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Quantization of formal classical dynamical r-matrices: the reductive case

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Abstract

In this paper we prove the existence of a formal dynamical twist quantization for any triangular and non-modified formal classical dynamical r-matrix in the reductive case. The dynamical twist is constructed as the image of the dynamical r-matrix by a L_∞-quasi-isomorphism. This quasi-isomorphism also allows us to classify formal dynamical twist quantizations up to gauge equivalence.

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0. Introduction

In [Fe], Felder introduced dynamical versions of both classical and quantum Yang–Baxter equations which has been generalized to the case of a non-abelian base in [EV] for the classical part and in [X3] for the quantum part. Naturally this leads to quantization problems which have been formulated in terms of twist quantization *à la* Drinfeld [Dr1] in [X2,X3,EE1,EE2].

Let us formulate this problem in the general context. Consider an inclusion $\mathfrak{h} \subset \mathfrak{g}$ of Lie algebras equipped with an element $Z \in (\wedge^3\mathfrak{g})^{\mathfrak{g}}$. A *(modified) classical dynamical*

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r-matrix for $(\mathfrak{g}, \mathfrak{h}, Z)$ is a regular (meaning C^∞, meromorphic, formal, ... depending on the context) $\mathfrak{h}$-equivariant map $\rho : \mathfrak{h}^* \to \wedge^2\mathfrak{g}$ which satisfies the (*modified*) *classical dynamical Yang–Baxter equation* (CDYBE)

$$\mathrm{CYB}(\rho) - \mathrm{Alt}(d\rho) = Z, \tag{1}$$

where $\mathrm{CYB}(\rho) := [\rho^{1,2}, \rho^{1,3}] + [\rho^{1,2}, \rho^{2,3}] + [\rho^{1,3}, \rho^{2,3}] = \frac{1}{2}[\rho, \rho]$ and

$$\mathrm{Alt}(d\rho) := \sum_i \left(h_i^1 \frac{\partial \rho^{2,3}}{\partial \lambda^i} - h_i^2 \frac{\partial \rho^{1,3}}{\partial \lambda^i} + h_i^3 \frac{\partial \rho^{1,2}}{\partial \lambda^i} \right).$$

Here (h_i) and (λ^i) are dual basis of $\mathfrak{h}$ and $\mathfrak{h}^*$.

Let $\Phi = 1 + O(\hbar^2) \in (U\mathfrak{g}^{\otimes 3})^{\mathfrak{g}}[[\hbar]]$ be an associator quantizing Z (of which the existence was proved in [Dr2, Proposition 3.10]). A *dynamical twist quantization* of a (modified) classical dynamical r-matrix ρ associated to Φ is a regular $\mathfrak{h}$-equivariant map $J = 1 + O(\hbar) \in \mathrm{Reg}(\mathfrak{h}^*, U\mathfrak{g}^{\otimes 2})[[\hbar]]$ such that $\mathrm{Alt}\frac{J-1}{\hbar} = \rho \bmod \hbar$ and which satisfies the (*modified*) *dynamical twist equation* (DTE)

$$J^{12,3}(\lambda) * J^{1,2}(\lambda + \hbar h^3) = \Phi^{-1} J^{1,23}(\lambda) * J^{2,3}(\lambda), \tag{2}$$

where $*$ denotes the PBW star-product of functions on $\mathfrak{h}^*$ and

$$J^{1,2}(\lambda + \hbar h^3) := \sum_{k \geqslant 0} \frac{\hbar^k}{k!} \sum_{i_1, \ldots, i_k} (\partial_{\lambda^{i_1}} \ldots \partial_{\lambda^{i_k}} J)(\lambda) \otimes (h_{i_1} \ldots h_{i_k}).$$

Now observe that many (modified) classical dynamical r-matrices can be viewed as formal ones by taking their Taylor expansion at 0. In this paper we are interested in the following conjecture:

Conjecture 1 (*[EE1]*). *Any* (*modified*) *formal classical dynamical r-matrix admits a dynamical twist quantization.*

Let us reformulate DTE in the formal framework. A formal (modified) dynamical twist is an element $J(\lambda) = 1 + O(\hbar) \in (U\mathfrak{g}^{\otimes 2}\hat{\otimes}\hat{S}\mathfrak{h})^{\mathfrak{h}}[[\hbar]]$ which satisfies DTE, and $J^{1,2}(\lambda + \hbar h^3) \in (U\mathfrak{g}^{\otimes 3}\hat{\otimes}\hat{S}\mathfrak{h})[[\hbar]]$ is equal to $(\mathrm{id}^{\otimes 2} \otimes \tilde{\Delta})(J)$ where $\tilde{\Delta} : \hat{S}\mathfrak{h} \to (U\mathfrak{g}\hat{\otimes}\hat{S}\mathfrak{h})[[\hbar]]$ is induced by $\mathfrak{h} \ni x \mapsto \hbar x \otimes 1 + 1 \otimes x$. Then define $K := J(\hbar\lambda) \in (U\mathfrak{g}^{\otimes 2} \otimes S\mathfrak{h})^{\mathfrak{h}}[[\hbar]]$ which we view as an element of $(U\mathfrak{g}^{\otimes 2} \otimes U\mathfrak{h})^{\mathfrak{h}}[[\hbar]]$ using the symmetrization map $S\mathfrak{h} \to U\mathfrak{h}$. Since J is a solution of DTE K satisfies the (*modified*) *algebraic dynamical twist equation* (ADTE)

$$K^{12,3,4} K^{1,2,34} = (\Phi^{-1})^{1,2,3} K^{1,23,4} K^{2,3,4}. \tag{3}$$

Moreover and by construction, $K = 1 + \sum_{n\geqslant 1} \hbar^n K_n$ has the *$\hbar$-adic valuation property*. Namely, $U\mathfrak{h}$ is filtered by $(U\mathfrak{h})_{\leqslant n} = \ker(\mathrm{id} - \eta\circ\varepsilon)^{\otimes n+1} \circ \Delta^{(n)}$ where $\varepsilon : U\mathfrak{h} \to \mathbf{k}$ and $\eta : \mathbf{k} \to U\mathfrak{h}$ are the counit and unit maps, and $K_n \in (U\mathfrak{h})_{\leqslant n-1}$. Conversely, any algebraic dynamical twist having the $\hbar$-adic valuation property can be obtained from a unique formal dynamical twist by this procedure.

This paper, in which we always assume $Z = 0$ and $\Phi = 1$ (non-modified case), is organized as follow.

In Section 1 we define two differential graded Lie algebras (`dgla`'s), respectively, associated to classical dynamical r-matrices and algebraic dynamical twists. Then we formulate the main theorem of this paper which states that if $\mathfrak{h}$ admits an $\mathrm{ad}\mathfrak{h}$-invariant complement (the *reductive* case) then these two `dgla`'s are L_∞-quasi-isomorphic and we prove that it implies Conjecture 1 in this case, which generalizes Theorem 5.3 of [X2]:

Theorem 2. *In the reductive case, any formal classical dynamical r-matrix for* $(\mathfrak{g}, \mathfrak{h}, 0)$ *admits a dynamical twist quantization* (*associated to the trivial associator*).

The second section is devoted to the proof of the main theorem of Section 1: using an equivariant formality theorem for homogeneous spaces which is obtained from [Do], we construct a L_∞-quasi-isomorphism which we then modify in order to obtain the desired one. We use this L_∞-quasi-isomorphism to classify formal dynamical twist quantizations up to gauge equivalence for the reductive case in Section 3. In Section 4 we prove that if $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ for $\mathfrak{h}$ abelian and $\mathfrak{m}$ a Lie subalgebra then the results of Sections 1 and 2 are still true in this situation. We conclude the paper with some open questions, and recall basic results for L_∞-algebras in an appendix.

