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The complete solution of a diophantine equation involving biquadrates

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Abstract

While parametric solutions of the diophantine equation $\sum_{i=1}^{s} x_i^4 = \sum_{i=1}^{s} y_i^4$ are known for any integral value of $s \geqslant 2$, the complete solution in integers is not known for any value of s. In this paper, we obtain the complete solution of this equation when $s \geqslant 13$.

Keywords: Biquadrates; Quartic diophantine equation

Parametric solutions of the quartic diophantine equation

$$\sum_{i=1}^{s} x_i^4 = \sum_{i=1}^{s} y_i^4 \tag{1}$$

have been obtained when $s = 2$ [1, 3, p. 201, 4] as well as when $s > 2$ [2, p. 653–657]. The complete solution of (1) has, however, not been obtained for any value of s. If $\{X_i,\ i = 1, 2, \ldots, s\}$, $\{Y_i,\ i = 1, 2, \ldots, s\}$ is an integer solution of (1) such that

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the sets $\{X_i^4\}$ and $\{Y_i^4\}$ are not disjoint, we may, by cancellation of common terms in the two sets, reduce this solution of (1) to a solution of the equation

$$\sum_{i=1}^{r} x_i^4 = \sum_{i=1}^{r} y_i^4 \tag{2}$$

with $r < s$. A solution of (1) which cannot be so reduced will be considered as a non-trivial solution of (1). Further, any integer solution $\{X_i,\ i = 1, 2, \ldots, s\}$, $\{Y_i,\ i = 1, 2, \ldots, s\}$ of (1) is called a primitive solution if $\gcd(X_1, X_2, \ldots, X_s, Y_1, Y_2, \ldots, Y_s) = 1$. In this paper, we obtain a non-trivial primitive parametric solution of (1) for any arbitrary $s \geqslant 3$, and we will show that this solution gives the complete non-trivial primitive solution of (1) for any integer $s \geqslant 13$. We present this parametric solution in two theorems, the first theorem gives the solution in terms of parameters satisfying certain linear conditions while the second theorem gives the solution explicitly in terms of arbitrary parameters.

Theorem 1. *When $s \geqslant 13$, the complete non-trivial primitive integral solution of (1) is given by*

$$\begin{aligned} \rho x_i &= \varepsilon_{1i}\{(\lambda_i + \mu_i)u + \alpha_i v\},\ i = 1, 2, \ldots, s, \\ \rho y_i &= \varepsilon_{2i}\{(-\lambda_i + \mu_i)u + \alpha_i v\},\ i = 1, 2, \ldots, s, \end{aligned} \tag{3}$$

where

(i) $\alpha_i,\ i = 1, 2, \ldots, s$, *are arbitrary integers, not all zero;*

(ii) $\lambda_i,\ i = 1, 2, \ldots, s$, *are any integers satisfying the linear condition*

$$\sum_{i=1}^{s} \alpha_i^3 \lambda_i = 0; \tag{4}$$

(iii) $\mu_i,\ i = 1, 2, \ldots, s$, *are any integers satisfying the linear condition*

$$\sum_{i=1}^{s} \alpha_i^2 \lambda_i \mu_i = 0; \tag{5}$$

(iv) ε_{1i} *and* ε_{2i}, $i = 1, 2, \ldots, s$, *are either* $+1$ *or* -1;

(v) *if* $\sum_{i=1}^{s}(\lambda_i^2 + \mu_i^2)\lambda_i\mu_i \neq 0$, *then* $u = -\sum_{i=1}^{s}(\lambda_i^2 + 3\mu_i^2)\alpha_i\lambda_i$ *and* $v = \sum_{i=1}^{s}(\lambda_i^2 + \mu_i^2)\lambda_i\mu_i$; *if* $\sum_{i=1}^{s}(\lambda_i^2+\mu_i^2)\lambda_i\mu_i = 0$ *and* $\sum_{i=1}^{s}(\lambda_i^2+3\mu_i^2)\alpha_i\lambda_i \neq 0$ *then* $u = 1,\ v = 0$; *if* $\sum_{i=1}^{s}(\lambda_i^2+\mu_i^2)\lambda_i\mu_i = 0$ *and* $\sum_{i=1}^{s}(\lambda_i^2+3\mu_i^2)\alpha_i\lambda_i = 0$ *then* u *and* v *are arbitrary integers;*

(vi) *and* ρ *is an integer so chosen that* $\gcd(x_1, x_2, \ldots, x_s, y_1, y_2, \ldots, y_s) = 1$.

