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The infinite dimensional Lagrange multiplier rule for convex optimization problems

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Abstract

In this paper an infinite dimensional generalized Lagrange multipliers rule for convex optimization problems is presented and necessary and sufficient optimality conditions are given in order to guarantee the strong duality. Furthermore, an application is presented, in particular the existence of Lagrange multipliers associated to the bi-obstacle problem is obtained.

Keywords: Lagrange multiplier rule; Convex optimization problems; Strong duality; Assumption S; Bi-obstacle problem

1. Introduction

The convex optimization problem we are concerned with is the following.

Let X be a linear topological space, let Y be a real normed space ordered by a convex cone C and let Z be a real normed space. Let S be a convex subset of X and let $f : S \to \mathbb{R}$ be a given functional and let $g : S \to Y$ be a given mapping and $h : S \to Z$ be an affine-linear mapping. Setting

$$K = \{x \in S\colon\ g(x) \in -C,\ h(x) = \theta_Z\}, \tag{1}$$

where θ_Z is the zero element in the space Z, we consider the optimization problem

$$\text{“}\mathit{find}\ x_0 \in K\ \mathit{such\ that}\ f(x_0) = \min_{x \in K} f(x)\text{”} \tag{2}$$

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and, as usual, we call its Lagrange dual problem the problem

$$\max_{u\in C^*,\, v\in Z^*} \inf_{x\in S}\bigl\{f(x)+\bigl\langle u, g(x)\bigr\rangle+\bigl\langle v, h(x)\bigr\rangle\bigr\} \tag{3}$$

where

$$C^*=\bigl\{u\in Y^*\colon\ \langle u,y\rangle\geqslant 0,\ \forall y\in C\bigr\}$$

is the dual cone of C.

In papers [22,15,2], the authors give sufficient conditions in order to have that the strong duality between a convex optimization problem in an infinite dimensional space and its Lagrange dual problem is guaranteed, i.e., the extremal values of the two problems are equals.

It is worth remarking that these usual conditions use concepts of interior, core, intrinsic core or strong quasi-relative interior which require the nonemptiness of the ordering cone which defines the cone constraints in convex optimization and variational inequalities. Since many infinite dimensional equilibrium problems have ordering cone empty, these usual conditions cannot be used to guarantee the strong duality. This is the case of all optimization problems or variational inequalities connected with network equilibrium problems, the obstacle problems, the elastic plastic torsion problems (see [1,5–9,11–14,16,19,20,23]) which use positive cones of $L^p(\Omega)$ or Sobolev spaces. Recently, in [9,8,10,18] the authors overcome this important difficulty by introducing a condition called *Assumption S* which ensures the strong duality.

Assumption S is the following. Firstly, we recall the concept of tangent cone.

Given a point $x\in X$ and a subset C of X, the set

$$T_C(x)=\Bigl\{h\in X\colon\ h=\lim_{n\to\infty}\lambda_n(x_n-x),\ \lambda_n\in\mathbb{R}\text{ and }\lambda_n>0,\ \forall n\in\mathbb{N},\\ x_n\in C,\ \forall n\in\mathbb{N}\text{ and }\lim_{n\to\infty}x_n=x\Bigr\}$$

is called the tangent cone to C at x. Of course, if $T_C(x)\neq\emptyset$, then $x\in cl\,C$. If $x\in cl\,C$ and C is convex, then we have

$$T_C(x)=cl\,cone\bigl(C-\{x\}\bigr),$$

where

$$cone(C)=\{\lambda x\colon\ x\in C,\ \lambda\in\mathbb{R},\ \lambda\geqslant 0\}$$

and *cl* denotes the closure.