1. Definitions and results

Let $\mathfrak{h} \subset \mathfrak{g}$ be an inclusion of Lie algebras.

1.1. Algebraic structures associated to CDYBE

Let us consider the following graded vector space

$$\mathrm{CDYB} := \wedge^*\mathfrak{g} \otimes S\mathfrak{h} = \bigoplus_{k\geqslant 0} \wedge^k\mathfrak{g} \otimes S\mathfrak{h}$$

equipped with the differential d defined by

$$\mathrm{d}(x_1 \wedge \cdots \wedge x_k \otimes h_1 \ldots h_l) := -\sum_{i=1}^{l} h_i \wedge x_1 \wedge \cdots \wedge x_k \otimes h_1 \ldots h_l \hat{h}_i. \tag{4}$$

With the exterior product $\wedge$ it becomes a differential graded commutative associative algebra. Moreover, one can define a graded Lie bracket of degree -1 on CDYB which is the Lie bracket of $\mathfrak{g}$ extended to CDYB in the following way:

$$[a, b \wedge c] = [a, b] \wedge c + (-1)^{(|a|-1)|b|} b \wedge [a, c]. \tag{5}$$

Thus one can observe that polynomial solutions to CDYBE are exactly elements $\rho \in$ CDYB of degree 2 such that $\mathrm{d}\rho + \frac{1}{2}[\rho, \rho] = 0$. We would like to say that such a ρ is a Maurer-Cartan element but (CDYB[1], d, [,]) is not a differential graded Lie algebra (`dgla`).

Instead, remember that we are interested in $\mathfrak{h}$-equivariant solutions of CDYBE (i.e., dynamical r-matrices) and thus consider the subspace $\mathfrak{g}_1 = (\mathrm{CDYB})^{\mathfrak{h}}$ of $\mathfrak{h}$-invariants with the same differential and bracket.

Proposition 3. $(\mathfrak{g}_1[1], \mathrm{d}, [,])$ *is a* `dgla`. *Moreover* $(\mathfrak{g}_1, \mathrm{d}, \wedge, [,])$ *is a Gerstenhaber algebra.*

Proof. Let $a = x_1 \wedge \cdots \wedge x_k \otimes h_1 \dots h_s$ and $b = y_1 \wedge \cdots \wedge y_l \otimes m_1 \dots m_t$ be $\mathfrak{h}$-invariant elements in $\mathfrak{g}_1$. We want to show that

$$\mathrm{d}[a, b] = [\mathrm{d}a, b] + (-1)^{k-1}[a, \mathrm{d}b]. \tag{6}$$

The l.h.s. of (6) is equal to

$$-\Big(\sum_{i=1}^{s} h_i \wedge [x_1 \wedge \cdots \wedge x_k, y_1 \wedge \cdots \wedge y_l] \otimes h_1 \dots h_s m_1 \dots m_t \hat{h}_i$$
$$+ \sum_{j=1}^{t} m_j \wedge [x_1 \wedge \cdots \wedge x_k, y_1 \wedge \cdots \wedge y_l] \otimes h_1 \dots h_s m_1 \dots m_t \hat{m}_j\Big).$$

The first term in the r.h.s. of (6) gives

$$\sum_{i=1}^{s} \big((-1)^{k-1} x_1 \wedge \cdots \wedge x_k \wedge [h_i, y_1 \wedge \cdots \wedge y_l] - h_i \wedge [x_1 \wedge \cdots \wedge x_k, y_1 \wedge \cdots \wedge y_l]\big)$$
$$\otimes h_1 \dots h_s m_1 \dots m_t \hat{h}_i$$

and for the second term we obtain

$$\sum_{j=1}^{t} \big((-1)^{k-1} [m_j, x_1 \wedge \cdots \wedge x_k] \wedge y_1 \wedge \cdots \wedge y_l - m_j \wedge [x_1 \wedge \cdots \wedge x_k, y_1 \wedge \cdots \wedge y_l]\big)$$
$$\otimes h_1 \dots h_s m_1 \dots m_t \hat{m}_j.$$

Thus, the difference between the l.h.s. and the r.h.s. of (6) is equal to

$$(-1)^k \Big(\sum_{i=1}^{k} x_1 \wedge \cdots \wedge x_k \wedge [h_i, y_1 \wedge \cdots \wedge y_l] \otimes h_1 \dots h_s m_1 \dots m_t \hat{h}_i$$
$$+ \sum_{j=1}^{l} [m_j, x_1 \wedge \cdots \wedge x_k] \wedge y_1 \wedge \cdots \wedge y_l \otimes h_1 \dots h_s m_1 \dots m_t \hat{m}_j \Big).$$

Then using $\mathfrak{h}$-invariance of a and b one obtains

$$(-1)^{k-1} \sum_{i,j} x_1 \wedge \cdots \wedge x_k \wedge y_1 \wedge \cdots \wedge y_l$$
$$\otimes \big(h_1 \dots h_s m_1 \dots m_t ([h_i, m_j] - [m_j, h_i]) \hat{h}_i \hat{m}_j \big) = 0.$$

The second statement of the proposition is obvious from the definition (5) of the bracket. $\square$

Let $\rho(\lambda) \in (\wedge^2 \mathfrak{g} \hat{\otimes} \hat{S}\mathfrak{h})^{\mathfrak{h}}$ be a formal classical dynamical r-matrix. Since ρ satisfies CDYBE, $\alpha := \hbar \rho(\hbar\lambda) \in \hbar \mathfrak{g}_1[[\hbar]]$ is a Maurer-Cartan element (i.e., $\mathrm{d}\alpha + \frac{1}{2}[\alpha, \alpha] = 0$).

1.2. Algebraic structures associated to ADTE

Let us now consider the graded vector space

$$\mathrm{ADT} := T^* U\mathfrak{g} \otimes U\mathfrak{h} = \bigoplus_{k \geqslant 0} \otimes^k U\mathfrak{g} \otimes U\mathfrak{h}$$

equipped with the differential b given by

$$b(P) := P^{2,\dots,k+2} + \sum_{i=1}^{k+1} (-1)^i P^{1,\dots,ii+1,\dots,k+2} \quad \text{for } P \in \otimes^k U\mathfrak{g} \otimes U\mathfrak{h}. \tag{7}$$

Remark 4. This is just the coboundary operator of Hochschild's cohomology with value in a comodule; and $b^2 = 0$ follows directly from an easy calculation.

