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Construction of Coreflectors

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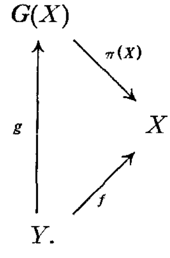
INTRODUCTION

Morita considered the construction of a reflector for the category of modules [6]. The purpose of this paper is to dualize some results of [6] and to construct the coreflectors for the category of modules.

The notions of V-dominant dimensions, for an injective module V, are introduced by Tachikawa [8] and are enlarged by Morita [5] for an arbitrary module V (cf. [3]). According to Morita [5], we say that a module X has V-dominant dimension $\geqq n$, denoted V-dom. dim $X \geqq n$, if there exists an exact sequence $0 \to X \to X_1 \to \cdots \to X_n$, where each X_i is isomorphic to a direct product of copies of V.

On the other hand, following Onodera [7], for modules U and Y, we say that Y has U-codominant dimension $\geqq n$, denoted U-cod. dim $Y \geqq n$, if there exists an exact sequence $Y_n \to \cdots \to Y_1 \to Y \to 0$, where each Y_i is isomorphic to a direct sum of copies of U.

Let A be a ring and $\mathscr{M}_A$ be the category of right A-modules. A full subcategory $\mathscr{N}$ of $\mathscr{M}_A$ is called a coreflective subcategory if there exists a functor $G\colon \mathscr{N} \to \mathscr{M}_A$ such that for every $X \in \mathscr{M}_A$ there exists a homomorphism $\pi(X)\colon G(X) \to X$, and for every homomorphism $f\colon Y \to X$ with $Y \in \mathscr{N}$ there exists a unique homomorphism $g\colon Y \to G(X)$ such that the following diagram is commutative



Then we call G a coreflector.

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A reflective subcategory and a reflector are the dual of the above.

Let ${}_A\mathscr{M}$ be the category of left A-modules and $V \in {}_A\mathscr{M}$. Let $\mathscr{D}(V)$ be the full subcategory of ${}_A\mathscr{M}$ consisting of all left A-modules of V-dominant dimension $\geqq 2$. Then, under some conditions, Morita [6] showed that $\mathscr{D}(V)$ is a reflective subcategory with a suitable reflector. The class of modules, studied in [6], which give reflectors for $\mathscr{D}(V)$ includes, for example, injective modules and modules of type FI (see [5], for definition of a module of type FI).

Take $U \in \mathscr{M}_A$. Let us denote by $\mathscr{C}(U)$ the full subcategory of $\mathscr{M}_A$ consisting of all right A-modules of U-codominant dimension $\geqq 2$. Onodera [7] investigated properties of $\mathscr{C}(U)$ when U is projective and also studied the equivalences over $\mathscr{C}(U)$.

We set the assumption for U which is satisfied whenever U is projective or type FP (see [5], for definition of a module of type FP). Then, in Section 2 we show that $\mathscr{C}(U)$ is a coreflective subcategory with $\tilde{F}$ its coreflector. Furthermore, we also construct coreflectors for modules of different type from the previous one in Section 5.

In Section 3 we investigate the conditions for $\tilde{F} = F$, where the definitions of $\tilde{F}$ and F are seen in Section 1.

In [5] Morita constructed a module of type FI from a module of type FP by a suitable method. In Section 4 we construct a module V given in [6] from a module U given in Section 2 by the same way as in [5].

Throughout this paper all rings have an identity element and all modules are unital.

1. Preliminaries

Let A be a ring and U be a right A-module. Put $B = \mathrm{End}(U_A)$ and $C = \mathrm{End}({}_BU)$. For any right A-module X we put;

$F(X) = \mathrm{Hom}_A(U, X) \otimes_B U_A$,

$\lambda(X): F(X) \to X[f \otimes u \mapsto fu (f \in \mathrm{Hom}(U, X), u \in U)]$,

$\bar{F}(X) = \sum \{\mathrm{Im}\, \phi;\ \phi \in \mathrm{Hom}(U, F(X)) \text{ such that } \lambda(X)\phi = 0\}$,

$\tilde{F}(X) = F(X)/\bar{F}(X)$,

$\eta(X): F(X) \to \tilde{F}(X)$ a canonical projection,

$\tilde{\lambda}(X): \tilde{F}(X) \to X$ such that $\lambda(X) = \tilde{\lambda}(X)\eta(X)$ (note that $\bar{F}(X) \subset \mathrm{Ker}\, \lambda(X)$).

Let X, Y be right A-modules and $f: X \to Y$ be an A-homomorphism. Put $F(f) = \mathrm{Hom}(I_U, f) \otimes I_U: F(X) \to F(Y)$. Then the following diagram is commutative

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle \lambda(X)} \downarrow & & \downarrow {\scriptstyle \lambda(Y)} \\ X & \xrightarrow{f} & Y. \end{array}$$

Take any $x \in \bar{F}(X)$. Then we have $x = \phi u$ for some $u \in U$ and $\phi \in \mathrm{Hom}(U, F(X))$ such that $\lambda(X)\phi = 0$. Put $\psi = F(f)\phi \in \mathrm{Hom}(U, F(X))$. Then we have $\lambda(Y)\psi = \lambda(Y)F(f)\phi = f\lambda(X)\phi = 0$ and $F(f)x = F(f)\phi u = \psi u$. Thus $F(f)x \in \bar{F}(Y)$. Therefore, there exists an A-homomorphism $\tilde{F}(f): \tilde{F}(X) \to \tilde{F}(Y)$ such that the diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle \eta(X)} \downarrow & & \downarrow {\scriptstyle \eta(Y)} \\ \tilde{F}(X) & \xrightarrow{\tilde{F}(f)} & \tilde{F}(Y) \end{array}$$

is commutative. Hence we have:

LEMMA 1.1. *$\tilde{F}$ is a covariant additive functor and $\tilde{\lambda}(X)$ is a natural homomorphism.*

Proof. The first statement is almost clear. We have $f\tilde{\lambda}(X)\eta(X) = f\lambda(X) = \lambda(Y)F(f) = \tilde{\lambda}(Y)\eta(Y)F(f) = \tilde{\lambda}(Y)\tilde{F}(f)\eta(X)$. Since $\eta(X)$ is an epimorphism, we have $f\tilde{\lambda}(X) = \tilde{\lambda}(Y)\tilde{F}(f)$. Thus $\tilde{\lambda}(X)$ is a natural homomorphism.

