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Collective fixed points and maximal elements with applications to abstract economies ☆

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Abstract

In this paper, we first establish collective fixed points theorems for a family of multivalued maps with or without assuming that the product of these multivalued maps is Φ-condensing. As an application of our collective fixed points theorem, we derive the coincidence theorem for two families of multivalued maps defined on product spaces. Then we give some existence results for maximal elements for a family of L_S-majorized multivalued maps whose product is Φ-condensing. We also prove some existence results for maximal elements for a family of multivalued maps which are not L_S-majorized but their product is Φ-condensing. As applications of our results, some existence results for equilibria of abstract economies are also derived. The results of this paper are more general than those given in the literature.

Keywords: Collective fixed points theorems; Maximal element theorems; Φ-condensing multivalued maps; Transfer compactly open valued maps; L_S-majorized maps; Abstract economies

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1. Introduction

In the last decade, the theory of fixed points and maximal elements for a family of multivalued maps defined on a product space has been investigated by many authors, see for example [1,2,4–6,10–12,19] and references therein. It has many applications in abstract economies, nonlinear analysis and other branches of mathematics.

In 1990, Mehta [14] used the Kuratowski measure of noncompactness to prove the existence of maximal elements for condensing preferences defined on a closed, bounded and convex subset of a Banach space. Chebli and Florenzano [4] established the maximal element theorems for Φ-condensing and L-majorized multivalued maps defined in a Hausdorff locally convex topological vector space.

In this paper, we first establish some collective fixed points theorems for a family of multivalued maps with or without assuming that the product of these multivalued maps is Φ-condensing. As an application of our collective fixed points theorems, we derive a coincidence theorem for two families of multivalued maps defined on product spaces. Secondly, we establish some existence results for maximal elements of a family of L_S-majorized multivalued maps whose product is Φ-condensing. We also prove some existence theorems for maximal elements of a family of multivalued maps which are not L_S-majorized but their product is Φ-condensing. Our definition of L_S-majorized multivalued maps is more general than the one given in [6] and therefore our results also more general than those given in [6]. As applications of our results, some existence results for equilibria of abstract economies are also derived. The results of this paper are more general than those given in the literature. Further applications of the results of this paper to the systems of generalized vector quasi-equilibrium problems are under consideration in the next paper.

2. Preliminaries

For a nonempty set D, we denote by 2^D (respectively, $\langle D\rangle$) the family of all subsets (respectively, the family of all nonempty finite subsets) of D. If D is a nonempty subset of a vector space, then $\operatorname{co} D$ denotes the convex hull of D. When D is a nonempty subset of a topological space, $\bar{D}$ or $\operatorname{cl} D$ and $\operatorname{int} D$ denote the closure and interior of D, respectively. Throughout the paper, I is any index set.

Let X and Y be nonempty sets. Let M be a nonempty subset of X and $T : X \to 2^Y$ a multivalued map. Then for all $x \in X$ and $y \in Y$, we have $T(M) = \bigcup\{T(x)\colon x \in M\}$ and $x \in T^{-1}(y)$ if and only if $y \in T(x)$. Also $T^{-1}(N) = \{x \in X\colon T(x) \cap N \neq \emptyset\}$ for all nonempty subset N of Y. The multivalued map $\operatorname{cl} T : X \to 2^Y$ is defined as $(\operatorname{cl} T)(x) = \operatorname{cl}(T(x))$ for all $x \in X$.

Let X and Y be two sets, $A : X \to 2^Y$ a multivalued map and $S : Y \to X$ a single-valued map. Then the composition map $A \circ S$ from Y to 2^Y is defined by $A \circ S(y) = A(S(y))$ for all $y \in Y$.

A nonempty subset D of a topological space X is said to be *compactly open* (respectively, compactly closed) if for every nonempty compact subset C of X, $D \cap C$ is open (respectively, closed) in C. The *compact interior* of D [7] is defined by

$$\operatorname{cint} D = \bigcup\{G\colon G \subseteq D \text{ and } G \text{ is compactly open in } X\}.$$

It is easy to see that $\operatorname{cint} D$ is a compactly open set in X and for each nonempty compact subset C of X with $D \cap C \neq \emptyset$, we have $(\operatorname{cint} D) \cap C = \operatorname{int}_C(D \cap C)$, where $\operatorname{int}_C(D \cap C)$ denotes the interior of $D \cap C$ in C. It is clear that a subset D of X is compactly open in X if and only if $\operatorname{cint} D = D$.

Let X and Y be topological spaces and $T : X \to 2^Y$ a multivalued map. Then T is said to be *transfer compactly open valued* (respectively, *transfer open valued*) on X (see [7]) if for every $x \in X$, $y \in T(x)$, there exists a point $\hat{x} \in X$ such that $y \in \operatorname{cint} T(\hat{x})$ (respectively, $y \in \operatorname{int} T(\hat{x})$). T is said to be *compact* if $\overline{T(X)}$ is compact.

Throughout this paper, all topological spaces are assumed to be Hausdorff.

The following lemma immediately follows from the definition of a transfer compactly open valued map.

Lemma 2.1. *Let X and Y be two topological spaces and let $G : X \to 2^Y$ be a multivalued map. Then G is transfer compactly open valued if and only if*

$$\bigcup_{x \in X} G(x) = \bigcup_{x \in X} \operatorname{cint} G(x).$$

By applying Lemma 2.1 and following the argument of Proposition 1 [13], we have the following lemma.

Lemma 2.2. *Let X and Y be two topological spaces and let $G : X \to 2^Y$ be a multivalued map. Then the following statements are equivalent*:

(i) $G^{-1} : Y \to 2^X$ *is transfer compactly open valued and for all $x \in X$, $G(x)$ is nonempty*;
(ii) $X = \bigcup_{y \in Y} \operatorname{cint} G^{-1}(y)$.

Following the same argument as in Lemma 5.1 [8], we have the following result.

Lemma 2.3. *Let X be a topological space, Y a topological vector space and $G : X \to 2^Y$ a multivalued map. Let $\psi : X \to 2^Y$ be defined as $\psi(x) = \operatorname{co} G(x)$. For all $y \in Y$, if $G^{-1}(y)$ is compactly open, then $\psi^{-1}(y)$ is compactly open.*

Definition 2.1. Let X be a topological space. For each $i \in I$, let Y_i be a nonempty convex subset of a topological vector space E_i. Let $Y = \prod_{i \in I} Y_i$ and $S : Y \to X$ be a single-valued map. For each $i \in I$, $Q_i : X \to 2^{Y_i}$ be a multivalued map. Then Q_i is said to be

(i) *of class* L_S if
  (a) Q_i is convex valued,
  (b) $y_i \notin Q_i(S(y))$ for each $y = (y_i)_{i \in I} \in Y$, where y_i is the ith projection of y,
  (c) $Q_i^{-1}(y_i)$ is compactly open for each $y_i \in Y_i$;
(ii) L_S*-majorized* if for each $x \in X$, there exist an open neighborhood $N(x)$ of x in X and a convex valued mapping $B_x : X \to 2^{Y_i}$ such that
  (a) $Q_i(z) \subseteq B_x(z)$ for each $z \in N(x)$,
  (b) $y_i \notin B_x(S(y))$ for each $y = (y_i)_{i \in I} \in Y$, where y_i is the ith projection of y,

(c) $B_x^{-1}(y_i)$ is compactly open in X for each $y_i \in Y_i$.
Here the mapping B_{ix} is called an L_S-*majoriant of* Q_i *at* x.

We shall denote by $M_S(X, Y_i)_{i\in I}$ (respectively, $L_S(X, Y_i)_{i\in I}$) the set of families $\{Q_i\}_{i\in I}$ such that for each $i \in I$, $Q_i : X \to 2^{Y_i}$ is L_S-majorized (respectively, of class L_S).