One can define on ADT an associative product $\cup$ (the *cup* product) which is given on homogeneous elements $P \in \otimes^k U\mathfrak{g} \otimes U\mathfrak{h}$ and $Q \in \otimes^l U\mathfrak{g} \otimes U\mathfrak{h}$ by

$$P \cup Q := P^{1,\dots,k,k+1\dots k+l+1} Q^{k+1,\dots,k+l+1}.$$

Proposition 5. $(\mathrm{ADT}, b, \cup)$ *is a differential graded associative algebra.*

Proof. The cup product is obviously associative. Thus, the only thing we have to check is that

$$b(P \cup Q) = bP \cup Q + (-1)^{|P|} P \cup bQ. \tag{8}$$

Let $k = |P|$ and $l = |Q|$. The l.h.s. of (8) is equal to

$$P^{2,\dots,k+1,k+2\dots k+l+2} Q^{k+2,\dots,k+l+2} + \sum_{i=1}^{k} (-1)^i P^{1,\dots,ii+1,\dots,k+1,k+2\dots k+l+2} Q^{k+2,\dots,k+l+2}$$
$$+ \sum_{i=k+1}^{k+l+1} (-1)^i P^{1,\dots,k,k+1\dots k+l+2} Q^{k+1,\dots,ii+1,\dots,k+l+2}.$$

The first line of this expression is equal to

$$bP \cup Q - (-1)^{k+1} P^{1,\dots,k,k+1\dots k+l+2} Q^{k+2,\dots,k+l+2}$$

and the last term of the same expression gives

$$(-1)^k \big(P \cup bQ - P^{1,\dots,k,k+1\dots k+l+2} Q^{k+2,\dots,k+l+2}\big).$$

The proposition is proved. $\square$

Recall that in the case $\mathfrak{h} = \{0\}$ one can define a brace algebra structure on $(T^*U\mathfrak{g})[1]$ (see [Ge]). Unfortunately we are not able to extend this structure to ADT in general. Since we deal with $\mathfrak{h}$-equivariant solutions of ADTE we can consider the subspace $\mathfrak{g}_2 = (\mathrm{ADT})^{\mathfrak{h}}$ of $\mathfrak{h}$-invariants. Let us now define a collection of linear homogeneous maps of degree zero $\{-|-,\dots,-\} : \mathfrak{g}_2[1] \otimes \mathfrak{g}_2[1]^{\otimes m} \to \mathfrak{g}_2[1]$ indexed by $m \geqslant 0$, and $\{P|Q_1,\dots,Q_m\}$ is given by

$$\sum_{\substack{0 \leqslant i_1, i_m+k_m \leqslant n \\ i_l+k_l \leqslant i_{l+1}}} (-1)^{\varepsilon} P^{1,\dots,i_1+1\dots i_1+k_1,\dots,i_m+1\dots i_m+k_m,\dots,n+1}$$
$$\times \prod_{s=i}^{m} Q_s^{i_s+1,\dots,i_s+k_s,i_s+k_s+1\dots n+1},$$

where $k_s = |Q_s|$, $n = |P| + \sum_s k_s - m$ and $\varepsilon = \sum_s (k_s - 1) i_s$.

Proposition 6. $(\mathfrak{g}_2[1], \{-|-,\dots,-\})$ *is a brace algebra.*

Proof. Since we work with $\mathfrak{h}$-invariant elements one can remark that if $i_s + k_s \leqslant i_t$ then $Q_s^{i_s+1,\dots,i_s+k_s,i_s+k_s+1\dots n+1}$ and $Q_t^{i_t+1,\dots,i_t+k_t,i_t+k_t+1\dots n+1}$ commute. Using this the proof becomes identical to the case when $\mathfrak{h} = 0$ (see [Ge] for example). $\square$

Now observe that since $m = 1^{\otimes 3} \in (\otimes^2 U\mathfrak{g} \otimes U\mathfrak{h})^{\mathfrak{h}}$ is such that $\{m|m\} = 0$ one obtains a B_∞-algebra structure [Ba] on $\mathfrak{g}_2$ (see [Kh]). More precisely, we have a differential graded bialgebra structure on the cofree tensorial coalgebra $T^c(\mathfrak{g}_2[1])$ of which structure maps $a^n, a^{p,q}$ are given by

- $a^1(P) = bP = (-1)^{|P|-1}[m, P]_G$, where

$$[P, Q]_G := \{P|Q\} - (-1)^{(|P|-1)(|Q|-1)}\{Q|P\}$$

- $a^2(P, Q) = \{m|P, Q\} = P \cup Q$.
- $a^{0,1} = a^{1,0} = \mathrm{id}$.
- $a^{1,n}(P; Q_1, \dots, Q_n) = \{P|Q_1, \dots, Q_n\}$ for $n \geqslant 1$.
- All other maps are zero.

In particular, we have

Proposition 7. $(\mathfrak{g}_2[1], b, [,]_G)$ *is a* dgla.

Remark 8. Since that for any graded vector space V, dg bialgebra structures on the cofree coassociative coalgebra T^cV are in one-to-one correspondence with dg Lie bialgebra structures on the cofree Lie coalgebra L^cV (see [Ta, Section 5]), then $L^c(\mathfrak{g}_2[1])$ becomes a dg Lie bialgebra with differential and Lie bracket given by maps $l^n, l^{p,q}$ such that $l^1 = b$ and $l^{1,1} = [,]_G$. Therefore $d_2 := \sum_{i\geqslant 0} l^i + \sum_{p,q\geqslant 0} l^{p,q} : C^c(L^c(\mathfrak{g}_2[1])) \to C^c(L^c(\mathfrak{g}_2[1]))$ defines a G_∞-algebra structure on $\mathfrak{g}_2$ ($d_2 \circ d_2 = 0$ since d_2 is just the Chevalley–Eilenberg differential on the dg Lie algebra $L^c(\mathfrak{g}_2[1])$).

1.3. Main result and proof of Theorem 2

First of all, observe that CDYB, $\mathfrak{g}_1$ and $\mathcal{G}_1 := C^c(\mathfrak{g}_1[2])$ have a natural grading induced by the one of $S\mathfrak{h}$. In the same way ADT, $\mathfrak{g}_2$ and $\mathcal{G}_2 := C^c(\mathfrak{g}_2[2])$ have a natural filtration induced by the one of $U\mathfrak{h}$. Our main goal is to prove the following theorem, which is sufficient to obtain algebraic dynamical twists from formal dynamical r-matrices.

Theorem 9. *In the reductive case, there exists a L_∞-quasi-isomorphism*

$$\Psi : (\mathcal{G}_1, \mathrm{d} + [,]) \to (\mathcal{G}_2, b + [,]_G)$$

with the following two filtration properties:

(F1) $\forall X \in (\mathfrak{g}_1)_k,\ \Psi^1(X) = (\mathrm{alt} \otimes \mathrm{sym})(X) \mod (\mathfrak{g}_2)_{\leqslant k-1}$.
(F2) $\forall X \in (\Lambda^n \mathfrak{g}_1)_k,\ \Psi^n(X) \in (\mathfrak{g}_2)_{\leqslant n+k-1}$.

Thus, we have

Proof of Theorem 2. Now consider a formal solution $\rho(\lambda) \in (\wedge^2 \mathfrak{g} \hat{\otimes} \hat{S}\mathfrak{h})^{\mathfrak{h}}$ to CDYBE. Let us define $\alpha := \hbar\rho(\hbar\lambda) \in \hbar\mathfrak{g}_1[[\hbar]]$ which is a Maurer-Cartan element in $\hbar\mathfrak{g}_1[[\hbar]]$. The L_∞-morphism property implies that $\widetilde{\alpha} := \sum_{n=1}^{\infty} \frac{1}{n!} \Psi^n(\Lambda^n \alpha)$ is a Maurer-Cartan element in $\hbar\mathfrak{g}_2[[\hbar]]$; this exactly means that $K := 1 + \widetilde{\alpha} \in (\otimes^2 U\mathfrak{g} \otimes U\mathfrak{h})^{\mathfrak{h}}[[\hbar]]$ satisfies ADTE. Moreover, due to (F2) the coefficient K_n of $\hbar^n$ in K lies in $(\mathfrak{g}_2)_{\leqslant n-1}$. It means that there exists $J \in (U\mathfrak{g}^{\otimes 2} \hat{\otimes} \hat{S}\mathfrak{h})^{\mathfrak{h}}[[\hbar]]$ satisfying DTE and such that $K = (\mathrm{id}^{\otimes 2} \otimes \mathrm{sym})(J(\hbar\lambda))$. Finally, property (F1) obviously implies that the semi-classical limit condition $\frac{J - J^{\mathrm{op}}}{\hbar} = \rho \bmod \hbar$ is satisfied. $\square$

2. Proof of Theorem 9

In this section we assume that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ with $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$. Let us denote by $\mathrm{p} : \mathfrak{g} \to \mathfrak{m}$ the projection on $\mathfrak{m}$ along $\mathfrak{h}$; it is $\mathfrak{h}$-equivariant.

