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Approximate solution of singular integral equations of the first kind with Cauchy kernel

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ABSTRACT

In this work a study of efficient approximate methods for solving the Cauchy type singular integral equations (CSIEs) of the first kind, over a finite interval, is presented. In the solution, Chebyshev polynomials of the first kind, $T_n(x)$, second kind, $U_n(x)$, third kind, $V_n(x)$, and fourth kind, $W_n(x)$, corresponding to respective weight functions $W^{(1)}(x) = (1-x^2)^{-\frac{1}{2}}$, $W^{(2)}(x) = (1-x^2)^{\frac{1}{2}}$, $W^{(3)}(x) = (1+x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}$ and $W^{(4)}(x) = (1+x)^{-\frac{1}{2}}(1-x)^{\frac{1}{2}}$, have been used to obtain systems of linear algebraic equations. These systems are solved numerically. It is shown that for a linear force function the method of approximate solution gives an exact solution, and it cannot be generalized to any polynomial of degree n. Numerical results for other force functions are given to illustrate the efficiency and accuracy of the method.

1. Introduction

A singular integral equation of the form

$$\int_{-1}^{1} \frac{K_0(x,t)\varphi(t)}{t-x}\,dt + \int_{-1}^{1} K(x,t)\varphi(t)\,dt = f(x), \quad -1 < x < 1, \tag{1}$$

where $K_0(x,t)$, $K(x,t)$ and $f(x)$ are given real valued functions belonging to the Hölder class and $\varphi(t)$ is to be determined, occurs in a variety of mixed boundary value problems of mathematical physics (see [3,5]), for isotropic elastic bodies involving cracks (see [8]), and for other related problems. The integral is considered as a Cauchy principal value integral. This kind of integral can be found in many references [3,7,11]. Chakrabarti and Berge [2] have proposed an approximate method for solving CSIEs (1) using polynomial approximation of degree n and collocation points chosen to be the zeros of the Chebyshev polynomial of the first kind for all cases. They showed that the approximate method is exact when the force function $f(t)$ is linear. Abdou and Naser [1] considered CSIEs of the second kind and used orthogonal polynomials (Legendre) to find the approximate solution, and described the physical meaning of the stated problems. Rashed [10] presented two numerical methods for solution of non-singular integral equations of the first kind using Chebyshev polynomials of the first kind to approximate the kernel and unknown function. Kim [4] solved CSIEs by using Gaussian quadrature and chose the zeros of Chebyshev polynomials of the first and second kinds as the collocation and abscissae points. Srivastav and Zhang [12]used general quadrature–collocation nodes to solve CSIE.

In this work we present the approximate solution of CSIE (1) when $K_0(x,t) = 1$ and $K(x,t) = 0$, which is called the characteristic singular integral equation:

$$\int_{-1}^{1} \frac{\varphi(t)}{t-x}\,dt = f(x), \quad -1 < x < 1. \tag{2}$$

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It is well known that the complete analytical solutions of Eq. (2), for four cases, are given by the following expressions (see [6], pp. 5):

Case (I): The solution is unbounded at both end-points $x = \pm 1$,

$$\varphi(x) = -\frac{1}{\pi^2\sqrt{1-x^2}}\int_{-1}^{1}\frac{\sqrt{1-t^2}f(t)}{t-x}\,\mathrm{d}t + \frac{C}{\pi\sqrt{1-x^2}}, \tag{3}$$

where

$$\int_{-1}^{1}\varphi(t)\,\mathrm{d}t = C. \tag{4}$$

Case (II): The solution is bounded at both end-points $x = \pm 1$,

$$\varphi(x) = -\frac{\sqrt{1-x^2}}{\pi^2}\int_{-1}^{1}\frac{f(t)}{\sqrt{1-t^2}(t-x)}\,\mathrm{d}t, \tag{5}$$

provided that

$$\int_{-1}^{1}\frac{f(x)}{\sqrt{1-x^2}}\,\mathrm{d}x = 0. \tag{6}$$

Case (III): The solution is bounded at the point $x = -1$,

$$\varphi(x) = -\frac{1}{\pi^2}\sqrt{\frac{1+x}{1-x}}\int_{-1}^{1}\sqrt{\frac{1-t}{1+t}}\frac{f(t)}{t-x}\,\mathrm{d}t. \tag{7}$$