Proof. To solve (1), we substitute

$$\begin{aligned} x_i &= (\lambda_i + \mu_i)u + \alpha_i v, \; i = 1, 2, \ldots, s, \\ y_i &= (-\lambda_i + \mu_i)u + \alpha_i v, \; i = 1, 2, \ldots, s, \end{aligned} \tag{6}$$

in (1) which reduces, on simplification, to

$$\begin{aligned} &u^4 \textstyle\sum_{i=1}^{s} (\lambda_i^2 + \mu_i^2)\lambda_i \mu_i + u^3 v \sum_{i=1}^{s} (\lambda_i^2 + 3\mu_i^2)\alpha_i \lambda_i \\ &\quad + 3u^2 v^2 \textstyle\sum_{i=1}^{s} \alpha_i^2 \lambda_i \mu_i + u v^3 \sum_{i=1}^{s} \alpha_i^3 \lambda_i = 0. \end{aligned} \tag{7}$$

We now choose $\lambda_i, \; i = 1, 2, \ldots, s$, as any integers satisfying the linear condition (4), and thereafter, we choose $\mu_i, \; i = 1, 2, \ldots, s$, as any integers satisfying the linear condition (5) when (7) reduces to the equation

$$u^3 \left\{ u \sum_{i=1}^{s} (\lambda_i^2 + \mu_i^2)\lambda_i \mu_i + v \sum_{i=1}^{s} (\lambda_i^2 + 3\mu_i^2)\alpha_i \lambda_i \right\} = 0. \tag{8}$$

Now it is readily seen that with the values of u and v given under the conditions stated in the theorem, Eq. (8) is satisfied, and this leads to the primitive parametric solution (3) of (1). This solution is non-trivial when $s \geqslant 3$.

We will now show that (3) gives the complete non-trivial primitive solution of (1) when $s \geqslant 13$. For this, we will show that there exist suitable integer values of the parameters $\alpha_i, \lambda_i, \mu_i, \varepsilon_{1i}, \varepsilon_{2i}$, such that with these values of the parameters, the solution (3) generates any arbitrary non-trivial primitive integer solution of (1).

Let $x_i = \xi_i, \; i = 1, 2, \ldots, s, \; y_i = \eta_i, \; i = 1, 2, \ldots, s$ be an arbitrary non-trivial primitive solution of (1) so that

$$\sum_{i=1}^{s} \xi_i^4 = \sum_{i=1}^{s} \eta_i^4. \tag{9}$$

It follows from (9) that if we take $X_i = \varepsilon_{1i}\xi_i, \; Y_i = \varepsilon_{2i}\eta_i$, where, for each i, $\varepsilon_{1i}, \varepsilon_{2i}$ take the values $+1$ or -1, then

$$\sum_{i=1}^{s} X_i^4 = \sum_{i=1}^{s} Y_i^4. \tag{10}$$

At present we assume that with a suitable choice of values of ε_{1i} and ε_{2i}, there exist integers α_i', not all zero, such that

$$\sum_{i=1}^{s} (X_i^2 - Y_i^2)\alpha_i'^2 = 0, \tag{11}$$

$$\sum_{i=1}^{s}(X_i - Y_i)\alpha_i'^3 = 0 \tag{12}$$

and we choose

$$\begin{aligned} \lambda_i' &= m(X_i - Y_i),\ i = 1, 2, \ldots, s, \\ \mu_i' &= m(X_i + Y_i - \alpha_i'),\ i = 1, 2, \ldots, s, \end{aligned} \tag{13}$$

