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Physics Letters B 584 (2004) 186–191
PHYSICS LETTERS B
www.elsevier.com/locate/physletb

Conformal field theories with Z_N and Lie algebra symmetries

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Received 24 October 2003; accepted 14 January 2004

Editor: L. Alvarez-Gaumé

Abstract

We construct two-dimensional conformal field theories with a Z_N symmetry, based on the second solution of Fateev–Zamolodchikov for the parafermionic chiral algebra. Primary operators are classified according to their transformation properties under the dihedral group ($Z_N \times Z_2$, where Z_2 stands for the Z_N charge conjugation), as singlets, $\lfloor (N-1)/2 \rfloor$ different doublets, and a disorder operator. In an assumed Coulomb gas scenario, the corresponding vertex operators are accommodated by the Kac table based on the weight lattice of the Lie algebra $B_{(N-1)/2}$ when N is odd, and $D_{N/2}$ when N is even. The unitary theories are representations of the coset $SO_n(N) \times SO_2(N)/SO_{n+2}(N)$, with $n = 1, 2, \ldots$. We suggest that physically they realize the series of multicritical points in statistical systems having a Z_N symmetry.

Conformal field theory (CFT) has been instrumental in classifying the critical behavior of two-dimensional systems enjoying local scale invariance [1]. The conformal symmetry is encoded in the stress–energy tensor $T(z)$ which plays the rôle of the conserved current. Its mode operators generate the Virasoro algebra, involving the central charge c whose value characterizes the corresponding CFT. There exists a countably infinite set of values $c = 1 - 6/p(p+1)$, with $p = 3, 4, \ldots$, for which the CFT is unitary and minimal; by minimality is meant that all local fields are generated by a finite number of so-called primary fields. The scaling dimensions of these fields can be inferred by looking for degenerate representations of the Virasoro algebra.

In a number of cases conformal invariance can be married with other local symmetries. The mode operator algebra of such extended CFTs is based on $T(z)$ and on the chiral currents corresponding to the extra symmetries. It thus contains the Virasoro algebra as a sub-algebra. The primary fields are obtained by demanding the degeneracy of its representations. Among the first examples of such theories was the W_3 algebra [2]. Later work showed that, for each classical Lie algebra, one can construct an extended CFT by supplementing $T(z)$ by an appropriate set of extra bosonic and fermionic currents [3]. The corresponding chiral algebras are called W-algebras and have been much studied in the mathematical physics literature.

While (unitary, minimal) CFTs based on the Virasoro algebra have $c < 1$, the representations of extended CFTs allow for $c > 1$. Indeed, the need for $c > 1$ theories in string theory and statistical physics

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0370-2693
doi:10.1016/j.physletb.2004.01.033

has served as a strong motivation for constructing such theories since the mid-1980's.

Further extended CFTs were discovered by letting the chiral algebra represent the group Z_N [4]. Since this requires semi-locality in the chiral algebra (exchanging the positions of two currents produces a complex phase), the corresponding theories are known as parafermionic CFTs. Consistency requirements lead to constraints on the dimensions Δ_k of the parafermionic currents $\Psi^k(z)$. Thus, in the simplest such theory one has $\Delta_k = \Delta_{-k} = k(N-k)/N$ for $k = 1, 2, \ldots, \lfloor N/2 \rfloor$ (by $\lfloor x \rfloor$ we denote the integer part of x).

This first parafermionic theory has found wide applications in condensed matter [5], statistical physics [6], and string theory [7], because of its relation to Z_N, and because its unitary theories represent the coset $SU_N(2)/U(1)$. These parafermions also describe the critical behavior of an integrable Z_N symmetric lattice model [8] and the antiferromagnetic phase transition in the Potts model [6].

There are several reasons to search for generalizations of the above parafermionic theory. First, this CFT is somewhat poor in the sense that $c = 2 - 6/(N+2)$ is fixed just by requiring associativity of the chiral algebra [4]. In particular, no infinite series of minimal models exists. On the other hand, it seems natural to suppose that the Z_N lattice models [8] should have an infinite series of higher multicritical points, such as is the case for the Ising model [9].

In the appendix of Ref. [4], a second associative solution of the parafermionic chiral algebra was given. In this theory, the dimensions of the currents $\Psi^k(z)$ are

$$\Delta_k = \Delta_{-k} = \frac{2k(N-k)}{N}, \tag{1}$$

and c is not fixed by associativity alone. This second parafermionic theory is therefore a good candidate for the supposed multicritical points described above. An infinite series of minimal models for the case $N = 3$ was given in Ref. [10], and the first minimal model could indeed be identified with the tricritical Z_3 model.

