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On some applications of Ruscheweyh derivative

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ABSTRACT

In this paper, we discuss a subclass $\mathcal{V}_k(\beta, b, \delta)$ of analytic functions, which was introduced and discussed by Latha and Nanjunda Rao (1994) [3]. Some results such as inclusion relationship, coefficient inequality and radius of convexity for this class are proved. We also observe that this class is preserved under the Bernardi integral transform.

1. Introduction

Let A denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the unit disc $E = \{z : |z| < 1\}$. Also let $S^*(\beta)$ and $C(\beta)$ denote the well known classes of starlike and convex of order β respectively. For any two analytic functions $f(z)$ and $g(z)$ with

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \quad \text{for } z \in E,$$

the convolution (Hadamard product) is given by

$$(f \star g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad \text{for} z \in E.$$

Let $f(z) \in A$. Denote by $D^{\delta} : A \to A$, the operator defined by

$$D^{\delta} f(z) = \frac{z}{(1-z)^{\delta+1}} * f(z) = z + \sum_{n=2}^{\infty} \varphi_n(\delta) a_n z^n, \ (\delta > -1)$$

with

$$\varphi_n(\delta) = \frac{(\delta+1)_{n-1}}{(n-1)!}, \tag{1.2}$$

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where $(\rho)_n$ is a Pochhammer symbol given as

$$(\rho)_n = \begin{cases} 1, n = 0, \\ \rho(\rho+1)(\rho+2)\cdots(\rho+n-1), & n \in \mathbb{N}. \end{cases}$$

It is obvious that $D^0 f(z) = f(z)$, $D^1 f(z) = zf'(z)$ and

$$D^n f(z) = \frac{z(z^{n-1}f(z))^{(n)}}{n!}, \quad \text{for all } \delta = n \in N_0 = \{0, 1, 2, \ldots\}.$$

The following identity can easily be established.

$$(\delta+1)D^{\delta+1}f(z) = \delta D^{\delta} f(z) + z(D^{\delta} f(z))'. \tag{1.3}$$

The operator $D^{\delta} f(z)$ is called the Ruscheweyh derivative of $f(z)$, see [1].

Let $P_k(\beta)$ be the class of analytic functions $p(z)$ defined in E satisfying the properties $p(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{\mathrm{Re}p(z) - \beta}{1-\beta} \right| d\theta \le k\pi, \tag{1.4}$$

where $z = re^{i\theta}$, $k \ge 2$ and $0 \le \beta < 1$. When $\beta = 0$, we obtain the class P_k defined in [2] and for $k = 2$, $\beta = 0$, we have the class P of functions with positive real part. We can write (1.4) as

$$p(z) = \frac{1}{2}\int_0^{2\pi} \frac{1+(1-2\beta)ze^{-i\theta}}{1-ze^{-i\theta}} d\mu(\theta),$$

where $\mu(\theta)$ is a function with bounded variation on $[0, 2\pi]$ such that

$$\int_0^{2\pi} d\mu(\theta) = 2\pi \text{ and } \int_0^{2\pi} |d\mu(\theta)| \le k\pi.$$

Also, for $p(z) \in P_k(\beta)$, we can write from (1.4)

$$p(z) = \left(\frac{k}{4}+\frac{1}{2}\right)p_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)p_2(z), \quad z \in E, \tag{1.5}$$

where $p_1(z), p_2(z) \in P(\beta)$, $P(\beta)$ is the class of functions with positive real part greater than β.

We now consider the following class.

Definition 1.1. A function $f(z) \in A$ of the form (1.1) is in the class $\mathcal{V}_k(\beta, b, \delta)$ if and only if

$$\left(1 - \frac{2}{b} + \frac{2}{b}\frac{D^{\delta+1}f(z)}{D^{\delta}f(z)}\right) \in P_k(\beta), \quad z \in E,$$

where $k \ge 2$, $\delta > -1$, $0 \le \beta < 1$ and $b \in \mathbb{C} - \{0\} = \mathbb{C}^*$.