2.1. Resolutions

Let us first observe that the bilinear map $[,]_{\mathfrak{m}} := (\wedge^{\cdot}\mathrm{p}) \circ [,]$ defines a graded Lie bracket of degree -1 on $(\wedge^* \mathfrak{m})^{\mathfrak{h}}$. Then we prove

Proposition 10. *The natural map* $p_1 : (\mathfrak{g}_1[1], \mathrm{d}, [,]) \to ((\wedge^* \mathfrak{m})^{\mathfrak{h}}[1], 0, [,]_{\mathfrak{m}})$ *is a morphism of* `dgla`*'s. Moreover, there exists an operator* $\delta : \mathfrak{g}_1^* \to \mathfrak{g}_1^{*-1}$ *such that* $\delta\mathrm{d} + \mathrm{d}\delta = \mathrm{id} - p_1$, $\delta \circ \delta = 0$ *and* $\delta\big((\mathfrak{g}_1)_k\big) \subset (\mathfrak{g}_1)_{k+1}$. *In particular,* p_1 *induces an isomorphism in cohomology.*

Proof. The projection $p_1 := (\wedge^{\cdot}\mathrm{p}) \otimes \varepsilon : (\mathrm{CDYB}, \mathrm{d}) \to (\wedge^* \mathfrak{m}, 0)$ is a $\mathfrak{h}$-equivariant morphism of complexes, and it obviously restricts to a morphism of (differential) graded Lie algebras at the level of $\mathfrak{h}$-invariants.

Moreover, $\wedge^n \mathfrak{g} \otimes S\mathfrak{h} \cong \bigoplus_{p+q=n} \wedge^p \mathfrak{m} \otimes \wedge^q \mathfrak{h} \otimes S\mathfrak{h}$ as a $\mathfrak{h}$-module; and under this identification d becomes $-\mathrm{id} \otimes d_K$, where $d_K : \wedge^* \mathfrak{h} \otimes S\mathfrak{h} \to \wedge^{*+1} \mathfrak{h} \otimes S\mathfrak{h}$ is Koszul's coboundary operator, and p_1 corresponds to the projection on the part of zero antisymmetric and symmetric degrees in $\mathfrak{h}$. Let us define $\delta = \mathrm{id} \otimes \delta_K$ with $\delta_K : \wedge^* \mathfrak{h} \otimes S^* \mathfrak{h} \to \wedge^{*-1} \mathfrak{h} \otimes S^{*+1} \mathfrak{h}$ defined by

$$\delta_K(x_1 \wedge \cdots \wedge x_n \otimes h_1 \ldots h_m) = \begin{cases} \dfrac{1}{m+n} \sum_i (-1)^i x_1 \wedge \cdots \hat{x_i} \cdots \wedge x_n \otimes h_1 \ldots h_m x_i & \text{if } m+n \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Finally, remark that δ is a $\mathfrak{h}$-equivariant homotopy operator: $\delta\mathrm{d} + \mathrm{d}\delta = \mathrm{id} - p_1$ and $\delta \circ \delta = 0$. The proposition is proved. $\square$

Now we prove a similar result for $\mathfrak{g}_2$. Let us first define $U\mathfrak{m} := \mathrm{sym}(S\mathfrak{m}) \subset U\mathfrak{g}$; this is a sub-coalgebra of $U\mathfrak{g}$ and thus $T^*U\mathfrak{m}$ equipped with its Hochschild's coboundary operator $b_{\mathfrak{m}}$ becomes a cochain subcomplex of the Hochschild complex $(T^*U\mathfrak{g}, b_{\mathfrak{g}})$ of $U\mathfrak{g}$. We also have the following

Lemma 11. *$U\mathfrak{g} = U\mathfrak{g}\cdot\mathfrak{h} \oplus U\mathfrak{m}$ as a filtered $\mathfrak{h}$-module. Moreover $[,]_{G,\mathfrak{m}} := (\otimes^{\cdot}\mathrm{p})\circ[,]$ defines a graded Lie bracket of degree -1 on $(T^*U\mathfrak{m})^{\mathfrak{h}}$.*

Proof. See [He, Chapter II Section 4.2] for the first statement. The second statement follows from a direct computation. □

Then we prove

Proposition 12. *The natural map $p_2 : (\mathfrak{g}_2[1], b, [,]_G) \to ((T^*U\mathfrak{m})^{\mathfrak{h}}[1], b_{\mathfrak{m}}, [,]_{G,\mathfrak{m}})$ is a morphism of* `dgla`*'s. Moreover, there exists an operator $\kappa : \mathfrak{g}_2^* \to \mathfrak{g}_2^{*-1}$ such that $\kappa b + b\kappa = \mathrm{id} - p_2$, $\kappa\circ\kappa = 0$ and $\kappa\big((\mathfrak{g}_2)_{\leqslant k}\big) \subset (\mathfrak{g}_2)_{\leqslant k+1}$. In particular, p_2 induces an isomorphism in cohomology.*

Proof. The projection $p_2 := (\otimes^{\cdot}\mathrm{p}) \otimes \varepsilon : (\mathrm{ADT}, b) \to (T^*U\mathfrak{m}, b_{\mathfrak{m}})$ is a $\mathfrak{h}$-equivariant morphism of complexes, and it obviously restricts to a morphism of `dgla`'s at the level of $\mathfrak{h}$-invariants (by Lemma 11).

Remember that $\mathfrak{g}_2$ has a natural filtration induced by the one of $U\mathfrak{h}$. Then one obtains a spectral sequence of which we compute the first terms:

$$E_0^{*,*} = (T^*U\mathfrak{g} \otimes S^*\mathfrak{h})^{\mathfrak{h}}, \quad d_0 = b_{\mathfrak{g}} \otimes \mathrm{id},$$

$$E_1^{*,*} = (\wedge^*\mathfrak{g} \otimes S^*\mathfrak{h})^{\mathfrak{h}}, \quad d_1 = \mathrm{d},$$

$$E_2^{*,*} = E_2^{*,0} = (\wedge^*\mathfrak{m})^{\mathfrak{h}}, \quad d_2 = 0.$$

Then the proposition follows from Proposition 10. □

2.2. *Inverting p_2*

In this subsection, taking our inspiration from [Mo, appendix], we prove the following

Proposition 13. *There exists a L_∞-quasi-isomorphism*

$$\mathcal{Q} : (C^c((T^*U\mathfrak{m})^{\mathfrak{h}}[2]), b_{\mathfrak{m}} + [,]_{G,\mathfrak{m}}) \to (C^c(\mathfrak{g}_2[2]), b + [,]_G)$$

such that $\mathcal{Q}^1$ is the natural inclusion and $\mathcal{Q}^n$ takes values in $(\mathfrak{g}_2)_{\leqslant n-1}$.