Definition 1. Given three functions f, g, h and a set K as in (1), we say that Assumption S is fulfilled at a point $x_0\in K$ if and only if

$$T_{\widetilde{M}}(0,\theta_Y,\theta_Z)\cap\,]-\infty,0[\,\times\{\theta_Y\}\times\{\theta_Z\}=\emptyset, \tag{4}$$

where

$$\widetilde{M}=\bigl\{\bigl(f(x)-f(x_0)+\alpha, g(x)+y, h(x)\bigr)\colon\ x\in S\setminus K,\ \alpha\geqslant 0,\ y\in C\bigr\}.$$

A clear geometrical meaning of *Assumption S* is that the tangent cone to the subset $\widetilde{M}$ of $\mathbb{R} \times Y \times Z$ at the point $(0, \theta_Y, \theta_Z)$ does not contain $]-\infty, 0[\times \{\theta_Y\} \times \{\theta_Z\}$. $\widetilde{M}$ is a particular type of conic extension of the image of the optimization problem (2) in the image space $\mathbb{R} \times Y \times Z$. From an analytic point of view the meaning of *Assumption S* is that $f(x_n) - f(x_0) + \alpha_n$, with $\alpha_n \geqslant 0$ for all $n \in \mathbb{N}$, positively converges to zero when x_n does not belong to K but the limits of the constraint sequences $\lambda_n(g(x_n) + y_n)$ and $\lambda_n(h(x_n))$ with $y_n \in C$ and $\lambda_n > 0$ for all $n \in \mathbb{N}$, vanish. Then *Assumption S* essentially required to show that a particular limit is nonnegative. After all the calculus of a limit could not be an exorbitant price to pay considering the importance to have a necessary and sufficient condition for the strong duality.

Now, we recall the main theorem on strong duality theory.

Theorem 1. *(See [10].) Assume that the functions $f : S \to \mathbb{R}$, $g : S \to Y$ are convex and that $h : S \to Z$ is an affine-linear mapping. Assume that Assumption S is fulfilled at the optimal solution $x_0 \in K$ to* (2)*. Then also problem* (3) *is solvable and if $\bar{u} \in C^*$, $\bar{v} \in Z^*$ are the optimal solutions to* (3)*, we have*

$$\langle \bar{u}, g(x_0) \rangle = 0$$

and the optimal values of the two problems coincide, namely

$$f(x_0) = \max_{u \in C^*,\, v \in Z^*} \inf_{x \in S} \big\{ f(x) + \langle u, g(x) \rangle + \langle v, h(x) \rangle \big\}.$$

Assumption S is also a necessary condition for the strong duality, in fact the following corollary holds.

Corollary 1. *If the strong duality between problems* (2) *and* (3) *holds, then Assumption S is fulfilled.*

Proof. See Corollary 3.1 of [3]. □

An important consequence of the strong duality is the usual relationship between a saddle point of the so-called Lagrange functional

$$\mathcal{L}(x, u, v) = f(x) + \langle u, g(x) \rangle + \langle v, h(x) \rangle, \quad \forall x \in S,\ \forall u \in C^*,\ \forall v \in Z^*,$$

and the solution to (2) and (3). In fact one has the following theorem.

Theorem 2. *(See [8] and [9].) Let the assumptions of Theorem* 1 *be fulfilled. Then $x_0 \in \mathbb{K}$ is an optimal solution to problem* (2) *if and only if there exist $\bar{u} \in C^*$ and $\bar{v} \in Z^*$ such that $(x_0, \bar{u}, \bar{v})$ is a saddle point of the Lagrangean functional, namely*

$$\mathcal{L}(x_0, u, v) \leqslant \mathcal{L}(x_0, \bar{u}, \bar{v}) \leqslant \mathcal{L}(x, \bar{u}, \bar{v}), \quad \forall x \in S,\ \forall u \in C^*,\ \forall v \in Z^*$$

and, moreover, it results that

$$\langle \bar{u}, g(x_0) \rangle = 0.$$

In Section 2 of this paper we investigate a generalized Lagrange multipliers rule for the optimization problem (2) and formulate a multiplier rule as necessary and sufficient optimality conditions.

In details we will prove the following theorem.