PROPOSITION 1.2. *If $Y \in \mathcal{M}_B$, then $Y \otimes_B U_A \in \mathscr{C}(U_A)$.*

Proof. Let $\oplus B \to \oplus B \to Y \to 0$ be a free resolution of Y. Then we have an exact sequence $\oplus U \to \oplus U \to Y \otimes_B U \to 0$. Thus $Y \otimes_B U \in \mathscr{C}(U_A)$.

COROLLARY 1.3. *$F(X) \in \mathscr{C}(U_A)$ for any $X \in \mathcal{M}_A$.*

PROPOSITION 1.4. $U_A -$ cod. $\dim \bar{F}(X) \geqq 1$.

Proof. It is almost clear.

PROPOSITION 1.5. *$\lambda(U^n)$: $U^n \cong F(U^n)$, $\tilde{F}(U^n) \cong F(U^n)$. If $X \cong \oplus U$, then $\lambda(X)$ is an epimorphism.*

Proof. These follow from definitions.

Let $f \in \mathrm{Hom}(U^n, X)$. Put $\sigma^n(X)f = F(f)\lambda(U^n)^{-1}$: $U^n \to F(X)$, $\tilde{\sigma}^n(X)f = \tilde{F}(f)\eta(U^n)\lambda(U^n)^{-1}$: $U^n \to \tilde{F}(X)$. Then we have;

$$f = \lambda(X)\sigma^n(X)f = \tilde{\lambda}(X)\tilde{\sigma}^n(X)f,$$
$$\tilde{\sigma}^n(X)f = \eta(X)\sigma^n(X)f.$$

Hence we have:

PROPOSITION 1.6.

$$\mathrm{Hom}(I_{U^n}, \lambda(X))\, \sigma^n(X) = I_{\mathrm{Hom}(U^n, X)}$$

$$\mathrm{Hom}(I_{U^n}, \tilde{\lambda}(X))\, \tilde{\sigma}^n(X) = I_{\mathrm{Hom}(U^n, X)}\,.$$

Proof. Take $f \in \mathrm{Hom}(U^n, X)$. Then we have $\mathrm{Hom}(I, \lambda(X))\, \sigma^n(X) f = \lambda(X)\, \sigma^n(X) f = \lambda(X) F(f)\, \lambda(U^n)^{-1} = f\lambda(U^n)\, \lambda(U^n)^{-1} = f$.
The latter is obtained similarly.

PROPOSITION 1.7. *Let* $f \in \mathrm{Hom}(U^n, X)$, $h \in \mathrm{Hom}(U, U^n)$. *Then we have* $\sigma^1(X)\,(fh) = (\sigma^n(X) f)\, h$.

Proof. We have $\sigma^1(X)\,(fh) = F(fh)\, \lambda(U)^{-1} = F(f)\, F(h)\, \lambda(U)^{-1} = F(f)\, \lambda(U^n)^{-1}\, \lambda(U^n) F(h)\, \lambda(U)^{-1} = (\sigma^n(X) f)\, h$.

2. COREFLECTOR $\tilde{F}$

Let $U \in \mathscr{M}_A$. Put $B = \mathrm{End}(U_A)$, $C = \mathrm{End}({}_BU)$, $\phi\colon A \to C$ a canonical ring homomorphism. Now we assume the following condition.

Condition (*). There exists a subring R of C such that

(a) $\phi(A) \subset R \subset C$

(b) ${}_BU_R \cong {}_BU \otimes_A R_R$

(c) U_R is projective.

Let U_A be type FP or projective. If we put $R = C$ or A, then U_A satisfies (*).

LEMMA 2.1 (cf. [5, Lemma 3.1]). *Let S, T be rings and ${}_SG_T$ be an S–T-bimodule. Assume that G_T is finitely generated quasi-projective. Put*

$P\colon \mathscr{M}_T \to \mathscr{M}_S$, $P(X) = \mathrm{Hom}_T(G, X)$, $X \in \mathscr{M}_T$,

$Q\colon \mathscr{M}_S \to \mathscr{M}_T$, $Q(Y) = Y \otimes_S G$, $Y \in \mathscr{M}_S$, *and*

$\Gamma(X)\colon QP(X) \to X$, $X \in \mathscr{M}_T$,

$\Delta(Y)\colon Y \to PQ(Y)$, $Y \in \mathscr{M}_S$, *canonically.*

Let U be a right T-module such that $\Gamma(U)$ is an isomorphism. Put $M = P(U)$. Then we have

$P\colon \mathscr{C}(U_T) \to \mathscr{C}(M_S)$, $Q\colon \mathscr{C}(M_S) \to \mathscr{C}(U_T)$, *and*

$QP = I$ *on* $\mathscr{C}(U_T)$, $PQ = I$ *on* $\mathscr{C}(M_S)$.

Proof. Since P and Q commute with direct sums and are right exact, the first of the lemma is easily obtained.