This class was introduced by Latha and Nanjunda Rao in [3]. It contains several well known classes of analytic and univalent functions studied earlier.

We note the following special cases.

(i) $\mathcal{V}_2(\beta, 1, 1) = C(\beta)$, $\mathcal{V}_2(\beta, 2, 0) = S^*(\beta)$,
(ii) $\mathcal{V}_k(\beta, 1, 1) = V_k(\beta)$, $\mathcal{V}_k(\beta, 2, 0) = R_k(\beta)$,

where $V_k(\beta)$ and $R_k(\beta)$ denote the class of bounded boundary and bounded radius rotation of order β, for further advancement see [4–8].

2. Preliminary results

We need the following results to obtain our results.

Lemma 2.1 ([9]). *Let $u = u_1 + iu_2$, $v = v_1 + iv_2$ and $\Psi(u, v)$ be a complex valued function satisfying the conditions:*

(i) *$\Psi(u, v)$ is continuous in a domain $D \subset \mathbb{C}^2$,*
(ii) *$(1, 0) \in D$ and $\mathrm{Re}\Psi(1, 0) > 0$,*
(iii) *$\mathrm{Re}\Psi(iu_2, v_1) \le 0$, whenever $(iu_2, v_1) \in D$ and $v_1 \le -\frac{1}{2}\left(1 + u_2^2\right)$.*

If $h(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is a function that is analytic in E such that $(h(z), zh'(z)) \in D$ and $\mathrm{Re}\Psi(h(z), zh'(z)) > 0$ hold for all $z \in E$, then $\mathrm{Re}h(z) > 0$ in E.

Lemma 2.2 ([2]). *Let $h(z) \in P_k$. Then, for $|z| = r < 1$, we have*

$$\frac{1-kr+r^2}{1-r^2} \le \operatorname{Re} h(z) \le |h(z)| \le \frac{1+kr+r^2}{1-r^2}.$$

Lemma 2.3. *Let $h(z) \in P_k$. Then, for $|z| = r < 1$, we have*

$$|zh'(z)| \le \frac{r(k+4r+kr^2)\operatorname{Re} h(z)}{(1-r^2)(1+kr+r^2)}.$$

The result follows directly by using Lemma 2.2 and (1.5).

3. Main results

Theorem 3.1. *Let $f(z) \in \mathcal{V}_k(\beta, b, \delta)$ with $b \in \mathbb{C}^*$, $0 \le \beta < 1$, $\delta > -1$. Then*

$$|a_n| \le \frac{(\sigma)_{n-1}}{(n-1)!\varphi_n(\delta)}, \quad \text{for } n \ge 2, \tag{3.1}$$

where $\sigma = \frac{k|b|(1-\beta)(\delta+1)}{2}$ and $\varphi_n(\delta)$ is given by (1.2).

This result is sharp.

Proof. Set

$$1 - \frac{2}{b} + \frac{2}{b}\frac{D^{\delta+1}f(z)}{D^{\delta}f(z)} = p(z), \tag{3.2}$$

so that $p(z) \in P_k(\beta)$. Let $p(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$. Then (3.2) can be written as

$$2(D^{\delta+1}f(z) - D^{\delta}f(z)) = bD^{\delta}f(z)\sum_{n=1}^{\infty} b_n z^n,$$

which implies that

$$\frac{2\varphi_n(\delta)(n-1)a_n}{(\delta+1)} = b(b_{n-1} + \varphi_2(\delta)a_2 b_{n-2} + \cdots + \varphi_{n-1}(\delta)a_{n-1}b_1).$$

Using the coefficient estimates $|b_n| \le k(1-\beta)$ for the class $P_k(\beta)$, we obtain

$$|a_n| \le \frac{k|b|(1-\beta)(\delta+1)}{2(n-1)\varphi_n(\delta)}(1 + \varphi_2(\delta)|a_2| + \cdots + \varphi_{n-1}(\delta)|a_{n-1}|).$$

For $n = 2$, $|a_2| \le \frac{k|b|(1-\beta)}{2}$.