In the case $X = \prod_{i\in I} X_i$ and $Y_i = X_i$ and the map $S = I_X$ the identity mapping on X, we shall denote $M(X, X_i)_{i\in I}$ (respectively, $L(X, X_i)_{i\in I}$) in place of $M_{I_X}(X, X_i)_{i\in I}$ (respectively, $L_{I_X}(X, X_i)_{i\in I}$).

Remark 2.1. The definitions of $L_S(X, Y_i)_{i\in I}$ and $M_S(X, Y_i)_{i\in I}$ are more general than those given in [6].

Following the argument of Lemma 5 in [6], we have the following result.

Lemma 2.4. *Let X be a regular paracompact topological space. For each $i \in I$, let Y_i be a nonempty convex subset of a topological vector space E_i. Let $Y = \prod_{i\in I} Y_i$ and let $S : Y \to X$ be a single-valued map and $\{Q_i\}_{i\in I} \in M_S(X, Y_i)_{i\in I}$. Then there exists a family $\{B_i\}_{i\in I} \in L_S(X, Y_i)_{i\in I}$ such that $Q_i(x) \subseteq B_i(x)$ for all $x \in X$ and for each $i \in I$.*

Definition 2.2 [9]. Let E be a topological vector space and let C be a lattice with a minimal element, denoted by $\mathbf{0}$. A mapping $\Phi : 2^E \to C$ is called *measure of noncompactness* provided that the following conditions hold for any $M, N \in 2^E$:

(a) $\Phi(\overline{\text{co}}\, M) = \Phi(M)$, where $\overline{\text{co}}\, M$ denotes the closed convex hull of M.
(b) $\Phi(M) = \mathbf{0}$ if and only if M is precompact.
(c) $\Phi(M \cup N) = \max\{\Phi(M), \Phi(N)\}$.

Definition 2.3 [9]. Let E be a topological vector space, $X \subseteq E$, and let Φ be a measure of noncompactness on E. A multivalued map (correspondence) $T : X \to 2^E$ is called *Φ-condensing* provided that if $M \subseteq X$ with $\Phi(T(M)) \geqslant \Phi(M)$ then M is relative compact, that is, $\bar{M}$ is compact.

Remark 2.2. Note that every multivalued map defined on a compact set is Φ-condensing for any measure of noncompactness Φ. If E is locally convex, then a compact multivalued map (i.e., $T(X)$ is precompact) is Φ-condensing for any measure of noncompactness Φ. Obviously, if $T : X \to 2^E$ is Φ-condensing and $T' : X \to 2^E$ satisfies $T'(x) \subseteq T(x)$ for all $x \in X$, then T' is also Φ-condensing.

Lemma 2.5 [14]. *Let X be a nonempty, closed and convex subset of a topological vector space E. Let Φ be a measure of noncompactness on X and let $T : X \to 2^X$ be a Φ-condensing multivalued map. Then there exists a nonempty compact convex subset K of X such that $T(K) \subseteq K$.*

Remark 2.3. In [14], E is assumed to be a locally convex topological vector space, but Lemma 2.4 is true for any topological vector space as we can see in the proof.

3. Collective fixed points theorems

The following collective fixed points theorem is one of the main results of this paper.

Theorem 3.1. *Let X be a topological space. For each $i \in I$, let Y_i be a nonempty convex subset of a topological vector space E_i. Let $Y = \prod_{i\in I} Y_i$ and let $S : Y \to X$ be a continuous single-valued map. For each $i \in I$, let $P_i, Q_i : X \to 2^{Y_i}$ be multivalued maps satisfying the following conditions*:

(a) *For all $x \in X$, $P_i(x)$ is nonempty and* co $P_i(x) \subseteq Q_i(x)$.
(b) *For each $y_i \in X_i$, $P_i^{-1}(y_i)$ is compactly open.*
(c) *If X is not compact, then there exist a nonempty compact subset K of X and a nonempty compact convex subset D_i of Y_i for each $i \in I$ such that for all $x \in X\setminus K$, $P_i(x) \cap D_i \neq \emptyset$ for all $i \in I$.*

Then there exists $\bar{y} = (\bar{y}_i)_{i\in I} \in Y$ such that $\bar{y}_i \in Q_i(S(\bar{y}))$ for all $i \in I$.

Proof. Since $P_i(x)$ is nonempty for all $x \in X$ and for each $i \in I$, we have

$$X = \bigcup \{P_i^{-1}(y_i):\ y_i \in Y_i\}, \quad \text{for each } i \in I.$$

Since K is a nonempty compact subset of X, for each $i \in I$, there exists a finite subset M_i of Y_i such that

$$X \cap K = K \subseteq \bigcup \{P_i^{-1}(y_i):\ y_i \in M_i\}. \tag{3.1}$$

For each $i \in I$, let $L_{M_i} = \text{co}\{M_i \cup D_i\}$. Then L_{M_i} is a compact convex subset of Y_i. Let $L_M = \prod_{i\in I} L_{M_i}$. Then L_M is a compact convex subset of Y. Since for each $i \in I$, $M_i \subseteq L_{M_i}$ from (3.1) we have

$$X \cap K \subseteq \bigcup \{P_i^{-1}(y_i):\ y_i \in L_{M_i}\}, \quad \text{for each } i \in I. \tag{3.2}$$

From condition (c) for each $x \in X \setminus K$, there exists $y_i \in D_i \subseteq L_{M_i}$ such that $y_i \in P_i(x)$ for each $i \in I$ and so

$$x \in \bigcup \{P_i^{-1}(y_i):\ y_i \in D_i\} \subseteq \bigcup \{P_i^{-1}(y_i):\ y_i \in L_{M_i}\}, \quad \text{for each } i \in I.$$

Therefore, for each $i \in I$

$$X \setminus K \subseteq \bigcup \{P_i^{-1}(y_i):\ y_i \in L_{M_i}\}. \tag{3.3}$$

By (3.2) and (3.3), we get $X = \bigcup\{P_i^{-1}(y_i):\ y_i \in L_{M_i}\}$ for each $i \in I$. Now for each $i \in I$

$$Y = S^{-1}(X) = S^{-1}\Big(\bigcup \{P_i^{-1}(y_i):\ y_i \in L_{M_i}\}\Big) = \bigcup \{S^{-1}P_i^{-1}(y_i):\ y_i \in L_{M_i}\}.$$

Since $L_M \subseteq Y$, we have

$$L_M \subseteq \bigcup \{S^{-1}P_i^{-1}(y_i):\ y_i \in L_{M_i}\}, \quad \text{for each } i \in I. \tag{3.4}$$

Also $L_M \subseteq S^{-1}(S(L_M))$ and from (3.4) we have for each $i \in I$

$$
\begin{aligned}
L_M &\subseteq \Big(\bigcup\big\{S^{-1}P_i^{-1}(y_i)\colon\ y_i \in L_{M_i}\big\}\Big) \cap \big(S^{-1}\big(S(L_M)\big)\big)\\
&= \bigcup\big\{\big(S^{-1}P_i^{-1}(y_i) \cap S^{-1}\big(S(L_M)\big)\big)\colon\ y_i \in L_{M_i}\big\}\\
&= \bigcup\big\{S^{-1}\big(P_i^{-1}(y_i) \cap S(L_M)\big)\colon\ y_i \in L_{M_i}\big\}.
\end{aligned}
$$

Since L_M is compact and $S:Y \to X$ is continuous, $S(L_M)$ is compact and $S(L_M) \cap P_i^{-1}(y_i)$ is open in $S(L_M)$ because each $P_i^{-1}(y_i)$ is compactly open. Therefore for each $i \in I$ and for all $y \in L_{M_i}$, $(S^{-1}[P_i^{-1}(y_i) \cap S(L_M)]) \cap L_M$ is open in L_M and

$$
L_M = \bigcup\big\{L_M \cap \big(S^{-1}\big[P_i^{-1}(y_i) \cap S(L_M)\big]\big)\colon\ y_i \in L_{M_i}\big\}, \quad \text{for each } i \in I.
$$

