We suppose that the Fréchet space E carries a differentiable representation β of H', denoted by $(h', e) \mapsto \beta(h')e : H' \times E \to E$. We then have a representation of H' in the dual space E' provided with the strong dual topology, viz., $(h', e') \mapsto {}^t\beta(h'^{-1})e' =: \beta^*(h')e' : H' \times E' \to E'$. According to [B, Proposition 2.4], (β^*, E') is a differentiable representation of H'.

Since H' acts on X as well as in E, we have an induced action of H' in $C_c^\infty(X:E)$, viz., $(h' \cdot f)(h' \cdot x) = \beta(h')f(x)$. In other words,

$$h' \cdot f = \beta(h') \circ l(h')f \quad (h' \in H',\ f \in C_c^\infty(X:E)).$$

Via $\langle h' \cdot T, h' \cdot f \rangle = \langle T, f \rangle$ this leads to an induced action of H' in $\mathcal{D}'(X:E)$. We say that $T \in \mathcal{D}'(X:E)$ is H'-invariant if $h' \cdot T = T$, for all $h' \in H'$; and we write

$$\mathcal{D}'(X:E)^{H'}, \qquad \mathcal{D}_O'(X:E)^{H'}, \qquad \text{and} \qquad \mathcal{D}_O'^{(r)}(X:E)^{H'} \quad (r \in \mathbf{Z}_{\geq 0}),$$

for the linear subspace in $\mathcal{D}'(X : E)$ of H'-invariant E-distributions on X, those with support in O, and those of transverse order $\leq r$ on all of O, respectively. These are the spaces of our main interest.

In addition to this, we assume that we are given an H-homogeneous C^∞ vector bundle F over O of finite rank m. In applications we shall have $F = M^{(r)'}$. We denote the action of H on F by α. Then α induces an action of H, also denoted by α, on the space of sections of F over open sets U in O

$$\alpha(h) : \Gamma(U : F) \to \Gamma(h \cdot U : F), \qquad \Gamma_c^\infty(U : F) \to \Gamma_c^\infty(h \cdot U : F).$$

The Fréchet sheaf $F \otimes E$ over O now acquires in a natural fashion the structure of an H'-module for the action $\alpha|_{H'} \otimes \beta$ of H'; if $U \subset O$ is an H'-invariant open set

$$(3.1) \qquad h' \cdot t = (\alpha \otimes \beta)(h') \circ l(h')t \quad (t \in \Gamma(U : F \otimes E)).$$

We write $\mathcal{D}'(U : F \otimes E)^{H'}$ for the linear subspace of H'-invariant $F \otimes E$-distributions defined on U. The global nature of $\sigma^{(r)}(T)$ and its uniqueness guarantee that $\sigma^{(r)}(T)$ inherits all the invariance properties of T:

Lemma 3.1. *Let $Y \subset X$ be open and H'-invariant, and $U = Y \cap O$. Then U is closed in Y, and we have, in the notation of Theorem* 2.1,

$$\sigma^{(r)} : \mathcal{D}_U^{\prime(r)}(Y : E)^{H'} \to \mathcal{D}'(U : M^{(r)'} \otimes E)^{H'} \quad (r \in \boldsymbol{Z}_{\geq 0}).$$

We now begin the study of $\mathcal{D}'(U : F \otimes E)^{H'}$. Consider the sheaf $F \otimes E$ over O. Let $x \in O$ be an arbitrary but fixed point, and introduce

$$\pi : H \to O \qquad \text{given by} \qquad h \mapsto h \cdot x.$$

We set $W_0 = F_x \otimes E$, and we let W be the trivial bundle over H whose fibers are isomorphic to W_0, thus

$$W = H \times W_0 = H \times (F_x \otimes E).$$

We shall introduce the structure of an $H' \times H_x$-module in the space of sections of W. Let U be an H'-invariant open set in O and let $V = \pi^{-1}(U) \subset H$. Then V is left H'-invariant and right H_x-invariant. Observe that F_x if not an H'-module in a natural fashion, whereas E is. Therefore we define the structure of an H'-module in the space of sections of W over V by

$$(3.2) \qquad (h' \cdot s)(h) = (\mathrm{id} \otimes \beta)(h')s(h'^{-1}h) \quad (h' \in H',\, s \in \Gamma_c^\infty(V : W),\, h \in H).$$

Recall that F is an H-module for the action α. Hence W_0 is an H_x-module for the action given by $\alpha|_{H_x} \otimes \mathrm{id}$. Let

$$(3.3) \qquad \delta_{H_x} = |\det(\mathrm{Ad}|_{\mathfrak{h}_x})| : H_x \to \boldsymbol{R}_{>0}.$$

Since V is a right H_x-invariant open set in H, the space of sections of W over V acquires the structure of an H_x-module for the action R by setting

$$(3.4) \qquad (R(\xi)s)(h) = (\alpha \otimes \mathrm{id})(\xi)s(h\xi) \quad (\xi \in H_x,\, s \in \Gamma_c^\infty(V : W)).$$

Since the actions of H' and of H_x in W commute, we have an $H' \times H_x$-module.

Lifting $F \otimes E$-distributions on H/H_x to $F_x \otimes E$-distributions on H. Let notation be as above. Our aim is to prove the following theorem.

Theorem 3.2. *There exists an injective continuous mapping of H'-modules*

$$\sharp : \mathcal{D}'(U : F \otimes E) \to \mathcal{D}'(V : W),$$

linear over $C^\infty(O) \cong C^\infty(H)^{H_x}$, *satisfying* $\delta_{H_x}(\xi)^{-1}R(\xi) \circ \sharp = \sharp$, *for* $\xi \in H_x$, *and* $\operatorname{supp}(\sharp\tau) \subset \pi^{-1}(\operatorname{supp}(\tau))$. *In particular, if* $\tau \in \mathcal{D}'(U : F \otimes E)^{H'}$, *we have*

$$h' \cdot (\sharp\tau) = \delta_{H_x}(\xi)^{-1}R(\xi)(\sharp\tau) = \sharp\tau \quad (h' \in H', \xi \in H_x). \tag{3.5}$$

This result will follow by duality from the following

Proposition 3.3. *There exists a surjective continuous mapping of H'-modules*

$$\flat : \Gamma_c^\infty(V : W) \to \Gamma_c^\infty(U : F \otimes E),$$

linear over $C^\infty(O) \cong C^\infty(H)^{H_x}$, *satisfying* $\flat \circ \delta_{H_x}(\xi)^{-1}R(\xi) = \flat$, *for* $\xi \in H_x$, *and* $\operatorname{supp}(\flat s) \subset \pi(\operatorname{supp}(s))$.

Theorem 3.2 is immediate from Proposition 3.3, if we define

$$\langle \sharp\tau, s\rangle = \langle \tau, \flat s\rangle \quad (\tau \in \mathcal{D}'(U : F \otimes E), s \in \Gamma_c^\infty(V : W)).$$

We shall now proceed to the proof of Proposition 3.3. This requires a little preparation. We shall define $\flat$ as $\flat_2 \circ \flat_1$ where $\flat_1$ and $\flat_2$ are defined as follows. For any open set V in H with

$$V = H'VH_x, \tag{3.6}$$

consider $\Gamma(V : W)^{H_x}$, the linear subspace of elements in $\Gamma(V : W)$ that are invariant under the action of H_x

$$s \in \Gamma(V : W)^{H_x} \iff s(h) = (\alpha \otimes \mathrm{id})(\xi)s(h\xi) \quad (h \in H, \xi \in H_x). \tag{3.7}$$

Since the actions of H' and H_x on $\Gamma(V : W)$ commute, $\Gamma(V : W)^{H_x}$ inherits the structure of an H'-module.

Lemma 3.4. *For every H'-invariant open set U in O and $V = \pi^{-1}(U)$, we have a topological isomorphism of H'-modules*

$$\flat_2 : \Gamma^\infty(V : W)^{H_x} \to \Gamma^\infty(U : F \otimes E).$$

Proof. If $y \in U$ is arbitrary, there exists $h \in H$ such that $h \cdot x = y$. Now define, for $s \in \Gamma(V : W)^{H_x}$

$$(\flat_2 s)(y) = (\alpha \otimes \mathrm{id})(h)s(h) \in (\alpha \otimes \mathrm{id})(h)(F_x \otimes E) = F_y \otimes E.$$

The definition of $(\flat_2 s)(y)$ is independent of the choice of $h \in H$ satisfying

$h \cdot x = y$; this is immediate from (3.7). Thus $\flat_2 s \in \Gamma(U : F \otimes E)$ is well-defined. Conversely, for $t \in \Gamma(U : F \otimes E)$ and $h \in H$, we have $t(h \cdot x) \in F_{h \cdot x} \otimes E$; and so we obtain

$$s(h) := (\alpha \otimes \mathrm{id})(h^{-1})t(h \cdot x) \in F_x \otimes E.$$

We then have, for $\xi \in H_x$, that $s(h\xi) = (\alpha \otimes \mathrm{id})(\xi^{-1})s(h)$; and according to (3.7) that proves $s \in \Gamma(V : W)^{H_x}$. Moreover $\flat_2 s = t$, and therefore $\flat_2 : \Gamma(V : W)^{H_x} \to \Gamma(U : F \otimes E)$ is a linear isomorphism.

One verifies easily that $\flat_2$ is a mapping of H'-modules. That $\flat_2$ is smooth and continuous is essentially a local result which can be proved by standard arguments. □

We define $\Gamma^{\infty}_{(c)}(V : W)^{H_x}$ as the H'-submodule of sections in $\Gamma^{\infty}(V : W)^{H_x}$ which have a support in $V \subset H$ that is compact modulo H_x. Then we have the induced topological isomorphism of H'-modules

$$\flat_2 : \Gamma^{\infty}_{(c)}(V : W)^{H_x} \to \Gamma^{\infty}_{c}(U : F \otimes E) \quad \text{with} \quad \pi(\mathrm{supp}(s)) = \mathrm{supp}(\flat_2 s).$$

Next we select a left invariant Haar measure $d_l\eta$ on H_x, and we define the mapping (cf. (3.4))

$$\flat_1 : \Gamma^{\infty}_{c}(V : W) \to \Gamma^{\infty}_{(c)}(V : W) \qquad \text{by} \qquad \flat_1 s = \int_{H_x} R(\eta) s \, d_l\eta.$$

Lemma 3.5. *The map $\flat_1$ is a surjective continuous mapping of H'-modules*

$$\flat_1 : \Gamma^{\infty}_{c}(V : W) \to \Gamma^{\infty}_{(c)}(V : W)^{H_x},$$

satisfying $\flat_1 \circ \delta_{H_x}(\xi)^{-1} R(\xi) = \flat_1$, for $\xi \in H_x$, and $\mathrm{supp}(\flat_1 s) \subset \mathrm{supp}(s) H_x$.