Therefore (3.1) holds for $n = 2$.

Assume that (3.1) is true for $n = m$ and consider

$$\begin{aligned}
|a_{m+1}| &\le \frac{k|b|(1-\beta)(\delta+1)}{2m\varphi_{m+1}(\delta)}(1 + \varphi_2(\delta)|a_2| + \cdots + \varphi_{n-1}(\delta)|a_m|) \\
&\le \frac{k|b|(1-\beta)(\delta+1)}{2m\varphi_{m+1}(\delta)}\left\{1 + \frac{k|b|(1-\beta)(\delta+1)}{2!}\left(1 + \frac{k|b|(1-\beta)(\delta+1)}{2}\right)\right. \\
&\quad \left. + \cdots + \frac{k|b|(1-\beta)(\delta+1)}{(m-1)!}\prod_{j=1}^{m-2}\left(1 + \frac{k|b|(1-\beta)(\delta+1)}{2j}\right)\right\} \\
&= \frac{k|b|(1-\beta)(\delta+1)}{2m\varphi_{m+1}(\delta)}\prod_{j=1}^{m-1}\left(1 + \frac{k|b|(1-\beta)(\delta+1)}{2j}\right) \\
&= \frac{(\sigma)_m}{(m)!\varphi_{m+1}(\delta)}.
\end{aligned}$$

Therefore, the result is true for $n = m + 1$. Using mathematical induction, (3.1) holds true for all $n \ge 2$.

This result is sharp for $\delta > -1$, $0 \le \beta < 1$, $b \in \mathbb{C}^*$ and $k \ge 2$ as can be seen from the functions $f_0(z)$ which are given as

$$1 - \frac{2}{b} + \frac{2}{b}\frac{D^{\delta+1}f_0(z)}{D^{\delta}f_0(z)} = (1-\beta)\left[\left(\frac{k}{4} + \frac{1}{2}\right)\frac{1+z}{1-z} - \left(\frac{k}{4} - \frac{1}{2}\right)\frac{1-z}{1+z}\right] + \beta.$$

For different values of β, b, δ, we obtain the following corollaries [10]. □

Corollary 3.2. *If $f(z) \in V_k(\beta, 2, 0) = R_k(\beta)$, then*

$$|a_n| \leq \frac{(k(1-\beta))_{n-1}}{(n-1)!}, \quad \text{for } n \geq 2.$$

This result is sharp.

Corollary 3.3. *If $f(z) \in \mathcal{V}_k(\beta, 1, 1) = V_k(\beta)$, then*

$$|a_n| \leq \frac{(k(1-\beta))_{n-1}}{n!}, \quad \text{for } n \geq 2.$$

This result is sharp.

Theorem 3.4. *For $b > 0$, $\mathcal{V}_k(\alpha, b, \delta + 1) \subseteq \mathcal{V}_k(\beta_1, b + 1, \delta)$, $z \in E$, where*

$$\beta_1 = \frac{-\eta + \sqrt{\eta^2 + 4(\delta+1)(b+1-(1-b)(\delta - b + \delta b\alpha + 2b\alpha + 1))}}{2(\delta+1)(b+1)}, \tag{3.3}$$

with $\eta = \eta(b, \alpha, \delta) = (1 - b(\delta\alpha + 2\alpha - \delta))$.

Proof. Suppose $f(z) \in \mathcal{V}_k(\alpha, b, \delta)$ and set

$$p(z) = 1 - \frac{2}{b+1} + \frac{2}{b+1}\frac{D^{\delta+1}f(z)}{D^{\delta}f(z)}, \tag{3.4}$$

where $p(z)$ is analytic in E with $p(0) = 1$. Then simple computations, together with (1.3) and (3.4), yield

$$1 - \frac{2}{b} + \frac{2}{b}\frac{D^{\delta+2}f(z)}{D^{\delta+1}f(z)} = (1-\mu_1) + \mu_1\left[p(z) + \frac{\mu_2 zp'(z)}{p(z)+\mu_3}\right], \tag{3.5}$$