Theorem 3. *Let X be a linear topological space, let Y be a real normed space ordered by a convex cone C and let Z be a real normed space. Let S be a convex subset of X and let $f : S \to \mathbb{R}$ be a given convex functional and let $g : S \to Y$ be a given convex mapping and $h : S \to Z$ be an affine-linear mapping. Assume that f, g, h have a directional derivative at $x_0 \in K$ solution to problem* (2) *in every direction $x - x_0$ with arbitrary $x \in S$. Moreover assume that Assumption S is fulfilled at the minimal point $x_0 \in K$. Then there exist $\bar{u} \in C^*$, $\bar{v} \in Z^*$ such that*

$$f'(x_0) + \bar{u}\big(g'(x_0)\big) + \bar{v}\big(h'(x_0)\big)(x - x_0) \geqslant 0, \quad \forall x \in S \tag{5}$$

and

$$\bar{u}\big(g(x_0)\big) = 0. \tag{6}$$

Vice versa, if (5) *and* (6) *hold, then x_0 is the minimal solution of problem* (2) *and Assumption S is verified.*

It is worth to compare Theorem 3 with well-known results presented in the literature, as, for example, with Theorem 5.3 and Corollary 5.4 of [17] for the necessary conditions and with Theorem 5.14 of [17] for the sufficient conditions. In fact, let us observe that our main result, Theorem 3, generalize Theorem 5.3 of [17], with regard to the case when h is an affine-linear mapping. Our assumptions are very general and the Kurcyusz–Robinson–Zowe regularity condition (see [21] and [25]):

$$\begin{pmatrix} g'(x_0) \\ h'(x_0) \end{pmatrix} cone\big(S - \{x_0\}\big) + cone \begin{pmatrix} C + \{g(x_0)\} \\ \{\theta_Z\} \end{pmatrix} = Y \times Z,$$

in our theorem, is replaced by *Assumption S*.

Finally, Section 3 is devoted to the application of *Assumption S* to the study of the bi-obstacle problem.

2. Proof of Theorem 3

Let us start remarking that, in virtue of Theorems 1 and 2, there exist $\bar{u} \in C^*$, $\bar{v} \in Z^*$ solutions to dual problem (3) and one has that

$$\langle \bar{u}, g(x_0) \rangle = 0$$

and

$$f(x_0) = \max_{u \in C^*,\, v \in Z^*} \inf_{x \in S} \big\{ f(x) + \langle u, g(x) \rangle + \langle v, h(x) \rangle \big\}.$$

Moreover, setting

$$\mathcal{L}(x,u,v) = f(x) + \langle u, g(x)\rangle + \langle v, h(x)\rangle, \quad \forall x \in S, \forall u \in C^*, \forall v \in Z^*,$$

it results that $(x_0, \bar{u}, \bar{v})$ is a saddle point of the Lagrangean functional, namely

$$\mathcal{L}(x_0,u,v) \leqslant \mathcal{L}(x_0,\bar{u},\bar{v}) \leqslant \mathcal{L}(x,\bar{u},\bar{v}), \quad \forall x \in S, \forall u \in C^*, \forall v \in Z^*. \tag{7}$$

Let us consider now the right-hand side of (7), we get

$$f(x_0) + \langle \bar{u}, g(x_0)\rangle + \langle \bar{v}, h(x_0)\rangle \leqslant f(x) + \langle \bar{u}, g(x)\rangle + \langle \bar{v}, h(x)\rangle, \quad \forall x \in S.$$

Taking into account that $\langle \bar{u}, g(x_0)\rangle = 0$ and $h(x_0) = 0$, we obtain

$$f(x) + \langle \bar{u}, g(x)\rangle + \langle \bar{v}, h(x)\rangle \geqslant f(x_0), \quad \forall x \in S.$$

So we have that x_0 is a minimal point of functional $f(x) + \langle \bar{u}, g(x)\rangle + \langle \bar{v}, h(x)\rangle$ in S.

In virtue of well-known theorems (see for example Theorem 3.8 of [17]), since the functional $f(x) + \langle \bar{u}, g(x)\rangle + \langle \bar{v}, h(x)\rangle$ has directional derivative at x_0 in every direction $x - x_0$ with arbitrary $x \in S$, one has the thesis

$$\big(f'(x_0) + \langle \bar{u}, g'(x_0)\rangle + \langle \bar{v}, h'(x_0)\rangle\big)(x - x_0) \geqslant 0, \quad \forall x \in S.$$