Proof. We have, for $\xi \in H_x$

$$R(\xi) \circ \flat_1 = \int_{H_x} R(\xi\eta) d_l\eta = \flat_1;$$

and therefore $\flat_1 s \in \Gamma^{\infty}_{(c)}(V : W)^{H_x}$. Obviously $\flat_1$ is a mapping of H'-modules, since the actions of H' and of H_x in W commute. In view of (3.4) we obtain, for $\xi \in H_x$

$$\flat_1 \circ \delta_{H_x}(\xi)^{-1} R(\xi) = \delta_{H_x}(\xi)^{-1} \int_{H_x} R(\eta\xi) d_l\eta = \int_{H_x} R(\eta) d_l\eta = \flat_1.$$

Here we have used the following transformation formula for integrals, valid for $\phi \in C^{\infty}_c(H_x)$ and $\xi \in H_x$ (cf. (3.3))

$$(3.8) \qquad \int_{H_x} \phi(\eta\xi) d_l\eta = \int_{H_x} \phi(\xi\eta\xi^{-1}\xi) |\det(\mathrm{Ad}\,\xi|_{\mathfrak{h}_x})| d_l\eta = \delta_{H_x}(\xi) \int_{H_x} \phi(\eta) d_l\eta.$$

Finally we construct a continuous right inverse for $\flat_1$, this will show that $\flat_1$ is surjective. Using C^{∞} partitions of unity on H, we see that it is sufficient to do so on open sets V in H that are sufficiently small. So we may assume that $V =$

$V_0 = \pi^{-1}(U_0)$, where $U_0 \subset O$ has a C^∞ cross section $\theta : U_0 \to H$. Then we have a C^∞ diffeomorphism

(3.9) $\quad U_0 \times H_x \to V_0 \qquad$ given by $\qquad (y, \xi) \mapsto \theta(y)\xi.$

Next we select $\phi \in C_c^\infty(H_x)$ such that $\int_{H_x} \phi(\eta)\, d_l\eta = 1$. Given $s' \in \Gamma_{(c)}^\infty(V_0 : W)^{H_x}$, we now define $s \in \Gamma_c^\infty(V_0 : W)$ by $s(\theta(y)\xi) = (\alpha \otimes \mathrm{id})(\xi^{-1})s'(\theta(y))\phi(\xi)$. Then it is an easy verification that $(\flat_1 s)(\theta(y)\xi) = s'(\theta(y)\xi)$. This proves that $\flat_1$ is surjective locally. □

Proposition 3.3 is now immediate if we define

$$\flat = \flat_2 \circ \flat_1 : \Gamma_c^\infty(V : W) \to \Gamma_{(c)}^\infty(V : W)^{H_x} \to \Gamma_c^\infty(U : F \otimes E).$$

Indeed, $\flat$ is a surjective continuous mapping of H'-modules satisfying

$$\flat = \flat \circ \delta_{H_x}(\xi)^{-1} R(\xi) \quad (\xi \in H_x),$$
$$\mathrm{supp}(\flat s) \subset \pi(\mathrm{supp}\, \flat_1 s) \subset (\mathrm{supp}(s) H_x) = \pi(\mathrm{supp}(s)).$$

***H'*-invariant distributions.** Let $V = H'V$ be open in H and let $\zeta \in \mathcal{D}'(V : W)^{H'}$. We can write V as a union of sets $V_M = H'M$ where M is a C^∞ manifold in H with

$$V_M = H'M \simeq H' \times M.$$

It is our intention to study the structure of ζ on the $V_M \subset V$. We have a well-defined projection mapping onto the first factor $\gamma : V_M \to H'$. Next we extend the action β of H' in E to the mapping β_M of V_M into $\boldsymbol{GL}(E)$ given by $\beta_M = \beta \circ \gamma$. For every $h' \in H'$ and $h \in V_M$, we have $h'^{-1}\gamma(h) = \gamma(h'^{-1}h)$; and therefore $\beta(h'^{-1})\beta_M(h) = \beta_M(h'^{-1}h)$. Hence $(\mathrm{id} \otimes \beta_M)(h) = (\mathrm{id} \otimes \beta(h'))(\mathrm{id} \otimes \beta_M(h'^{-1}h))$, and so we have, for $h' \in H'$ and $s \in \Gamma_c^\infty(V_M : W)$ (cf. (3.2))

(3.10) $\quad (\mathrm{id} \otimes \beta_M)(l(h')s) = (\mathrm{id} \otimes \beta)(h')l(h')((\mathrm{id} \otimes \beta_M)s) = h' \cdot ((\mathrm{id} \otimes \beta_M)s).$

Lemma 3.6. *To each $\zeta \in \mathcal{D}'(V_M : W)^{H'}$ there corresponds a unique $\omega \in \mathcal{D}'(M : W)$ such that, for all $s \in \Gamma_c^\infty(V_M : W)$*

(3.11) $\quad \langle \zeta, s \rangle = \langle \omega_m, \int_{H'} (\mathrm{id} \otimes \beta)(h'^{-1}) s(h'm)\, d_l h' \rangle.$

In particular $\mathrm{supp}(\zeta) = H' \cdot \mathrm{supp}(\omega)$.

Proof. First we introduce $\theta \in \mathcal{D}'(V_M : W)$ by

(3.12) $\quad \langle \theta, s \rangle = \langle \zeta, (\mathrm{id} \otimes \beta_M)s \rangle \quad (s \in \Gamma_c^\infty(V_M : W)).$

From (3.10) we obtain, for $h' \in H'$

$$\langle \theta, l(h')s \rangle = \langle \zeta, h' \cdot ((\mathrm{id} \otimes \beta_M)s) \rangle = \langle \theta, s \rangle.$$

Thus

(3.13) $\quad l(h')\theta = \theta \quad (h' \in H').$

For the moment we fix $w \in \Gamma_c^\infty(M : W)$. Then, for every $\psi \in C_c^\infty(H')$, we have $\psi \otimes w \in \Gamma_c^\infty(V_M : W)$, given by $(\psi \otimes w)(h'm) = \psi(h')w(m)$. Notice that $l(h')(\psi \otimes w) = l(h')\psi \otimes w$. Now consider $\tilde{\theta} \in \mathcal{D}'(H')$ given by $\langle \tilde{\theta}, \psi \rangle = \langle \theta, \psi \otimes w \rangle$. In view of (3.13) we get

$$\langle \tilde{\theta}, l(h')\psi \rangle = \langle \theta, l(h')(\psi \otimes w) \rangle = \langle \tilde{\theta}, \psi \rangle \quad (h' \in H').$$

That is, $\tilde{\theta}$ is a left H'-invariant distribution on H'. Therefore it is a left Haar measure on H'. More precisely, there exists a uniquely determined constant $\omega(w) \in \boldsymbol{C}$ such that

$$\langle \theta, \psi \otimes w \rangle = \omega(w) \int_{H'} \psi(h') d_l h'.$$

From the uniqueness of $\omega(w)$ we get that $w \mapsto \omega(w) : \Gamma_c^\infty(M : W) \to \boldsymbol{C}$ is linear; and by choosing $\psi \in C_c^\infty(H')$ satisfying $\int_{H'} \psi(h') d_l h' = 1$, we obtain the continuous linear mapping $w \mapsto \psi \otimes w \mapsto \langle \theta, \psi \otimes w \rangle = \omega(w)$. Therefore $\omega \in \mathcal{D}'(M : W)$, and we can write

$$\langle \theta, \psi \otimes w \rangle = \langle \omega_m, \int_{H'} (\psi \otimes w)(h'm) d_l h' \rangle.$$

Since $C_c^\infty(H') \otimes \Gamma_c^\infty(M : W)$ is dense in $\Gamma_c^\infty(V_M : W)$, we have obtained

$$(3.14) \quad \langle \theta, s \rangle = \langle \omega_m, \int_{H'} s(h'm) d_l h' \rangle \quad (s \in \Gamma_c^\infty(V_M : W)).$$

Combining (3.12) and (3.14), we find (3.11). □

We now take up invariance under right H_x-action. From now on we assume

$$H' \subset H \quad \text{is a normal closed subgroup.}$$

For any $x \in O$, we set

$$H'_x = H' \cap H_x.$$

Then H'_x is a normal closed subgroup in H_x, and it is the stabilizer subgroup in H' of x. The relation

$$h'm\xi = h'(m\xi m^{-1})m \quad (m \in M,\ \xi \in H'_x)$$

shows that $V_M H'_x = V_M$ and so the behaviour of ζ under the right action of at least H'_x can be studied without leaving V_M. For any $x, y \in O$, there exists $h \in H$ with $y = h \cdot x$, and therefore $H_y = hH_xh^{-1}$; and so we get $H'_y = H' \cap hH_xh^{-1} = h(H' \cap H_x)h^{-1} = hH'_xh^{-1}$. That is, all the stabilizer subgroups H'_y in H' of points $y \in O$, are conjugate by elements of H. We introduce the homomorphism

$$(3.15) \quad \chi_x : H'_x \to \boldsymbol{R} \qquad \text{by} \qquad \chi_x(\xi) = \frac{\delta_{H'}(\xi)}{\delta_{H'_x}(\xi)} = \frac{\delta_{H'}(\xi)}{\delta_{H_x}(\xi)},$$

since $\delta_{H'_x} = \delta_{H_x}|_{H'_x}$ because of the normality of H'_x in H_x. Moreover, for any $m \in M$, we define the action β^m of H' in $E^m := E$ by

$$\beta^m(h') = \beta(h'^m) \quad (h' \in H').$$

Lemma 3.7. *Suppose* $\zeta \in \mathcal{D}'(V_M : W)^{H'}$ *satisfies* $\delta_{H_x}(\xi)^{-1}R(\xi)\zeta = \zeta$, *for all* $\xi \in H'_x$. *Let* $\omega \in \mathcal{D}'(M : W)$ *be as in the preceding lemma, then it satisfies, for all* $w \in \Gamma_c^\infty(M : W)$ *and* $\xi \in H'_x$

$$\langle \omega, w \rangle = \langle \omega_m, \chi_x(\xi)(\alpha \otimes \beta^m)(\xi) w(m) \rangle. \tag{3.16}$$

Proof. For $s \in \Gamma_c^\infty(V : W)$ and $\xi \in H'_x$

$$\langle \zeta, s \rangle = \langle \omega_m, \delta_{H_x}(\xi)^{-1}(\alpha \otimes \mathrm{id})(\xi) \int_{H'} (\mathrm{id} \otimes \beta)(h'^{-1}) s(h'm\xi)\, d_l h' \rangle. \tag{3.17}$$

We have, for $h' \in H'$, $m \in M$, $\xi \in H'_x$, that $h'm\xi = h'\xi^m m$, with $\xi^m = m\xi m^{-1} \in H'$. In view of the analogue of (3.8) for H' we now obtain

$$\begin{cases} \int_{H'} (\mathrm{id} \otimes \beta)(h'^{-1}) s(h'm\xi)\, d_l h' \\ \quad = \delta_{H'}(\xi^m)(\mathrm{id} \otimes \beta)(\xi^m) \int_{H'} (\mathrm{id} \otimes \beta)(h'^{-1}) s(h'm)\, d_l h'. \end{cases} \tag{3.18}$$

Combining (3.11), (3.17) and (3.18) we see, for all $\xi \in H'_x$

$$\begin{cases} \langle \omega_m, \int_{H'} (\mathrm{id} \otimes \beta)(h'^{-1}) s(h'm)\, d_l h' \rangle \\ \quad = \langle \omega_m, \dfrac{\delta_{H'}(\xi^m)}{\delta_{H_x}(\xi)} (\alpha(\xi) \otimes \beta(\xi^m)) \int_{H'} (\mathrm{id} \otimes \beta)(h'^{-1}) s(h'm)\, d_l h' \rangle. \end{cases} \tag{3.19}$$

Since $\mathfrak{h}'$ is normalized by Ad m, for $m \in M$, we obtain (cf. (3.3))

$$\delta_{H'}(\xi^m) = |\det(\mathrm{Ad}\, m\xi m^{-1}|_{\mathfrak{h}'})| = |\det(\mathrm{Ad}\, \xi|_{\mathfrak{h}'})| = \delta_{H'}(\xi).$$

In view of (3.15) we now have

$$\frac{\delta_{H'}(\xi^m)}{\delta_{H_x}(\xi)} = \frac{\delta_{H'}(\xi)}{\delta_{H'_x}(\xi)} = \chi_x(\xi) \quad (\xi \in H'_x).$$

Now apply (3.19) with $s \in \Gamma_c^\infty(V:W)$ of the form $s(h'm) = (\mathrm{id} \otimes \beta)(h')\psi(h')w(m)$, with $\psi \in C_c^\infty(H')$ satisfying $\int_{H'} \psi(h')\, d_l h' = 1$, and $w \in \Gamma_c^\infty(M : W)$. We then obtain (3.16). □

Three special cases. In order to analyze condition (3.16) on $\omega \in \mathcal{D}'(M : W)$, we shall treat three cases separately:

(i) The representation β of H' in E extends to a representation of H in E.
(ii) $H' = H$ (this actually is a special case of (i)).
(iii) $\dim E < \infty$.