with $\mu_1 = \frac{\delta+1}{\delta+2}\frac{b+1}{b}$, $\mu_2 = \frac{2}{(\delta+1)(b+1)}$, $\mu_3 = \frac{2}{b+1} - 1$. Since $f(z) \in \mathcal{V}_k(\alpha, b, \delta)$, it follows that

$$\left[(1-\mu_1) + \mu_1\left(p(z) + \frac{\mu_2 zp'(z)}{p(z)+\mu_3}\right)\right] \in P_k(\alpha),$$

or, equivalently,

$$\left[\frac{(1-\alpha-\mu_1)}{(1-\alpha)} + \frac{\mu_1}{(1-\alpha)}\left(p(z) + \frac{\mu_2 zp'(z)}{p(z)+\mu_3}\right)\right] \in P_k. \tag{3.6}$$

Define

$$\varphi(z) = \frac{1}{(1+\mu_3)}\frac{z}{(1-z)^{\mu_2}} + \frac{\mu_3}{(1+\mu_3)}\frac{z}{(1-z)^{\mu_2+1}},$$

and by using convolution techniques (see [11]) together with (1.5) we have

$$p(z) + \frac{\mu_2 zp'(z)}{p(z)+\mu_3} = \left(\frac{k}{4}+\frac{1}{2}\right)\left(p_1(z) + \frac{\mu_2 zp_1'(z)}{p_1(z)+\mu_3}\right) - \left(\frac{k}{4}-\frac{1}{2}\right)\left(p_2(z) + \frac{\mu_2 zp_2'(z)}{p_2(z)+\mu_3}\right).$$

By using (3.6), we see that

$$\left[\frac{(1-\alpha-\mu_1)}{(1-\alpha)} + \frac{\mu_1}{(1-\alpha)}\left(p_i(z) + \frac{\mu_2 zp_i'(z)}{p_i(z)+\mu_3}\right)\right] \in P, \quad z \in E, i = 1, 2.$$

We want to show that $p_i(z) \in P(\beta_1)$, where β_1 is given by (3.3).

Let

$$p_i(z) = (1-\beta_1)h_i(z) + \beta_1, \quad i = 1, 2.$$

Then, for $z \in E$

$$\left[\frac{(1-\alpha-\mu_1)}{(1-\alpha)} + \frac{\mu_1}{(1-\alpha)}\left((1-\beta_1)h_i(z) + \beta_1 + \frac{\mu_2(1-\beta_1)zh_i'(z)}{(1-\beta_1)h_i(z)+\mu_3+\beta_1}\right)\right] \in P.$$

We now form the function $\Psi(u, v)$ by taking $u = h_i(z)$, $v = zh_i'(z)$ as

$$\Psi(u, v) = \frac{(1-\alpha-\mu_1)}{(1-\alpha)} + \frac{\mu_1}{(1-\alpha)}\left[(1-\beta_1)u + \beta_1 + \frac{\mu_2(1-\beta_1)v}{(1-\beta_1)u+\mu_3+\beta_1}\right].$$

The first two conditions of Lemma 2.1 are clearly satisfied. We verify condition (iii) as.

$$\begin{aligned}\mathrm{Re}\Psi(iu_2, v_1) &= \frac{(1-\alpha-\mu_1)}{(1-\alpha)} + \frac{\mu_1}{(1-\alpha)}\left[\beta_1 + \frac{\mu_2(1-\beta_1)(\mu_3+\beta)v_1}{(\mu_3+\beta_1)^2+(1-\beta_1)^2u_2^2}\right] \\ &\le \frac{(1-\alpha-\mu_1)}{(1-\alpha)} + \frac{\mu_1}{(1-\alpha)}\left[\beta_1 - \frac{\mu_2(1-\beta_1)(\mu_3+\beta_1)(1+u_2^2)}{2\left[(\mu_3+\beta_1)^2+(1-\beta_1)^2u_2^2\right]}\right] \\ &= \frac{A+Bu_2^2}{C},\end{aligned}$$