Since L_M is compact, for each $i \in I$, there exists a finite set $N_i = \{y_i^{(1)}, \ldots, y_i^{n_i+1}\}$ of L_{M_i} for some $n_i \in \mathbb{N}$ such that

$$
L_M \subseteq \bigcup_{j=1}^{n_i+1} \big(L_M \cap \big(S^{-1}\big(\big[P_i^{-1}\big(y_i^{(j)}\big) \cap S(L_M)\big]\big)\big)\big).
$$

Since L_M is compact, there also exists a continuous partition of unity $\{\beta_i^{(1)}, \ldots, \beta_i^{(n_i+1)}\}$ subordinated to the open covering $\{L_M \cap (S^{-1}([P_i^{-1}(y_i^{(j)}) \cap S(L_M)]))\}_{j=1}^{n_i+1}$, that is, for each $j = 1, \ldots, n_i + 1$, $\beta_i^{(j)}:L_M \to [0,1]$ is continuous such that for all $x \in L_M$, $\sum_{j=1}^{n_i+1} \beta_i^{(j)}(x) = 1$ and for each $j = 1, \ldots, n_i + 1$, $\beta_i^{(j)}(x) = 0$ for $x \notin L_M \cap (S^{-1}[P_i^{-1}(y_i^{(j)}) \cap S(L_M)])$. In other words, $\beta_i^{(j)}(x) \neq 0$ implies $x \in S^{-1}P_i^{-1}(y_i^{(j)})$, that is, $y_i^{(j)} \in P_i(S(x))$ for all $j = 1, \ldots, n_i + 1$ and for each $i \in I$.

Let $\phi_i : L_M \to \Delta_{n_i}$ be a map defined by

$$
\varphi_i(x) = \sum_{j=1}^{n_i+1} \beta_i^{(j)}(x) e_i^{(j)}, \quad \text{for all } x \in L_M,
$$

where $e_i^{(j)}$ is the jth unit vector in $\mathbb{R}^{n_i+1}$ and Δ_{n_i} denotes the standard n_i-simplex.

For each $i \in I$, let $g_i : \Delta_{n_i} \to \operatorname{co} H_i \subseteq L_{M_i}$ be defined by

$$
g_i\left(\sum_{j=1}^{n_i+1} \alpha_i^{(j)} e_i^{(j)}\right) = \sum_{j=1}^{n_i+1} \alpha_i^{(j)} y_i^{(j)},
$$

where $\operatorname{co} H_i = \operatorname{co}\{y_i^{(1)}, \ldots, y_i^{(n_i+1)}\}$, $\alpha_i^{(j)} \geqslant 0$ for all $i \in I$ and $1 \leqslant j \leqslant n_i + 1$ and $\sum_{j=1}^{n_i+1} \alpha_i^{(j)} = 1$. Then clearly $\varphi_i : L_M \to \Delta_{n_i}$ and $g_i : \Delta_{n_i} \to L_{M_i}$ are continuous functions.

Let $J_i(x) = \{1 \leqslant j \leqslant n_i + 1\colon\ \beta_i^{(j)}(x) \neq 0\}$. Then for each $x \in L_M$,

$$
g_i\phi_i(x) = \sum_{j=1}^{n_i+1} \beta_i^{(j)}(x) y_j = \sum_{j \in J_i(x)} \beta_i^{(j)}(x) y_j \in \operatorname{co} P_i\big(S(x)\big) \subseteq Q_i\big(S(x)\big)\big|_{L_M}.
$$

For each $i \in I$, let E_i be the finite dimensional vector space containing Δ_{n_i} and let $C = \prod_{i \in I} \Delta_{n_i}$. Then C is a compact convex subset of the locally convex Hausdorff topological vector space. Let $G : C \to L_M$ be defined by

$$G(z) = \big(g_i(z_i)\big)_{i \in I} \quad \text{for } z \in C,$$

where z_i is the ith projection of z. Let $\Psi : L_M \to C$ be defined by

$$\Psi(x) = \big(\phi_i(x)\big)_{i \in I} \quad \text{for } x \in L_M.$$

Let $F = \Psi \circ G$. Then $F : C \to C$ is a continuous function. By Tychonoff's fixed point theorem that there exists $\bar{u} \in C$ such that $\bar{u} = F(\bar{u}) = \Psi \circ G(\bar{u})$. Let $\bar{y} = (\bar{y}_i)_{i \in I} = G(\bar{u})$. Then $\bar{y}_i = g_i \phi_i(\bar{y}) \in Q_i(S(\bar{y}))$ for all $i \in I$. □

As a simple consequence of Theorem 3.1, we have the following result.

Corollary 3.1. *Let X be a topological space. For each $i \in I$, let Y_i be a nonempty convex subset of a topological vector space E_i. Let $Y = \prod_{i \in I} Y_i$ and let $S : Y \to X$ be a continuous single-valued map. For each $i \in I$, let $P_i, Q_i : X \to 2^{Y_i}$ be multivalued maps satisfying the following conditions*:

(a) *For each $x \in X$, $\operatorname{co} P_i(x) \subseteq Q_i(x)$.*
(b) $X = \bigcup\{\operatorname{cint} P_i^{-1}(y_i)\colon y_i \in Y_i\}$.
(c) *If X is not compact, then there exist a nonempty compact subset K of X and a nonempty compact convex subset D_i of Y_i for each $i \in I$ such that for all $x \in X \setminus K$, there exists $\tilde{y}_i \in D_i$ such that $x \in \operatorname{cint} P_i^{-1}(\tilde{y}_i)$ for all $i \in I$.*

Then there exists $\bar{y} = (\bar{y}_i)_{i \in I}$ such that $\bar{y}_i \in Q_i(S(\bar{y}))$ for all $i \in I$.

Proof. By condition (b) for each $i \in I$ and for all $x \in X$, there exists $y_i \in Y_i$ such that $x \in \operatorname{cint} P_i^{-1}(y_i)$. For each $i \in I$ and for all $x \in X$, define a multivalued map $F_i : X \to 2^{Y_i}$ by

$$F_i(x) = \big\{y_i \in Y_i\colon x \in \operatorname{cint} P_i^{-1}(y_i)\big\}.$$

Then for each $i \in I$ and for all $y_i \in Y_i$, $F_i^{-1}(y_i) = \operatorname{cint} P_i^{-1}(y_i)$ is compactly open. Again by condition (b), for each $i \in I$ and for all $x \in X$, $F_i(x)$ is nonempty and $\operatorname{co} F_i(x) \subseteq \operatorname{co} P_i(x) \subseteq Q_i(x)$. By condition (c), for each $x \in X$, there exists $\tilde{y}_i \in D_i$ such that $x \in \operatorname{cint} P_i^{-1}(\tilde{y}_i)$, we have $\tilde{y}_i \in P_i(x)$ and hence $F_i(x) \cap D_i \neq \emptyset$. Thus the conclusion follows from Theorem 3.1. □

Remark 3.1. Corollary 3.1 generalizes Theorem 3.1 in [1] and thus Theorem 2.1 in [10].

Remark 3.2. Conditions (a) and (b) in Corollary 3.1 can be replaced by the following conditions:

(a′) For each $x \in X$, $P_i(x)$ is nonempty and $\operatorname{co} P_i(x) \subseteq Q_i(x)$.
(b′) P_i^{-1} is transfer compactly open valued on Y_i.

Proof. By conditions (a′) and (b′), and Lemma 2.2, we have

$$X = \bigcup \bigl\{ \operatorname{cint} P_i^{-1}(y_i):\ y_i \in Y_i \bigr\}. \qquad \square$$

Remark 3.3. For $Y = X$ and $S(x) = x$ for all $x \in X$, Corollary 3.1 along with Remark 3.2 generalizes Corollary 3.1 of Lin et al. [11].