Proof. Let $(N, b_N) \subset (\mathfrak{g}_2, b)$ be the kernel of the surjective morphism of complexes $p_2 : (\mathfrak{g}_2, b) \to ((T^*U\mathfrak{m})^{\mathfrak{h}}, b_{\mathfrak{m}})$. It follows from Proposition 12 that there exists an operator $H : N^* \to N^{*-1}$ such that $H \circ H = 0$, $b_N H + H b_N = \mathrm{id}$ and $H(N_{\leqslant n}) \subset N_{\leqslant n+1}$.

Now let us construct a L_∞-isomorphism

$$\mathcal{F} : \left(C^c(\mathfrak{g}_2[2]), b + [\,,]_G\right) \xrightarrow{\sim} \left(C^c((T^*U\mathfrak{m})^{\mathfrak{h}}[2] \oplus N[2]), b_{\mathfrak{m}} + b_N + [\,,]_{G,\mathfrak{m}}\right)$$

with structure maps $\mathcal{F}^n : \Lambda^n \mathfrak{g}_2 \to ((T^*U\mathfrak{m})^{\mathfrak{h}} \oplus N)[1-n]$ such that

- $\mathcal{F}^1$ is the sum of p_2 with the projection on N along $(T^*U\mathfrak{m})^{\mathfrak{h}}$ (in some sense $\mathcal{F}^1$ is the identity),
- for any $n > 1$ and $X \in (\Lambda^n \mathfrak{g}_2)_{\leqslant k}$, $\mathcal{F}^n(X) \in N_{\leqslant n+k-1}$.

Let us prove it by induction on n. First $\mathcal{F}^1$ is a morphism of complexes by definition. Then let us define $\mathcal{K}_2 : \Lambda^2 \mathfrak{g}_2 \to ((T^*U\mathfrak{m})^{\mathfrak{h}} \oplus N)[1]$ by

$$\mathcal{K}_2(x \Lambda y) = [\mathcal{F}^1(x), \mathcal{F}^1(y)]_{G,\mathfrak{m}} - \mathcal{F}^1([x, y]_G).$$

It takes values in $N[1]$ and is such that $b_N \mathcal{K}_2(x, y) + \mathcal{K}_2(bx, y) + \mathcal{K}_2(x, by) = 0$. Consequently, $\mathcal{F}^2 := H \circ \mathcal{K}_2 : \Lambda^2 \mathfrak{g}_2 \to N$ is such that

$$b_N \mathcal{F}^2(x, y) - \mathcal{F}^2(bx, y) - \mathcal{F}^2(x, by) = \mathcal{K}_2(x, y) \quad (L_\infty\text{-condition for } \mathcal{F}^2)$$

and for any $X \in (\Lambda^2 \mathfrak{g}_2)_{\leqslant k}$, $\mathcal{F}^2(X) \in N_{\leqslant k+1}$. After this, suppose we have constructed $\mathcal{F}^1, \ldots, \mathcal{F}^n$ and let us define

$$\mathcal{K}_{n+1} := [\,,]_{G,\mathfrak{m}} \circ \mathcal{F}^{\leqslant n} - \mathcal{F}^{\leqslant n} \circ [\,,]_G : \Lambda^2 \mathfrak{g}_2 \to ((T^*U\mathfrak{m})^{\mathfrak{h}} \oplus N)[1].$$

It obviously takes values in $N[1]$ and is such that $b_N K_{n+1} + K_{n+1} b = 0$. Consequently, $\mathcal{F}^{n+1} := H \circ K_{n+1}$ satisfies the L_∞-condition

$$b_N \mathcal{F}^{n+1} - \mathcal{F}^{n+1} b = b_N H K_{n+1} - H K_{n+1} b = (b_N H + H b_N) K_{n+1} = K_{n+1}$$

and for any $X \in (\Lambda^n \mathfrak{g}_2)_{\leqslant n+1}$, $\mathcal{F}^{n+1}(X) \in N_{\leqslant n+k}$ (since $\mathcal{K}_{n+1}(X) \in N_{\leqslant n+k-1}$).

Now let $\mathcal{H}$ be the inverse of the isomorphism $\mathcal{F}$, it is such that for any $n \geqslant 1$ and $X \in (\Lambda^n \mathfrak{g}_2)_{\leqslant k}$, $\mathcal{H}^n(X) \in N_{\leqslant n+k-1}$. Finally, we obtain $\mathcal{Q}$ by composing $\mathcal{H}$ with the inclusion of `dgla`'s $(T^*U\mathfrak{m})^{\mathfrak{h}}[1] \hookrightarrow ((T^*U\mathfrak{m})^{\mathfrak{h}} \oplus N)[1]$. $\square$

2.3. End of the proof

Recall from [He, Chapter II Section 4.2] that $(T^*U\mathfrak{m})^{\mathfrak{h}} = \mathrm{Diff}^*(G/H)^G$ and $(\wedge^* \mathfrak{m})^{\mathfrak{h}} = \Gamma(G/H, \wedge^* T(G/H))^G$ as `dgla`'s. Remember also from [No, Chapter II Section 8] that G-invariant connections on G/H are in one-to-one correspondence with $\mathfrak{h}$-equivariant linear maps $\alpha : \mathfrak{m} \otimes \mathfrak{m} \to \mathfrak{m}$, and that the torsion tensor is given by $\alpha - \alpha^{21} - \mathrm{p} \circ [\,,]$. Thus G/H is equipped with a G-invariant torsion free connection ∇, corresponding to the map $\alpha := \frac{1}{2} \mathrm{p} \circ [\,,]$. Then using a theorem of Dolgushev, see [Do, Theorem 5], we obtain a G-equivariant L_∞-quasi-isomorphism ϕ :

$\Gamma(G/H, \wedge^* T(G/H)) \to \mathrm{Diff}^*(G/H)$ with first structure map $\phi^1 = \mathrm{alt}$, which restricts to a L_∞-quasi-isomorphism at the level of G-invariants. Let us define $\psi := \mathcal{Q} \circ \phi \circ p_1 : (C^c(\mathfrak{g}_1[2]), \mathrm{d} + [,]) \to (C^c(\mathfrak{g}_2[2]), b + [,]_G)$; it is a L_∞-quasi-isomorphism with first structure map $\psi^1 = (\mathrm{alt} \otimes 1) \circ (\wedge^{\cdot}\mathrm{p} \otimes \varepsilon)$.

Finally, define $V := (\mathrm{alt} \otimes \mathrm{sym}) \circ \delta : \mathfrak{g}_1 \to \mathfrak{g}_2[-1]$ and use Lemma 22 to construct a L_∞-quasi-morphism $\Psi : (C^c(\mathfrak{g}_1[2]), \mathrm{d}+[,]) \to (C^c(\mathfrak{g}_2[2]), b+[,]_G)$ with first structure map $\Psi^1 = \psi^1 + b \circ V + V \circ \mathrm{d}$. Since for any $X \in (\mathcal{G}_1)_k$, then

$$b \circ (\mathrm{alt} \otimes \mathrm{sym})(X) = (\mathrm{alt} \otimes \mathrm{sym}) \circ \mathrm{d}(X) \mod (\mathfrak{g}_2)_{\leqslant k-1}$$

and thus

$$\begin{aligned} \Psi^1(X) &= \psi^1(X) + bV(X) + V(\mathrm{d}X) \\ &= (\mathrm{alt} \otimes \mathrm{sym}) \circ (p_1 + \mathrm{d}\delta + \delta\mathrm{d})(X) \mod (\mathfrak{g}_2)_{\leqslant k-1} \\ &= (\mathrm{alt} \otimes \mathrm{sym})(X) \mod (\mathfrak{g}_2)_{\leqslant k-1}. \end{aligned}$$

Consequently, Ψ satisfies (F1). Moreover, it follows from Remark 23 that Ψ also satisfies (F2). $\square$

3. Classification

Theorem 9 implies a stronger result than just the existence of the twist quantization. Namely, since Ψ is a L_∞-quasi-isomorphism there is a bijection between the moduli spaces of Maurer-Cartan elements of `dgla`'s $(\mathfrak{g}_1[1])[[\hbar]]$ and $(\mathfrak{g}_2[1])[[\hbar]]$.