If $X \in \mathscr{C}(U_T)$, then there exists an exact sequence $Z_2 \to Z_1 \to X \to 0$ with each $Z_i \cong \oplus U$. Thus we have the following commutative diagram

$$\begin{array}{ccccccc} Z_2 & \longrightarrow & Z_1 & \longrightarrow & X & \longrightarrow & 0 \\ {\scriptstyle \Gamma(Z_2)}\uparrow & & {\scriptstyle \Gamma(Z_1)}\uparrow & & {\scriptstyle \Gamma(X)}\uparrow & & \\ QP(Z_2) & \longrightarrow & QP(Z_1) & \longrightarrow & QP(X) & \longrightarrow & 0 \end{array}$$

with exact rows. Since each $\Gamma(Z_i)$ is an isomorphism, $\Gamma(X)$ is an isomorphism.

If $Y \in \mathscr{C}(M_S)$, then there exists an exact sequence $Y_2 \to Y_1 \to Y \to 0$ with each $Y_i \cong \oplus M. \cong \oplus P(U) \cong P(\oplus U) = P(X_i)$ with $X_i \cong \oplus U$, we have the following commutative diagram

$$\begin{array}{ccccccc} P(X_2) & \longrightarrow & P(X_1) & \longrightarrow & Y & \longrightarrow & 0 \\ {\scriptstyle \Delta(P(X_2))}\downarrow & & {\scriptstyle \Delta(P(X_1))}\downarrow & & {\scriptstyle \Delta(Y)}\downarrow & & \\ PQP(X_2) & \longrightarrow & PQP(X_1) & \longrightarrow & PQ(Y) & \longrightarrow & 0 \\ {\scriptstyle P(\Gamma(X_2))}\downarrow & & {\scriptstyle P(\Gamma(X_1))}\downarrow & & & & \\ P(X_2) & \longrightarrow & P(X_1) & & & & \end{array}$$

with exact rows. Since each $P(\Gamma(X_i))\,\Delta(P(X_i))$ and $P(\Gamma(X_i))$ are isomorphisms, each $\Delta(P(X_i))$ is also an isomorphism. Hence $\Delta(Y)$ is an isomorphism. This completes the proof.

Now, if we put $S = A$, $T = R$, ${}_SG_T = {}_AR_R$ in Lemma 2.1, then we have:

LEMMA 2.2. *If $X, X' \in \mathscr{C}(U_A)$, $f \in \mathrm{Hom}_A(X, X')$, then we have $X, X' \in \mathscr{C}(U_R)$ and f is an R-homomorphism.*

PROPOSITION 2.3. U_R — cod. dim $\bar{F}(X) \geqq 1$ *for any* $X \in \mathscr{M}_A$.

Proof. $F(X) \in \mathscr{C}(U_A)$ by Corollary 1.3. Since $U \in \mathscr{C}(U_A)$, each $g \in \mathrm{Hom}_A(U, F(X))$ is an R-homomorphism by Lemma 2.2. Thus $\bar{F}(X)$ is an R-module. Hence U_R — cod. dim $\bar{F}(X) \geqslant 1$ by Proposition 1.4.

LEMMA 2.4. $\tilde{F}(X) \in \mathscr{C}(U_R)$ *for any* $X \in \mathscr{M}_A$.

Proof. Consider the following diagram in $\mathscr{M}_R$

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \uparrow & & & & \\ 0 & \longrightarrow & \bar{F}(X) & \longrightarrow & F(X) & \xrightarrow{\eta(X)} & \tilde{F}(X) & \longrightarrow & 0, \\ & & & & {\scriptstyle g}\uparrow & & & & \\ & & & & Z & & & & \end{array}$$

where row and column are exact and $Z \cong \bigoplus U$. Let $K = \operatorname{Ker} \eta(X) g = g^{-1}(\bar{F}(X))$. We shall prove that U generates K. Take $x \in K$. Then $gx \in \bar{F}(X)$. By Proposition 2.3 there exist $h \in \operatorname{Hom}_R(U, F(X))$ and $u \in U$ such that $hu = gx$. Since U_R is projective, there exists $h' \in \operatorname{Hom}_R(U, Z)$ such that $h = gh'$. Thus $\operatorname{Im} h' \subset K$ and $x - h'u \in \operatorname{Ker} g$. There exist $h'' \in \operatorname{Hom}_R(U, \operatorname{Ker} g)$ and $u' \in U$ such that $h''u' = x - h'u$. Therefore, U generates K. Hence $\tilde{F}(X) \in \mathscr{C}(U_R)$.

In the following, we write $\sigma(X)$, $\tilde{\sigma}(X)$ instead of $\sigma^1(X)$, $\tilde{\sigma}^1(X)$, respectively.

LEMMA 2.5. *$\tilde{\sigma}(X)$: $\operatorname{Hom}_A(U, X) \cong \operatorname{Hom}_A(U, \tilde{F}(X))$, $\operatorname{Hom}(I_U, \tilde{\lambda}(X))$: $\operatorname{Hom}_A(U, \tilde{F}(X)) \cong \operatorname{Hom}_A(U, X)$ for any $X \in \mathscr{M}_A$.*

Proof. By Proposition 1.6 we need to show that $\tilde{\sigma}(X)$ is an epimorphism. Let $g \in \operatorname{Hom}(U, \tilde{F}(X))$. Then g is an R-homomorphism by Lemma 2.2. Consider the following diagram

$$\begin{array}{ccccc} & & U & & \\ & & \downarrow{\scriptstyle g} & & \\ F(X) & \xrightarrow{\eta(X)} & \tilde{F}(X) & \longrightarrow & 0 \quad \text{(exact)} \\ & \searrow{\scriptstyle \lambda(X)} & \downarrow{\scriptstyle \tilde{\lambda}(X)} & & \\ & & X. & & \end{array}$$

Since U_R is projective, there exists $h \in \operatorname{Hom}(U, F(X))$ such that $g = \eta(X) h$. Put $f = \tilde{\lambda}(X) g$. We have $f = \lambda(X) h$. On the other hand, $\lambda(X)(h - \sigma(X) f) = 0$ by $f = \lambda(X)(\sigma(X) f)$. Thus $\eta(X)(h - \sigma(X) f) = 0$ by definition of $\eta(X)$. Therefore, we have $g = \eta(X) h = \eta(X)(\sigma(X) f) = \tilde{\sigma}(X) f$. Hence $\tilde{\sigma}(X)$ is an epimorphism.