Vice versa, let us assume that (5) and (6) hold. The functional $f(x) + \langle \bar{u}, g(x)\rangle + \langle \bar{v}, h(x)\rangle$, in virtue of assumptions on $f, g, h, \bar{u}, \bar{v}$, is convex. In fact, $f(x)$ is a convex functional and $\bar{v}(h(x))$ is affine because $\bar{v}$ is linear and h is an affine-linear mapping. Moreover g is a convex mapping with respect to the ordering cone C, namely $\forall x, y \in S$, $\forall \lambda, \mu \in \mathbb{R}$ one has

$$g(\lambda x + \mu y) - \big(\lambda g(x) + \mu g(y)\big) \in -C.$$

Since $\bar{u} \in C^*$, we get

$$\bar{u}\big(g(\lambda x + \mu y) - \big(\lambda g(x) + \mu g(y)\big)\big) \leqslant 0.$$

Hence, it follows that $\bar{u}(g(x))$ is convex, because

$$\bar{u}\big(g(\lambda x + \mu y)\big) \leqslant \bar{u}\big(\lambda g(x) + \mu g(y)\big) = \lambda \bar{u}\big(g(x)\big) + \mu \bar{u}\big(g(y)\big).$$

In virtue of Theorem 3.8, case b, of [17], we have that x_0 is the minimal point of the functional $f(x) + \langle \bar{u}, g(x)\rangle + \langle \bar{v}, h(x)\rangle$, namely:

$$f(x_0) + \langle \bar{u}, g(x_0)\rangle + \langle \bar{v}, h(x_0)\rangle = \min_{x \in S}\big(f(x) + \langle \bar{u}, g(x)\rangle + \langle \bar{v}, h(x)\rangle\big).$$

In particular, from (6), for every $x \in K$, we get

$$f(x_0) \leqslant f(x) + \langle \bar{u}, g(x)\rangle + \langle \bar{v}, h(x)\rangle \leqslant f(x), \tag{8}$$

since $\langle \bar{u}, g(x)\rangle \leqslant 0$ and $h(x) = 0$ for all $x \in K$.

Now, let us show that the strong duality and *Assumption S* hold true. In fact, from (8), one has

$$f(x_0) \leqslant \inf_{x \in S}\bigl(f(x) + \langle \bar{u}, g(x)\rangle + \langle \bar{v}, h(x)\rangle\bigr). \tag{9}$$

Furthermore, for all $u \in C^*$ and $v \in Z^*$ and taking into account that $\langle u, g(x_0)\rangle \leqslant 0$, we get

$$\begin{aligned}\inf_{x \in S}\bigl(f(x) + \langle u, g(x)\rangle + \langle v, h(x)\rangle\bigr) &\leqslant f(x_0) + \langle u, g(x_0)\rangle + \langle v, h(x_0)\rangle \leqslant f(x_0)\\ &\leqslant \inf_{x \in S}\bigl(f(x) + \langle \bar{u}, g(x)\rangle + \langle \bar{v}, h(x)\rangle\bigr).\end{aligned}$$

Then

$$\begin{aligned}\sup_{u \in C^*,\, v \in Z^*} \inf_{x \in S}\bigl(f(x) + \langle u, g(x)\rangle + \langle v, h(x)\rangle\bigr) &\leqslant f(x_0) \leqslant \inf_{x \in S}\bigl(f(x) + \langle \bar{u}, g(x)\rangle + \langle \bar{v}, h(x)\rangle\bigr)\\ &\leqslant \sup_{u \in C^*,\, v \in Z^*} \inf_{x \in S}\bigl(f(x) + \langle \bar{u}, g(x)\rangle + \langle \bar{v}, h(x)\rangle\bigr),\end{aligned}$$

namely

$$\begin{aligned}\max_{u \in C^*,\, v \in Z^*} \inf_{x \in S}\bigl(f(x) + \langle u, g(x)\rangle + \langle v, h(x)\rangle\bigr) &= \inf_{x \in S}\bigl(f(x) + \langle \bar{u}, g(x)\rangle + \langle \bar{v}, h(x)\rangle\bigr)\\ &= f(x_0) + \langle \bar{u}, g(x_0)\rangle + \langle \bar{v}, h(x_0)\rangle\\ &= f(x_0) = \min_{x \in S} f(x).\end{aligned}$$