Case (IV): The solution is bounded at the point $x = 1$,

$$\varphi(x) = -\frac{1}{\pi^2}\sqrt{\frac{1-x}{1+x}}\int_{-1}^{1}\sqrt{\frac{1+t}{1-t}}\frac{f(t)}{t-x}\,\mathrm{d}t. \tag{8}$$

In obtaining the approximate solution we do the following:

1. The Chebyshev polynomials of the first, second, third and fourth kinds corresponding to the weight functions $W^{(j)}(x), j = 1, 2, 3, 4$, are used to approximate the density function in all cases considered (I–IV).
2. The collocation points are chosen as the zeros of the Chebyshev polynomials.
3. The singular integrals which are obtained by substituting the Chebyshev polynomials with corresponding weight functions into density function $\varphi(t)$ have been computed analytically.
4. For the unbounded case the condition $\int_{-1}^{1}\varphi(t)\,\mathrm{d}t = 0$ is imposed to obtain the unique solution.
5. FORTRAN code is developed to obtain the numerical results.

2. The method of the approximate solution

The Chebyshev polynomials of the first kind, $T_i(x)$, second kind, $U_{i-1}(x)$, third kind, $V_i(x)$, and fourth kind, $W_i(x)$, $i = 0, 1, 2, \ldots$, are defined by [9]

$$T_i(x) = \cos\left[\mathrm{i}\cos^{-1}(x)\right], \qquad U_{i-1}(x) = \frac{\sin[\mathrm{i}\cos^{-1}(x)]}{\sin[\cos^{-1}(x)]},$$

$$V_i(x) = \frac{\cos\left[\frac{2\mathrm{i}+1}{2}\cos^{-1}(x)\right]}{\cos\left[\frac{1}{2}\cos^{-1}(x)\right]}, \qquad W_i(x) = \frac{\sin\left[\frac{2\mathrm{i}+1}{2}\cos^{-1}(x)\right]}{\sin\left[\frac{1}{2}\cos^{-1}(x)\right]}.$$

The corresponding weight functions are

$$W^{(1)}(x) = \frac{1}{\sqrt{1-x^2}}, \qquad W^{(2)}(x) = \sqrt{1-x^2},$$

$$W^{(3)}(x) = \sqrt{\frac{1+x}{1-x}}, \qquad W^{(4)}(x) = \sqrt{\frac{1-x}{1+x}}.$$

Let the unknown function φ in Eq. (2) be approximated by the polynomial function φ_n:

$$\varphi_n(x) = W^{(j)}(x)\sum_{i=0}^{n}\beta_i^{(j)}G_i^{(j)}(x), \quad (j = 1, 2, 3, 4), \tag{9}$$

where $\beta_i^{(j)}$, $i = 0, 1, 2, \ldots, n$, are unknown coefficients and in Case (I): $W^{(1)}(x) = \frac{1}{\sqrt{1-x^2}}$ and $G_i^{(1)}(x) = T_i(x)$, in Case (II): $W^{(2)}(x) = \sqrt{1-x^2}$ and $G_i^{(2)}(x) = U_i(x)$, in Case (III): $W^{(3)}(x) = \sqrt{\frac{1+x}{1-x}}$ and $G_i^{(3)}(x) = V_i(x)$ and in Case (VI): $W^{(4)}(x) = \sqrt{\frac{1-x}{1+x}}$ and $G_i^{(4)}(x) = W_i(x)$.

Substituting the approximate solution (9) for the unknown function $\varphi(t)$ into (2) yields

$$\sum_{i=0}^{n} \beta_i^{(j)} \int_{-1}^{1} \frac{W^{(j)}(t) G_i^{(j)}(t)}{t-x}\, dt = f(x), \quad -1 < x < 1, (j = 1, 2, 3, 4). \tag{10}$$

Rewrite Eq. (10) as

$$\sum_{i=0}^{n} \beta_i^{(j)} \gamma_i^{(j)}(x) = f(x), \quad -1 < x < 1, (j = 1, 2, 3, 4) \tag{11}$$

where

$$\gamma_i^{(j)}(x) = \int_{-1}^{1} \frac{W^{(j)}(t) G_i^{(j)}(t)}{t-x}\, dt, \quad (j = 1, 2, 3, 4). \tag{12}$$