LEMMA 2.6. *If $X \in \mathscr{M}_A$, $Y \in \mathscr{C}(U_A)$, then there exists a natural isomorphism* $\operatorname{Hom}(I_Y, \tilde{\lambda}(X))$: $\operatorname{Hom}_A(Y, \tilde{F}(X)) \cong \operatorname{Hom}_A(Y, X)$.

Proof. Since $Y \in \mathscr{C}(U_A)$, there exists an exact sequence $Y_2 \to Y_1 \to Y \to 0$, where each Y_i is a direct sum of copies of U. Then we have a commutative diagram with exact rows;

$$\begin{array}{ccccccc} 0 & \longrightarrow & \operatorname{Hom}(Y, \tilde{F}(X)) & \longrightarrow & \operatorname{Hom}(Y_1, \tilde{F}(X)) & \longrightarrow & \operatorname{Hom}(Y_2, \tilde{F}(X)) \\ & & \downarrow{\scriptstyle \operatorname{Hom}(I, \tilde{\lambda}(X))} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} \\ 0 & \longrightarrow & \operatorname{Hom}(Y, X) & \longrightarrow & \operatorname{Hom}(Y_1, X) & \longrightarrow & \operatorname{Hom}(Y_2, X). \end{array}$$

By Lemma 2.5 α_1, α_2 are isomorphisms. Thus $\operatorname{Hom}(I, \tilde{\lambda}(X))$ is an isomorphism.

THEOREM 2.7. *Let U be a right A-module which satisfies (*). Then $\mathscr{C}(U_A)$ is a coreflective subcategory of $\mathscr{M}_A$ and $\tilde{F}$ is its coreflector.*

Proof. Theorem follows from Lemma 2.6.

COROLLARY 2.8. *$\tilde{\lambda}(X): \tilde{F}(X) \to X$ is an isomorphism if and only if $X \in \mathscr{C}(U_A)$.*

Proof. If $\tilde{\lambda}(X)$ is an isomorphism, then $X \in \mathscr{C}(U_A)$ by Lemma 2.4.

Conversely, if $X \in \mathscr{C}(U_A)$, then there exists $\psi \in \mathrm{Hom}(X, \tilde{F}(X))$ such that $\tilde{\lambda}(X)\psi = I_X$ by Lemma 2.6. We have $\tilde{\lambda}(X) = I_X \tilde{\lambda}(X) = \tilde{\lambda}(X)(\psi\tilde{\lambda}(X))$. Thus $I_{\tilde{F}(X)} = \psi\tilde{\lambda}(X)$ by Lemma 2.6. Hence $\tilde{\lambda}(X)$ is an isomorphism.

3. CONDITIONS FOR $F = \tilde{F}$

For a right A-module U which satisfies (*), we easily get the following by Corollary 2.8.

PROPOSITION 3.1. *Let $X \in \mathscr{M}_A$. Then we have $\lambda(X): X \simeq F(X) \Leftrightarrow X \in \mathscr{C}(U_A)$ and $\tilde{F}(X) = F(X)$.*

Now, we consider some conditions for $\tilde{F}(X) = F(X)$. Let $U, F, \tilde{F}$, etc. be the same as in Section 1. U does not satisfy (*) unless specifically stated.

THEOREM 3.2. *Consider the following conditions for $X \in \mathscr{M}_A$.*

(a) $\tilde{F}(X) = F(X)$.

(b) $\mathrm{Hom}(I, \lambda(X)): \mathrm{Hom}_A(U, F(X)) \simeq \mathrm{Hom}_A(U, X)$.

(c) $\lambda(\tilde{F}(X)): F(\tilde{F}(X)) \simeq \tilde{F}(X)$.

(d) $\lambda(F(X)): F(F(X)) \simeq F(X)$.

(e) $\lambda(F^i(X)): F^i(X) \simeq F^{i+1}(X)$ *for some integer* $\geqq 1$.

(f) $\lambda(F^i(X)): F^i(X) \simeq F^{i+1}(X)$ *for all integer* $\geqq 1$.

Then it holds that (a) $\Leftrightarrow$ (b), (b) $\Rightarrow$ (c), (b) $\Rightarrow$ (d), *and* (d) $\Leftrightarrow$ (e) $\Leftrightarrow$ (f). *Furthermore, if U_A satisfies* (*), *then all the conditions are equivalent.*

Proof. (a) $\Leftrightarrow$ (b). Assume $F(X) = \tilde{F}(X)$. Since $\bar{F}(X) = 0$, it holds that $f = 0$ whenever $\lambda(X)f = 0$ for any $f \in \mathrm{Hom}(U, F(X))$. This means that $\mathrm{Hom(I}, \lambda(X))$ is monic. Hence $\mathrm{Hom}(I, \lambda(X))$ is an isomorphism by Proposition 1.6.

Conversely, consider an exact sequence $0 \to \mathrm{Hom}(U, \bar{F}(X)) \to \mathrm{Hom}(U, F(X)) \xrightarrow{\mathrm{Hom}(I,\eta(X))} \mathrm{Hom}(U, \tilde{F}(X))$. Then $\mathrm{Hom}(U, \bar{F}(X)) = 0$, since $\mathrm{Hom}(I, \eta(X))$ is monic by (b) and $\mathrm{Hom}(I, \lambda(X)) = \mathrm{Hom}(I, \tilde{\lambda}(X))\,\mathrm{Hom}(I, \eta(X))$. Hence by Proposition 1.4 $\bar{F}(X) = 0$, i.e., $F(X) = \tilde{F}(X)$.