where m is an arbitrary non-zero integer. It is easily verified that the integers $\alpha_i',\ \lambda_i',\ \mu_i'$, $i = 1, 2, \ldots, s$ satisfy the conditions

$$\sum_{i=1}^{s}\alpha_i'^3\lambda_i' = 0, \tag{14}$$

$$\sum_{i=1}^{s}\alpha_i'^2\lambda_i'\mu_i' = 0. \tag{15}$$

From (13) we get,

$$\begin{aligned} \lambda_i' + \mu_i' + m\alpha_i' &= 2mX_i,\ i = 1, 2, \ldots, s, \\ -\lambda_i' + \mu_i' + m\alpha_i' &= 2mY_i,\ i = 1, 2, \ldots, s, \end{aligned} \tag{16}$$

so it follows from (10) that

$$\sum_{i=1}^{s}(\lambda_i' + \mu_i' + m\alpha_i')^4 - \sum_{i=1}^{s}(-\lambda_i' + \mu_i' + m\alpha_i')^4 = 0 \tag{17}$$

and, in view of the conditions (14) and (15), the relation (17) reduces to

$$\sum_{i=1}^{s}(\lambda_i'^2 + \mu_i'^2)\lambda_i'\mu_i' = -m\sum_{i=1}^{s}(\lambda_i'^2 + 3\mu_i'^2)\alpha_i'\lambda_i' = mk. \tag{18}$$

As we have chosen $\alpha_i',\ \lambda_i',\ \mu_i',\ i = 1, 2, \ldots, s$ so as to satisfy all the conditions stipulated in the theorem, when we assign to the parameters $\alpha_i,\ \lambda_i,\ \mu_i$, the values $\alpha_i',\ \lambda_i',\ \mu_i',\ i = 1, 2, \ldots, s$ respectively, (3) generates a solution of (1). There are now two possibilities:

(i) if $k \neq 0$, according to the solution in the theorem, we have to take $u = k,\ v = mk$, so that using the relations (16), we get from (3),

$$\begin{aligned} \rho x_i &= 2\varepsilon_{1i}mkX_i,\ i = 1, 2, \ldots, s, \\ \rho y_i &= 2\varepsilon_{2i}mkY_i,\ i = 1, 2, \ldots, s \end{aligned} \tag{19}$$

and taking $\rho = 2mk$, we finally get

$$\begin{aligned} x_i &= \xi_i, \ i = 1, 2, \ldots, s, \\ y_i &= \eta_i, \ i = 1, 2, \ldots, s. \end{aligned} \tag{20}$$

(ii) if $k = 0$, then u and v are arbitrary integers, hence we may take $u = 1, \ v = m$, and again using the relations (16), we get from (3),

$$\begin{aligned} \rho x_i &= 2\varepsilon_{1i} m X_i, \ i = 1, 2, \ldots, s, \\ \rho y_i &= 2\varepsilon_{2i} m Y_i, \ i = 1, 2, \ldots, s, \end{aligned} \tag{21}$$

so that on taking $\rho = 2m$, we finally get (20). Thus in both cases the assigned values of the parameters generate any arbitrarily chosen non-trivial primitive solution of (1).

It only remains to prove the existence of the integers α'_i, not all zero, satisfying the relations (11) and (12). We consider (11) and (12) as simultaneous equations in the variables α'_i and note that since the sets $\{X_i^4\}$ and $\{Y_i^4\}$ are disjoint, the integers $(X_i^2 - Y_i^2), \ (X_i - Y_i), \ i = 1, 2, \ldots, s$, are all non-zero. Moreover, in view of (10), all of the integers $(X_i^2 - Y_i^2)$ cannot be of the same sign. It therefore follows from a theorem of Wooley [5, p. 319] that the Eqs. (11) and (12) will have a non-trivial solution in integers if $s \geqslant 13$ and these equations have a non-trivial real solution. We accordingly take $s \geqslant 13$ and we will prove the existence of a non-trivial real solution of the simultaneous equations (11) and (12).