Corollary 3.2. *Let X be a topological space. For each $i \in I$, let Y_i be a nonempty convex subset of a topological vector space E_i. Let $Y = \prod_{i \in I} Y_i$ and let $S: Y \to X$ be a continuous single-valued map. For each $i \in I$, let $P_i, Q_i : X \to 2^{Y_i}$ be multivalued maps satisfying the following conditions*:

(a) *For all $x \in X$, $P_i(x)$ is nonempty and* $\operatorname{co} P_i(x) \subseteq Q_i(x)$.
(b) *For each $y_i \in Y_i$, $P_i^{-1}(y_i)$ contains compactly open subset O_{y_i}* (*may be empty*) *of X and*

$$X = \bigcup \{O_{y_i}:\ y_i \in Y_i\}.$$

(c) *If X is not compact, then there exist a nonempty compact subset K of X and a nonempty compact convex subset D_i of Y_i for each $i \in I$ such that for all $x \in X \setminus K$, there exists $\tilde{y}_i \in D_i$ such that $x \in O_{\tilde{y}_i}$ for all $i \in I$.*

Then there exists $\bar{y} = (\bar{y}_i)_{i \in I} \in Y$ such that $\bar{y}_i \in Q_i(S(\bar{y}))$ for all $i \in I$.

Proof. By condition (b), we have for each $i \in I$

$$X = \bigcup \{O_{y_i}:\ y_i \in Y_i\} \subseteq \bigcup \bigl\{ \operatorname{cint} P_i^{-1}(y_i):\ y_i \in Y_i \bigr\} \subseteq X.$$

By condition (c) for each $x \in X \setminus K$, there exists $\tilde{y}_i \in D_i$ such that $x \in O_{\tilde{y}_i} \subseteq \operatorname{cint} P_i^{-1}(\tilde{y}_i)$ for each $i \in I$. Then the conclusion follows from Corollary 3.1. $\square$

Remark 3.4. Condition (c) of Corollary 3.2 can be replaced by the following condition:

(c′) If X is not compact, then for each $i \in I$ there exist a nonempty compact convex subset D_i of Y_i and a finite subset $\{y_i^1, \ldots, y_i^{(n)}\}$ of Y_i such that

$$\bigcap_{y_i \in D_i} O_{y_i}^c \subseteq \bigcup_{j=1}^{n} O_{y_i^{(j)}}$$

where $O_{y_i}^c$ denotes the complement of O_{y_i}.

Proof. Set $C_i = \operatorname{co}\{D_i \cup \{y_i^{(1)}, \ldots, y_i^{(n)}\}\}$ for each $i \in I$. Then C_i is a nonempty compact convex subset of Y_i and by condition (c′) we have

$$\bigcap_{u_i \in C_i} O_{u_i}^c \subseteq \Biggl(\bigcap_{y_i \in D_i} O_{y_i}^c \Biggr) \cap \Biggl(\bigcap_{j=1}^{n} O_{y_i}^c \Biggr) = \emptyset \subseteq K$$

for any compact set K. Therefore for any $x \in X\backslash K$, there exists $u_i \in C_i$ such that $x \in O_{u_i}$. □

Remark 3.5. Corollary 3.2 improves and generalizes Theorem 1 in [15] Theorem 2 in [16], Theorem 2.1 in [17], and Theorem 1 in [8].

Corollary 3.3. *Let X be a topological vector space. For each $i \in I$, let Y_i be a nonempty convex subset of a topological vector space E_i. Let $Y = \Pi_{i\in I} Y_i$ and let $S : Y \to X$ be a continuous single-valued map. For each $i \in I$, let $P_i, Q_i : X \to 2^{Y_i}$ be multivalued maps satisfying the following conditions*:

(a) *For all $x \in X$, $\operatorname{co} P_i(x) \subseteq Q_i(x)$.*
(b) $X = \bigcup\{\operatorname{int} P_i^{-1}(y_i)\colon\ y_i \in Y_i\}$.
(c) *If X is not compact, then there exist a nonempty compact subset K of X and a nonempty compact convex subset D_i of Y_i for each $i \in I$ such that for all $x \in X\backslash K$, there exists $\tilde{y}_i \in D_i$ such that $x \in \operatorname{int} P_i^{-1}(\tilde{y}_i)$ for all $i \in I$.*

Then there exists $\bar{y} = (\bar{y}_i)_{i\in I} \in Y$ such that $\bar{y}_i \in Q_i(S(\bar{y}))$ for all $i \in I$.

Proof. By condition (b),

$$X = \bigcup\bigl\{\operatorname{int} P_i^{-1}(y_i)\colon\ y_i \in Y_i\bigr\} \subseteq \bigcup\bigl\{\operatorname{cint} P_i^{-1}(y_i)\colon\ y_i \in Y_i\bigr\} \subseteq X.$$

Therefore $X = \bigcup\{\operatorname{cint} P_i^{-1}(y_i)\colon\ y_i \in Y_i\}$ and hence the conclusion follows from Corollary 3.1. □

Next we shall establish a collective fixed points theorem for a family of multivalued maps whose product is Φ-condensing.

Theorem 3.2. *For each $i \in I$, let X_i be a nonempty closed convex subset of a topological vector space E_i. Let $X = \prod_{i\in I} X_i$ and let Φ be a measure of noncompactness on $E = \prod_{i\in I} E_i$. For each $i \in I$, let $P_i, Q_i : X \to 2^{X_i}$ be multivalued maps satisfying the following conditions*:

(a) *For all $x \in X$, $P_i(x)$ is nonempty and $\operatorname{co} P_i(x) \subseteq Q_i(x)$.*
(b) *P_i^{-1} is transfer compactly open valued.*
(c) *The multivalued map $Q : X \to 2^X$ defined by $Q(x) = \prod_{i\in I} Q_i(x)$ is Φ-condensing.*

Then there exists $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ such that $\bar{x}_i \in Q_i(\bar{x})$ for all $i \in I$.

Proof. Since for each $i \in I$, X_i is a nonempty closed convex subset of E_i, we have $X = \prod_{i\in I} X_i$ is a nonempty closed convex subset of $E = \prod_{i\in I} E_i$. Since $Q : X \to 2^X$ is Φ-condensing, it follows from Lemma 2.5 that there exists a nonempty compact convex subset K of X such that $Q(K) \subseteq K$. Let $K = \prod_{i\in I} K_i$, where K_i is the ith projection of K. Then K_i is a compact convex subset of X_i and for each $x \in K$, $\operatorname{co} P_i(x) \subseteq Q_i(x) \subseteq K_i$ and the conclusion follows from Corollary 3.1. □

Remark 3.6. Theorem 3.2 along with Lemma 2.2 generalizes Theorem 1 in [12].

As an application of Corollary 3.1, we have the following coincidence theorem for two families of multivalued maps defined on product spaces.

Theorem 3.3. *For each $i \in I$, let X_i and Y_i be nonempty convex subsets of topological vector spaces E_i and $\tilde{E}_i$, respectively. Let $X = \prod_{i\in I} X_i$ and $Y = \prod_{i\in I} Y_i$. For each $i \in I$, let $A_i : X \to 2^{Y_i}$ and $B_i : Y \to 2^{X_i}$ be multivalued maps. Suppose that there exist nonempty compact subset $L \subseteq Y$ and $K \subseteq X$, and nonempty compact convex subset $C_i \subseteq Y_i$ and $D_i \subseteq X_i$ for each $i \in I$ such that*

(a) *for each $i \in I$, A_i^{-1}, B_i^{-1} are transfer compactly open valued on Y_i and X_i, respectively,*
(b) *for each $i \in I$ and for all $x \in X$ and $y \in Y$, $A_i(x)$ and $B_i(y)$ are nonempty convex sets,*
(c) *for each $(x, y) \in X \times Y \backslash K \times L$, there exist $y_i \in C_i$, $x_i \in D_i$ such that $x \in \operatorname{cint} A_i^{-1}(y_i)$ and $y \in \operatorname{cint} B_i^{-1}(x_i)$ for all $i \in I$.*

Then there exists $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ and $\bar{y} = (\bar{y}_i)_{i\in I} \in Y$ such that $\bar{y}_i \in A_i(\bar{x})$ and $\bar{x}_i \in B_i(\bar{y})$ for all $i \in I$.