(b) $\Rightarrow$ (*c*). By the above (b) implies (a). Hence (c) follows easily.

(b) $\Rightarrow$ (d), (d) $\Rightarrow$ (f), (f) $\Rightarrow$ (e) are almost clear.

(e) $\Rightarrow$ (d). We assume that $F^{i+1}(X) \cong F^{i+2}(X)$ for some integer $i \geqq 1$. Put $Y = F^i(X)$. Then $F(F(Y)) \cong F(Y)$. Thus $\bar{F}(Y) = 0$, i.e., $F(Y) = \tilde{F}(Y)$. Since $Y \in \mathscr{C}(U_A)$, we have $Y \cong F(Y)$ by [7, Lemma 2]. Hence $F^i(X) \cong F^{i+1}(X)$. By repeating this process we finally obtain $F(F(X)) \cong F(X)$.

Now, we assume (*) for U_A.

(c) $\Rightarrow$ (a). Consider the following diagram;

$$\begin{array}{ccc} \tilde{F}(X) & \xrightarrow{\tilde{\lambda}(X)} & X \\ \uparrow{\scriptstyle \lambda(\tilde{F}(X))} & \nwarrow{\scriptstyle \eta(X)} & \uparrow{\scriptstyle \lambda(X)} \\ F(\tilde{F}(X)) & \xrightarrow[F(\tilde{\lambda}(X))]{} & F(X). \end{array}$$

Since $\tilde{\lambda}(X)\,\eta(X)\,F(\tilde{\lambda}(X)) = \lambda(X)\,F(\tilde{\lambda}(X)) = \tilde{\lambda}(X)\,\lambda(\tilde{F}(X))$, we have $\eta(X)\,F(\tilde{\lambda}(X)) = \lambda(\tilde{F}(X))$ by Lemma 2.6. On the other hand, $F(\tilde{\lambda}(X))$ is an isomorphism by Lemma 2.5. Thus we have (c) $\Rightarrow$ (a).

(d) $\Rightarrow$ (c). Consider the following diagram;

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bar{F}(X) & \longrightarrow & F(X) & \longrightarrow & \tilde{F}(X) & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \lambda(\bar{F}(X))} & & \uparrow{\scriptstyle \lambda(F(X))} & & \uparrow{\scriptstyle \lambda(\tilde{F}(X))} & & \\ & & F(\bar{F}(X)) & \longrightarrow & F(F(X)) & \longrightarrow & F(\tilde{F}(X)) & \longrightarrow & 0. \end{array}$$

The two rows are exact, since U_R is projective. $\lambda(\bar{F}(X))$ is an epimorphism by Proposition 2.3. Thus, by a diagram chase, if $\lambda(F(X))$ is an isomorphism, then $\lambda(\tilde{F}(X))$ is, too. This completes the proof.

THEOREM 3.3. *Assume* (*) *for* U_A. *Let* X_A *be a factor module of* U^n *for some integer* $n \geqq 1$. *Then we have* $\lambda(X)\colon X \cong F(X)$ *if and only if* $X \in \mathscr{C}(U_A)$.

Proof. If $X \cong F(X)$, then clearly $X \in \mathscr{C}(U_A)$.

Conversely, we have an exact sequence $Z \to U^n \to X \to 0$ with $Z \cong \oplus U$. Since U_R is projective, we have the following commutative diagram with exact rows

$$\begin{array}{ccccccc} Z & \longrightarrow & U^n & \longrightarrow & X & \longrightarrow & 0 \\ \uparrow{\scriptstyle \lambda(Z)} & & \uparrow{\scriptstyle \lambda(U^n)} & & \uparrow{\scriptstyle \lambda(X)} & & \\ F(Z) & \longrightarrow & F(U^n) & \longrightarrow & F(X) & \longrightarrow & 0. \end{array}$$

By Proposition 1.5 $\lambda(Z)$ is an epimorphism and $\lambda(U^n)$ is an isomorphism. Hence $\lambda(X)$ is an isomorphism.

THEOREM 3.4. *Assume* (*) *for* U_A. *If* $F(X)$ *is isomorphic to a factor module of* U^n *for some integer* $n \geqq 1$, *then* $\tilde{F}(X) = F(X)$:

Proof. It is clear by Theorem 3.2 and 3.3.

PROPOSITION 3.5. *Consider the following conditions for* $X \in \mathcal{M}_A$.

(a) *There exists an epimorphism* $g: U^n \rightarrow F(X)$ *for some integer* $n \geqq 1$.

(b) $\mathrm{Hom}_A(U, X)$ *is a finitely generated* B*-module.*

(c) *There exist some integer* $n \geqq 1$ *and an* A*-homomorphism* $\psi: U^n \rightarrow X$ *such that, for every* $f \in \mathrm{Hom}_A(U, X)$, *there exists* $h \in \mathrm{Hom}_A(U, U^n)$ *with* $f = \psi h$.

Then (*c*) $\Leftrightarrow$ (*b*) $\Rightarrow$ (a). *Furthermore, if* U_A *satisfies* (*), *then* (a) $\Rightarrow$ (*c*) *holds.*

Proof. (*c*) $\Rightarrow$ (b) and (b) $\Rightarrow$ (a) are almost clear.

(b) $\Rightarrow$ (*c*). Let $f_1, \ldots, f_n$ be B-generators for $\mathrm{Hom}(U, X)$. Define $\psi: U^n \rightarrow X$ with $\psi(u_i) = \sum f_i u_i$. Take any $f \in \mathrm{Hom}(U, X)$. Put $f = \sum f_i b_i$ ($b_i \in B$). Define $h \in \mathrm{Hom}(U, U^n)$ with $hu = (b_i u)$. Then we have $f = \psi h$.

