A Bound for the Permanent of the Laplacian Matrix

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ABSTRACT

It is shown that if G is a simple connected graph on n vertices, then $\operatorname{per} L(G) \geqslant 2(n-1)\kappa(G)$, where $L(G)$ is the Laplacian matrix of G and $\kappa(G)$ is the complexity of G.

1. INTRODUCTION

We consider graphs without loops but which may have multiple edges. A graph is simple if its has no multiple edges. Let $G=(V, E)$ be a simple graph with vertex set $V=\{v_1,\dots,v_n\}$ and edge set E. We assume throughout that $n \geqslant 2$. The adjacency matrix $A=A(G)=((a_{ij}))$ of G is an $n\times n$ symmetric 0-1 matrix with $a_{ij}=1$ if and only if v_i and v_j are adjacent. Suppose the degree of vertex v_i is d_i, $i=1,2,\dots,n$; and let $D=D(G)=\operatorname{diag}\{d_1,\dots,d_n\}$. The matrix $L=L(G)=D(G)-A(G)$ is called the Laplacian matrix of G. The matrix $L(G)$ is symmetric, is positive semidefinite, and has row and column sums equal to zero. We will denote the permanent of a square matrix Z by $\operatorname{per} Z$.

The purpose of this note is to prove that if G is a simple connected graph on n vertices, then $\operatorname{per} L(G) \geqslant 2(n-1)\kappa(G)$, where $\kappa(G)$ is the complexity (= the number of spanning trees) of G. This strengthens a previous result of Merris [3], as conjectured in [2], and which has also been proved by Brualdi and Goldwasser [1], that for any simple connected graph G, $\operatorname{per} L(G) \geqslant 2(n-1)$.

We will use the Binet-Cauchy formula for permanents. In order to state the formula, the following notation needs to be introduced.

For positive integers n, m we denote by $G_{n,m}$ denote the set of all nondecreasing sequences $\alpha=(\alpha_1,\dots,\alpha_n)$ of length n of integers from

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$1, 2, \ldots, m$. If $\alpha \in G_{n,m}$ then $\lambda_t(\alpha)$ is the number of times t appears in α, $t = 1, 2, \ldots, m$, and $\mu(\alpha) = \prod_{t=1}^{m} \lambda_t(\alpha)!$. Note that $G_{n,m}$ has $\binom{m+n-1}{n}$ members, and suppose they are ordered lexicographically. Let A be an $n \times m$ matrix. We define a row vector $\hat{A}$ of order $\binom{m+n-1}{n}$ as follows. Let the ith element of $G_{n,m}$ be α, and let $A(\alpha)$ be the matrix formed by those columns of A whose indices are in α. Then the ith entry of $\hat{A}$ is $\operatorname{per} A(\alpha)/\sqrt{\mu(\alpha)}$. The following result is the Binet-Cauchy formula for permanents [4, p. 17].

Lemma 1. *If A, B are $n \times m$ matrices, then* $\operatorname{per} AB^t = \hat{A}\hat{B}^t$.

Note that if A is positive semidefinite, then $A = XX^t$ for some X and so $\operatorname{per} A = \operatorname{per} XX^t = \hat{X}\hat{X}^t \geqslant 0$.

Suppose $G = (V, E)$ is a simple directed graph with $V = \{v_1, \ldots, v_n\}$ and $E = \{e_1, \ldots, e_m\}$. The incidence matrix $Q = Q(G) = ((q_{ij}))$ is an $n \times m$ matrix defined as follows:

$$q_{ij} = \begin{cases} 0 & \text{if} \quad v_i \text{ and } e_j \text{ are not incident,} \\ 1 & \text{if} \quad e_j \text{ originates at } v_i, \\ -1 & \text{if} \quad e_j \text{ terminates at } v_i. \end{cases}$$

If G is a simple graph, we may give an arbitrary orientation to each of its edges, and it is a simple matter to check that $L(G) = Q(G)Q(G)^t$.

If G is a simple graph, we denote by G^2 the graph obtained from G by adding to it a copy of each of its edges.

For later use we recall the Frobenius-Konig theorem, which asserts that the permanent of a nonnegative $n \times n$ matrix is zero if and only if its has a zero submatrix of order $r \times s$ with $r + s = n + 1$. We actually require only the "if" part of the theorem.

2. THE MAIN RESULT

The following remark will be useful later.

Remark 2. Suppose H is a connected directed graph with k vertices and k edges. Then H must be a tree to which an edge has been added. Therefore H has a unique cycle (disregarding orientation). Denote the length of the cycle by l. We claim the following:

$$\operatorname{per} Q(H) = \begin{cases} 0 & \text{if} \quad l \text{ is odd,} \\ 2 \text{ or } -2 & \text{if} \quad l \text{ is even.} \end{cases} \tag{1}$$

To prove (1) first note that if some vertex v_i of H has degree one, then $\operatorname{per} Q(H)$ may be expanded along the ith row and we essentially work with the graph obtained by deleting v_i and the adjacent edge from H. Continuing this process, we may assume that H itself is a cycle, and then the result is obvious.