3.1. Classification of algebraic and formal dynamical twists

Following [EE1], two dynamical twists $J(\lambda)$ and $J'(\lambda)$ are said to be *gauge equivalent* if there exists a regular $\mathfrak{h}$-equivariant map $T(\lambda) = \exp(q) + O(\hbar) \in \mathrm{Reg}(\mathfrak{h}^*, U\mathfrak{g})^{\mathfrak{h}}[[\hbar]]$, with $q \in \mathrm{Reg}(\mathfrak{h}^*, \mathfrak{g})^{\mathfrak{h}}$ such that $q(0) = 0$, and satisfying

$$J'(\lambda) = T^{12}(\lambda) * J(\lambda) * T^2(\lambda)^{-1} * T^1(\lambda + \hbar h^2)^{-1}. \tag{9}$$

Dealing with formal functions one can easily derive an equivalence relation for the corresponding algebraic dynamical twists $K = J(\hbar\lambda)$ and $K' = J'(\hbar\lambda)$:

$$K' = Q^{12,3} K (Q^{2,3})^{-1} (Q^{1,23})^{-1} \tag{10}$$

in $(U\mathfrak{g}^{\otimes 2} \otimes U\mathfrak{h})^{\mathfrak{h}}[[\hbar]]$, with $Q = 1 + O(\hbar) \in (U\mathfrak{g} \otimes U\mathfrak{h})^{\mathfrak{h}}[[\hbar]]$ given by $T(\hbar\lambda)$.

Assume now we are in the reductive case.

Since the composition $\mathcal{Q}_2 \circ \phi : (C^c((\wedge\mathfrak{m})^{\mathfrak{h}}[2]), [,]_{\mathfrak{m}}) \to (C^c(\mathfrak{g}_2[2]), b + [,]_G)$ in the previous section is a L_∞-quasi-isomorphism then we have a bijective correspondence

$$\frac{\{\pi \in \hbar(\wedge^2\mathfrak{m})^{\mathfrak{h}}[[\hbar]] \text{ s.t. } [\pi,\pi]_{\mathfrak{m}} = 0\}}{G_0} \longleftrightarrow \frac{\{\text{algebraic dynamical twists}\}}{\text{gauge equivalence (10)}}, \tag{11}$$

where G_0 is the prounipotent group corresponding to the Lie algebra $\hbar\,\mathfrak{m}^{\mathfrak{h}}[[\hbar]]$. Moreover, since the structure maps $\mathcal{Q}_2^n$ take values in $(\mathfrak{g}_2)_{\leqslant n-1}$ then it appears that any algebraic dynamical twist is gauge equivalent to a one with the $\hbar$-adic valuation property and thus we have a bijection

$$\frac{\{\text{algebraic dynamical twists}\}}{\text{gauge equivalence (10)}} \longleftrightarrow \frac{\{\text{formal dynamical twists}\}}{\text{gauge equivalence (9)}}. \tag{12}$$

3.2. Classical counterpart

Assume that we are in the reductive case. Since p_1 is a L_∞-quasi-isomorphism by Proposition 10 then we have a bijection

$$\frac{\{\alpha \in \hbar(\wedge^2\mathfrak{g}\otimes S\mathfrak{h})^{\mathfrak{h}}[[\hbar]] \text{ s.t. } \mathrm{d}\alpha + \frac{1}{2}[\alpha,\alpha] = 0\}}{G_1}$$
$$\longleftrightarrow \frac{\{\pi \in \hbar(\wedge^2\mathfrak{m})^{\mathfrak{h}}[[\hbar]] \text{ s.t. } [\pi,\pi]_{\mathfrak{m}} = 0\}}{G_0},$$

where G_1 is a prounipotent group and its action (by affine transformations) is given by the exponentiation of the infinitesimal action of its Lie algebra $\hbar(\mathfrak{g}\otimes S\mathfrak{h})^{\mathfrak{h}}[[\hbar]]$:

$$q \cdot \alpha = \mathrm{d}q + [q,\alpha] \quad \big(q \in \hbar(\mathfrak{g}\otimes S\mathfrak{h})^{\mathfrak{h}}[[\hbar]]\big). \tag{13}$$

Then going along the lines of Section 2.2 one can prove the following

Proposition 14. *There exists a L_∞-quasi-isomorphism*

$$\mathcal{Q}_1 : (C^c((\wedge^*\mathfrak{m})^{\mathfrak{h}}[2]), [,]_{\mathfrak{m}}) \to (C^c(\mathfrak{g}_1[2]), \mathrm{d} + [,])$$

such that $\mathcal{Q}_1^1$ is the natural inclusion and $\mathcal{Q}_1^n$ takes values in $(\mathfrak{g}_1)_{\leqslant n-1}$.

Consequently, any Maurer-Cartan element in $(\mathfrak{g}_1[1])[[\hbar]]$ is equivalent to a one of the form $\hbar\rho_\hbar(\hbar\lambda)$, where $\rho_\hbar \in (\wedge^2\mathfrak{g}\hat{\otimes}\hat{S}\mathfrak{h})^{\mathfrak{h}}[[\hbar]]$ satisfies CDYBE. In other words $\rho_\hbar$ is a $\hbar$-dependant formal dynamical r-matrix. On such a $\rho_\hbar$ the infinitesimal action (13) becomes

$$q \cdot \rho_\hbar = -\sum_i h_i \wedge \frac{\partial q}{\partial \lambda^i} + [q, \rho_\hbar] \quad (q \in \mathfrak{g}\hat{\otimes}\hat{S}\mathfrak{h})^{\mathfrak{h}}[[\hbar]]. \tag{14}$$

This action integrates in an affine action of some group $\widetilde{G_1}$ of $\mathfrak{h}$-equivariant formal maps with values in the Lie group G of $\mathfrak{g}$. And then we have a bijection

$$\frac{\{\pi \in \hbar(\wedge^2\mathfrak{m})^{\mathfrak{h}}[[\hbar]] \text{ s.t. } [\pi,\pi]_{\mathfrak{m}} = 0\}}{G_0} \longleftrightarrow \frac{\{\text{form. dynam. } r\text{-mat.}/\mathbb{R}[[\hbar]]\}}{\widetilde{G_1}}. \tag{15}$$

Remark 15. This bijection has to be compared with Proposition 2.13 in [X2] and Section 3 of [ES]

Finally, combining (15), (11) and (12) we obtain the following generalization of Theorem 6.11 in [X2] to the case of a non-abelian base:

Theorem 16. *Let* $\pi \in (\wedge^2\mathfrak{m})^{\mathfrak{h}}$ *such that* $[\pi,\pi]_{\mathfrak{m}} = 0$. *Then there are bijective correspondences between*

(1) *the set of* $\hbar$*-dependant and* G*-invariant Poisson structures* $\pi_\hbar = \hbar\pi \bmod \hbar^2$ *on* G/H, *modulo the action of* G_0,
(2) *the set of* $\hbar$*-dependant formal dynamical r-matrices* $\rho_\hbar(\lambda)$ *such that* $\rho_\hbar(0) = \pi \bmod \hbar$ *in* $\wedge^2(\mathfrak{g}/\mathfrak{h})[[\hbar]]$, *modulo the action* (14) *of* $\widetilde{G_1}$,
(3) *the set of formal dynamical twists* $J(\lambda)$ *satisfying* $\mathrm{Alt}\frac{J(0)-1}{\hbar} = \pi \bmod \hbar$ *in* $\wedge^2(\mathfrak{g}/\mathfrak{h})[[\hbar]]$, *modulo gauge equivalence* (9).