(a) $\Rightarrow$ (*c*). Put $\psi = \lambda(X) g$. We have $f = \lambda(X)(\sigma(X) f)$ for any $f \in \mathrm{Hom}(U, X)$. By Lemma 2.2 $\sigma(X) f$ is an R-homomorphism. Thus there exists $h \in \mathrm{Hom}(U, U^n)$ such that $\sigma(X) f = gh$, since U_R is projective. Hence $f = \lambda(X) gh = \psi h$.

COROLLARY 3.6. *If* U_A *is projective, then all the conditions of Proposition* 3.5 *are equivalent to the following condition* (d): *There exists* $f \in \mathrm{Hom}_A(U^n, X)$ *such that* $\mathrm{Im}\, \lambda(X) = \mathrm{Im}\, f$.

Proof. (a) $\Rightarrow$ (d). Put $f = \lambda(X) g: U^n \rightarrow X$. Then we have (d).

(d) $\Rightarrow$ (c). Let $\psi \in \mathrm{Hom}(U^n, X)$ with $\mathrm{Im}\, \lambda(X) = \mathrm{Im}\, \psi$. For any $f \in \mathrm{Hom}(U, X)$, since $\mathrm{Im}\, f \subset \mathrm{Im}\, \lambda(X) = \mathrm{Im}\, \psi$ and U_A is projective, there exists $h \in \mathrm{Hom}(U, U^n)$ such that $f = \psi h$.

Remark 1. By the above, if U_A is projective, then the existence of $g: U^n \rightarrow X \rightarrow 0$ (exact) means the existence of $g': U^n \rightarrow F(X) \rightarrow 0$ (exact).

Remark 2. As for Theorem 3.3, Proposition 3.5, and Corollary 3.6, we can easily see from the proof that these statements hold whenever U_A is quasi-projective.

THEOREM 3.7. *Let* X *be a right* A*-module which satisfies* (c) *of Proposition* 3.5. *Then* $\lambda(X)$ *is a monomorphism if and only if* U_A *generates* $\mathrm{Ker}\, \psi$, *where* $\psi: U^n \rightarrow X$ *is one which is given in* (c) *of Proposition* 3.5.

Proof. Let $\psi(u_i) = \sum f_i u_i$ ($f_i \in \mathrm{Hom}_A(U, X)$; $1 \leqq i \leqq n$). Define $\phi: U^n \rightarrow F(X)$ with $\phi(u_i) = \sum f_i \otimes u_i$. Then $\psi = \lambda(X) \phi$. We shall prove $\mathrm{Ker}\, \phi =$

$\{(u_1, ..., u_n); u_i = \sum_k b_{ik}u'_k$ for some $u'_1, ..., u'_p \in U$, $b_{ik} \in B$ $(1 \leqq i \leqq n; 1 \leqq k \leqq p)$ such that $\sum_i f_i b_{ik} = 0$ for every $k\}$. We write K instead of the right-hand side. If $(u_i) \in K$, then $\phi(u_i) = \sum f_i \otimes u_i = \sum f_i \otimes (\sum b_{ik}u'_k) = \sum (\sum f_i b_{ik}) \otimes u'_k = 0$. Thus $(u_i) \in \operatorname{Ker} \phi$.

Conversely, since $\operatorname{Hom}(U, X) = \sum f_i B$, we have an exact sequence $0 \to \operatorname{Ker} \pi \to^{v} B^n \to^{\pi} \operatorname{Hom}(U, X) \to 0$. Then we have an exact sequence $\operatorname{Ker} \pi \otimes_B U \to^{v \otimes I} U^n \to^{\pi \otimes I} F(X) \to 0$, where $\pi \otimes I = \phi$. Thus if $(u_i) \in \operatorname{Ker} \phi$, then there exists $\sum_k ((b_{ik}) \otimes u'_k) \in \operatorname{Ker} \pi \otimes_B U$ such that $u_i = \sum_k b_{ik}u'_k$. Since $(b_{ik}) \in \operatorname{Ker} \pi$, we have $\sum_i f_i b_{ik} = 0$ for any k. Hence $(u_i) \in K$. Since ϕ is an epimorphism, $\lambda(X)$ is a monomorphism $\Leftrightarrow \operatorname{Ker} \phi = \operatorname{Ker} \psi \Leftrightarrow U$ generates $\operatorname{Ker} \psi$.

4. Construction of a Module V

The purpose of this section is to construct a left A-module V with assumption (**) (which is denoted by (*) in [6]) from a right A-module U with assumption (*).

Let V be a left A-module. Put $D = \operatorname{End}({}_AV)$, $C = \operatorname{End}(V_D)$, and $\phi' : A \to C$ a canonical ring homomorphism. We assume that V satisfies the following condition.

Condition (**). There exists a subring R of C such that

(a) $\phi'(A) \subset R \subset C$,

(b) ${}_RV_D \cong \operatorname{Hom}_A({}_AR_R, {}_AV_D)$,

(c) ${}_RV$ is injective.

In this situation Morita [6] showed that $\mathscr{D}({}_AV)$ is a reflective subcategory of ${}_A\mathscr{M}$ with a suitable reflector.

Now, since $B = \operatorname{End}(U_R)$ and U_R is projective, it holds that $TU = U$ where T is a trace ideal of ${}_BU$, and ${}_BU \otimes_R \operatorname{Hom}_B(U, X) \cong {}_BX$, canonically, for every $X \in \mathscr{G} = \{X \in {}_B\mathscr{M}; {}_BU$ generates ${}_BX\}$ (see [4, Lemma 2.2; Lemma 4; 9, Theorem 3.2]).