Case (i) The representation β of H' in E extends to a representation of H in E. We denote the action of H in E also by β. Then we get $\beta^m(h') = \beta(m)\beta(h')\beta(m^{-1})$, for $m \in M$ and $h' \in H'$. Thus (3.16) takes the form, if we write $\beta^{(-1)}(m) = \beta(m^{-1})$

$$\langle \omega, w \rangle = \langle \omega, \chi_x(\xi)(\mathrm{id} \otimes \beta)(\alpha \otimes \beta)(\xi)(\mathrm{id} \otimes \beta^{(-1)}) w \rangle \quad (\xi \in H'_x).$$

If we now define $\mu \in \mathcal{D}'(M : W)$ by

$$\text{(3.20)} \quad \langle \mu, w \rangle = \langle \omega, (\mathrm{id} \otimes \beta) w \rangle,$$

we obtain

$$\text{(3.21)} \quad \langle \mu, w \rangle = \langle \mu, \chi_x(\xi)(\alpha \otimes \beta)(\xi) w \rangle \quad (\xi \in H'_x).$$

Next we identify

$$\begin{aligned} w &\in \Gamma_c^\infty(M : W) \quad \text{with} \quad w \otimes 1 \in \Gamma_c^\infty(M : W) \otimes \mathbf{C}_x, \\ \omega &\in \mathcal{D}'(M : W) \quad \text{with} \quad \omega \otimes 1 \in \mathcal{D}'(M : W) \otimes \mathbf{C}'_x, \end{aligned}$$

where $\mathbf{C}_x$, and $\mathbf{C}'_x$, denote the one-dimensional H'_x-module $\mathbf{C}$ determined by the character χ_x, and $\chi_x^* = \chi_x^{-1}$ resp., of H'_x. Then (3.21) gives

$$\text{(3.22)} \quad (\alpha^* \otimes \beta^* \otimes \chi_x^*)(\xi)\mu = \mu \quad (\xi \in H'_x).$$

Because of (2.2) we have the canonical isomorphism $\mathcal{D}'(M : W) \simeq \mathcal{D}'(M) \mathbin{\overline{\otimes}} (F_x \otimes E)'$, and therefore

$$\mu \in \mathcal{D}'(M) \mathbin{\overline{\otimes}} (F'_x \otimes E' \otimes \mathbf{C}'_x) \simeq \mathcal{D}'(M : (F'_x \otimes E' \otimes \mathbf{C}'_x)').$$

Notice that in condition (3.22) no action of H'_x on the base space M is involved, therefore we find, using (3.11) and (3.20) the following

Proposition 3.8. *Suppose $H' \subset H$ is normal and the action β of H' in E extends to a continuous representation β of H in E. Let $\tau \in \mathcal{D}'(U : F \otimes E)^{H'}$ and let $V_M \subset V = \pi^{-1}(U)$. Then there exists a unique*

$$\mu \in \mathcal{D}'(M : ((F'_x \otimes E' \otimes \mathbf{C}'_x)^{H'_x})')$$

such that $\sharp\tau = (\mathrm{id} \otimes \beta^*)(d_l h' \otimes \mu)$, *that is, for every* $s \in \Gamma_c^\infty(V_M : W)$

$$\langle \sharp\tau, s \rangle = \langle \mu_m, \int_{H'} (\mathrm{id} \otimes \beta)((h'm)^{-1}) s(h'm) d_l h' \rangle.$$

In particular, $\mathrm{supp}_{V_M}(\sharp\tau) \subset H' \, \mathrm{supp}(\mu)$.

Theorem 3.9. *Suppose $H' \subset H$ is normal and β extends to a continuous representation of H in E. Then, for all H'-invariant open sets $U \subset O$,*

$$(F'_x \otimes E' \otimes \mathbf{C}'_x)^{H'_x} = (0) \implies \mathcal{D}'(U : F \otimes E)^{H'} = (0).$$

Proof. This is immediate from the preceding proposition. □

The condition in Theorem 3.9 is independent of the choice of the point $x \in O$. In fact, we have, for $y = h \cdot x$ with $h \in H$, $\eta \in H'_y$, $\xi = h^{-1}\eta h$

$$\begin{aligned} \delta_{H'_x}(\xi) &= |\det(\mathrm{Ad}\, h^{-1}\eta h \,|_{\mathfrak{h}'_x})| = |\det(\mathrm{Ad}\, \eta \,|_{\mathrm{Ad}\, h(\mathfrak{h}'_x)})| = |\det(\mathrm{Ad}\, \eta \,|_{\mathfrak{h}'_y})| \\ &= \delta_{H'_y}(\eta). \end{aligned}$$

Similarly we get $\delta_{H'}(\xi) = \delta_{H'}(\eta)$, using that H' is normal in H; and therefore $\chi_y(\eta) = \chi_x(\xi)$. The independence remarked above follows from

Lemma 3.10. *Suppose that H acts on E through β. Then there exists a linear isomorphism, for every $x, y \in O$*

$$I_x := (F'_x \otimes E' \otimes C'_x)^{H'_x} \simeq I_y := (F'_y \otimes E' \otimes C'_y)^{H'_y}.$$

Proof. Select $h \in H$ such that $y = h \cdot x$. Then $\alpha^*(h) : F'_x \to F'_y$ is a linear isomorphism, and by assumption $\beta^*(h) : E' \to E'$ is a linear isomorphism as well. Hence we have the linear isomorphism $\gamma = (\alpha^* \otimes \beta^* \otimes \mathrm{id})(h)$

$$\gamma : F'_x \otimes E' \otimes C'_x \to F'_y \otimes E' \otimes C'_y. \tag{3.23}$$

It is easily checked that the mapping in (3.23) induces a linear isomorphism: $I_x \to I_y$. □

Case (ii) $H' = H$. A special case of the results above occurs if $H' = H$. Then $M = \{1\}$, and from Proposition 3.8 we obtain, for $\tau \in \mathcal{D}'(O : F \otimes E)^H$

$$\langle \sharp\tau, \psi \otimes w \rangle = \langle \mu, w \rangle \int_H \psi(h) d_l h \quad (\psi \in C_c^\infty(H),\ w \in F_x \otimes E).$$

Moreover we may consider μ as a vector (instead of a distribution) in a fixed vector space, viz.

$$\mu \in (F'_x \otimes E' \otimes C'_x)^{H_x}.$$

In this case we have the following complete description of $\mathcal{D}'(O : F \otimes E)^H$.

Theorem 3.11. *Suppose $H = H'$. Then, for every $\tau \in \mathcal{D}'(O : F \otimes E)^H$, there exists a unique $\mu \in (F'_x \otimes E' \otimes C'_x)^{H_x}$ such that $\sharp\tau = d_l h \otimes \mu$.*

We drop the condition of β being extendable to H, and we return to the problem of analyzing condition (3.16) on $\omega \in \mathcal{D}'(M : W)$. We set

$$L_\xi(m) := \mathrm{id} - \chi_x(\xi)(\alpha \otimes \beta^m)(\xi) \in \mathrm{End}(W_0), \tag{3.24}$$

and we write condition (3.16) as

$$\langle \omega, L_\xi w \rangle = 0 \quad (\xi \in H'_x,\ w \in \Gamma_c^\infty(M : W)). \tag{3.25}$$

Determining $\mathrm{supp}(\omega) \subset M$ from (3.25), obviously is tied up with finding the range of the linear operators L_ξ in $W_0 = F_x \otimes E$; and therefore complications of a functional-analytic nature may arise. Hence we now assume that we are in

Case (iii) $\dim E < \infty$.

Lemma 3.12. *If $\dim E < \infty$, then $\mathrm{supp}(\omega) \subset \{m \in M \mid (F'_x \otimes (E^m)' \otimes C'_x)^{H'_x} \neq 0\}$, and the latter set is closed.*

Proof. First we shall establish

$$(3.26)\qquad \operatorname{supp}(\omega) \subset \left\{ m \in M \,\middle|\, \sum_{\xi \in H'_x} L_\xi(m)\, W_0 \neq W_0 \right\};$$

here the latter set consists of the $m \in M$ such that linear span of the ranges of the $L_\xi(m)$, for all $\xi \in H'_x$, is different from W_0. Indeed, suppose $m_0 \in M$ satisfies

$$\sum_{\xi \in H'_x} L_\xi(m_0)\, W_0 = W_0.$$

Since $\dim W_0 < \infty$, there are $\xi_i \in H'_x$, for $1 \leq i \leq s < \infty$, with $(\sum_{1 \leq i \leq s} L_{\xi_i}(m_0)) W_0 = W_0$. But then

$$L(m) := \sum_{1 \leq i \leq s} L_{\xi_i}(m) \in \operatorname{End}(W_0) \quad (m \in M)$$

is invertible if $m = m_0$, and therefore also if m varies in M_0, an open neighborhood of m_0 in M. Now let $\gamma \in C_c^\infty(M_0)$ and $w \in \Gamma_c^\infty(M : W)$, and define $w' \in \Gamma_c^\infty(M : W)$ by $w'(m) = \gamma(m) L(m)^{-1} w(m)$. Then we obtain from (3.25) that $0 = \langle \omega, Lw' \rangle = \langle \omega, \gamma w \rangle$; and this implies $\omega\,|_{M_0} = 0$. Therefore $m_0 \notin \operatorname{supp}(\omega)$, and moreover a full neighborhood of m_0 in M is not contained in the RHS in (3.26). From (3.26) it follows that, for every $m \in \operatorname{supp}(\omega)$, there exists $0 \neq \lambda \in W_0'$, such that $\lambda(L_\xi(m)\, W_0) = 0$, for all $\xi \in H'_x$; and so, in view of (3.24)

$$\langle \lambda, w - \chi_x(\xi^{-1})(\alpha \otimes \beta^m)(\xi^{-1}) w \rangle = 0 \quad (w \in W_0,\ \xi \in H'_x).$$

Therefore $\lambda \otimes 1 = (\alpha^* \otimes (\beta^m)^\vee \otimes \chi_x^*)(\xi)(\lambda \otimes 1)$, for all $\xi \in H'_x$. In other words

$$0 \neq \lambda \otimes 1 \in (F'_x \otimes (E^m)' \otimes C'_x)^{H'_x}. \qquad \square$$

Lemma 3.13. *For every $h' \in H'$, there exists a linear isomorphism*

$$(F'_x \otimes E' \otimes C'_x)^{H'_x} \simeq (F'_{h' \cdot x} \otimes E' \otimes C'_{h' \cdot x})^{H'_{h' \cdot x}}.$$