So, the strong duality holds and in virtue of Corollary 1 also *Assumption S* is fulfilled. □

Corollary 2. *If $S = X$, then*

$$f'(x_0) + \bar{u}\bigl(g'(x_0)\bigr) + \bar{v}\bigl(h'(x_0)\bigr)(h) = 0, \quad \forall h \in X. \tag{10}$$

Furthermore, if f and g are Gateaux differentiable on X, then we also get from (10) *that*

$$f'(x_0) + \bar{u}\bigl(g'(x_0)\bigr) + \bar{v}\bigl(h'(x_0)\bigr) = 0.$$

3. Application to the bi-obstacle problem

Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain, either convex or with $C^{1,1}$ boundary. Let us consider the linear elliptic operator of second order

$$Lu = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial}{\partial x_j}\right) + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu \tag{11}$$

with associated bilinear form on $H_0^1(\Omega) \times H_0^1(\Omega)$ given by

$$a(u, v) = \int_{\Omega}\left(\sum_{i,j=1}^{n} a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} v + cuv\right) dx \tag{12}$$

where

$$\begin{cases} \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geqslant a|\xi|^2 \quad \text{a.e. on } \Omega,\ \forall \xi \in \mathbb{R}^n, \\ a > 0, \quad a_{ij} \in C^1(\overline{\Omega}),\ b_i \in L^\infty(\Omega), \\ c > 0 \quad \text{such large that } a(u,u) \geqslant \alpha \|u\|^2_{H_0^1(\Omega)},\ \alpha > 0,\ \forall u \in H_0^1(\Omega). \end{cases}$$

Let $\psi(x), \psi^*(x) \in H^1(\Omega)$, $\psi(x) \leqslant \psi^*(x)$ a.e. in Ω, $\psi(x) \leqslant 0 \leqslant \psi^*(x)$ a.e. on $\partial\Omega$ and consider the set $K = \{u \in L^2(\Omega)\colon\ \psi \leqslant u \leqslant \psi^* \text{ a.e. in } \Omega\}$.

Then the following result holds true (see [4, Corollaire I.1] and [24]).

Theorem 4. *Assume that $L\psi$ and $L\psi^*$ are measures with $(L\psi - f)^+$ and $(L\psi^* - f)^- \in L^p(\Omega)$, $p \geqslant 2$. Then, for every $f \in L^p(\Omega)$ there exists $u \in K \cap W^{2,p}(\Omega) \cap H_0^1(\Omega)$ unique solution to the variational inequality*

$$\int_\Omega Lu(v-u)\,dx \geqslant \int_\Omega f(v-u)\,dx, \quad \forall v \in K$$

such that

$$\|u\|_{W^{2,p}(\Omega)} \leqslant c\big(\|f\|_{L^p} + \big\|(L\psi - f)^+\big\|_{L^p} + \big\|\big(L\psi^* - f\big)^-\big\|_{L^p}\big).$$

Now, in this section, we would like to apply the infinite dimensional Lagrange multiplier rule of the previous section to the variational inequality

$$\int_\Omega (Lu - f)(v-u)\,dx \geqslant 0, \quad \forall v \in K \tag{13}$$

where

$$K = \big\{u \in L^2(\Omega)\colon\ \psi \leqslant u \leqslant \psi^* \text{ a.e. in } \Omega\big\}.$$

Firstly, let us show that *Assumption S* is verified. To this aim let us rewrite variational inequality (13) as an optimization problem. Setting

$$f(v) = \int_\Omega (Lu - f)(v-u)\,dx, \quad \forall v \in K$$

we get

$$f(v) \geqslant 0, \quad \forall v \in K$$

and u is a minimal point of the problem

$$\min_{v \in K} f(v) = f(u) = 0. \tag{14}$$

We can show the following lemma.