Let $x_k^{(j)}$, $j = 1, 2, 3, 4$, be the zeros of $U_n(x)$, $T_{n+2}(x)$, $W_{n+1}(x)$ and $V_{n+1}(x)$, respectively. Then

$$\left.\begin{array}{ll} x_k^{(1)} = \cos\left(\dfrac{k\pi}{(n+1)}\right), & x_k^{(2)} = \cos\left(\dfrac{(2k-1)\pi}{2(n+2)}\right), \\ \quad (k = 1, 2, \ldots, n) & \quad (k = 1, 2, \ldots, n+2) \\ x_k^{(3)} = \cos\left(\dfrac{2k\pi}{(2n+3)}\right), & x_k^{(4)} = \cos\left(\dfrac{(2k-1)\pi}{(2n+3)}\right). \\ \quad (k = 1, 2, \ldots, n+1) & \quad (k = 1, 2, \ldots, n+1) \end{array}\right\} \tag{13}$$

It is known (see [9]) that

$$\left.\begin{array}{ll} \displaystyle\int_{-1}^{1} \frac{T_i(t)}{\sqrt{1-t^2}(t-x)}\, dt = \pi\, U_{i-1}(x), & \displaystyle\int_{-1}^{1} \frac{\sqrt{1-t^2}\, U_i(t)}{t-x}\, dt = -\pi\, T_{i+1}(x), \\ \displaystyle\int_{-1}^{1} \sqrt{\frac{1+t}{1-t}}\, \frac{V_i(t)}{t-x}\, dt = \pi\, W_i(x), & \displaystyle\int_{-1}^{1} \sqrt{\frac{1-t}{1+t}}\, \frac{W_i(t)}{t-x}\, dt = -\pi\, V_i(x), \\ \displaystyle\int_{-1}^{1} \frac{T_0(t)}{\sqrt{1-t^2}}\, dt = \pi, & \displaystyle\int_{-1}^{1} \frac{T_i(t)}{\sqrt{1-t^2}}\, dt = 0, \quad (i = 1, 2, \ldots). \end{array}\right\} \tag{14}$$

Substituting the collocation points (13) into (11) and using (14) we obtain the following systems of linear equations:

$$\left.\begin{array}{l} \displaystyle\pi \sum_{i=1}^{n} \beta_i^{(1)}\, U_{i-1}(x_k^{(1)}) = f(x_k^{(1)}), \quad k = 1, 2, \ldots, n, \\ \displaystyle\sum_{i=0}^{n} \beta_i^{(1)} \int_{-1}^{1} W^{(1)}(t) T_i(t)\, dt = \pi \beta_0^{(1)} = 0. \end{array}\right\}, \tag{15}$$

$$-\pi \sum_{i=0}^{n} \beta_i^{(2)}\, T_{i+1}(x_k^{(2)}) = f(x_k^{(2)}), \quad k = 1, 2, \ldots, n+1, \tag{16}$$

$$\pi \sum_{i=0}^{n} \beta_i^{(3)}\, W_i(x_k^{(3)}) = f(x_k^{(3)}), \quad k = 1, 2, \ldots, n+1, \tag{17}$$

$$-\pi \sum_{i=0}^{n} \beta_i^{(4)}\, V_i(x_k^{(4)}) = f(x_k^{(4)}), \quad k = 1, 2, \ldots, n+1. \tag{18}$$

Solving the systems (15)–(18) for the unknown coefficients $\beta_i^{(j)}$, $j = 1, 2, 3, 4$, and substituting the values of $\beta_i^{(j)}$ into (9) we obtain the approximate solutions of Eq. (2) in the form of

$$\varphi(x) \approx W^{(j)}(x) \sum_{i=0}^{n} \beta_i^{(j)} G_i^{(j)}(x), \quad (j = 1, 2, 3, 4). \tag{19}$$

Proposition 1. *If $f(x)$ in Eq. (2) is a linear function, then the approximate solution (19) is exact.*

Proof. Let $f(x)$ in (2) be a linear function

$$f(x) = ax + b, \quad (-1 < x < 1), \tag{20}$$

where a and b are known constants.

Substituting (20) into (11) yields

$$\sum_{i=0}^{n} \beta_i^{(j)} \gamma_i^{(j)}(x) = ax + b, \quad (j = 1, 2, 3, 4). \tag{21}$$

Due to (14), one can easily find results as given in Table 1.

Table 1
Solutions of (12).