Consider the integers $(X_i^2 - Y_i^2), \ i = 1, 2, \ldots, s$. We have already noted that all of them cannot be of the same sign, and in fact, there is no loss of generality in taking at least two of the integers $(X_i^2 - Y_i^2)$ as negative (if this is not so, we may simply interchange the sets $\{X_i\}$ and $\{Y_i\}$). Thus, by suitably renaming $X_i, \ i = 1, 2, \ldots, s$, and $Y_i, \ i = 1, 2, \ldots, s$, if necessary, we can always take

$$X_1^2 - Y_1^2 > 0, \ X_2^2 - Y_2^2 < 0, \ \text{ and } \ X_3^2 - Y_3^2 < 0.$$

There are now two possibilities:

(i) at least one of the integers $(X_i^2 - Y_i^2), \ i = 4, 5, \ldots, s$, is positive;
(ii) all of the integers $(X_i^2 - Y_i^2), \ i = 4, 5, \ldots, s$, are negative.

When (i) holds, we take $\alpha'_1 = (-X_2^2 + Y_2^2)^{1/2} t, \ \alpha'_2 = (X_1^2 - Y_1^2)^{1/2} t$, and we easily choose non-zero real values of $\alpha'_i, \ i = 3, 4, \ldots, s$, such that $\sum_{i=3}^{s} (X_i^2 - Y_i^2) \alpha_i'^2 = 0$, so that (11) holds for arbitrary t, and substituting in (12) the values of $\alpha'_i, \ i = 1, 2, \ldots, s$, already chosen, we get an equation in t. With a suitable choice of values of ε_{1i} and ε_{2i}, this equation can be readily solved for t to get a real solution. Thus we can obtain real values of α'_i satisfying (11) and (12), and it follows from the aforementioned theorem of Wooley that there exist integral values of α'_i, not all zero, such that (11) and (12) are satisfied.

When (ii) holds, we take $\alpha'_1 = -(X_2 - Y_2)^{1/3} t, \ \alpha'_2 = (X_1 - Y_1)^{1/3} t$, and we easily choose non-zero real values of $\alpha'_i, \ i = 3, 4, \ldots, s$, such that $\sum_{i=3}^{s} (X_i - Y_i) \alpha_i^3 = 0$ so that (12) holds for arbitrary t, and substituting in (11) the values of α_i', $i=1, 2, \ldots, s$, already chosen, we get the following equation in t:

$$\begin{aligned}&\left\{(X_1^2-Y_1^2)(X_2-Y_2)^{2/3}+(X_2^2-Y_2^2)(X_1-Y_1)^{2/3}\right\}t^2\\&\quad=-\textstyle\sum_{i=3}^{s}(X_i^2-Y_i^2)\alpha_i'^2,\end{aligned} \tag{22}$$

where the right-hand side is positive since $(X_i^2-Y_i^2)<0$ for $i=3, 4, \ldots, s$. Now

$$\begin{aligned}&(X_1^2-Y_1^2)^3(X_2-Y_2)^2+(X_2^2-Y_2^2)^3(X_1-Y_1)^2\\&\quad=(X_1-Y_1)^2(X_2-Y_2)^2\phi(X_1, X_2, Y_1, Y_2),\end{aligned} \tag{23}$$

where

$$\begin{aligned}\phi(X_1, X_2, Y_1, Y_2) = {}& X_1^4-Y_1^4+X_2^4-Y_2^4+2X_1Y_1(X_1^2-Y_1^2)\\&+2X_2Y_2(X_2^2-Y_2^2).\end{aligned} \tag{24}$$