Proof. The proof is similar to that of Lemma 3.10. $\square$

We have obtained the following

Support Theorem 3.14. *Suppose* $\dim E < \infty$. *Then, for every* $\tau \in \mathcal{D}'(U : F \otimes E)^{H'}$, *we have*

$$\operatorname{supp}(\tau) \subset \{ y \in U \mid (F'_y \otimes E' \otimes C'_y)^{H'_y} \neq 0 \}.$$

Finally we can combine all three situations and prove the following:

Vanishing Theorem 3.15. *We assume that we are in one of the cases* (i)–(iii) *and also that, for all $y \in O$ and $r \in \mathbf{Z}_{\geq 0}$*

$$(3.27)\qquad (M_y^{(r)} \otimes E' \otimes C'_y)^{H'_y} = (0).$$

Then

$$\mathcal{D}'_O(X:E)^{H'} = (0).$$

If the action β of H' in E extends to an action β of H in E, then it is sufficient to require condition (3.27) *for a single $y \in O$. Otherwise it is sufficient to require the condition for one representative y of every H'-orbit in O. If E is a Hilbert space and the representation β is unitary, then condition* (3.27) *takes the form*

$$(3.28) \qquad (M_y^{(r)'} \otimes E \otimes \boldsymbol{C}_y)^{H'_y} = (0).$$

Proof. If $T \in \mathcal{D}'_O(X:E)^{H'}$ and $T \neq 0$, there exist $x \in O$ and $r \in \boldsymbol{Z}_{\geq 0}$ such that $T \neq 0$ in arbitrary small neighborhoods of x and such that the transverse order of T at x is equal to r. We define

$$U = \{y \in O \mid \text{transverse order of } T \text{ at } y \text{ is } r\}.$$

Then U is an open neighborhood of x in O (see infra formula (2.5)) and U is H'-invariant since T is H'-invariant. We may write $U = Y \cap O$ where Y is an H'-invariant open set in X; U is closed in Y. Now consider the restriction $T^{(r)} = T\,|_Y$. Then Theorem 2.1 and Lemma 3.1 imply

$$0 \neq \tau^{(r)} := \sigma^{(r)}(T^{(r)}) \in \mathcal{D}'(U : M^{(r)'} \otimes E)^{H'}.$$

Now we obtain a contradiction if we apply Theorem 3.9 or the Support Theorem 3.14 to $\tau^{(r)}$. □

Finally we identify in geometric terms the line bundle Λ over the H'-orbit $O(x)$ of x in O, formed by the one-dimensional H'_y-modules $\boldsymbol{C}_y$ determined by the characters χ_y, for $y \in O(x)$. According to (3.15) we have

$$\chi_y = \frac{\delta_{H'}}{\delta_{H'_y}} = \left| \frac{\det(\mathrm{Ad}\,|_{\mathfrak{h}'})}{\det(\mathrm{Ad}\,|_{\mathfrak{h}'_y})} \right| = |\det(\mathrm{Ad}\,|_{\mathfrak{h}'/\mathfrak{h}'_y})|.$$

Since $O(y) \simeq H'/H'_y$, we have $T_y O(y) \simeq \mathfrak{h}'/\mathfrak{h}'_y$. So a C^∞ section $y \mapsto s(y)$ of this bundle Λ has the following properties. Any diffeomorphism $\eta \in H'_y$ induces an action in the fiber $\Lambda_y = \boldsymbol{C}_y$ by means of $(\eta \cdot s)(\eta \cdot y) = \chi_y(\eta) s(y)$, and therefore

$$(\eta \cdot s)(y) = |\det(\mathrm{Ad}\,\eta\,|_{\mathfrak{h}'/\mathfrak{h}'_y})| s(y) \quad (y \in O(x)).$$

Hence the section s transforms like a density on $O(x)$.

4. APPLICATIONS

Application in Case (i): Whittaker theory. As mentioned in the introduction, in the years 1971–82 Harish-Chandra developed a complete theory of spectral decomposition of $L^2(G/N_0, \chi)$, the so-called Whittaker theory. Here G is a connected semisimple Lie group with finite center and N_0 is the unipotent radical of a minimal parabolic subgroup, moreover χ is a unitary character of N_0 which is regular (see below) and $L^2(G/N_0, \chi)$ is the space on which the representation of G induced by χ acts. The entire theory was based on a rather remarkable vanishing theorem on distributions on G. His proof of this theorem involves calculations that are hard to follow. We shall now present a more transparent proof that makes it a special case of the Vanishing Theorem 3.15. For another approach to the spectral theory of $L^2(G/N_0, \chi)$, see Wallach [W, Chapter 15].

We now use the definitions and results in [Ha2, § 3 and 6]. In particular, G denotes a reductive Lie group. Fix a minimal p-pair (P_0, A_0) in G, and let $P_0 = M_0N_0$ be the Langlands decomposition. Let χ be a finite-dimensional representation of N_0. Fix a p-pair $(P, A) \succ (P_0, A_0)$ with $P = MN$, and put $\bar{N} = \theta N$, where θ is a Cartan involution of $\mathfrak{g}$ as well as G. Note that $\bar{N}MN$ is an open subset of G. We shall study distributions T on G that are left $\bar{N}$-invariant and that transform according to χ under the right action of N_0; thus

$$(4.1) \qquad l(\bar{n})T = T \quad (\bar{n} \in \bar{N}), \qquad r(n_0)T = \chi(n_0)T \quad (n_0 \in N_0).$$

We shall say that χ is regular if none of the characters (= one-dimensional representations) of N_0 that occur in a composition series for χ is identically 1 on each simple root subgroup of N_0. We shall give a new proof of the following known result.

Theorem 4.1 [Harish-Chandra]. *Suppose χ is regular and T is a distribution on G satisfying the conditions in* (4.1). *Then $T = 0$ if $T = 0$ on the big Bruhat cell $\bar{P}N_0$.*

Using the composition series for χ one comes down immediately to the case when χ is one-dimensional. Let us therefore suppose this. In order to put the problem in the context of Case (i) in Section 3, we set

$$X = G; \quad H = \bar{P} \times N_0, \quad h = (\bar{p}, n_0); \quad H' = \bar{N} \times N_0, \quad h' = (\bar{n}, n_0).$$

$\bar{N}$ being normal in $\bar{P}$, the closed subgroup H' is normal in H; and actually H is a semidirect product of H' and M. The action of H on X is given by

$$(4.2) \qquad h \cdot x = (\bar{p}, n_0) \cdot x = \bar{p}xn_0^{-1} = l(\bar{p})r(n_0)x.$$

We set

$$E = C \qquad \text{and} \qquad \beta = \mathrm{id} \otimes \chi.$$

Notice that this action of H' in E can be extended to an action, also denoted by β, of H in E. Moreover H' acts in $C_c^\infty(X : E) = C_c^\infty(G)$ by

$$(\bar{n}, n_0) \cdot f = \chi(n_0)l(\bar{n})r(n_0)f \quad (f \in C_c^\infty(G)).$$

The induced action of H' in $\mathcal{D}'(X : E) = \mathcal{D}'(G)$ therefore is given by

$$\langle h' \cdot T, f \rangle = \langle T, \chi(n_0^{-1})l(\bar{n}^{-1})r(n_0^{-1})f \rangle = \langle \chi(n_0^{-1})l(\bar{n})r(n_0)T, f \rangle.$$

Hence

$$(\bar{n}, n_0) \cdot T = \chi(n_0^{-1})l(\bar{n})r(n_0)T \quad (T \in \mathcal{D}'(G)).$$

In particular (cf. (4.1))

$$T \in \mathcal{D}'(G)^{H'} \iff l(\bar{n})r(n_0)T = \chi(n_0)T \quad ((\bar{n}, n_0) \in \bar{N} \times N_0).$$

According to the Bruhat decomposition the orbits O for the H-action on X are of the form (with $x_s \in G$ a representative of $s \in \mathfrak{w}(G/A_0)$)

$$O_s = \bar{P}x_sN_0 \quad (s \in \mathfrak{w}(G/A_0)).$$

We notice that the orbits for the H'-action on X are parametrized (redundantly) by the set $M \times \mathfrak{w}(G/A_0)$, and this set does not satisfy Bruhat's criterion of being at most countable. This is why we are bringing in the H-action.

Computation of $(M_y^{(r)} \otimes E' \otimes C_y')^{H_y'}$. According to the Vanishing Theorem 3.15 it is sufficient to compute this space for a single $y \in O$, and in view of the Bruhat decomposition we select $y = x_s$ with $s \in \mathfrak{w}(G/A_0)$.

In the cases that conjugation by x_s is independent of the choice of x_s as a representative of $s \in \mathfrak{w}(G/A_0)$, we write

$$s \cdot g = x_s g x_s^{-1}, \qquad s \cdot X = \operatorname{Ad} x_s X \quad (g \in G,\ X \in \mathfrak{g}).$$

We have $(\bar{p}, n_0) \in H_s \Leftrightarrow \bar{p}x_s n_0^{-1} = x_s \Leftrightarrow n_0 = x_s^{-1}\bar{p}x_s$. Hence

$$H_s = \{(\bar{p}, s^{-1}\cdot\bar{p}) \mid \bar{p} \in \bar{P} \cap s\cdot N_0\},\ \ \mathfrak{h}_s = \{(X, s^{-1}\cdot X) \mid X \in \bar{\mathfrak{p}} \cap s\cdot\mathfrak{n}_0\} \simeq \bar{\mathfrak{p}} \cap s\cdot\mathfrak{n}_0,$$
$$H_s' = \{(\bar{n}, s^{-1}\cdot\bar{n}) \mid \bar{n} \in \bar{N} \cap s\cdot N_0\},\ \ \mathfrak{h}_s' = \{(X, s^{-1}\cdot X) \mid X \in \bar{\mathfrak{n}} \cap s\cdot\mathfrak{n}_0\} \simeq \bar{\mathfrak{n}} \cap s\cdot\mathfrak{n}_0.$$

Since H' and H_s' are connected unipotent groups, their respective modular functions are identically 1. Therefore the line bundle Λ is trivial, and in particular we have that $E \otimes \Lambda_{x_s} \simeq E$. Also, because of the connectedness of H_s', it is sufficient to compute

$$\operatorname{Hom}_{\mathfrak{h}_s'}(E, M_s^{(r)}) \simeq (M_s^{(r)} \otimes E' \otimes C_s')^{H_s'}.$$

Because $\bar{p}x_s n_0 = x_s(s^{-1}\cdot\bar{p})n_0$, it is immediate from (4.2) that

$$T_s O_s = s^{-1}\cdot\bar{\mathfrak{p}} + \mathfrak{n}_0 = s^{-1}\cdot(\bar{\mathfrak{p}} + s\cdot\mathfrak{n}_0).$$

We set

$$\mathfrak{n}^+ := \mathfrak{n} \cap s\cdot\mathfrak{n}_0, \qquad \mathfrak{n}^- := \mathfrak{n} \cap s\cdot\overline{\mathfrak{n}_0}.$$