	$i = 0$	$i = 1$	$i = 2$	$i = 3$	...
$\gamma_i^{(1)}$	0	π	$2\pi x$	$\pi(4x^2 - 1)$	...
$\gamma_i^{(2)}$	$-\pi x$	$-\pi(2x^2 - 1)$	$-\pi(4x^3 - 3x)$	$-\pi(8x^4 - 8x^2 + 1)$	...
$\gamma_i^{(3)}$	π	$\pi(2x + 1)$	$\pi(4x^2 + 2x - 1)$	$\pi(8x^3 + 4x^2 - 4x - 1)$	...
$\gamma_i^{(4)}$	$-\pi$	$-\pi(2x - 1)$	$-\pi(4x^2 - 2x - 1)$	$-\pi(8x^3 - 4x^2 - 4x + 1)$	...

The unknown coefficients $\beta_i^{(j)}$ can be determined without solving the system of equations in (15)–(18). Instead, we compare the coefficients of various powers of x from both sides of Eq. (21). The $\beta_i^{(j)}$ obtained are substituted into (9) to give the approximate solution $\varphi_n(x)$ of Eq. (2).

Let us illustrate this for Case (II) by choosing $n = 3$; in view of (6) the constant b must be equal to zero in (21), so

$$\left[\beta_0^{(2)} \gamma_0^2(x) + \beta_1^{(2)} \gamma_1^{(2)}(x) + \beta_2^{(2)} \gamma_2^{(2)}(x) + \beta_3^{(2)} \gamma_3^{(2)}(x)\right] = ax. \tag{22}$$

From Table 1, Eq. (22) becomes

$$-\pi\left[\beta_0^{(2)} x + \beta_1^{(2)}(2x^2 - 1) + \beta_2^{(2)}(4x^3 - 3x) + \beta_3^{(2)}(8x^4 - 8x^2 + 1)\right] = ax, \tag{23}$$

which yields

$$\pi\left[\beta_1^{(2)} - \beta_3^{(2)}\right] = 0,$$
$$\pi\left[3\beta_2^{(2)} - \beta_0^{(2)}\right] = a,$$
$$\pi\left[8\beta_3^{(2)} - 2\beta_1^{(2)}\right] = 0,$$
$$-4\pi\beta_2^{(2)} = 0,$$
$$-8\pi\beta_3^{(2)} = 0.$$

Thus we have

$$\beta_0^{(2)} = -\frac{a}{\pi}, \qquad \beta_i^{(2)} = 0, \quad i = 1, 2, 3. \tag{24}$$

Substituting (24) into (9) we obtain the approximate solution

$$\varphi_n(x) = -a\frac{\sqrt{1 - x^2}}{\pi} \tag{25}$$

which is the exact solution obtained by (5).

In Case (III) for $n = 3$, from Table 1, Eq. (21) becomes

$$\pi\left[\beta_0^{(3)} + \beta_1^{(3)}(2x + 1) + \beta_2^{(3)}(4x^2 + 2x - 1) + \beta_3^{(3)}(8x^3 + 4x^2 - 4x - 1)\right] = ax + b, \tag{26}$$

and so

$$\pi\left[\beta_0^{(3)}+\beta_1^{(3)}-\beta_2^{(3)}-\beta_3^{(3)}\right]=b,$$
$$\pi\left[2\beta_1^{(3)}+2\beta_2^{(3)}-4\beta_3^{(3)}\right]=a,$$
$$4\pi\left[\beta_2^{(3)}+\beta_3^{(3)}\right]=0,$$
$$8\pi\beta_3^{(3)}=0,$$

which yields

$$\beta_0^{(3)}=\frac{2b-a}{2\pi},\qquad \beta_1^{(3)}=\frac{a}{2\pi},\qquad \beta_i^{(3)}=0,\quad i=2,3. \tag{27}$$

Substituting (27) into (9) we obtain the approximate solution

$$\varphi_n(x)=\frac{1}{\pi}\sqrt{\frac{1+x}{1-x}}\left[a(x-1)+b\right], \tag{28}$$

This approximate solution coincides with exact solution obtained by (7). Similarly, doing the same operations as we did for Case (II) and Case (III), one can solve for Case (I) and Case (IV). □