Lemma 1. *Let $u \in K$ be a solution to variational inequality* (13). *Let us set*

$$\begin{aligned} \Omega_+ &= \{x \in \Omega\colon\ u(x) = \psi(x)\}, \\ \Omega_0 &= \{x \in \Omega\colon\ \psi(x) < u(x) < \psi^*(x)\}, \\ \Omega_- &= \{x \in \Omega\colon\ u(x) = \psi^*(x)\}. \end{aligned}$$

Then one has

$$\begin{aligned} Lu - f &\geqslant 0 \quad \text{a.e. in } \Omega_+, \\ Lu - f &= 0 \quad \text{a.e. in } \Omega_0, \\ Lu - f &\leqslant 0 \quad \text{a.e. in } \Omega_-. \end{aligned}$$

Proof. Let us observe that we have

$$\begin{aligned} f(v) &= \int_\Omega (Lu - f)(v - u)\,dx \\ &= \int_{\Omega_+} (Lu - f)(v - \psi)\,dx + \int_{\Omega_0} (Lu - f)(v - u)\,dx \\ &\quad + \int_{\Omega_-} (Lu - f)\big(v - \psi^*\big)\,dx \geqslant 0, \quad \forall v \in K. \end{aligned}$$

Let us assume as test function:

$$v \begin{cases} = w & \text{in } \Omega_+\colon\ \psi < w < \psi^*,\ \forall w \in L^2(\Omega_+), \\ = u & \text{in } \Omega_0, \\ = \psi^* & \text{in } \Omega_-, \end{cases}$$

then

$$f(w) = \int_{\Omega_+} (Lu - f)(w - \psi)\,dx \geqslant 0, \quad \forall \psi < w < \psi^*. \tag{15}$$

Since $w - \psi > 0$ in Ω_+ then $Lu - f \geqslant 0$ a.e. in Ω_+. In fact, if, by contradiction, there exists a subset E of Ω_+ with $m(E) > 0$ such that

$$Lu - f < 0 \quad \text{in } E,$$

choosing

$$w \begin{cases} = \psi & \text{in } \Omega_+/E, \\ = s & \text{in } E\colon\ \psi < s < \psi^* \end{cases}$$

we get

$$f(s) = \int_E (Lu - f)(s - \psi)\,dx < 0$$

that contradicts (15). Hence

$$Lu - f \geqslant 0 \quad \text{a.e. in } \Omega_+.$$

In the same way we can show other cases. □

Now we prove the following lemma.

Lemma 2. *Problem* (14) *verifies Assumption S at the minimal point $u \in K$.*

Proof. In our case, we have

$$X = S = L^2(\Omega), \qquad Y = L^2(\Omega) \times L^2(\Omega),$$

the dual cone of the ordering cone C of Y is $C^* = \{(\alpha, \beta) \in L^2(\Omega) \times L^2(\Omega)\colon \alpha \geqslant 0,\ \beta \geqslant 0$ a.e. in $\Omega\}$ and $g(v) = (g_1(v), g_2(v)) = (\psi - v, v - \psi^*)$. Of course in our case $C = C^*$. Furthermore

$$\widetilde{M} = \big\{\big(f(v) + \alpha, \psi - v + y_1, v - \psi^* + y_2\big)\colon\ v \in L^2(\Omega) \setminus K,\ \alpha \geqslant 0,\ y = (y_1, y_2) \in C\big\}$$

and

$$\begin{aligned}
&T_{\widetilde{M}}(0, \theta_{L^2(\Omega)}, \theta_{L^2(\Omega)}) \\
&\quad = \Big\{y\colon\ y = \lim_{n\to+\infty} \lambda_n\big[\big(f(v_n) + \alpha_n, \psi - v_n + y_{1n}, v_n - \psi^* + y_{2n}\big) - (0, \theta_{L^2(\Omega)}, \theta_{L^2(\Omega)})\big], \\
&\qquad \text{with } \lambda_n > 0,\ \lim_{n\to+\infty}\big(f(v_n) + \alpha_n\big) = 0,\ \lim_{n\to+\infty} \lambda_n(\psi - v_n + y_{1n}) = \theta_{L^2(\Omega)}, \\
&\qquad \lim_{n\to+\infty} \lambda_n\big(v_n - \psi^* + y_{2n}\big) = \theta_{L^2(\Omega)},\ \lim_{n\to+\infty}(\psi - v_n + y_{1n}) = \theta_{L^2(\Omega)}, \\
&\qquad \lim_{n\to+\infty}\big(v_n - \psi^* + y_{2n}\big) = \theta_{L^2(\Omega)},\ v_n \in L^2(\Omega) \setminus K,\ \alpha_n \geqslant 0,\ y_n = (y_{1n}, y_{2n}) \in C\Big\}.
\end{aligned}$$