Since $\theta\mathfrak{n}^- = \bar{\mathfrak{n}} \cap s\cdot\mathfrak{n}_0$, we have $\mathfrak{n} = \mathfrak{n}^+ \oplus \mathfrak{n}^-$ and $s\cdot\mathfrak{n}_0 = \mathfrak{n}^+ \oplus \theta\mathfrak{n}^- \oplus (s\cdot\mathfrak{n}_0 \cap \mathfrak{m})$. Therefore $\bar{\mathfrak{p}} + s\cdot\mathfrak{n}_0 = \bar{\mathfrak{p}} \oplus \mathfrak{n}^+$. Since $\mathfrak{g} = \bar{\mathfrak{p}} \oplus \mathfrak{n} = \bar{\mathfrak{p}} \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^-$, we get

$$\mathfrak{g} = s^{-1}\cdot(\bar{\mathfrak{p}} \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^-) = s^{-1}\cdot(\bar{\mathfrak{p}} + s\cdot\mathfrak{n}_0) \oplus s^{-1}\cdot\mathfrak{n}^- = T_s O_s \oplus s^{-1}\cdot\mathfrak{n}^-.$$

Thus $s^{-1}\cdot\mathfrak{n}^-$ is a transverse subspace of $\mathfrak{g}$ with respect to $T_s O_s$. If $h' = (\bar{n}, s^{-1}\cdot\bar{n}) \in H_s'$ and $Z \in \mathfrak{n}^-$, then

$$\begin{aligned} h'\cdot(s^{-1}\cdot Z) &= \frac{d}{dt}\Big|_{t=0} h'\cdot x_s \exp(ts^{-1}\cdot Z) \\ &= \frac{d}{dt}\Big|_{t=0} \bar{n}x_s(x_s^{-1}\exp tZ\, x_s)(x_s^{-1}\bar{n}x_s)^{-1} \\ &= \frac{d}{dt}\Big|_{t=0} x_s(x_s^{-1}\bar{n})\exp tZ(\bar{n}^{-1}x_s) = s^{-1}\cdot\operatorname{Ad}\bar{n}Z. \end{aligned}$$

For $\bar{n} \in \bar{N} \cap s \cdot N_0$ and $Z \in \mathfrak{n}^-$, the vector $\mathrm{Ad}\,\bar{n}Z$ does not necessarily belong to $\mathfrak{n}^-$. We denote by

$$p^- : \mathfrak{g} \to \mathfrak{n}^- \quad \text{the linear projection along} \quad \bar{\mathfrak{p}} \oplus \mathfrak{n}^+.$$

Then $p^-(\mathrm{Ad}\,\bar{n}Z) \in \mathfrak{n}^-$, for $Z \in \mathfrak{n}^-$. Hence the induced action of $h' = (\bar{n}, s^{-1} \cdot \bar{n}) \in H'_s$, and of $(X_0, s^{-1} \cdot X_0) \in \mathfrak{h}'_s$, on the transverse subspace $s^{-1} \cdot \mathfrak{n}^-$ is given by

$$s^{-1} \cdot Z \mapsto s^{-1} \cdot (p^- \circ \mathrm{Ad}\,\bar{n})Z, \qquad \text{and}$$
$$s^{-1} \cdot Z \mapsto s^{-1} \cdot (p^- \circ \mathrm{ad}\,X_0)Z, \qquad \text{resp.}$$

In order to simplify our notation, we make some identifications. We shall write

$$\mathfrak{h}'_s \simeq \bar{\mathfrak{n}} \cap s \cdot \mathfrak{n}_0.$$

Since H' acts in E through $\beta = \mathrm{id} \otimes \chi$, we have $\beta(\bar{n}, s^{-1} \cdot \bar{n}) = \chi(s^{-1} \cdot \bar{n})$; and thus $X_0 \in \mathfrak{h}'_s$ acts in E as

$$d\chi'(X_0) = d\chi(s^{-1} \cdot X_0).$$

Here $d\chi$ denotes the infinitesimal character of χ acting in E. Further we identify $M_s^{(1)}$, the space of first-order differential operators at x_s in directions transversal to $T_s O_s$, with $\mathfrak{n}^-$, thus

$$M_s^{(1)} \simeq \mathfrak{n}^-, \qquad \text{hence} \qquad M_s^{(r)} \simeq S^{(r)}(\mathfrak{n}^-) \quad (r \in \mathbf{Z}_{\geq 0}).$$

Then the induced action of $X_0 \in \mathfrak{h}'_s$ in $M_s^{(1)}$ is through the map $D(X_0)$ acting in $\mathfrak{n}^-$ that is given by

$$D(X_0) := p^- \circ \mathrm{ad}\,X_0.$$

Since $M^{(r)}$ is the r-th symmetric power of $M^{(1)}$, it follows that the induced action of X_0 in $M_s^{(r)}$ is the restriction to $M_s^{(r)}$ of the derivation of $S(\mathfrak{n}^-)$ that extends $D(X_0)$, denoted again by $D(X_0)$. We have obtained the following

Lemma 4.2. $\mathrm{Hom}_{\mathfrak{h}'_s}(E, M_s^{(r)}) \simeq \mathrm{Hom}_{\bar{\mathfrak{n}} \cap s \cdot \mathfrak{n}_0}(d\chi', S^{(r)}(\mathfrak{n}^-))$. *Here*

$$d\chi'(X_0) = d\chi(s^{-1} \cdot X_0) \quad (X_0 \in \bar{\mathfrak{n}} \cap s \cdot \mathfrak{n}_0).$$

Since $\mathfrak{n}^-$ is an $\mathfrak{a}_0$-module, $S(\mathfrak{n}^-)$ also is an $\mathfrak{a}_0$-module; and $\mathfrak{a}_0$ acts semisimply on $S(\mathfrak{n}^-)$ preserving the gradation. For $\nu \in \mathfrak{a}^*_{0c}$, we write $S_\nu^{(r)}(\mathfrak{n}^-)$ for the set of all $q \in S^{(r)}(\mathfrak{n}^-)$ such that

$$H \cdot q = \nu(H)q \quad (H \in \mathfrak{a}_0).$$

Lemma 4.3 [Harish-Chandra]. *Suppose there exists* $\nu_0 \in \mathfrak{a}^*_{0c}$ *such that* $X_0 \in \bar{\mathfrak{n}} \cap s \cdot \mathfrak{n}_0$ *satisfies*

$$[H, X_0] = \nu_0(H)X_0 \quad (H \in \mathfrak{a}_0).$$

Then

$$[\mathrm{ad}\,H, D(X_0)] = \nu_0(H)D(X_0) \quad (H \in \mathfrak{a}_0).$$

In particular

$$D_0 := D(X_0) : S_\nu^{(r)}(\mathfrak{n}^-) \to S_{\nu+\nu_0}^{(r)}(\mathfrak{n}^-) \quad (\nu \in \mathfrak{a}_{0c}^*).$$

If $0 \neq \nu_0 \in \mathfrak{a}_{0c}^*$, *then* $D(X_0)$ *is a derivation which acts in a nilpotent fashion on all of* $S^{(r)}(\mathfrak{n}^-)$. *In particular, in this case* $D(X_0)$ *has no eigenvalues different from* 0.

Proof. Fix $H \in \mathfrak{a}_0$. Since D_0 and $d := \operatorname{ad} H$ are both derivations of $S(\mathfrak{n}^-)$, the same holds for $[d, D_0] = d \circ D_0 - D_0 \circ d$. Put

$$D := [d, D_0] - \nu_0(H) D_0.$$

Then D is also a derivation of $S(\mathfrak{n}^-)$. We claim $D = 0$. In order to prove this, it would be enough to verify that $DZ = 0$, for $Z \in \mathfrak{n}^-$. Now

$$DZ = [H, p^-[X_0, Z]] - p^-[X_0, [H, Z]] - \nu_0(H) p^-[X_0, Z].$$

Since p^- commutes with ad H, we conclude that

$$\begin{aligned} DZ &= p^-(\operatorname{ad} H \operatorname{ad} X_0 - \operatorname{ad} X_0 \operatorname{ad} H - \nu_0(H) \operatorname{ad} X_0) Z \\ &= p^-(\operatorname{ad}[H, X_0] - \nu_0(H) \operatorname{ad} X_0) Z = 0. \end{aligned}$$

Now suppose $q \in S_\nu^{(r)}(\mathfrak{n}^-)$, then

$$\begin{aligned} \operatorname{ad} H(D_0 q) &= [d, D_0] q + D_0 dq \\ &= \nu_0(H) D_0 q + \nu(H) D_0 q = (\nu + \nu_0)(H) D_0 q. \qquad \square \end{aligned}$$

Lemma 4.4 [Harish-Chandra]. *Suppose* $s \notin \mathfrak{w}_M$. *Then there exists a simple root* α_0 *of* (P_0, A_0) *such that* $s \cdot \mathfrak{n}_0(\alpha_0) \subset \bar{\mathfrak{n}}$.

Proof. For otherwise $s \cdot \mathfrak{n}_0(\alpha) \subset \mathfrak{m} + \mathfrak{n} = \mathfrak{p}$, for every simple root α of (P_0, A_0). Hence, if β is any root of (P_0, A_0), the restriction of $s \cdot \beta$ on $\mathfrak{a}$ is ≥ 0. Thus $s \cdot P_0 = (M \cap s \cdot P_0) N$. Therefore $M \cap s \cdot P_0$ is a minimal parabolic subgroup of M. Hence we can choose $t \in \mathfrak{w}_M$ such that $M \cap s \cdot P_0 = t \cdot (M \cap P_0)$. Let k be a representative for t in $K \cap M$. Then $M \cap x \cdot P_0 = M \cap P_0$, where $x = k^{-1} x_s$. Therefore $x \cdot P_0 = (M \cap P_0) N = P_0$. This proves that $t^{-1} s \cdot P_0 = P_0$, and therefore $t^{-1} s = 1$. But then $s = t \in \mathfrak{w}_M$, a contradiction. $\square$

Corollary 4.5. *If* $s \notin \mathfrak{w}_M$, *then* $\operatorname{Hom}_{\mathfrak{h}_s'}(E, M_s^{(r)}) = 0$.