where

$$\begin{aligned} A &= (\mu_3+\beta_1)\left[2\left(\mu_3+\beta_1\right)\left(1-\alpha-\mu_1+\mu_1\beta_1\right) - \mu_1\mu_2(1-\beta_1)\right], \\ B &= (1-\beta_1)[2(1-\beta_1)(1-\alpha-\mu_1+\mu_1\beta_1) - \mu_1\mu_2(\mu_3+\beta_1)], \\ C &= 2(1-\alpha)[(\mu_3+\beta_1)^2 + (1-\beta_1)^2u_2^2] > 0. \end{aligned}$$

We notice that $\mathrm{Re}\Psi(iu_2, v_1) \le 0$ if and only if $A \le 0$ and $B \le 0$. From $A \le 0$, we obtain β_1 as defined by (3.3) and $B \le 0$ gives us $0 < \beta_1 < 1$. This proves that $h_i(z) \in P$, $i = 1, 2$ and hence $p(z) \in P_k(\beta_1)$.

If we take $b = 1$ and $\delta = 0$, we obtain the following result [11]. □

Corollary 3.5. *Let $f(z) \in V_k(\alpha)$. Then $f(z) \in R_k(\beta_1)$, where*

$$\beta_1 = \frac{1}{4}[-(1-2\alpha) + \sqrt{(1-2\alpha)^2+8}].$$

For $\alpha = 0$ and $k = 2$ in Corollary 3.5, we have the following well known result

$$V_2(0) = C \subseteq R_2\left(\frac{1}{2}\right) = S^*\left(\frac{1}{2}\right), \quad \text{for } z \in E.$$

For a function $f(z) \in A$, we consider the integral operator

$$F(z) = I_\gamma(f(z)) = \frac{(\gamma+1)}{z^\gamma}\int_0^z t^{\gamma-1}f(t)dt, \quad \gamma > -1. \tag{3.7}$$

The operator I_γ, when $\gamma \in \mathbb{N}$ was introduced by Bernardi [12]. In particular, the operator I_1 was studied earlier by Libera [13] and Livingston [14].

Theorem 3.6. *Let $f(z) \in \mathcal{V}_k(\alpha, b, \delta)$ and let $F(z)$ be defined by (3.7). Then $F(z) \in \mathcal{V}_k(\beta_2, b, \delta)$, where $0 < \beta_2 < 1$, $b > 0$ and*

$$\beta_2 = \frac{1}{4}[-(2\mu_5 - 2\alpha + \mu_4) + \sqrt{(2\mu_5 - 2\alpha + \mu_4)^2 + 8(2\alpha\mu_5 + \mu_4)}]. \tag{3.8}$$

with $\mu_4 = 2$ and $\mu_5 = \frac{2(1+\gamma)}{(\delta+1)b} - 1$.

The proof follows by using the same technique as in Theorem 3.4.

Theorem 3.7. *If $f(z)$ is of the form (1.1) belongs to $\mathcal{V}_k(\beta, b, \delta)$ and $F(z) = z + \sum_{n=2}^{\infty} b_n z^n$, where $F(z)$ is the integral operator defined by (3.7), then*

$$|b_n| \le \frac{(\gamma+1)}{(\gamma+n)}\frac{(\sigma)_{n-1}}{(n-1)!\varphi_n(\delta)}, \quad \textit{for } n \ge 2.$$