In order to achieve *Assumption S*, we must show that, if we have

$$\begin{aligned}
&(l, \theta_{L^2(\Omega)}, \theta_{L^2(\Omega)}) \\
&\quad = \Big(\lim_{n\to+\infty} \lambda_n\big(f(v_n) + \alpha_n\big),\ \lim_{n\to+\infty} \lambda_n(\psi - v_n + y_{1n}),\ \lim_{n\to+\infty} \lambda_n\big(v_n - \psi^* + y_{2n}\big)\Big)
\end{aligned}$$

belongs to $T_{\widetilde{M}}(0, \theta_{L^2(\Omega)}, \theta_{L^2(\Omega)})$, then l must be nonnegative.

It results

$$
\begin{aligned}
l &= \lim_{n\to+\infty} \lambda_n\big(f(v_n)+\alpha_n\big) = \lim_{n\to+\infty} \lambda_n\Big[\int_\Omega (Lu-f)(v_n-u)\,dx+\alpha_n\Big] \\
&= \lim_{n\to+\infty} \lambda_n\Big[\int_{\Omega_+} (Lu-f)(v_n-\psi)\,dx+\int_{\Omega_0} (Lu-f)(v_n-u)\,dx \\
&\quad + \int_{\Omega_-} (Lu-f)\big(v_n-\psi^*\big)\,dx+\alpha_n\Big] \\
&= \lim_{n\to+\infty} \lambda_n\Big[\int_{\Omega_+} (Lu-f)(v_n-\psi-y_{1n})\,dx+\int_{\Omega_+} (Lu-f)y_{1n}\,dx \\
&\quad + \int_{\Omega_-} (Lu-f)\big(v_n-\psi^*+y_{2n}\big)\,dx+\int_{\Omega_-} (Lu-f)(-y_{2n})\,dx+\alpha_n\Big].
\end{aligned}
$$

Taking into account that

$$
\lim_{n\to+\infty} \int_{\Omega_+} (Lu-f)\lambda_n(v_n-\psi-y_{1n})\,dx = 0,
$$

$$
\lim_{n\to+\infty} \int_{\Omega_-} (Lu-f)\lambda_n\big(v_n-\psi^*+y_{2n}\big)\,dx = 0
$$

and

$$
(Lu-f)\lambda_n y_{1n} \geqslant 0 \quad \text{in } \Omega_+, \qquad (Lu-f)\lambda_n(-y_{2n}) \geqslant 0 \quad \text{in } \Omega_-,
$$

and $\alpha_n \geqslant 0$, it follows that $l \geqslant 0$. Hence *Assumption S* holds. □

Since the other assumptions, required by Theorem 3 and Corollary 2, are fulfilled, then there exists $(\bar\lambda, \bar\mu) \in C^*$ such that:

(i)
$$
\int_\Omega \bar\lambda(\psi-u)\,dx = 0 \quad \Leftrightarrow \quad \bar\lambda(\psi-u)=0 \quad \text{a.e. in } \Omega
$$

and

$$
\int_\Omega \bar\mu\big(u-\psi^*\big)\,dx = 0 \quad \Leftrightarrow \quad \bar\mu\big(u-\psi^*\big)=0 \quad \text{a.e. in } \Omega,
$$

(ii)
$$
(Lu-f)-\bar\lambda+\bar\mu = 0 \quad \text{a.e. in } \Omega.
$$

In particular, we can obtain explicitly the values of Lagrangean multipliers $\bar{\lambda}$ and $\bar{\mu}$. In fact, when $\bar{\lambda} > 0$ one has $u = \psi$, $\bar{\mu} = 0$ and $\bar{\lambda} = L\psi - f$. Whereas, when $\bar{\mu} > 0$ one has $u = \psi^*$, $\bar{\lambda} = 0$ and $\bar{\mu} = -(L\psi^* - f)$.

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