Proof. Since the character χ is assumed to be regular, we have $d\chi|_{\mathfrak{n}_0(\beta)} \neq 0$, for all simple roots β. Thus, in particular, with α_0 as in Lemma 4.4

$$d\chi'|_{s \cdot \mathfrak{n}_0(\alpha_0)} \neq 0, \quad \text{that is,} \quad d\chi'(X_0) \neq 0, \quad \text{for some} \quad X_0 \in s \cdot \mathfrak{n}_0(\alpha_0).$$

But now

$$0 \neq \nu_0 := s\alpha_0 \in \mathfrak{a}_0^*; \qquad \text{and} \qquad [H, X_0] = \nu_0(H) X_0 \quad (H \in \mathfrak{a}_0).$$

According to Lemma 4.3, $D(X_0)$ acts as a nilpotent derivation in the $S^{(r)}(\mathfrak{n}^-)$, for $r \in \boldsymbol{Z}_{\geq 0}$. In particular, $D(X_0)$ has no eigenvalues different from 0. So this means that

$$\mathrm{Hom}_{\bar{\mathfrak{n}}\cap s\cdot\mathfrak{n}_0}(d\chi', S^{(r)}(\mathfrak{n}^-)) = (0) \quad (r \in \boldsymbol{Z}_{\geq 0}). \qquad \square$$

Proof of Harish-Chandra's Theorem 4.1. Let C be the smallest closed set which is a union of double cosets $\bar{P}x_s N_0$ containing the support of T. Since $T = 0$ on $\bar{P}N_0$, the set C is disjoint from $\bar{P}N_0$. We wish to argue that $C = \emptyset$. Otherwise we can find an s such that $\bar{P}x_s N_0$ is open in C; $s \notin \mathfrak{w}_M$ obviously. We can therefore find an open set $X \subset G$ such that $\bar{P}XN_0 = X$ and $X \cap C = \bar{P}x_s N_0 = O$. We have $\mathrm{supp}(T) \subset O$ and hence, by the corollary above, $T = 0$ on X. Thus $\mathrm{supp}(T) \subset C\backslash\bar{P}x_s N_0$, contradicting the minimality of C. In view of the Vanishing Theorem 3.15 this implies $T = 0$. $\square$

Application in Case (ii): Bruhat–Harish-Chandra theory of parabolically induced representations. In the representation theory of semisimple Lie groups a central question has always been the irreducibility of representations induced from a parabolic subgroup. This has been studied by many techniques but one of the first such results goes back to Bruhat [B] who reduced it to the computation of the dimensions of certain spaces of invariant distributions. He was able to treat the case when the inducing subgroup was a minimal parabolic subgroup. Later on, with Harish-Chandra's determination of the discrete series and his approach to the Plancherel formula, it became necessary to determine the irreducibility of representations induced from other parabolic subgroups. In [Ha3, §41] he was able to treat the irreducibility of the unitary representations induced from a cuspidal parabolic subgroup and discrete series representations. With wider applications in mind he was however after an irreducibility criterion where the inducing subgroup could be any parabolic subgroup. In an unpublished manuscript going back to the 1970's, he took up Bruhat's method and pushed it through in this general context for those inducing representations with 'real' infinitesimal characters, such as limits of discrete series. He mentioned this result to many people, including G. van Dijk [Ha6]. It is our intention here to present a brief treatment of Harish-Chandra's theorem, as a consequence of the Vanishing Theorem 3.15.

Before formulating and proving Harish-Chandra's theorem we discuss briefly the general situation of inducing possibly infinite-dimensional representations of an arbitrary unimodular Lie group (cf. [B]).

Let G be a unimodular Lie group. We fix a right (and left) Haar measure dx on G; thus, for $\phi \in C_c(G)$ and $g \in G$ (cf. (3.8))

$$(4.3) \qquad \int_G \phi(gx)\,dx = \int_G \phi(xg)\,dx = \int_G \phi(x)\,dx.$$

If $P \subset G$ is a closed subgroup, we select a right Haar measure $d_r p$ on P, and we have similarly to (3.3)

$$(4.4) \qquad \delta_P : P \to \boldsymbol{R}_{>0}.$$

Let π_i be two continuous representations of G in Fréchet spaces E_i, for $i = 1, 2$.

A separately continuous bilinear form $B : E_1 \times E_2 \to \boldsymbol{C}$ is said to be a (π_1, π_2)-intertwining form if

$$(4.5) \qquad B \circ (\pi_1 \otimes \pi_2)(g) = B \quad (g \in G).$$

We denote the linear space of these forms by $\mathrm{I}(\pi_1, \pi_2)$, and

$$i(\pi_1, \pi_2) = \dim \mathrm{I}(\pi_1, \pi_2)$$

is said to be the intertwining number of π_1 and π_2.

Let π be a continuous representation of G in a Fréchet space E. We say that $e \in E$ is a differentiable vector in E if $\tilde{e} \in \mathrm{C}^\infty(G : E)$, where $\tilde{e}(x) = \pi(x)e$. We define $E^0 \subset E$ to be the linear subspace of differentiable vectors in E. It is immediate that E^0 is π-invariant, and that, if $g \in G$

$$\widetilde{\pi(g)e}(x) = \pi(x)\pi(g)e = \pi(xg)e = (r(g)\tilde{e})(x).$$

Hence the mapping $e \mapsto \tilde{e} : E^0 \to \mathrm{C}^\infty(G : E)$ is (π, r)-intertwining. We have the following properties:

(i) E^0 is dense in E;

(ii) E^0 is a closed subspace in the Fréchet topology of $\mathrm{C}^\infty(G : E)$, and therefore it inherits a topology from $\mathrm{C}^\infty(G : E)$ that makes it into a Fréchet space;

(iii) $E^{00} = E^0$, that is, E^0 is differentiable.

The restriction π^0 of π to E^0 is called the differentiable representation associated with π.

Suppose now that σ is a differentiable representation of P in a Fréchet space E. We define $\mathcal{D}^\sigma(E)$ as the linear space of all functions $f \in \mathrm{C}^\infty(G : E)$ satisfying

(i) $l(p^{-1})f(x) = \delta_P(p)^{1/2}\sigma(p)f(x)$, for all $p \in P$ and $x \in G$;

(ii) $\mathrm{supp}(f)$ is compact modulo P.

We introduce $\pi = \mathrm{Ind}_P^G \sigma$, the representation of G induced by the representation σ of P, by letting G act in $\mathcal{D}^\sigma(E)$ by right translation

$$\pi(g)f(x) = f(xg) \quad (g, x \in G, f \in \mathcal{D}^\sigma(E)).$$

For every compact subset $L \subset G$ we have the linear subspace $\mathcal{D}_L^\sigma(E)$ of functions in $\mathcal{D}^\sigma(E)$ with support contained in the set PL. We put on $\mathcal{D}_L^\sigma(E)$ the topology induced by $\mathrm{C}^\infty(G : E)$, and then we equip $\mathcal{D}^\sigma(E)$ with the strict inductive limit topology of the topologies on the $\mathcal{D}_L^\sigma(E)$. This turns $\mathcal{D}^\sigma(E)$ into an LF-space as E is a Fréchet space. Moreover, π is a differentiable representation of G in $\mathcal{D}^\sigma(E)$.

Similar to Lemma 3.5, and with a similar proof, we obtain the following

Lemma 4.6. *We have a surjective continuous linear mapping*

$$\flat : \mathrm{C}_c^\infty(G : E) \to \mathcal{D}^\sigma(E) \quad \textit{given by} \quad \flat = \int_P \delta_P(q)^{-1/2}\sigma(q^{-1})l(q^{-1})\,d_r q,$$

satisfying $\flat \circ l(p) \circ r(g) = \delta_P(p)^{1/2}\pi(g) \circ \flat \circ \sigma(p^{-1})$, *for* $p \in P$, $g \in G$.

Theorem 4.7. *We set* $\pi_i = \mathrm{Ind}_{P_i}^G \sigma_i$, *for* $i = 1, 2$. *There exists a linear isomorphism between the linear space* $\mathrm{I}(\pi_1, \pi_2)$ *of* (π_1, π_2)*-intertwining forms, and the linear space* $\mathcal{D}'(G : E_1 \overline{\otimes} E_2)^{P_1 \times P_2}$ *of* $P_1 \times P_2$*-invariant* $E_1 \overline{\otimes} E_2$*-distributions on* G. *Here the invariance in* $\mathcal{D}'(G : E_1 \overline{\otimes} E_2)$ *is by duality coming from the action of* $P_1 \times P_2$ *on* $\mathcal{D}(G : E_1 \overline{\otimes} E_2)$ *given by*

$$(p_1, p_2) \cdot f = \theta(p_1, p_2)\sigma_1(p_1) \otimes \sigma_2(p_2) l(p_1) r(p_2) f,$$

while the homomorphism θ *is defined by*

$$\theta : (p_1, p_2) \mapsto (\delta_{P_1}(p_1)\delta_{P_2}(p_2))^{-1/2} : P_1 \times P_2 \to \mathbf{R}_{>0}. \tag{4.6}$$

Proof. Consider $B \in \mathrm{I}(\pi_1, \pi_2)$. Then we obtain a separately continuous bilinear form

$$B^\sharp := B \circ (\flat_1 \otimes \flat_2) : C_c^\infty(G : E_1) \times C_c^\infty(G : E_2) \to \mathbf{C}. \tag{4.7}$$

Because of Lemma 4.6 the mapping $B \mapsto B^\sharp$ is injective and linear. We may therefore identify $B^\sharp$ with an $E_1 \overline{\otimes} E_2$-distribution on $G \times G$

$$T^\sharp \in \mathcal{D}'(G \times G : E_1 \overline{\otimes} E_2).$$

It is easy to check that the distribution $T^\sharp$ satisfies

$$\begin{aligned} &T^\sharp \circ l(p_1, p_2) \circ r(g, g) \\ &\quad = \theta(p_1^{-1}, p_2^{-1}) T^\sharp \circ (\sigma_1(p_1^{-1}) \otimes \sigma_2(p_2^{-1})) \qquad (p_i \in P,\, g \in G). \end{aligned}$$

Next we observe that $\mu : (x, y) \mapsto (xy^{-1}, y) : G \times G \to G \times G$ is a C^∞ diffeomorphism, and so we can introduce the push forward $\mu_* T^\sharp$. This is a distribution on $G \times G$ which is right G-invariant with respect to the second variable. As in the proof of Lemma 3.6 we now find a unique distribution

$$T \in \mathcal{D}'(G : E_1 \overline{\otimes} E_2),$$

such that, for all $\psi \in C_c^\infty(G \times G : E_1 \overline{\otimes} E_2)$

$$\langle T, \psi' \rangle = \langle \mu_* T^\sharp, \psi \rangle \qquad \text{with} \qquad \psi'(x) = \int_G \psi(x, g)\,dg \quad (x \in G). \tag{4.8}$$

We leave it to the reader to check that the map $B \mapsto T$ is the isomorphism in question. □

Theorem 4.7 is also valid for a G that is not unimodular if we replace (4.6) by

$$\theta : (p_1, p_2) \mapsto (\delta_G(p_1 p_2^{-1})\delta_{P_1}(p_1)\delta_{P_2}(p_2))^{-1/2} : P_1 \times P_2 \to \mathbf{R}_{>0}.$$

Study of $(M_s^{(r)} \otimes (E_1^0 \overline{\otimes} E_2^0)' \otimes C_s')^{(P_1 \times P_2)_s}$**.** Let us now return to the case when G is a connected semisimple Lie group with finite center. Our notation is as in the Whittaker theory, but now we consider two p-pairs $(P_i, A_i) \succ (P_0, A_0)$, with $P_i = M_i N_i$ and $\mathfrak{p}_i = \mathfrak{m}_i + \mathfrak{n}_i = {}^0\mathfrak{m}_i + \mathfrak{a}_i + \mathfrak{n}_i$, where $i = 1, 2$. We apply the results of Section 3 with

$$X = G, \qquad H = H' = P_1 \times P_2, \qquad (p_1, p_2) \cdot x = p_1 x p_2^{-1}.$$

It follows from [B–T, 5.20] that the orbits for the $P_1 \times P_2$-action on G are of the form, with $x_s \in G$ a representative of $s \in \tilde{\mathfrak{w}}(G/A_0) := \mathfrak{w}(M_1/A_0)\backslash\mathfrak{w}(G/A_0)/\mathfrak{w}(M_2/A_0)$

$$O_s = P_1 x_s P_2 \quad (s \in \tilde{\mathfrak{w}}(G/A_0)).$$

We have

$$(P_1 \times P_2)_s = \{\gamma_p := (p, s^{-1} \cdot p) \mid p \in P_1 \cap s \cdot P_2\} \simeq P_s := P_1 \cap s \cdot P_2,$$

where the identification is via $\gamma_p \leftrightarrow p$. We shall find it convenient to work with the differentiable representation σ_i^0 and the corresponding induced representation.