Proof. We follow the argument of Theorem 10 in [6]. Let $W = K \times L$. For each $i \in I$, let $V_i = C_i \times D_i$ and define $W_i : X \times Y \to 2^{Y_i \times X_i}$ by $W_i(x, y) = A_i(x) \times B_i(y)$ for each $(x, y) \in X \times Y$. Let $S : Y \times X \to X \times Y$ be defined by $S(y, x) = (x, y)$ for each $(y, x) \in Y \times X$. Then all the conditions of Corollary 3.1 are satisfied and it follows that there exists $(\bar{x}, \bar{y}) = (\bar{x}_i, \bar{y}_i)_{i\in I} \in X \times Y$ such that $(\bar{y}_i, \bar{x}_i) \in W_i(S(\bar{y}, \bar{x})) = W_i(\bar{x}, \bar{y}) = A_i(\bar{x}) \times B_i(\bar{y})$ for all $i \in I$. Thus $\bar{y}_i \in A_i(\bar{x})$ and $\bar{x}_i \in B_i(\bar{y})$ for all $i \in I$. □

Remark 3.7. Theorem 3.3 improves and generalizes Theorem 10 in [6] and Theorem 4.3 in [5].

4. Maximal elements for a family of multivalued maps

We recall that a point $x \in X$ is a *maximal element* of a multivalued map T from a topological space X to another topological space Y if $T(x) = \emptyset$.

For each $i \in I$, Let X_i be a nonempty subset of a topological space E_i and $T_i : X = \prod_{i\in I} X_i \to 2^{X_i}$ a multivalued map. Then a point $x = (x_i)_{i\in I} \in X$ is called a *maximal element* for the family of multivalued maps $\{T_i\}_{i\in I}$ if $T_i(x) = \emptyset$ for all $i \in I$.

In the recent past the existence theorems for a maximal element for a family of multivalued maps have been used to prove the existence of a solution of system of variational inequalities and system of equilibrium problems, see for example [2,10–12,19] and references therein. It can be easily seen that the maximal elements theory for the family of multivalued maps is useful to study the following qualitative game.

A *qualitative game* is a family $\Gamma = (X_i, P_i)_{i \in I}$ of ordered pairs (X_i, P_i) where for each $i \in I$, X_i is a topological space and $P_i : X = \prod_{i \in I} X_i \to 2^{X_i}$ is an irreflexive preference correspondence, that is, $x_i \notin P_i(x)$ for all $x \in X$. A point $x \in X$ is said to be an *equilibrium point* of the qualitative game Γ if $P_i(x) =$ for all $i \in I$.

For further detail on qualitative games, we refer to [19] and reference therein.

We establish the following proposition which plays an important role throughout this section.

Proposition 4.1. *Let X and Y be two nonempty subsets of a topological vector space and $T : X \to 2^Y$ a multivalued map. Then the following two statements are equivalent*:

(a) *For each $x \in X$ such that $T(x) \neq \emptyset$, there exists $y \in Y$ such that $x \in \operatorname{cint} T^{-1}(y)$.*
(b) *T^{-1} is transfer compactly open valued on Y.*

Proof. (a) $\Rightarrow$ (b). Let $x \in X$ such that $x \in T^{-1}(y)$ for some $y \in Y$, then $y \in T(x) \neq \emptyset$. By (a), there exists $y' \in Y$ such that $x \in \operatorname{cint} T^{-1}(y')$. Hence T^{-1} is transfer compactly open valued on Y.

Conversely, let $x \in X$ such that $T(x) \neq \emptyset$. Therefore, $x \in T^{-1}(y)$ for some $y \in Y$. By (b), there exists $y' \in Y$ such that $x \in \operatorname{cint} T^{-1}(y')$. □

Following the argument of proof of Theorem 8 in [6], we have the following result.

Theorem 4.1. *Let X be a regular and paracompact topological space. For each $i \in I$, let Y_i be a nonempty convex subset of a topological vector space E_i. Let $Y = \prod_{i \in I} Y_i$, $S : Y \to X$ a continuous single-valued map and $Q_i \in M_S(X, Y_i)_{i \in I}$. Suppose that there exists a nonempty compact subset K of X and a nonempty compact convex subset C_i of Y_i for each $i \in I$ such that for all $x \in X \backslash K$, there exists $i \in I$ such that $Q_i(x) \cap C_i \neq \emptyset$. Then there exists $\bar{x} \in K$ such that $Q_i(\bar{x}) = \emptyset$ for all $i \in I$.*

Remark 4.1. Since our definition of $M_S(X, Y_i)_{i \in I}$ is more general than the one given in [6], Theorem 4.1 generalizes Theorem 8 in [6].

As a particular case of above theorem, we have the following result.

Corollary 4.1. *For each $i \in I$, let X_i be a nonempty convex subset of a topological vector space E_i. Let $X = \prod_{i \in I} X_i$ and $Q_i \in M(X, X_i)_{i \in I}$. Suppose that X is regular and paracompact and there exist a nonempty compact subset K of X and a nonempty compact convex subset C_i of X_i for each $i \in I$ such that for all $x \in X \backslash K$, there exists $i \in I$ with $Q_i(x) \cap C_i \neq \emptyset$. Then there exists $\bar{x} \in K$ such that $Q_i(\bar{x}) = \emptyset$ for all $i \in I$.*

Theorem 4.2. *For each $i \in I$, let X_i be a nonempty closed convex subset of a topological vector space E_i. Let $X = \prod_{i \in I} X_i$ and let Φ be a measure of noncompactness on $E = \prod_{i \in I} E_i$. For each $i \in I$, let $Q_i : X \to 2^{X_i}$ be a multivalued map satisfying the following conditions*:

(a) $\{Q_i\}_{i\in I} \in M(X, X_i)_{i\in I}$.
(b) *The multivalued map* $Q : X \to 2^X$ *defining by* $Q(x) = \prod_{i\in I} Q_i(x)$ *is* Φ*-condensing.*

Then there exists $\bar{x} \in X$ *such that* $Q_i(\bar{x}) = \emptyset$ *for all* $i \in I$.

Proof. Since for each $i \in I$, X_i is a nonempty closed convex subset of E_i, we have $X = \prod_{i\in I} X_i$ is a nonempty closed convex subset of E. Since $Q : X \to 2^X$ is Φ-condensing, it follows from Lemma 2.5 that there exists a nonempty compact convex subset K of X such that $Q(K) \subseteq K$. Since K is compact, K is regular and paracompact. By condition (a), $\{Q_i\}_{i\in I} \in M(X, X_i)_{i\in I}$ and it is easy to see that $\{Q_i|_K\}_{i\in I} \in M(K, K_i)_{i\in I}$ where K_i is the ith projection of K. Then the conclusion follows from Corollary 4.1. □

Remark 4.2. Theorem 4.2 improves Proposition 2 in [4] in th following ways:

(i) For each $i \in I$, E_i need not be locally convex.
(ii) Theorem 4.2 does not have the following condition: *the set* $\{x \in C\colon\ Q_i(x) \neq \emptyset\}$ *is open in* C*, for every nonempty compact subset* C *of* X.