Now $X_1^4-Y_1^4+X_2^4-Y_2^4=-\sum_{i=3}^{s}(X_i^4-Y_i^4)>0$, and so with a suitable choice of values of ε_{1i} and ε_{2i}, we can easily ensure that $\phi(X_1, X_2, Y_1, Y_2)>0$. It now follows that the left-hand side of (23) is positive. We note that if a and b are two real numbers, $a^3+b^3>0$ implies that $a+b>0$. The coefficient of t^2 in the left-hand side of (22) is the sum of two real numbers, and since the sum of the cubes of these real numbers has been shown to be positive, the coefficient of t^2 is also positive, and hence (22) can be solved for t to get a real solution. Thus we can obtain real values of α_i' satisfying (11) and (12) and, as before, it follows from the theorem of Wooley that there exist integral values of α_i', not all zero, such that (11) and (12) are satisfied. This completes the proof of Theorem 1. □

The solution of (1) given by Theorem 1 involves the parameters λ_i, $i=1, 2, \ldots, s$ and μ_i, $i=1, 2, \ldots, s$ which must satisfy the conditions (4) and (5). We can obtain the complete solution of (1) explicitly in terms of independent parameters by suitably expressing λ_i, μ_i in terms of arbitrary parameters such that the conditions (4) and (5) are satisfied. This complete parametric solution of (1) is given in the next theorem.

Theorem 2. *When $s\geqslant 13$, the complete non-trivial primitive integral solution of* (1) *is given by*

$$\begin{aligned}\rho x_i &= \varepsilon_{1i}\{(\lambda_i+\mu_i)u+\alpha_i v\},\ i=1, 2, \ldots, s,\\ \rho y_i &= \varepsilon_{2i}\{(-\lambda_i+\mu_i)u+\alpha_i v\},\ i=1, 2, \ldots, s,\end{aligned} \tag{25}$$

where

(i) α_i, $i=1, 2, \ldots, s$, *are arbitrary integers such that* $\alpha_s\neq 0$;

(ii) $\lambda_i,\ \mu_i$ *are defined as follows*:

$$
\begin{aligned}
\lambda_i &= \alpha_s^3\beta_i\delta, \quad i=1,\,2,\,\ldots,\,s-1,\\
\lambda_s &= -\delta\textstyle\sum_{i=1}^{s-1}\alpha_i^3\beta_i,\\
\mu_i &= \gamma_i\textstyle\sum_{i=1}^{s-1}\alpha_i^3\beta_i, \quad i=1,\,2,\,\ldots,\,s-1,\\
\mu_s &= \alpha_s\textstyle\sum_{i=1}^{s-1}\alpha_i^2\beta_i\gamma_i,
\end{aligned}
\tag{26}
$$

where $\beta_i,\ i=1,\,2,\,\ldots,\,s-1,$ *and* δ *are arbitrary non-zero integers, and* $\gamma_i,\ i=1,\,2,\,\ldots,\,s-1,$ *are arbitrary integers*;

(iii) ε_{1i} *and* $\varepsilon_{2i},\ i=1,\,2,\,\ldots,\,s,$ *are either* $+1$ *or* -1;

(iv) *if* $\sum_{i=1}^{s}(\lambda_i^2+\mu_i^2)\lambda_i\mu_i \neq 0,$ *then* $u=-\sum_{i=1}^{s}(\lambda_i^2+3\mu_i^2)\alpha_i\lambda_i$ *and* $v=\sum_{i=1}^{s}(\lambda_i^2+\mu_i^2)\lambda_i\mu_i$; *if* $\sum_{i=1}^{s}(\lambda_i^2+\mu_i^2)\lambda_i\mu_i=0$ *and* $\sum_{i=1}^{s}(\lambda_i^2+3\mu_i^2)\alpha_i\lambda_i\neq 0$ *then* $u=1,\ v=0$; *if* $\sum_{i=1}^{s}(\lambda_i^2+\mu_i^2)\lambda_i\mu_i=0$ *and* $\sum_{i=1}^{s}(\lambda_i^2+3\mu_i^2)\alpha_i\lambda_i=0$ *then* u *and* v *are arbitrary integers*;

(v) *and* ρ *is an integer so chosen that* $\gcd(x_1,\,x_2,\,\ldots,\,x_s,\,y_1,\,y_2,\,\ldots,\,y_s)=1.$

Proof. It is easily verified that $\alpha_i,\ \lambda_i,\ \mu_i$ as defined in this theorem satisfy the conditions stipulated in Theorem 1, and hence the solution stated in Theorem 2 follows immediately from the solution obtained in Theorem 1.