We have a natural action of $(P_1 \times P_2)_s$ on $M_s^{(r)}$ which is induced by the action of $P_1 \times P_2$ on G, while $(P_1 \times P_2)_s$ acts on $E_1^0 \overline{\otimes} E_2^0$ through $\theta(\sigma_1^0 \otimes \sigma_2^0)$, and on $\mathbf{C}_s$ through χ_s as in formula (3.15). We shift θ to the last factor, and make the identification

$$(M_s^{(r)} \otimes (E_1^0 \overline{\otimes} E_2^0)' \otimes \mathbf{C}_s')^{(P_1 \times P_2)_s} \simeq (M_s^{(r)\prime} \otimes (E_1^0 \overline{\otimes} E_2^0) \otimes \mathbf{C}_s)'^{(P_1 \times P_2)_s},$$

where $p \in P_s$ acts on $E_1^0 \overline{\otimes} E_2^0$ through $\sigma_1^0(p) \otimes \sigma_2^0(s^{-1} \cdot p)$, and on $\mathbf{C}_s$ through $(\theta\chi_s)(\gamma_p)$.

From the formulae (4.6) and (3.15) we obtain

$$\theta\chi_s(\gamma_p) = \left(\frac{\delta_{P_1}}{\delta_{P_1 \cap s \cdot P_2}} \frac{s \cdot \delta_{P_2}}{s \cdot \delta_{P_2 \cap s^{-1} \cdot P_1}}\right)^{1/2} (p) \quad (p \in P_s).$$

To evaluate $\theta\chi_s$ is a matter of describing the quotients $\mathfrak{n}_1/(\mathfrak{n}_1 \cap s \cdot \mathfrak{n}_2)$ and $\mathfrak{n}_2/(\mathfrak{n}_2 \cap s^{-1} \cdot \mathfrak{n}_1)$ which is done using standard root space manipulations. Let

$$(4.9) \quad \begin{cases} {}^*P_1 = M_1 \cap s \cdot P_2, & {}^*M_1 = M_1 \cap s \cdot M_2, \qquad {}^*N_1 = M_1 \cap s \cdot N_2, \\ {}^*\rho_1 = \frac{1}{2} \operatorname{tr}(\operatorname{ad}|_{{}^*\mathfrak{n}_1}). \end{cases}$$

Then ${}^*P_1$ is a parabolic subgroup of M_1 with Levi decomposition ${}^*M_1{}^*N_1$, and the split component of ${}^*P_1$ is given by ${}^*A_1 = A_1 s \cdot A_2 \subset M_1$, since $s \cdot A_2 \subset A_0 \subset M_1$. We introduce ${}^*P_2$ with similar properties, thus ${}^*P_2 = M_2 \cap s^{-1} \cdot P_1$, and ${}^*\rho_2 = \frac{1}{2}\operatorname{tr}(\operatorname{ad}|_{{}^*\mathfrak{n}_2})$. Then

$$(4.10) \quad \theta\chi_s(\gamma_a) = (\delta_{{}^*P_1} s \cdot \delta_{{}^*P_2})^{1/2}(a) = e^{({}^*\rho_1 + s \cdot {}^*\rho_2)(\log a)} \quad (a \in A_0).$$

Let Σ be the set of positive roots β of $(\mathfrak{g}, \mathfrak{a}_0)$ and $\mathfrak{g}_{\pm\beta}$ the root space corresponding to $\pm\beta$, and let $\Sigma_i = \{\beta \in \Sigma \mid \mathfrak{g}_\beta \subset \mathfrak{p}_i\}$ and $\Sigma_i' = \{\beta \in \Sigma_i \mid \beta|_{\mathfrak{a}_i} \neq 0\}$. The following lemma and its corollary are straightforward.

Lemma 4.8. *A transverse subspace of* $\mathfrak{g}$ *with respect to* $T_s O_s$ *is given by*

$$s^{-1} \cdot \overline{\mathfrak{n}_1} \cap \overline{\mathfrak{n}_2} = \sum_{\beta \in s^{-1} \cdot \Sigma_1' \cap \Sigma_2'} \mathfrak{g}_{-\beta}.$$

The action of $(P_1 \times P_2)_s$ *on* $T_s G$ *is given by* $\gamma_p \cdot X = \operatorname{Ad}(s^{-1} \cdot p) X$, *for* $p \in P_s$ *and* $X \in \mathfrak{g}$.

Corollary 4.9. *γ_n acts unipotently in $M_s^{(r)'} \otimes C_s$, for $n \in {}^*N_1$. The weights of γ_a acting in $M_s^{(r)'} \otimes C_s$, for $a \in A_0$, are of the form*

$$e^{(\mu_1 + \cdots + \mu_r + {}^*\rho_1 + s \cdot {}^*\rho_2)(\log a)} \quad (\mu_i \in \Sigma_1' \cap s \cdot \Sigma_2').$$

Next we specify the representations σ_i of P_i that we consider. The Killing form is positive-definite on $\mathfrak{a}_0$, and using it we have the identifications $\mathfrak{a}_0^* \simeq \mathfrak{a}_0$ and $\mathfrak{a}_{0c}^* \simeq \mathfrak{a}_{0c}$. In the following we adopt the following convention. If $\mathfrak{b} \subset \mathfrak{a}_0$ is a linear subspace, then linear functionals on $\mathfrak{b}$ are always regarded as defined on $\mathfrak{a}_0$, being 0 on $\mathfrak{b}^\perp$.
Consider representations σ_i of $P_i = {}^0M_i A_i N_i$, for $i = 1, 2$, of the form

$$\sigma_i = {}^0\sigma_i \otimes e^{(-1)^{1/2}\lambda_i} \otimes \mathrm{id} \quad (\lambda_i \in \mathfrak{a}_{ic}^*).$$

We assume ${}^0\sigma_i$ is unitary and $\lambda_i \in \mathfrak{a}_i^*$; and further that ${}^0\sigma_i$ is irreducible. The infinitesimal character of ${}^0\sigma_i$ is given by ${}^0\Lambda_i \in \mathfrak{h}_c^*$, where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ extending $\mathfrak{a}_0$. Note that $\mathfrak{h}$ is a Cartan subalgebra of each of the reductive components $\mathfrak{m}_i = {}^0\mathfrak{m}_i + \mathfrak{a}_i$. Note that ${}^0\Lambda_i$ is not uniquely determined, but it is unique up to operations from $\mathfrak{w}(\mathfrak{m}_{ic}/\mathfrak{h}_c)$.

We make the assumption that ${}^0\Lambda_i$ is 'real' in the sense that $\langle {}^0\Lambda_i, \beta \rangle \in \boldsymbol{R}$, for all roots β of $(\mathfrak{m}_{ic}, \mathfrak{h}_c)$. We define

$$\Lambda_i = {}^0\Lambda_i + (-1)^{1/2}\lambda_i.$$

Harish-Chandra's vanishing Theorem 4.10. *Let the notation be as above. Assume that $\langle \lambda_i, \beta \rangle \neq 0$, for all $\beta \in \Sigma_i'$, and*

$$(M_s^{(r)'} \otimes (E_1^0 \mathbin{\overline{\otimes}} E_2^0) \otimes C_s)'^{(P_1 \times P_2)_s} \neq (0).$$

Then

$$s \cdot (\mathfrak{a}_2, \lambda_2) = (\mathfrak{a}_1, -\lambda_1) \qquad \textit{and} \qquad r = 0.$$

Proof. Select

$$0 \neq T \in (M_s^{(r)'} \otimes (E_1^0 \mathbin{\overline{\otimes}} E_2^0) \otimes C_s)'^{(P_1 \times P_2)_s}.$$

Then there exists $v \in E_2^0$ such that $T_v \neq 0$, where

$$T_v \in (M_s^{(r)'} \otimes E_1^0 \otimes C_s)' \qquad \text{with} \qquad T_v(u) = T(u \otimes v).$$

Since T is invariant under $(P_1 \times P_2)_s$, the vector T_v inherits transformation properties under this group. We will study the action of γ_{an} on T_v, for the choice $a \in s \cdot A_2 \subset {}^*A_1$, $n \in {}^*N_1$ (cf. (4.9)). Since the inducing representation σ_2^0 is trivial on N_2, we have $\gamma_n \cdot v = \sigma_2^0(s^{-1} \cdot n)v = v$. Hence $\gamma_n \cdot T_v(u) = T(\gamma_n^{-1} \cdot u \otimes v) = T(u \otimes \gamma_n \cdot v) = T_v(u)$; and thus $\gamma_n \cdot T_v = T_v$, for $n \in {}^*N_1$. Furthermore $\gamma_a \cdot T_v(u) = T(u \otimes \sigma_2^0(s^{-1} \cdot a)v) = e^{(is \cdot \lambda_2)(\log a)} T(u \otimes v)$; hence

$$\gamma_{an} \cdot T_v = e^{(is \cdot \lambda_2)(\log a)} T_v \quad (a \in s \cdot A_2,\ n \in {}^*N_2). \tag{4.11}$$

According to the description in Corollary 4.9 of the action of $s \cdot A_2{}^*N_1$, we can find a filtration of $s \cdot A_2{}^*N_1$-modules

$$M_s^{(r)'} \otimes \boldsymbol{C}_s = L_1 \supset L_2 \supset \cdots \supset L_k \supset L_{k+1} = (0),$$

such that ${}^*N_1$ acts as the identity on the successive subquotients. Thus we can find an integer q such that $T_v \in (L_q \otimes E_1^0)'$, while $T_v \notin (L_{q+1} \otimes E_1^0)'$. We set

$$\ell \otimes t_v = \text{image of } T_v \text{ in } (L_q/L_{q+1} \otimes E_1^0)' \simeq t_v \in E_1^{0'}.$$

The transformation properties of t_v under the action of $s \cdot A_2{}^*N_1$ now are:

(i) ${}^*N_1$ leaves t_v invariant as it acts trivially both on T_v and on $\ell \in L_q/L_{q+1}$.

(ii) For $a \in s \cdot A_2$, the element γ_a acts on t_v with a weight of the form

$$\text{(4.12)} \qquad e^{(is\cdot\lambda_2 - \mu_1 - \cdots - \mu_r - {}^*\rho_1 - s\cdot{}^*\rho_2)(\log a)} \quad (\mu_i \in \Sigma_1' \cap s \cdot \Sigma_2').$$

Indeed, formula (4.11) gives the action on T_v, and Corollary 4.9 gives it on $\ell \in L_q/L_{q+1}$.