On the other hand, by (b) of (*) we have ${}_BU \otimes_R \operatorname{Hom}_B(U, X) \cong {}_BU \otimes_A R \otimes_R \operatorname{Hom}_B(U, X) \cong {}_BU \otimes_A \operatorname{Hom}_B(U, X)$. Thus ${}_BX \cong {}_BU \otimes_A \operatorname{Hom}_B(U, X)$ for every $X \in \mathscr{G}$.

In these circumstances, it is well known that $X \in \mathscr{G}$ if and only if $TX = X$. Let W' be an injective cogenerator for ${}_B\mathscr{M}$ such that ${}_BU \subset_B (W')^n$ for some integer $n \geqq 1$. Take any $X \in \mathscr{G}$ and $f \in \operatorname{Hom}_B(X, W')$. Then $(X)f = (TX)f = T(X)f \subset TW'$. Hence, if we put $W = TW'$, then $W \in \mathscr{G}$, W is an injective cogenerator for the category $\mathscr{G}$, and ${}_BU \subset {}_BW^n$ for some integer $n \geqq 1$.

Put ${}_AV = \operatorname{Hom}_B({}_BU_A, {}_BW)$ and $D = \operatorname{End}({}_BW)$.

LEMMA 4.1. *The notations are as above. It holds that* $D = \mathrm{End}({}_AV), C = \mathrm{End}(V_D)$, *and* ${}_AV$ *satisfies* (**) *for the ring R in* (*).

Proof. Since ${}_BW \in \mathcal{G}$, we have $\mathrm{End}({}_AV) = \mathrm{Hom}_A(\mathrm{Hom}_D({}_BU_A, {}_BW)$ $\mathrm{Hom}_D({}_BU_A, {}_BW)) \cong \mathrm{Hom}_B({}_BU \otimes_A \mathrm{Hom}_B({}_BU_A, {}_BW), {}_BW) \cong \mathrm{Hom}_B({}_BW, {}_BW$ $= D$. Put $C' = \mathrm{End}(V_D)$. Let ϕ be as in (*) and ϕ' be a canonical ring homomorphism $A \to C'$. Define a ring homomorphism $\psi: C \to C'$ with $(\psi c)\, v = cv$ Since V is a C-D-bimodule, ψ is well-defined. If $u \in U$, $v \in V$, and $a \in A$, then $u((\psi(\phi a))\, v) = (ua)\, v = u((\phi' a)\, v)$. Thus $\psi\phi = \phi'$. Let $\psi c = 0$. Then, for any $u \in U$, $v \in V$, we have $0 = u(cv) = (uc)\, v$. Thus $uc \in \bigcap_{v \in V} \mathrm{Ker}\, v = 0$. Therefore $c = 0$. Hence ψ is a monomorphism. We regard C as a subring of C' through ψ Then $\phi(A) = \phi'(A) \subset R \subset C \subset C'$.

We shall show that V satisfies (**). (a) is almost clear. $\mathrm{Hom}_A({}_AR_R, {}_AV_D) = \mathrm{Hom}_A({}_AR_R, {}_A\mathrm{Hom}_B({}_BU_A, {}_BW_D)) \cong \mathrm{Hom}_B({}_BU \otimes_A R_R, {}_BW_D) \cong \mathrm{Hom}_B({}_BU_R, {}_BW_D) = {}_RV_D$. Hence (b) holds. For any $X \in {}_R\mathcal{M}$, we have $\mathrm{Hom}_R({}_RX, {}_RV) = \mathrm{Hom}_R({}_RX, \mathrm{Hom}_B({}_BU_R, {}_BW)) \cong \mathrm{Hom}_B({}_BU \otimes_R X, {}_BW)$. Since ${}_BU \otimes_R X \in \mathcal{G}$, ${}_BW$ is injective in $\mathcal{G}$, and U_R is flat, the functor $\mathrm{Hom}_R(-, {}_RV)$ is exact on ${}_R\mathcal{M}$. Hence ${}_RV$ is injective.

Since V is a C-D-bimodule, $D = \mathrm{End}({}_CV)$. Thus $\mathrm{End}({}_AV) = \mathrm{End}({}_RV) = \mathrm{End}({}_CV) = D$. Since ${}_BU \subset {}_BW^n$, there exists $v_1, \ldots, v_n \in V$ such that $\cap\, \mathrm{Ker}\, v_i = 0$. Define $\alpha: {}_CC \to {}_CV^n$ with $c\alpha = (cv_1, \ldots, cv_n)$. Then α is a C-monomorphism. Fix any $v \in V$. Define $\beta: {}_RC \to {}_RV$ with $c\beta = cv$. Then there exists $(d_1, \ldots, d_n) \in D^n$ such that $\alpha(d_1, \ldots, d_n) = \beta$, since ${}_RV$ is injective. Thus $v = \sum v_i d_i$. Therefore, $V_D = \sum v_i D$.

Define $\bar{\alpha}: {}_{C'}C' \to {}_{C'}V^n$ with $c'\bar{\alpha} = (c'v_1, \ldots, c'v_n)$. If $c'\bar{\alpha} = 0$, then $c' = 0$ by $V = \sum v_i D$. Thus $\bar{\alpha}$ is a monomorphism. Hence we can consider that $\bar{\alpha}$ is an extension of α to C'. We denote it also α. Then $({}_CC)\, \alpha \subset ({}_CC')\, \alpha$. By the similar manner as in [5, proof of Theorem 2.3, (b)], we obtain the following;

$$({}_CC')\, \alpha = \{(v_1', \ldots, v_n') \in V^n;\ (v_i')f = 0 \text{ for every } f \in \mathrm{Hom}_C(V^n, V) \text{ such that } (C)\, \alpha f = 0\}. \tag{\#}$$

Since ${}_BW$ is a cogenerator in $\mathcal{G}$, we have an exact sequence $0 \to {}_BU \xrightarrow{\gamma} {}_BW^n \to \prod {}_BW$, where $\gamma = (v_1, \ldots, v_n)$. Then we have an exact sequence $0 \to {}_CC \to {}_CV^n \to \prod {}_CV$. Thus ${}_CV^n/({}_CC)\, \alpha$ is cogenerated by ${}_CV$. Hence by (#) it holds that $(C)\, \alpha = (C')\, \alpha$, i.e., $C = C'$.