In this Letter, we obtain the representation theory and the series of minimal models for the parafermions (1) with $N \geqslant 5$. (Note that $N = 2$ has fixed $c = 1$, and that $N = 4$ factorizes trivially as two superconformal CFTs.) The representation theory is rather rich, with a number of sectors equal to the number of selfdual representations of Z_N, plus a Z_2 disorder sector. Moreover, these CFTs contain a Lie algebra structure, which was not significant for $N = 3$. Partial results for odd N have already appeared [11]; here we complete the solution and present it in a unified way for N odd and even.

Let us first recall the fusion rules of the currents [4], which read

$$\begin{aligned}\Psi^k(z)\Psi^{k'}(z') &= \frac{\lambda^{k,k'}_{k+k'}}{(z-z')^{\Delta_k+\Delta_{k'}-\Delta_{k+k'}}}\left\{\Psi^{k+k'}(z') \right.\\ &\quad \left. + (z-z')\frac{\Delta_{k+k'}+\Delta_k-\Delta_{k'}}{2\Delta_{k+k'}}\partial\Psi^{k+k'}(z') + \cdots\right\}\end{aligned}$$

for $k + k' \neq 0$, and otherwise

$$\Psi^k(z)\Psi^{-k}(z') = \frac{1}{(z-z')^{2\Delta_k}}\left\{1 + (z-z')^2\frac{2\Delta_k}{c}T(z') + \cdots\right\}.$$

Associativity fixes the structure constants $\lambda^{k,k'}_{k+k'}$ as functions of a single free parameter v [4]

$$\begin{aligned}&\left(\lambda^{k,k'}_{k+k'}\right)^2 \\ &= \frac{\Gamma(k+k'+1)\Gamma(N-k+1)\Gamma(N-k'+1)}{\Gamma(k+1)\Gamma(k'+1)\Gamma(N-k-k'+1)\Gamma(N+1)} \\ &\quad \times \frac{\Gamma(k+k'+v)\Gamma(N+v-k)\Gamma(N+v-k')\Gamma(v)}{\Gamma(N+v-k-k')\Gamma(k+v)\Gamma(k'+v)\Gamma(N+v)},\end{aligned}$$

and the central charge c of the Virasoro algebra

$$c = (N-1)\left(1 - \frac{N(N-2)}{p(p+2)}\right), \tag{2}$$

agrees with that of the coset [12]

$$\frac{SO_n(N) \times SO_2(N)}{SO_{n+2}(N)}, \quad n = 2v = 2 + p - N. \tag{3}$$

Here $SO_n(N)$ is the orthogonal group, with level n for its affine current algebra. Note that in the above the Z_N charges k and their sums $k + k'$ are defined modulo N.

The structure of the modules of physical operators (representation fields) can be inferred by considering first the module of the identity operator; see Fig. 1. The first descendent in each Z_N charge sector $q \neq 0$ is the current Ψ^q; the level corresponds to the conformal dimensions Δ_k. More general singlet operator

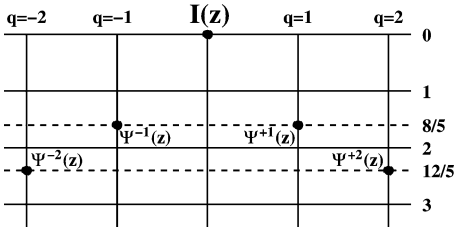

Fig. 1. Module of the identity operator for $N = 5$.

modules are obtained by replacing I at the summit by Φ^0 and filling the levels in a more general fashion; within each charge sector, the level spacing is one, due to the action of the Virasoro algebra. Finally, the structure of doublet modules $\{\Phi^{\pm q}\}$ is obtained by taking sub-modules.

The currents $\{\Psi^k\}$ can be decomposed into mode operators, whose action on the representation fields changes the Z_N charge:

$$\Psi^k(z)\Phi^q(0) = \sum_n \frac{1}{(z)^{\Delta_k - \delta^q_{k+q}+n}} A^k_{-\delta^q_{k+q}+n}\Phi^q(0). \tag{4}$$

The gap $\delta^q_k = 2(q^2-k^2)/N \bmod 1$ is the first level in the module of the doublet q corresponding to the Z_N charge sector k. As usual, primary fields are defined by $A^k_{-\delta^q_{k+q}+n}\Phi^q = 0$ for $n > 0$.

The action of zero modes between the summits in doublet modules permit to define the eigenvalues $\{h_q\}$:

$$A_0^{\mp 2q}\Phi^{\pm q}(0) = h_q\Phi^{\mp q}(0). \tag{5}$$

Note that the representations Φ^q are characterized by both $\{h_q\}$ and the conformal dimension Δ_q, the latter being just the eigenvalue of the usual Virasoro zero mode L_0.