4. Another case when the twist quantization exists

In this section we assume that $\mathfrak{h}$ is abelian and admits a Lie subalgebra $\mathfrak{m}$ as complement.

Note that since $\mathfrak{h}$ is abelian and $\mathfrak{m}$ a Lie subalgebra, the projection $p : \mathfrak{g} \to \mathfrak{g}$ on $\mathfrak{m}$ along $\mathfrak{h}$ extends to a morphsim of graded Lie algebras $\wedge^\cdot p : (\wedge\mathfrak{g})^{\mathfrak{h}} \to (\wedge\mathfrak{g})^{\mathfrak{h}}$ at the level of $\mathfrak{h}$-invariants. And thus $\wedge^\cdot p \otimes \varepsilon : (\mathfrak{g}_1[1], \mathrm{d}, [\,,\,]) \to ((\wedge\mathfrak{g})^{\mathfrak{h}}[1], 0, [\,,\,])$ is a morphism of `dgla`'s. Then the natural inclusion $\mathrm{id} \otimes 1 : (T^*U\mathfrak{g})^{\mathfrak{h}} \to \mathfrak{g}_2$ obviously allows one to consider $(T^*U\mathfrak{g})^{\mathfrak{h}}[1]$ as a sub-`dgla` of $\mathfrak{g}_2[1]$. Finally recall from [Ca, Section 3.3] that there exists a L_∞-quasi-isomorphism $\mathcal{F} : C^c((\wedge^*\mathfrak{g})^{\mathfrak{h}}[2]) \to C^c((T^*U\mathfrak{g})^{\mathfrak{h}}[2])$ with $\mathcal{F}^1 = \mathrm{alt}$. By composing these maps one obtains a L_∞-morphism

$$\widetilde{\mathcal{F}} : (\mathcal{G}_1, \mathrm{d} + [\,,\,]) \to (\mathcal{G}_2, b + [\,,\,]_G),$$

with values in $(\mathcal{G}_2)_{\leqslant 0}$ and first structure map $\widetilde{\mathcal{F}}^1 = (\mathrm{alt} \otimes 1) \circ (\wedge^\cdot p \otimes \varepsilon)$.

Theorem 17. *There exists a L_∞-quasi-isomorphism*

$$\Psi : (\mathcal{G}_1, \mathrm{d} + [,]) \to (\mathcal{G}_2, b + [,]_G)$$

with properties (F1) *and* (F2) *of Theorem* 9.

Proof. First observe that since $\mathfrak{h}$ is abelian then $\mathfrak{g}_1 \cong ((\wedge\mathfrak{g})^{\mathfrak{h}} \cap \wedge\mathfrak{m}) \otimes \wedge\mathfrak{h} \otimes S\mathfrak{h}$ as a vector space. Thus, if δ_K is as in the proof of Proposition 10 then $\delta := \mathrm{id} \otimes \delta_K$ is a homotopy operator: $\delta\mathrm{d} + \mathrm{d}\delta = \mathrm{id} - \wedge^{\cdot} p \otimes \varepsilon$ and $\delta \circ \delta = 0$.

Now we proceed like in Section 2.3: use Lemma 22 to construct Ψ with first structure map $\Psi^1 = \widetilde{\mathcal{F}}^1 + b \circ V + V \circ \mathrm{d}$, where $V := (\mathrm{alt} \otimes \mathrm{sym}) \circ \delta : \mathfrak{g}_1 \to \mathfrak{g}_2[-1]$.

It remains to prove that Ψ is a quasi-isomorphism. It follows from the first observation in this proof that $H^*(\mathfrak{g}_1, \mathrm{d}) = (\wedge\mathfrak{g})^{\mathfrak{h}} \cap \wedge\mathfrak{m}$, which also equals $H^*(\mathfrak{g}_2, b)$ due to the spectral sequence argument. Consequently, $\widetilde{\mathcal{F}}^1$ is a quasi-isomorphism of complexes, and so is Ψ^1. $\square$

Finally, using the same argumentation as in the proof of Theorem 2 (Section 1.3) one obtains the

Theorem 18. *If* $\mathfrak{h}$ *is an abelian subalgebra of* $\mathfrak{g}$ *with a Lie subalgebra as a complement, then any formal classical dynamical r-matrix for* $(\mathfrak{g}, \mathfrak{h}, 0)$ *admits a dynamical twist quantization* (*associated to the trivial associator*).

Example 19. In particular, this allows us to quantize dynamical r-matrices arising from semi-direct products $\mathfrak{g} = \mathfrak{m} \ltimes \mathbb{C}^n$ like in [EN, Example 3.7].

5. Concluding remarks

Let us first observe that if $\mathfrak{h}$ is abelian then $(\wedge^*\mathfrak{g})^{\mathfrak{h}} \cap \wedge^*\mathfrak{m}[1]$ (resp. $(T^*U\mathfrak{g})^{\mathfrak{h}} \cap T^*\mathrm{sym}(S\mathfrak{m})[1]$) inherits a `dgla` structure from the one of $\mathfrak{g}_1[1]$ (resp. $\mathfrak{g}_2[1]$) and $H^*(\mathfrak{g}_1, \mathrm{d}) = (\wedge^*\mathfrak{g})^{\mathfrak{h}} \cap \wedge^*\mathfrak{m} = H^*(\mathfrak{g}_2, b)$, for any complement $\mathfrak{m}$ of $\mathfrak{h}$. Thus I conjecture that there exists a L_∞-quasi-isomorphism between $(\wedge^*\mathfrak{g})^{\mathfrak{h}} \cap \wedge^*\mathfrak{m}[1]$ and $(T^*U\mathfrak{g})^{\mathfrak{h}} \cap T^*\mathrm{sym}(S\mathfrak{m})[1]$ which generalizes together ϕ of Section 2.3 and $\mathcal{F}$ of Section 4. In particular, this would imply Conjecture 1 in the abelian (and non-modified) case.

Let us then mention that one can consider a non-triangular (i.e., non-antisymmetric) version of non-modified classical dynamical r-matrices. Namely, $\mathfrak{h}$-equivariant maps $r \in \mathrm{Reg}(\mathfrak{h}^*, \mathfrak{g} \otimes \mathfrak{g})$ such that $\mathrm{CYB}(r) - \mathrm{Alt}(dr) = 0$. According to [X3], a quantization of such a r is a $\mathfrak{h}$-equivariant map $R = 1 + \hbar r + O(\hbar^2) \in \mathrm{Reg}(\mathfrak{h}^*, U\mathfrak{g}^{\otimes 2})[[\hbar]]$ that satisfies the *quantum dynamical Yang–Baxter equation* (QDYBE)

$$R^{1,2}(\lambda) * R^{1,3}(\lambda + \hbar h^2) * R^{2,3}(\lambda) = R^{2,3}(\lambda + \hbar h^1) * R^{1,3}(\lambda) * R^{1,2}(\lambda + \hbar h^3). \quad (16)$$

Question 1. *Does such a quantization always exist?*

The most famous example of non-triangular dynamical r-matrices was found in [AM] by Alekseev and Meinrenken, then extended successively to a more general context in [EV,ES,EE1], and quantized in [EE1].