(iii) t_v is a smooth vector in $E_1^{0'}$ (which is a differentiable representation, see beginning of Section 3). Since $-\Lambda_1$ is the infinitesimal character of $\sigma_1^{0^*}$, we have

$$z \cdot t_v = -\chi_{\Lambda_1}(z) t_v \quad (z \in \mathfrak{Z}(\mathfrak{m}_1)).$$

We now recall the standard fact (cf. [V, Chapter II.6]) that for every $z \in \mathfrak{Z}(\mathfrak{m}_1)$, there is an element $z_1 =: \gamma_{\mathfrak{m}_1/{}^*\mathfrak{m}_1}(z) \in \mathfrak{Z}({}^*\mathfrak{m}_1)$ such that

$$z \equiv z_1' \bmod U(\mathfrak{m}_1){}^*\mathfrak{n}_1 \qquad \text{(here ${}'$ denotes a suitable shift).}$$

Moreover we write $\mathfrak{Z}_1 = \mathrm{im}\,(\gamma_{\mathfrak{m}_1/{}^*\mathfrak{m}_1}) \subset \mathfrak{Z}({}^*\mathfrak{m}_1)$. It is well-known that $\mathfrak{Z}(\mathfrak{m}_1)$ is a finitely generated module over $\mathfrak{Z}_1$. Since ${}^*\mathfrak{n}_1$ annihilates t_v, we find

$$z \cdot t_v = z_1 \cdot t_v.$$

But this implies $t_v = t_1 + \cdots + t_l$, where the t_j are nonzero generalized eigenvectors for the action of $\mathfrak{Z}({}^*\mathfrak{m}_1)$ associated with eigenhomomorphisms determined by elements $\nu_j \in \mathfrak{h}_c^*$. Hence

$$\nu_j \in \mathfrak{w}(\mathfrak{m}_{1c}/\mathfrak{h}_c)\text{-orbit of } -\Lambda_1.$$

We may assume that Λ_1 is fixed such that $\nu_1 = -\Lambda_1$. On the other hand, $s \cdot \mathfrak{a}_2$ is contained in the split component of ${}^*\mathfrak{m}_1$, and therefore $s \cdot \mathfrak{a}_2 \subset \mathfrak{Z}({}^*\mathfrak{m}_1)$. In particular, this means that t_1 is a generalized eigenvector for $s \cdot \mathfrak{a}_2$ of weight $\nu_1 - {}^*\rho_1 |_{s\cdot\mathfrak{a}_2}$. But in formula (4.12) we computed already that $s \cdot \mathfrak{a}_2$ acts on t_v, and therefore on t_1, with weight

$$is \cdot \lambda_2 - \mu_1 - \cdots - \mu_r - {}^*\rho_1 - s \cdot {}^*\rho_2.$$

Hence $is \cdot \lambda_2 - \mu_1 - \cdots - \mu_r - {}^*\rho_1 - s \cdot {}^*\rho_2 = -\Lambda_1 - {}^*\rho_1$ on $s \cdot \mathfrak{a}_2$, and it follows that

$$\text{(4.13)} \qquad is \cdot \lambda_2 = \mu_1 + \cdots + \mu_r + s \cdot {}^*\rho_2 - \Lambda_1 \qquad \text{on} \qquad s \cdot \mathfrak{a}_2.$$

Equating imaginary parts we see

$$\Lambda_1 = -is \cdot \lambda_2 \qquad \text{on} \qquad s \cdot \mathfrak{a}_2.$$

On the other hand, we are given that $\Lambda_1 = i\lambda_1$ on $\mathfrak{a}_1$, and Λ_1 is real-valued on the orthogonal complement $\mathfrak{a}_1^{\perp}$ of $\mathfrak{a}_1$ in $\mathfrak{a}_0$. We extend λ_1 to $\mathfrak{a}_0$ by making it 0 on $\mathfrak{a}_1^{\perp}$. Then

$$\Lambda_1 = i\lambda_1 + \Lambda \qquad \text{on} \qquad \mathfrak{a}_0,$$

where Λ is real-valued everywhere on $\mathfrak{a}_0$, and $\Lambda \neq 0$ only on $\mathfrak{a}_1^{\perp}$. Comparing the two expressions for Λ_1, we obtain $-s \cdot \lambda_2 = \lambda_1$ on $s \cdot \mathfrak{a}_2$. We have found

$$\text{if} \quad \mu = s^{-1} \cdot \lambda_1 + \lambda_2, \qquad \text{then} \quad \mu = 0 \quad \text{on} \quad \mathfrak{a}_2.$$

Now we repeat all the arguments above, with the roles of σ_1^0 and σ_2^0 interchanged. Then s has to be replaced by s^{-1}, and we get $-s^{-1} \cdot \lambda_1 = \lambda_2$ on $s^{-1} \cdot \mathfrak{a}_1$. Thus $\mu = 0$ on $s^{-1} \cdot \mathfrak{a}_1$. Accordingly $\mu = 0$ on $\mathfrak{a}_2 + s^{-1} \cdot \mathfrak{a}_1$. Because of the identifications discussed earlier $\mu = 0$ on $\mathfrak{a}_2^{\perp} \cap (s^{-1} \cdot \mathfrak{a}_1)^{\perp}$. Therefore $\mu = 0$ on $\mathfrak{a}_0$, that is

$$s \cdot \lambda_2 = -\lambda_1 \qquad \text{on} \qquad \mathfrak{a}_0.$$

We now use the regularity of the λ_i, that is, $\langle \lambda_i, \beta \rangle \neq 0$ iff $\beta \in \Sigma_i'$. Hence

$$\beta \in \Sigma_2' \iff \langle \lambda_2, \beta \rangle \neq 0 \iff \langle \lambda_1, s\beta \rangle \neq 0 \iff s\beta \in \Sigma_1'.$$

Thus $\Sigma_1' = s \cdot \Sigma_2'$. Therefore Σ_1' is the set of roots that are nonvanishing on $s \cdot \mathfrak{a}_2$, thus $\mathfrak{a}_1 = s \cdot \mathfrak{a}_2$. We have obtained $s \cdot (\mathfrak{a}_2, \lambda_2) = (\mathfrak{a}_1, -\lambda_1)$. Finally $r = 0$ then follows by taking real parts in formula (4.13) and using that $s \cdot {}^*\rho_2$ vanishes on $s \cdot \mathfrak{a}_2$. □

We now proceed to study equivalence and irreducibility of these induced representations. For this we shall pass from intertwining forms to intertwining operators. Let π_2^* be the Hilbert space contragredient to π_2; the change from π_2 to π_2^* means changing λ_2 to $-\lambda_2$. Then $\dim \mathrm{Hom}_G(\pi_1, \pi_2) = i(\pi_1, \pi_2^*)$. On the other hand, we have $i(\pi_1, \pi_2^*) \leq i(\mathrm{Ind}_{P_1}^G \sigma_1^0, \mathrm{Ind}_{P_2}^G \sigma_2^{0*})$ since $\mathrm{Ind}_{P_i}^G \sigma_i^0$ is densely injected into $\pi_i = \mathrm{Ind}_{P_i}^G \sigma_i$. Theorem 4.7 and the Vanishing Theorem 3.15 together imply

$$i(\pi_1, \pi_2^*) \leq \sum_{s \in \tilde{\mathfrak{w}}(G/A_0),\, r \in \boldsymbol{Z}_{\geq 0}} \dim(M_s^{(r)} \otimes (E_1^0 \,\overline{\otimes}\, E_2^{0'})' \otimes C_s')^{(P_1 \times P_2)_s}.$$

Equivalence. If $\mathrm{Hom}_G(\pi_1, \pi_2) \neq 0$, then $i(\pi_1, \pi_2^*) > 0$ and so, for some $s \in \tilde{\mathfrak{w}}(G/A_0)$, we have $s \cdot (\mathfrak{a}_2, \lambda_2) = (\mathfrak{a}_1, \lambda_1)$ from Theorem 4.10. Hence P_1 and P_2 must be associate. In other words, if P_1 and P_2 are not associate, the induced representations are disjoint. It is a known result that the induced representations are equivalent when P_1 and P_2 are associate.

Irreducibility. We take $P_1 = P_2 = P$, $\sigma_1 = \sigma_2 = \sigma$, $\pi_1 = \pi_2 = \pi$, etc. It is a question of showing that

$$\sum_{s \in \tilde{\mathfrak{w}}(G/A_0),\, r \in \boldsymbol{Z}_{\geq 0}} \dim(M_s^{(r)} \otimes (E^0 \,\overline{\otimes}\, E^{0'})' \otimes C_s')^{(P \times P)_s} \leq 1.$$

If for some s the term in the above sum is $\neq 0$, then by Theorem 4.10 we find $r = 0$ and $s \cdot (\mathfrak{a}, \lambda) = (\mathfrak{a}, \lambda)$. We have already that this implies $s \in \mathfrak{w}(M/A_0)$, i.e. $s = 1$. But in this case $\theta^* \chi_s^* = \mathrm{id}$ and the contribution becomes $\leq \dim((\sigma \otimes \sigma^*)^* \otimes \mathrm{id})^{(P \times P)_1}$ which is ≤ 1 since σ is irreducible. We have thus obtained the following:

Harish-Chandra's irreducibility Theorem 4.11. *Let G be a connected semisimple real Lie group with finite center, $G = KA_0N_0$ an Iwasawa decomposition, with P_0 the corresponding minimal parabolic subgroup. Let (P_i, A_i) be two p-pairs with $(P_i, A_i) \succ (P_0, A_0)$, for $i = 1, 2$. Let $P_i = {}^0M_i A_i N_i$ be the Langlands decomposition of P_i, ${}^0\sigma_i$ an irreducible unitary representation of ${}^0M_i$, and $\mathrm{e}^{(-1)^{1/2}\lambda_i}$ a unitary character of A_i. Assume that*

(a) *${}^0\Lambda_i$, the parameter defining the infinitesimal character of ${}^0\sigma_i$, is real.*

(b) *$\langle \lambda_i, \beta \rangle \neq 0$ for all roots β of (P_i, A_i).*

Let $\pi_i = \mathrm{Ind}_{P_i}^G {}^0\sigma_i \otimes \mathrm{e}^{(-1)^{1/2}\lambda_i} \otimes \mathrm{id}$. Then

(i) *π_1 and π_2 are not disjoint only if P_1 and P_2 are associate and λ_1 and λ_2 are conjugate within G.*

(ii) *If $P_1 = P_2 = P$, $\sigma_1 = \sigma_2 = \sigma$, $\pi_1 = \pi_2 = \pi$, $\lambda_1 = \lambda_2 = \lambda$, then π is irreducible.*

Application in Case (iii): Bruhat theory of normally induced representations. As a last application of the Vanishing Theorem 3.15 we give a simple proof of the following:

Bruhat's irreducibility Theorem 4.12. *Let σ_i, for $i = 1, 2$, be two irreducible unitary finite-dimensional representations of a normal closed subgroup P of a Lie group G. Let $\pi_i = \mathrm{Ind}_P^G \sigma_i$. Then π_1 is irreducible if and only if σ_1 and $\sigma_1^x : p \mapsto \sigma_1(xpx^{-1})$ are nonequivalent representations of P for every $x \in G$ not belonging to P. Furthermore π_1 and π_2 are nonequivalent if and only if σ_1 and σ_2^x are nonequivalent for every $x \in G$.*

Proof. In order to put the problem in the context of Case (iii) in Section 3, we set

$$X = G, \qquad H = G \times G, \qquad H' = P \times P, \qquad (g_1, g_2) \cdot x = g_1 x g_2^{-1}.$$

Then $O = X = G$ is the single H-orbit, which implies $M_x^{(r)} = (0)$, for every $x \in O$ and $r \in \boldsymbol{Z}_{\geq 0}$. Since P is normal in G we have $x^{-1}Px = P$, for every $x \in O$, and thus

$$(P \times P)_x = \{(p, x^{-1}px) \in P \times P \,|\, p \in P\}, \qquad \delta\,|_P = \delta_P.$$

From formula (4.6) modified according to the remark following Theorem 4.7, and (3.15) we therefore get $(\theta\chi_x)(p, x^{-1}px) = 1$, for all $x \in G$ and $p \in P$. The argument now is similar to that in the case of parabolic induction. π_1 and π_2 are disjoint if and only if, for all $x \in G$

$$\dim((E_1 \otimes E_2')' \otimes \boldsymbol{C})^{(P \times P)_x} = (0).$$

In other words, no $x \in G$ should have fixed vectors for $p \mapsto \sigma_1(p) \otimes \sigma_2^*(x^{-1}px)$.

□

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