Corollary 4.2. *For each* $i \in I$*, let* X_i *be a nonempty closed convex subset of a topological vector space* E_i*. Let* $X = \prod_{i\in I} X_i$*,* $\{Q_i\}_{i\in I} \in L(X, X_i)_{i\in I}$ *and let* Φ *be a measure of noncompactness on* $E = \prod_{i\in I} E_i$*. Suppose that the multivalued map* $Q : X \to 2^X$ *defining by* $Q(x) = \prod_{i\in I} Q_i(x)$ *is* Φ*-condensing. Then there exists* $\bar{x} \in K$ *such that* $Q_i(\bar{x}) = \emptyset$ *for all* $i \in I$.

Proof. Since $\{Q_i\}_{i\in I} \in L(X, X_i)_{i\in I}$ and $L(X, X_i)_{i\in I} \subseteq M(X, X_i)_{i\in I}$, we have $\{Q_i\}_{i\in I} \in M(X, X_i)_{i\in I}$ and the result follows from Theorem 4.2. □

Theorem 4.3. *For each* $i \in I$*, let* X_i *be a nonempty closed convex subset of a topological vector space* E_i*. Let* $X = \prod_{i\in I} X_i$ *and let* Φ *be a measure of noncompactness on* $E = \prod_{i\in I} E_i$*. For each* $i \in I$*, assume that the multivalued map* $Q_i : X \to 2^{X_i}$ *satisfies the following conditions*:

(a) *For each* $x = (x_i)_{i\in I} \in X$*,* $x_i \notin \operatorname{co} Q_i(x)$.
(b) *For each* $x \in X$ *such that* $Q_i(x) \neq \emptyset$*, there exists* $y_i \in X_i$ *such that* $x \in \operatorname{cint} Q_i^{-1}(y_i)$.
(c) *The multivalued map* $Q : X \to 2^X$ *defined as* $Q(x) = \prod_{i\in I} Q_i(x)$ *for all* $x \in X$*, is* Φ*-condensing.*

Then there exists $\bar{x} \in X$ *such that* $Q_i(\bar{x}) = \emptyset$ *for all* $i \in I$.

Proof. Suppose that the conclusion of this theorem is not true. Then for every $x \in X$, there exists $j_x \in I$ such that $Q_{j_x}(x) \neq \emptyset$. For each $i \in I$, let $F_i : X \to 2^{X_i}$ be defined as $F_i(x) = \{y_i \in X_i\colon\ x \in \operatorname{cint} Q_i^{-1}(y_i)\}$. Then for each $y_i \in X_i$, $F_i^{-1}(y_i) = \operatorname{cint} Q_i^{-1}(y_i)$ is compactly open and also $F_i(x) \subseteq Q_i(x)$ for all $x \in X$. Since for each $x \in X$, $x_i \notin \operatorname{co} Q_i(x)$ and $F_i(x) \subseteq Q_i(x)$, we have $x_i \notin \operatorname{co} F_i(x)$. Therefore $\{F_i\}_{i\in I} \in L(X, X_i)_{i\in I} \subseteq M(X, X_i)_{i\in I}$. Since for each $x \in X$, $F_i(x) \subseteq Q_i(x)$ and $Q : X \to 2^X$ is Φ-condensing, it follows that

$F : X \to 2^X$ is also Φ-condensing, where $F(x) = \prod_{i\in I} F_i(x)$ for all $x \in X$. Then by Theorem 4.2 that there exists $\bar{x} \in X$ such that $F_i(\bar{x}) = \emptyset$ for all $i \in I$.

On the other hand, there exists $j_{\bar{x}} \in I$ such that $Q_{j_{\bar{x}}}(\bar{x}) \neq \emptyset$. Then by (b), there exists $y_{j_{\bar{x}}} \in X_{j_{\bar{x}}}$ such that $\bar{x} \in \operatorname{cint} Q_{j_{\bar{x}}}^{-1}(y_{j_{\bar{x}}})$. This shows that $y_{j_{\bar{x}}} \in F_{j_{\bar{x}}}(\bar{x}) = \emptyset$ which leads to a contradiction. Hence our supposition is wrong. □

Remark 4.3. Theorem 4.3 improves Corollary 4 in [4] and Corollary 4.1 in [11].

Corollary 4.3. *For each $i \in I$, let X_i be a nonempty convex subset of a topological vector space E_i. Let $X = \prod_{i\in I} X_i$ and let Φ be a measure of noncompactness on $E = \prod_{i\in I} E_i$. For each $i \in I$, let $Q_i, T_i : X \to 2^{X_i}$ be two multivalued maps satisfying the following conditions*:

(a) *For each $x \in X$,* $\operatorname{co} Q_i(x) \subseteq T_i(x)$.
(b) *For each $x = (\bar{x}_i)_{i\in I} \in X$, $x_i \notin T_i(x)$.*
(c) *For each $y_i \in X_i$, $Q_i^{-1}(y_i)$ is transfer compactly open in X.*
(d) *The multivalued map $T : X \to 2^X$ defined as $T(x) = \prod_{i\in I} T_i(x)$ for all $x \in X$, is Φ-condensing.*

Then there exists $\bar{x} \in X$ such that $Q_i(\bar{x}) = \emptyset$ for all $i \in I$.

Proof. By (b), for each $x = (x_i)_{i\in I} \in X$, $x_i \notin \operatorname{co} Q_i(x)$. Defined multivalued map $Q : X \to 2^X$ by $Q(x) = \prod_{i\in I} Q_i(x)$ for all $x \in X$. From (a), we have $Q(x) \subseteq T(x)$ for all $x \in X$. Since T is Φ-condensing, Q is also Φ-condensing. The result follows from Theorem 4.3 and Proposition 4.1. □

Remark 4.4. (i) Corollary 4.3 improves Theorem 4.2 in [11] in the following way: "For each $x \in X$, $I(x) = \{i \in I\colon\ Q_i(x) \neq \emptyset\}$ is finite" is not considered in Corollary 4.3.

(ii) If I is a singleton set, then Corollary 4.3 reduces to Corollary 2 in [12].

Following the argument of proof of Theorem 7 in [6], we have the following result which generalizes Theorem 7 in [6].

Theorem 4.4. *For each $i \in I$, let Y_i be a nonempty convex subset of a topological vector space E_i. Let $Y = \prod_{i\in I} Y_i$, X a Hausdorff topological space, $S : Y \to X$ a continuous single-valued map and $\{Q_i\}_{i\in I} \in L_S(X, Y_i)_{i\in I}$. Suppose that there exist a nonempty compact subset K of X and a nonempty compact convex subset C_i of Y_i for each $i \in I$ such that for all $x \in X\setminus K$, there exists $i \in I$ such that $Q_i(x) \cap C_i \neq \emptyset$. Then there exists $\bar{x} \in K$ such that $Q_i(\bar{x}) = \emptyset$ for all $i \in I$.*

As a simple consequence of Theorem 4.4, we have the following existence results for maximal elements.

Theorem 4.5. *Let X be a topological vector space. For each $i \in I$, let Y_i be a nonempty convex subset of a topological vector space E_i, $Q_i : X \to 2^{Y_i}$ a multivalued map and let*

$S: Y = \Pi_{i\in I} Y_i \to X$ be a continuous single-valued map. Assume that the following conditions hold:

(a) *For each $i \in I$ and for all $y \in Y$, $y_i \notin \operatorname{co} Q_i(S(y))$.*
(b) *For $x \in X$ such that $Q_i(x) \neq \emptyset$, there exists $y_i' \in Y_i$ such that $x \in \operatorname{cint} Q_i^{-1}(y_i')$.*
(c) *There exist a nonempty compact subset K of X and a nonempty compact convex subset C_i of Y_i for each $i \in I$ such that for all $x \in X\backslash K$, there exist $i \in I$ and $y_i \in C_i$ such that $x \in \operatorname{cint} Q_i^{-1}(y_i)$.*

Then there exists $\bar{x} \in K$ such that $Q_i(\bar{x}) = \emptyset$ for all $i \in I$.