3. Numerical results

Consider the singular integral equation

$$\int_{-1}^{1}\frac{\varphi(t)}{t-x}\,\mathrm{d}t=x^4+5x^3+2x^2+x-(11/8),\quad -1<x<1. \tag{29}$$

Using (3) through (8), one can obtain the exact solutions of Eq. (29), i.e.,

$$\textbf{Case(I)}:\quad \varphi(t)=\frac{1}{\pi\sqrt{1-t^2}}\left[t^5+5t^4+\frac{3}{2}\left(t^3-t^2\right)-\frac{5}{2}t-\frac{9}{8}\right]. \tag{30}$$

$$\textbf{Case(II)}:\quad \varphi(t)=\frac{-1}{\pi}\sqrt{1-t^2}\left[t^3+5t^2+\frac{5}{2}t+\frac{7}{2}\right]. \tag{31}$$

$$\textbf{Case(III)}:\quad \varphi(t)=\frac{1}{\pi}\sqrt{\frac{1+t}{1-t}}\left[t^4+4t^3-\frac{5}{2}t^2+t-\frac{7}{2}\right]. \tag{32}$$

$$\textbf{Case(IV)}:\quad \varphi(t)=\frac{-1}{\pi}\sqrt{\frac{1-t}{1+t}}\left[t^4+6t^3+\frac{15}{2}t^2+6t+\frac{7}{2}\right]. \tag{33}$$

The errors of approximate solutions of (29) are given by Tables 2–5.

Table 2
Error of approximate solution (19) compared with (30) when $j=1$.

	$n=20$
x	Error
−0.950	8.881784197001252E−016
−0.900	−6.661338147750939E−016
−0.700	7.494005416219807E−016
−0.500	−5.551115123125783E−016
−0.300	4.163336342344337E−016
−0.100	−3.330669073875470E−016
0.000	1.665334536937735E−016
0.100	5.551115123125783E−016
0.300	0.000000000000000E+000
0.500	−7.771561172376096E−016
0.700	0.000000000000000E+000
0.900	9.436895709313831E−016
0.950	8.881784197001252E−016

Table 3
Error of approximate solution (19) compared with (31) when $j = 2$.

$n = 20$	
x	Error
−0.950	1.665334536937735E−016
−0.900	3.330669073875470E−016
−0.700	1.776356839400250E−015
−0.500	−5.551115123125783E−016
−0.300	2.220446049250313E−016
−0.100	2.220446049250313E−016
0.000	6.661338147750939E−016
0.100	0.000000000000000E+000
0.300	6.661338147750939E−016
0.500	8.881784197001252E−016
0.700	0.000000000000000E+000
0.900	2.220446049250313E−016
0.950	4.440892098500626E−016

Table 4
Error of approximate solution (19) compared with (32) when $j = 3$.

$n = 20$	
x	Error
−0.950	5.551115123125783E−016
−0.900	7.771561172376096E−016
−0.700	2.220446049250313E−016
−0.500	0.000000000000000E+000
−0.300	−1.110223024625157E−016
−0.100	−8.881784197001252E−016
0.000	−4.440892098500626E−016
0.100	0.000000000000000E+000
0.300	0.000000000000000E+000
0.500	−6.661338147750939E−016
0.700	−2.220446049250313E−016
0.900	1.110223024625157E−015
0.950	−6.661338147750939E−016

Table 5
Error of approximate solution (19) compared with (33) when $j = 4$.

$n = 20$	
x	Error
−0.95	3.053113317719180E−015
−0.900	3.219646771412954E−015
−0.700	3.330669073875470E−016
−0.500	−2.220446049250313E−016
−0.300	1.443289932012704E−015
−0.100	6.661338147750939E−016
0.000	6.661338147750939E−016
0.100	2.220446049250313E−016
0.300	2.220446049250313E−016
0.500	4.440892098500626E−016
0.700	4.440892098500626E−016
0.900	−8.881784197001252E−016
0.95	−2.220446049250313E−016

4. Conclusion

The Chebyshev polynomial approximations with weight functions are used to solve the Cauchy type singular integral equations of the first kind. The collocation points x_k are chosen to be the zeros of U_n, T_{n+2}, W_{n+1} and V_{n+1} for Cases (I), (II), (III) and (IV), respectively. Numerical results (Tables 2–5) show that the errors of approximate solution with small value of n for all cases are very small. These show that the methods developed are very accurate and in fact for a linear function give the exact solution.

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