We will make use of the proof of Theorem 1 to establish that (25) gives the complete non-trivial primitive solution of (1). Let $x_i=\xi_i,\ i=1,\,2,\,\ldots,\,s,\ y_i=\eta_i,\ i=1,\,2,\,\ldots,\,s$ be an arbitrary non-trivial primitive solution of (1) so that (9) holds and we define $X_i,\,Y_i$ as before. We choose integers α_i', not all zero, such that (11) and (12) are satisfied. There is no loss of generality in assuming that $\alpha_s'\neq 0$. We now choose

$$
\begin{aligned}
\beta_i' &= X_i-Y_i, \quad i=1,\,2,\,\ldots,\,s-1,\\
\gamma_i' &= X_i+Y_i-\alpha_i', \quad i=1,\,2,\,\ldots,\,s-1,\\
\delta' &= -(X_s-Y_s),
\end{aligned}
\tag{27}
$$

and we denote by $\lambda_i',\ \mu_i',\ i=1,\,2,\,\ldots,\,s$ the values of λ_i and μ_i given by (26) when the parameters $\alpha_i,\ \beta_i,\ \gamma_i,\ \delta$ are assigned the values $\alpha_i',\ \beta_i',\ \gamma_i',\ \delta'$, respectively. Noting that $\sum_{i=1}^{s-1}\alpha_i'^3(X_i-Y_i)=-\alpha_s'^3(X_s-Y_s),$ we get

$$
\lambda_i'=-\alpha_s'^3(X_s-Y_s)(X_i-Y_i), \quad i=1,\,2,\,\ldots,\,s. \tag{28}
$$

Similarly,

$$
\mu_i'=-\alpha_s'^3(X_s-Y_s)(X_i+Y_i-\alpha_i'), \quad i=1,\,2,\,\ldots,\,s-1 \tag{29}
$$

and

$$\begin{aligned}\mu_s &= \alpha'_s \textstyle\sum_{i=1}^{s-1} \alpha'^2_i \beta'_i \gamma'_i \\ &= \alpha'_s \textstyle\sum_{i=1}^{s-1} \alpha'^2_i (X_i - Y_i)(X_i + Y_i - \alpha'_i)\end{aligned} \tag{30}$$

and, in view of the relations (11) and (12), we get

$$\mu_s = -\alpha'^3_s (X_s - Y_s)(X_s + Y_s - \alpha'_s). \tag{31}$$

From (28), (29) and (31), we get

$$\begin{aligned}\lambda'_i &= m(X_i - Y_i),\ i = 1, 2, \ldots, s, \\ \mu'_i &= m(X_i + Y_i - \alpha'_i),\ i = 1, 2, \ldots, s,\end{aligned} \tag{32}$$

where $m = -\alpha'^3_s(X_s - Y_s) \neq 0$. These values of λ'_i and μ'_i are exactly the same as chosen in (13) with a non-zero integral value of m and it follows, as in the proof of Theorem 1, that our choice of parameters generates the arbitrarily chosen non-trivial primitive solution of (1). This shows that the solution (25) indeed generates the complete solution of Eq. (1).

Finally we note that since (1) is a homogeneous equation, any rational solution of (1) leads, on clearing denominators, to a solution in integers. It follows that, in either of the solutions given by Theorem 1 or Theorem 2, if we take ρ to be an arbitrary rational parameter, we get the complete solution of Eq. (1) in rational numbers when $s \geqslant 13$. □

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