Proof. From (3.7), we obtain

$$1 + \gamma z + \sum_{n=2}^{\infty}(1+\gamma)a_n z^n = \gamma z + \sum_{n=2}^{\infty}\gamma b_n z^n + z + \sum_{n=2}^{\infty} n b_n z^n,$$

and thus

$$(n+\gamma)b_n = (1+\gamma)a_n, n \ge 2.$$

From the above we have

$$|b_n| \le \frac{(\gamma+1)}{(\gamma+n)}|a_n|, \quad n \ge 2.$$

Using the estimates from Theorem 3.1, we obtain the required result. □

Theorem 3.8. *Let $f(z) \in \mathcal{V}_k(0, b, \delta)$, $\delta > -1$, $b > 0$, $k \geq 2$ and $a = \frac{b(\delta+1)}{2} > 0$. Then $D^\delta f(z)$ maps $|z| < r_0$ onto a convex domain, where r_0 is the least positive root of the equation*

$$(4a^2 - 4a + 1)r^4 - (ka)r^3 - (k^2a^2 + 2)r^2 - (ka)r + 1 = 0. \tag{3.9}$$

This result is sharp.

Proof. Since $f(z) \in \mathcal{V}_k(0, b, \delta)$ then

$$\frac{D^{\delta+1}f(z)}{D^\delta f(z)} = \frac{b(p(z) - 1) + 2}{2}, \tag{3.10}$$

where $p(z) \in P_k(0)$. Using the identity (1.3), we have from (3.10)

$$\frac{z(D^\delta f(z))'}{D^\delta f(z)} = \frac{b(p(z) - 1)(\delta + 1) + 2}{2}. \tag{3.11}$$

Logarithmic differentiation of (3.11) yields

$$\frac{(z(D^\delta f(z))')'}{(D^\delta f(z))'} = ap(z) - a + 1 + \frac{zp'(z)}{p(z) - 1 + \frac{1}{a}},$$

where $a = \frac{b(\delta+1)}{2}$. Then we have

$$\mathrm{Re}\left(1 + \frac{z(D^\delta f(z))''}{(D^\delta f(z))'}\right) \geq a\,\mathrm{Re}p(z) + (1 - a) - \frac{|zp'(z)|}{\left|p(z) - 1 + \frac{1}{a}\right|},$$

and hence, by using Lemmas 2.2 and 2.3,

$$\begin{aligned}\mathrm{Re}\left(1 + \frac{z(D^\delta f(z))''}{(D^\delta f(z))'}\right) &\geq \mathrm{Re}p(z)\left\{a + \frac{(1-a)(1-r^2)}{r^2 + kr + 1} - \frac{r(kr^2 + 4r + k)a}{(r^2 + kr + 1)((2a - 1)r^2 - kar + 1)}\right\} \\ &= \mathrm{Re}p(z)\left\{\frac{(4a^2 - 4a + 1)r^4 - (ka)r^3 - (k^2a^2 + 2)r^2 - (ka)r + 1}{(r^2 + kr + 1)((2a - 1)r^2 - kar + 1)}\right\} > 0,\end{aligned}$$

provided

$$T(r) = (4a^2 - 4a + 1)r^4 - (ka)r^3 - (k^2a^2 + 2)r^2 - (ka)r + 1 > 0.$$

We have $T(0) = 1 > 0$ and $T(1) = -a(k + 2)((k - 2)a + 2) < 0$. Therefore $D^\delta f(z)$ maps $|z| < r_0$ onto a convex domain, where r_0 is the least positive root of the equation $T(r)$ lying in $(0, 1)$ and this gives (3.9).

For $D^\delta f_1(z)$ such that

$$\frac{D^{\delta+1}f_1(z)}{D^\delta f_1(z)} = \frac{b(p_k(z) - 1) + 2}{2},$$

where

$$p_k(z) = \left(\frac{k}{4} + \frac{1}{2}\right)\frac{1+z}{1-z} - \left(\frac{k}{4} - \frac{1}{2}\right)\frac{1-z}{1+z},$$

we have

$$\frac{(z(D^\delta f_1(z))')'}{(D^\delta f_1(z))'} = \frac{(4a^2 - 4a + 1)z^4 - (ka)z^3 - (k^2a^2 + 2)z^2 - (ka)z + 1}{(z^2 + kz + 1)((2a - 1)z^2 - kaz + 1)} = 0,$$

for $z = r_0$. Hence this radius r_0 is sharp. □

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