Following [EE1], remark that for any non-triangular dynamical r-matrix r such that $r + r^{\mathrm{op}} = t \in (S^2\mathfrak{g})^{\mathfrak{g}}$ (quasi-triangular case) one can define $\rho := r - t/2$ and $Z := \frac{1}{4}[t^{1,2}, t^{2,3}]$. Then ρ is a modified dynamical r-matrix for $(\mathfrak{g}, \mathfrak{h}, Z)$; moreover, the assignment $r \longmapsto \rho$ is a bijective correspondence between quasi-triangular dynamical r-matrices for $(\mathfrak{g}, \mathfrak{h}, t)$ and modified dynamical r-matrices for $(\mathfrak{g}, \mathfrak{h}, Z)$. Now observe that if $J(\lambda)$ is a dynamical twist quantizing ρ, then $R(\lambda) = J^{\mathrm{op}}(\lambda)^{-1} * e^{\hbar t/2} * J(\lambda)$ is a quantum dynamical R-matrix quantizing r.

In this paper we have constructed such a dynamical twist in the triangular case $t = 0$. One can ask

Question 2. *Does such a dynamical twist exist for any quasi-triangular dynamical r-matrix? At least in the reductive and abelian cases?*

This question seems to be more reasonable than the previous one.

More generally one can ask if Conjecture 1 (and its smooth and meromorphic versions) is true in general. A positive answer was given in [EE1] when $\mathfrak{h} = \mathfrak{g}$; but unfortunately it is not known in general, even for the non-dynamical case $\mathfrak{h} = \{0\}$ (which is the last problem of Drinfeld [Dr1]: quantization of coboundary Lie bialgebras).

Finally, let me mention that if $r(\lambda)$ is a triangular dynamical r-matrix for $(\mathfrak{g}, \mathfrak{h})$, then the bivector field

$$\pi := \overrightarrow{r(\lambda)} + \sum_i \frac{\partial}{\partial \lambda^i} \wedge \overrightarrow{h_i} + \pi_{\mathfrak{h}^*}$$

is a $G \times H$-biinvariant Poisson structure on $G \times \mathfrak{h}^*$ and the projection $p : G \times \mathfrak{h}^* \to \mathfrak{h}^*$ is a momentum map. Moreover, according to [X3] any dynamical twist quantization $J(\lambda)$ of $r(\lambda)$ allows us to define a $G \times H$-biinvariant star-product $*$ quantizing π on $G \times \mathfrak{h}^*$ as follows:

$$\begin{array}{ll} f * g = f *_{PBW} g & \text{if } f,\ g \in C^\infty(\mathfrak{h}^*), \\ f * g = fg & \text{if } f \in C^\infty(G),\ g \in C^\infty(\mathfrak{h}^*), \\ f * g = \exp\left(\hbar \sum_i \frac{\partial}{\partial \lambda^i} \otimes \overrightarrow{h_i}\right) \cdot (f \otimes g) & \text{if } f \in C^\infty(\mathfrak{h}^*),\ g \in C^\infty(G), \\ f * g = \overrightarrow{J(\lambda)}(f \otimes g) & \text{if } \quad f,\ g \in C^\infty(G). \end{array}$$

This way the map $p^* : (\mathrm{Fct}(\mathfrak{h}^*)[[\hbar]], *_{PBW}) \to (\mathrm{Fct}(G \times \mathfrak{h}^*)[[\hbar]], *)$ becomes a quantum momentum map in the sens of [X1].

So there may be a way to see momentum maps and their quantum analogues as Maurer-Cartan elements in `dgla`'s.

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Appendix: Homotopy Lie algebras

See [HS] for a detailed discussion of the theory.

Recall that a L_∞-*algebra* structure on a graded vector space $\mathfrak{g}$ is a degree 1 coderivation Q on the cofree cocommutative coalgebra $C^c(\mathfrak{g}[1])$ such that $Q \circ Q = 0$. By cofreeness, such a coderivation Q is uniquely determined by structure maps $Q^n : \Lambda^n \mathfrak{g} \to \mathfrak{g}[2-n]$ which satisfy an infinite collection of equations. In particular $(\mathfrak{g}, Q^1)$ is a cochain complex.

Example 20. Any `dgla` $(\mathfrak{g}, \mathrm{d}, [,])$ is canonically a L_∞-algebra. Namely, Q is given by structure maps $Q^1 = \mathrm{d}$, $Q^2 = [,]$ and $Q^n = 0$ for $n > 2$.

A L_∞-*morphism* between two L_∞-algebras $(\mathfrak{g}_1, Q_1)$ and $(\mathfrak{g}_2, Q_2)$ is a degree 0 morphism of coalgebras $F : C^c(\mathfrak{g}_1[1]) \to C^c(\mathfrak{g}_2[1])$ such that $F \circ Q_1 = Q_2 \circ F$. Again by cofreeness, such a morphism is uniquely determined by structure maps $F^n : \Lambda^n \mathfrak{g}_1 \to \mathfrak{g}_2[1-n]$ which satisfy an infinite collection of equations. In particular $F^1 : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a morphism of complexes; when it induces an isomorphism in cohomology we say that F is a L_∞-*quasi-isomorphism*.

Example 21. Any morphism of `dgla`'s is a L_∞-morphism with all structure maps equal to zero except the first one.

In this paper we use many times the following

Lemma 22 (*Dolgushev [Do]*). *Let $F : C^c(\mathfrak{g}_1[1]) \to C^c(\mathfrak{g}_2[1])$ be a L_∞-morphism. For any linear map $V : \mathfrak{g}_1 \to \mathfrak{g}_2[-1]$ there exists a L_∞-morphism $\Psi : C^c(\mathfrak{g}_1[1]) \to C^c(\mathfrak{g}_2[1])$ with first structure map $\Psi^1 = F^1 + Q_2^1 \circ V + V \circ Q_1^1$. Moreover, if F is a L_∞-quasi-isomorphism then Ψ is also.*

Proof. First remark that V extends uniquely to a linear map $C^c(\mathfrak{g}_1[1]) \to C^c(\mathfrak{g}_2[1])$ of degree -1 such that

$$\begin{aligned}\Delta_2 \circ V = \big(&F \otimes V + V \otimes F + \tfrac{1}{2} V \otimes (Q_2 \circ V + V \circ Q_1) \\ &+ \tfrac{1}{2}(Q_2 \circ V + V \circ Q_1) \otimes V\big) \circ \Delta_1,\end{aligned}$$

where Δ_1 and Δ_2 denote comultiplications in $C^c(\mathfrak{g}_1[1])$ and $C^c(\mathfrak{g}_2[1])$, respectively.

Then define $\Psi := F + Q_2 \circ V + V \circ Q_1$. $\square$

Remark 23. Assume that in the previous lemma $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are filtrated, F is such that F^n takes values in $(\mathfrak{g}_2)_{\leqslant n-1}$, and $V\big((\mathfrak{g}_1)_{\leqslant k}\big) \subset (\mathfrak{g}_2)_{\leqslant k+1}$. Then one can obviously check that for any $X \in (\Lambda^n \mathfrak{g}_1)_{\leqslant k}$, $F^n(X) \in (\mathfrak{g}_2)_{\leqslant n+k-1}$.

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