We now derive a formula for the permanent of the Laplacian matrix which expresses the permanent as a sum of certain positive terms. Since the terms are all positive, the formula will be useful in getting lower bounds for the permanent. This is in contrast to the expansion used by Brualdi and Goldwasser [1], which contains positive as well as negative terms, the negative terms being absent only when the graph is bipartite. The proof of Theorem 3 is similar to the one given by Merris in [3], the additional feature being an application of the result of Remark 2.

THEOREM 3. *If G is a simple graph with n vertices, then*

$$\operatorname{per} L(G) = \sum 2^{2c(H) - c_0(H)}, \tag{2}$$

where $c(H)$ and $c_0(H)$ denote the number of cycles in H and the number of cycles of length 2 in H respectively, and where the summation is over all subgraphs H of G^2 with n edges satisfying the following properties:

(i) *in every component of H the number of vertices equals the number of edges;*

(ii) *there are no cycles of odd length in H.*

Proof. Give each edge of G an arbitrary orientation so that $L(G) = Q(G)Q(G)^t$. Suppose the edge set of G is $\{e_1, \dots, e_m\}$.

By Lemma 1,

$$\operatorname{per} L(G) = \sum_{\alpha \in G_{n,m}} \frac{1}{\mu(\alpha)} \{\operatorname{per} Q(H(\alpha))\}^2, \tag{3}$$

where, if $\alpha = (\alpha_1, \dots, \alpha_n)$, then $H(\alpha)$ is formed by the edges $e_{\alpha_1}, \dots, e_{\alpha_n}$. We claim that if $\operatorname{per} Q(H(\alpha))$ is nonzero, then $H(\alpha)$ must be a subgraph of G^2

satisfying (i) and (ii). This is seen as follows. For convenience we will write H for $H(\alpha)$. As pointed out by Merris in [3], if an edge occurs in H for more than two times, then $Q(H)$ has a zero submatrix of order $(n-2)\times 3$, and by the Frobenius-Konig theorem, $\operatorname{per} Q(H)=0$. So we may assume that any edge occurs in H at most two times, so that H is a subgraph of G^2.

Suppose that a component of H has r vertices and s edges. Then after a rearrangement of rows and columns, $Q(H)$ has the following form:

$$Q(H)=\begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix},$$

where B is $r\times s$ and C is $(n-r)\times(n-s)$. Thus $Q(H)$ has zero submatrices of orders $r\times(n-s)$ and $(n-r)\times s$. If $\operatorname{per} Q(H)$ is nonzero, then again by the Frobenius-Konig theorem, $r+n-s\leqslant n$ and $n-r+s\leqslant n$. Thus $r=s$ and H satisfies (i).

If a component H' of H has an odd cycle, then by Remark 2, $\operatorname{per} Q(H')=0$ and hence $\operatorname{per} Q(H)=0$. Thus the claim is proved.

Now observe that if $H(\alpha)$ is a subgraph of G^2 satisfying (i) and (ii), then $\mu(\alpha)=2^{c_0(H)}$, and also, by applying Remark 2 to each component of $H(\alpha)$, we see that $|\operatorname{per} Q(H(\alpha))|=2^{c(H)}$. Now the result follows from (3). ∎

A simple graph with n vertices is called a star if it has one vertex of degree $n-1$ and the remaining vertices have degree 1. If G is a tree on n vertices, $n\geqslant 4$, then it is known (and not difficult to show) that G is a star if and only if the length of its longest path is 2.

Now we have the following:

THEOREM 4. *Let G be a simple connected graph on n vertices. Then*

$$\operatorname{per} L(G)\geqslant 2(n-1)\kappa(G), \tag{4}$$

where $\kappa(G)$ is the complexity of G.

Equality occurs in (4) *if and only if either $n\leqslant 3$ or if $n>3$ and G is a star.*

Proof. The result will be proved by restricting the summation on the right hand side of (2) to those subgraphs H of G^2 which are obtained by taking a spanning tree of G and then by adding to it a copy of one of its edges. Denote the class of such subgraphs by Ω. For any $H\in\Omega$, it is clear that $c_0(H)=c(H)=1$. Also, the cardinality of Ω is $(n-1)\kappa(G)$, since any spanning tree of G has $n-1$ edges and so it gives rise to $n-1$ graphs in Ω. Now the inequality (4) follows from Theorem 3.

Equality occurs in (4) if and only if every subgraph H of G^2 in the sum in (2) is obtained by doubling an edge of a spanning tree of G. This is clearly the case if G is a triangle or a star. Any other graph has two disjoint edges which lie in different components of a spanning forest. Doubling these edges results in a subgraph H which contributes to the sum in (2), so equality does not occur in (4). ∎

REMARK 5. It is possible to get stronger lower bounds for $\operatorname{per} L(G)$ by using more terms from the right hand side of (2) than those used in the proof of Theorem 4. For example, the following result may be proved along similar lines:

If G is a connected simple graph on $n = 2m$ vertices, then

$$\operatorname{per} L(G) \geqslant 2(n-1)\kappa(G) + 2^m \rho(G),$$

where $\rho(G)$ is the number of perfect matchings in G.

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