Proof. Suppose that the conclusion is false. Then for each $x \in X$, there exists $j_x \in I$ such that $Q_{j_x}(x) \neq \emptyset$. For each $i \in I$, let $F_i : X \to 2^{Y_i}$ be defined by $F_i(x) = \{y_i \in Y_i\colon x \in \operatorname{cint} Q_i^{-1}(y_i)\}$ for all $x \in X$. Then for each $i \in I$ and each $y_i \in Y_i$, $F_i^{-1}(y_i) = \operatorname{cint} Q_i^{-1}(y_i)$ is compactly open. By Lemma 2.3, for each $i \in I$ and $y_i \in Y_i$, $(\operatorname{co} F_i)^{-1}(y_i)$ is compactly open. Since for all $x \in X$ and for each $i \in I$, $F_i(x) \subseteq Q_i(x)$ and $y_i \notin \operatorname{co} Q_i(S(y))$ for all $y = (y_i)_{i\in I} \in Y$, we have $y_i \notin \operatorname{co} F_i(S(y))$ for all $y = (y_i)_{i\in I}$ and $i \in I$. Therefore $\{\operatorname{co} F_i\}_{i\in I} \in L_S(X, Y_i)_{i\in I}$. By (c), for all $x \in X\backslash K$, there exist $i \in I$ and $y_i \in C_i$ such that $x \in \operatorname{cint} Q_i^{-1}(y_i)$. Then $y_i \in F_i(x)$ and $F_i(x) \cap C_i \neq \emptyset$ and hence by Theorem 4.4, there exists $\bar{x} \in X$ such that $\operatorname{co} F_i(\bar{x}) = \emptyset$ for all $i \in I$.

On the other hand, for each $\bar{x}$ there exists $j_{\bar{x}} \in I$ such that $Q_{j_{\bar{x}}}(\bar{x}) \neq \emptyset$. Since $Q_{j_{\bar{x}}}(\bar{x}) \neq \emptyset$ and from condition (b), we have $\bar{x} \in \operatorname{cint} Q_{j_{\bar{x}}}^{-1}(y'_{j_{\bar{x}}})$ for some $y'_{j_{\bar{x}}} \in Y_{j_{\bar{x}}}$ and thus $y'_{j_{\bar{x}}} \in F_{j_{\bar{x}}}(\bar{x}) \neq \emptyset$. Therefore, $\operatorname{co} F_{j_{\bar{x}}}(\bar{x}) \neq \emptyset$. This leads to a contradiction. Hence our supposition is not true. □

Remark 4.5. Condition (b) of Theorem 4.5 can be replaced by the following condition:

(b′) For each $i \in I$, Q_i^{-1} is transfer compactly open valued on Y_i.

As a simple consequence of Theorem 4.4, we have the following corollary.

Corollary 4.4. *For each $i \in I$, let X_i be a nonempty convex subset of a topological vector space E_i and let $Q_i, T_i : X = \prod_{i\in I} X_i \to 2^{X_i}$ be multivalued maps satisfying the following conditions*:

(a) *For each $i \in I$ and for all $x \in X$, $\operatorname{co} Q_i(x) \subseteq T_i(x)$.*
(b) *For each $i \in I$ and for all $x = (x_i)_{i\in I} \in X$, $x_i \notin T_i(x)$, where x_i is the ith projection of x.*
(c) *For each $i \in I$ and for all $y_i \in X_i$, $Q_i^{-1}(y_i)$ is compactly open in X.*
(d) *There exists a nonempty compact subset K of X and a nonempty compact convex subset C_i of X_i for each $i \in I$ such that for all $x \in X\backslash K$, there exists $i \in I$ such that $Q_i(x) \cap C_i \neq \emptyset$.*

Then there exists $\bar{x} \in X$ such that $Q_i(\bar{x}) = \emptyset$ for all $i \in I$.

Proof. Let the multivalued map $G_i : X \to 2^{X_i}$ be defined by $G_i(x) = \operatorname{co} Q_i(x)$ for all $x \in X$. By (b), for each $i \in I$ and for all $x = (x_i)_{i \in J} \in X$, $x_i \notin G_i(x)$. By Lemma 2.3, for each $y_i \in X_i$, $G_i^{-1}(y_i)$ is compactly open in X and therefore $\{G_i\}_{i \in I} \in L(X, X_i)_{i \in I}$. By (d), for all $x \in X \setminus K$ there exists $i \in I$ such that $G_i(x) \cap C_i \neq \emptyset$. It follows from Theorem 4.4 that there exists $\bar{x} \in X$ such that $G_i(\bar{x}) = \emptyset$ for all $i \in I$. Therefore, $Q_i(\bar{x}) = \emptyset$ for all $i \in I$. □

Remark 4.6. In view of Proposition 4.1, condition (c) of Corollary 4.4 can be replaced by the following condition:

(c′) For each $i \in I$, multivalued map Q_i^{-1} is transfer compactly open valued on X_i.

Remark 4.7. Corollary 5.2 improves Theorem 4.1 in [11] in the way that the condition "for each $x \in X$, $I(x) = \{i \in I\colon S_i(x) \neq \emptyset\}$ is finite" is not considered in Corollary 5.2.

5. Equilibria of abstract economies

Let I be a any (finite or infinite) set of agents. An *abstract economy* is defined as a family of order quadruples $\Gamma = (X_i, A_i, B_i, P_i)_{i \in I}$ where for each $i \in I$, X_i is a topological space, $A_i, B_i : X = \prod_{i \in I} \to 2^{X_i}$ are constraint correspondences and $P_i : X \to 2^{X_i}$ is a preference correspondence. An *equilibrium* for Γ is a point $\bar{x} \in X$ such that for each $i \in I$, $\bar{x}_i \in \operatorname{cl} B_i(\bar{x})$ and $A_i(\bar{x}) \cap P_i(\bar{x}) = \emptyset$. When $A_i = B_i$ for all $i \in I$, above definitions of an abstract economy and an equilibrium coincide with the standard definitions, for example in [3] or in [18].

Theorem 5.1. *For each $i \in I$, let X_i be a nonempty convex subset of a topological vector space E_i. Let $X = \prod_{i \in I} X_i$ and let Φ be a measure of noncompactness on $E = \prod_{i \in I} E_i$. For each $i \in I$, let $\operatorname{cl} B_i : X \to 2^{X_i}$ be an upper semicontinuous multivalued map, $A_i : X \to 2^{X_i}$ a nonempty convex valued multivalued map such that for each $i \in I$, $A_i^{-1}(y_i)$ is open in X and $P_i : X \to 2^{X_i}$ a preference correspondence. Assume that the following conditions hold*:

(a) *For each $i \in I$ and for all $x \in X$, $\operatorname{co} A_i(x) \subseteq B_i(x)$.*
(b) $\{A_i \cap P_i\}_{i \in I} \in M(X, X_i)_{i \in I}$.
(c) *The multivalued map $A : X \to 2^X$ defined by $A(x) = \prod_{i \in I} A_i(x)$ is Φ-condensing.*

Then there exists $\bar{x} = (\bar{x}_i)_{i \in I} \in X$ such that $\bar{x}_i \in \operatorname{cl} B_i(\bar{x})$ and $A_i(\bar{x}) \cap P_i(\bar{x}) = \emptyset$ for all $i \in I$.

Proof. Since A is Φ-condensing, it follows from Lemma 2.45 that there exists a compact convex subset $K \subseteq X$ such that $A(K) \subseteq K$.

Let $\mathcal{F} = \{x = (x_i)_{i \in I} \in K\colon x_i \in \operatorname{cl} B_i(x)\}$. Then clearly $\mathcal{F}_i$ is a closed subset of K for each $i \in I$. By Lemma 2.4, there exists $\{T_i\}_{i \in I} \in L(K, X_i)_{i \in I}$ such that $A_i(x) \cap P_i(x) \subseteq T_i(x)$ for all $i \in I$ and $x \in K$.