COROLLARY 4.2. *Let U be a right A-module of type* FP. *Then V is a left A-module of type* FI *and* $\mathcal{C}(U_A) \cong \mathcal{M}_B$, *and* $\mathcal{D}({}_AV) \cong {}_B\mathcal{M}$.

Proof. Apply Lemma 2.1 to the case that $S = B$, $T = C$, ${}_SG_T = {}_BU_C$. Then $U_T = U_C$ and $M_S = B_B$. Therefore, $\mathcal{M}_B \cong \mathcal{C}(B_B) \cong \mathcal{C}(U_C)$. Similarly, considering a bimodule ${}_AC_C$, we have $\mathcal{C}(U_A) \cong \mathcal{C}(U_C)$. Hence $\mathcal{M}_B \cong \mathcal{C}(U_A)$. The other statements are obtained in [5, Theorem 4.1].

5. On Σ-quasi-projective Modules

If we closely examine the proofs of the statements in Sections 2 and 3, we can easily show that the condition (*) may be replaced by one that U_A is X_A-projective for any module X such that X is generated by U. Such a module is nothing but a $\sum$-quasi-projective module originated by Fuller [1]. Thus, following [1], we call a module U $\sum$-quasi-projective if $\oplus U$ is quasi-projective for any direct sum of copies of U.

Hence by the above we have:

THEOREM 5.1. *If U is a $\sum$-quasi-projective right A-module, then $\mathscr{C}(U_A)$ is a coreflective subcategory of $\mathscr{M}_A$ and $\tilde{F}$ is its coreflector.*

EXAMPLE 1. Let U be a finitely generated quasi-projective right A-module. Then by [2, Corollary 3.2] U is $\sum$-quasi-projective.

On the other hand, take any $X \in \mathscr{M}_A$ and let $\oplus B \to \oplus B \to \mathrm{Hom}(U, X) \to 0$ be a free resolution of $\mathrm{Hom}_A(U, X)_B$. Then we have the following commutative diagram

$$\begin{array}{ccccccc} \oplus B & \longrightarrow & \oplus B & \longrightarrow & \mathrm{Hom}(U, \mathrm{Hom}(U, X) \otimes U) & \longrightarrow & 0 \\ \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \mathrm{Hom}(I, \lambda(X))} & & \\ \oplus B & \longrightarrow & \oplus B & \longrightarrow & \mathrm{Hom}(U, X) & \longrightarrow & 0 \end{array}$$

with exact rows. Since α_1 and α_2 are isomorphisms, $\mathrm{Hom}(I, \lambda(X))$ is also an isomorphism. Thus by Theorem 3.2 we have $F = \tilde{F}$.

Hence we have:

THEOREM 5.2. *Let U_A be a finitely generated quasi-projective module. Then $\mathscr{C}(U_A)$ is a coreflective subcategory of $\mathscr{M}_A$ and $F = \tilde{F}$ is its coreflector.*

Furthermore, by Lemma 2.1 we obtain the following.

THEOREM 5.3 [7, Theorem 3]. *If U_A is a finitely generated quasi-projective module, then $\mathscr{M}_B \simeq \mathscr{C}(U_A)$.*

Proof. Put $S = B$, $T = A$, ${}_SG_T = {}_BU_A$, and $U_T = U_A$ in Lemma 2.1. Then $M_S = B_B$. Hence it holds that $\mathscr{C}(U_A) \simeq \mathscr{C}(B_B) \simeq \mathscr{M}_B$.

EXAMPLE 2. Let U be a right A-module which is projective as an $A/r_A(U)$-module, where $r_A(U) = \{a \in A; Ua = 0\}$. Then by [2, Proposition 2.1] U is $\prod U$-projective for any direct product of copies of U. Thus U is $\oplus U$-projective for any direct sum of copies of U, i.e., U is $\sum$-quasi-projective.

Let ${}_BW$ be an injective cogenerator for ${}_B\mathscr{M}$ such that ${}_BU \subset {}_BW^n$ for some integer $n \geqq 1$. Put ${}_AV = \mathrm{Hom}_B({}_BU, {}_BW)$ and $l_A(V) = \{a \in A; aV = 0\}$. Then we can easily show that $l_A(V) = r_A(U)$. Thus it holds that V is injective as an $A/l_A(V)$-module by the similar way as in the proof of Lemma 4.1. Hence by [1, Theorem 1.2] ${}_AV$ is $\prod$-quasi-injective in the sense of Fuller [1].

Now, all the statements in [6, Sections 2 and 3] are correct whenever we replace the assumption (*) in [6, Section 2] with one that V is $\prod V$-injective for any direct product of copies of V.

Thus, since V is $\prod V$-injective for any direct product of copies of V by the previous paragraph, we obtain the following theorem.

THEOREM 5.4. *Let U be a right A-module which is projective as an $A/r_A(U)$-module and V be as above. Then $\mathscr{C}(U_A)$ is a coreflective subcategory of $\mathscr{M}_A$ with $\tilde{F}$ as its coreflector and $\mathscr{D}({}_AV)$ is a reflective subcategory of ${}_A\mathscr{M}$ with $\tilde{D}$ as its reflector, where $\tilde{D}$ is one given in* [6].

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