Let the multivalued map $Q_i : K \to 2^{K_i}$ be defined as

$$Q_i(x) = \begin{cases} T_i(x) \cap A_i(x) & \text{if } x \in \mathcal{F}_i, \\ A_i(x) & \text{if } K \backslash \mathcal{F}_i, \end{cases}$$

where K_i is the ith projection of K. Since $\{T_i\}_{i\in I} \in L(K, X_i)$ for each $i \in I$ and for all $x = (x_i)_{i\in I} \in X$, we have $x_i \notin Q_i(x)$. It is easy to see that

$$Q_i^{-1}(y_i) = \left(T_i^{-1}(y_i) \cap (A_i)^{-1}(y_i)\right) \cup \left((K\backslash\mathcal{F}_i) \cap (A_i)^{-1}(y_i)\right)$$

is compactly open in K for each $y_i \in K_i$ and Q_i is convex valued multivalued map. Therefore $\{Q_i\}_{i\in I} \in L(K, K_i)_{i\in I} \subseteq M(K, K_i)_{i\in I}$. Since K is a compact set, $Q = \prod_{i\in I} Q_i : K \to 2^K$ is Φ-condensing. It follows from Theorem 4.2 that there exists $\bar{x} = (\bar{x}_i)_{i\in I} \in K$ such that $Q_i(\bar{x}) = \emptyset$ for each $i \in I$. If $\bar{x} \in X\backslash\mathcal{F}_j$ for some $j \in I$, then $A_j(\bar{x}) = Q_j(\bar{x}) = \emptyset$, which contradicts with $A_i(x)$ is nonempty for all $x \in X$ and for each $i \in I$. Therefore, $\bar{x} \in \mathcal{F}_i$ for all $i \in I$. Hence $\bar{x}_i \in \operatorname{cl} B_i(\bar{x})$ and $A_i(\bar{x}) \cap T_i(\bar{x}) = \emptyset$ for all $i \in I$. This shows that $A_i(\bar{x}) \cap P_i(\bar{x}) = \emptyset$ for all $i \in I$. □

Remark 5.1. Condition (b) of Theorem 5.1 can be replaced by the following conditions:

(b_1) For each $i \in I$ and for all $x = (x_i)_{i\in I}$, $x_i \notin P_i(\bar{x})$.
(b_2) For each $i \in I$ and for all $x \in X$, $P_i(x)$ is convex.
(b_3) For each $i \in I$ and for all $y_i \in X_i$, $P_i^{-1}(y_i)$ is compactly open.

As a particular case of Theorem 5.1, we have the following corollary.

Corollary 5.1. *Let $\Gamma = (X_i, A_i, P_i)_{i\in I}$ be an abstract economy satisfying for each $i \in I$:*

(i) *X_i is a nonempty closed and convex subset of a topological vector space E_i and $X = \prod_{i\in I} X_i$.*
(ii) *For each $x \in X$, $A_i(x)$ is nonempty and convex.*
(iii) *For all $y_i \in X_i$, $A_i^{-1}(y_i)$ is open.*
(iv) *$\operatorname{cl} A_i : X \to 2^{X_i}$ is upper semicontinuous.*
(v) *$(A_i \cap P_i)_{i\in I} \in M(X, X_i)_{i\in I}$.*
(vi) *$A(x) = \prod_{i\in I} A_i(x)$ is Φ-condensing.*

Then there exist $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ such that $\bar{x}_i \in \operatorname{cl} A_i(\bar{x})$ and $A_i(\bar{x}) \cap P_i(\bar{x}) = \emptyset$ for all $i \in I$.

Remark 5.2. Corollary 5.1 improves Proposition 3 in [4].

Theorem 5.2. *For each $i \in I$, let X_i be a nonempty convex subset of a topological vector space E_i. Let $X = \prod_{i\in I} X_i$ and let Φ be a measure of noncompactness on $E = \prod_{i\in I} E_i$. For each $i \in I$, let $\operatorname{cl} B_i : X \to 2^{X_i}$ be an upper semicontinuous multivalued map, $A_i : X \to 2^{X_i}$ a multivalued map with nonempty values such that for each $y_i \in X_i$, $A_i^{-1}(y_i)$ is compactly open in X, $P_i : X \to 2^{X_i}$ a preference correspondence and $G_i : X \to 2^{X_i}$ a multivalued map. Assume that the following conditions hold*:

(a) *For each* $i \in I$ *and for all* $x \in X$, $\text{co}\, A_i(x) \subseteq B_i(x)$.
(b) *For each* $i \in I$ *and for all* $y_i \in X_i$, $P_i^{-1}(y_i)$ *is compactly open in* X.
(c) *For each* $i \in I$ *and for all* $x = (x_i)_{i \in I} \in X$, $x_i \notin G_i(x)$.
(d) *For each* $i \in I$ *and for all* $x \in X$, $\text{co}\, P_i(x) \subseteq G_i(x)$.
(e) *The multivalued map* $A = \prod_{i \in I} A_i : X \to 2^X$ *defined as* $A(x) = \prod_{i \in I} A_i(x)$, *is* Φ-*condensing.*

Then there exists $\bar{x} = (\bar{x}_i)_{i \in I} \in X$ *such that* $\bar{x}_i \in \text{cl}\, B_i(\bar{x})$ *and* $A_i(\bar{x}) \cap P_i(\bar{x}) = \emptyset$ *for all* $i \in I$.

Proof. Since $A : X \to 2^X$ is Φ-condensing, it follows from Lemma 2.5 that there exists a nonempty compact convex set $K = \prod_{i \in I} K_i$ of X such that $A(K) \subseteq K$. For each $i \in I$, let $\mathcal{F}_i = \{x = (x_i)_{i \in I} \in K\colon\ x_i \in \text{cl}\, B_i(x)\}$. Then clearly $\mathcal{F}_i$ is closed for each $i \in I$. For each $i \in I$, define multivalued maps $Q_i, T_i : X \to 2^{X_i}$ by

$$Q_i(x) = \begin{cases} A_i(x) \cap P_i(x) & \text{if } x \in \mathcal{F}_i, \\ A_i(x) & \text{if } x \in K \setminus \mathcal{F}_i \end{cases}$$

and

$$T_i(x) = \begin{cases} \text{cl}\, B_i(x) \cap G_i(x)] & \text{if } x \in \mathcal{F}_i, \\ \text{cl}\, B_i(x) & \text{if } x \in K \setminus \mathcal{F}_i, \end{cases}$$

for all $x \in X$. By condition (a) and (d), for each $i \in I$ and for all $x \in X$, $\text{co}\, Q_i(x) \subseteq T_i(x)$. By condition (b)

$$Q_i^{-1}(y_i) = \left[A_i^{-1}(y_i) \cap P_i^{-1}(y_i)\right] \cup \left[(X \setminus \mathcal{F}_i) \cap A_i^{-1}(y_i)\right]$$

is compactly open in X for each $y_i \in X_i$. By condition (c) for each $i \in I$ and $x = (\bar{x}_i)_{i \in I} \in X$, $x_i \notin T_i(x)$. Since K is compact, $T = \prod_{i \in I} T_i : K \to 2^K$ is Φ-condensing. It follows from Corollary 4.3 that there exists $\bar{x} = (\bar{x}_i)_{i \in I} \in X$ such that $Q_i(\bar{x}) = \emptyset$ for all $i \in I$. If $\bar{x} \in K \setminus \mathcal{F}_j$ for some $j \in I$, then $A_j(\bar{x}) = Q_j(\bar{x}) = \emptyset$ which contradicts with $A_i(x)$ is nonempty for all $i \in I$ and $x \in X$. Therefore $\bar{x} \in \mathcal{F}_i$ for all $i \in I$. This shows that $\bar{x}_i \in \text{cl}\, B_i(\bar{x})$ and $A_i(\bar{x}) \cap P_i(\bar{x}) = \emptyset$ for all $i \in I$. □

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