To get a number of distinct sectors equal to the number of representations of Z_N one must in general consider doublet modules $\{\Phi^{\pm q}\}$ with $q \in \mathbb{Z}/2$. This can be argued on general grounds of selfduality [4] or be worked out explicitly [13]. Henceforth we adopt a more natural notation by setting $Q = 2q \in \mathbb{Z}$ and $K = 2k \in 2\mathbb{Z}$. Note that although the K charges are now defined mod $2N$, in each module only N distinct Z_N charge sectors will be occupied. The Q charges of primary fields, however, are still defined mod N, in order to stay consistent with the number of representations of Z_N. Thus, for N even, the $Q = N/2$ module is actually a singlet.

In summary, we have thus $2 - (N \bmod 2)$ singlet sectors and $\lfloor (N-1)/2 \rfloor$ doublet sectors. In addition, the Z_N charge conjugation is represented by a disorder operator R_a [10,11,14] with components $a = 1, 2, \ldots, N$. The non-Abelian monodromy of R_a with respect to Ψ^K leads to

$$\Psi^K(z)R_a(0) = \sum_n \frac{1}{(z)^{\Delta_K + n/2}} A^K_{n/2}R_a(0), \tag{6}$$

meaning that disorder modules have integer and half-integer levels.

Because of the connection with the coset (3) we shall suppose that the Kac table is based on the weight lattice of the Lie algebra B_r for $N = 2r+1$ odd, and D_r for $N = 2r$ even. The conformal dimensions of the primary operators are then assumed to take the Coulomb gas form

$$\Delta_{\vec\beta} = \Delta^{(0)}_{\vec\beta} + B = (\vec\beta - \vec\alpha_0)^2 - \vec\alpha_0^2 + B, \tag{7}$$

$$\vec\beta = \sum_{a=1}^{r}\left(\frac{1+n_a}{2}\alpha_+ + \frac{1+n'_a}{2}\alpha_-\right)\vec\omega_a, \tag{8}$$

$$\vec\alpha_0 = \frac{(\alpha_+ + \alpha_-)}{2}\sum_{a=1}^{r}\vec\omega_a, \tag{9}$$

where $\{\vec\omega_a\}$ are the fundamental weights of the Lie algebra. The position on the weight lattice is given by $\vec\beta = \vec\beta_{(n_1,n_2,\ldots,n_n)(n'_1,n'_2,\ldots,n'_n)}$, where $\{n_a\}$ (respectively $\{n'_a\}$) are the Dynkin labels on the α_+ (respectively α_-) side. The parameters α_+, α_- are defined as

$$\alpha_+ = \sqrt{\frac{p+2}{2}}, \qquad \alpha_- = -\sqrt{\frac{p}{p+2}}. \tag{10}$$

The constant B in Eq. (7) is the *boundary term*, which takes, in general, different values for the different sectors of the theory. We have already defined these sectors; it remains to work out the corresponding values of B, and to assign the proper sector label to each of the vectors $\vec\beta$.

The unitary theories correspond to $n \in \mathbb{Z}_+$ in Eq. (3). For a given n, the physical domain of the Kac table is delimited as follows:

$$\Sigma\big(\{n'_a\}\big) \leqslant p+1, \qquad \Sigma\big(\{n_a\}\big) \leqslant p-1, \tag{11}$$

where we have defined for future convenience

$$\Sigma\big(\{n_a\}\big) = n_1 + 2\sum_{a=2}^{r-2} n_a + \big(1 + (N \bmod 2)\big)n_{r-1} + n_r,$$

and $n'_a, n_a \in \mathbb{Z}_+$. This can be argued by invoking "ghosts" (reflections of primary submodule operators) situated outside the physical domain [11]. In correlation functions the ghosts decouple from physical operators.

We now define, for any $n \in \mathbb{Z}_+$, the *elementary cell* as the physical domain corresponding to $n = 0$ (whence $c = 0$). From Eq. (11) only the α_- side is nontrivial, so in the following we refer to the n'_a indices only. We then assume that to each sector corresponds exactly one independent operator in the elementary cell. These operators are *fundamental* in the sense that their modules are degenerate at the first possible levels.

Moreover, we assume that $\Delta_{\vec{\beta}} = 0$ for all operators in the elementary cell when $c = 0$. This fixes the available values of B, up to an overall normalization of $\{\vec{\omega}_a\}$.

We now need to

(1) fix the normalization of B;
(2) identify which operators inside the elementary cell are independent (and find the symmetry linking dependent operators); and
(3) assign the correct sector label to each independent operator.

To this end we have used two different techniques.

First, we have explicitly constructed the modules of several fundamental operators, by direct degeneracy calculations [11,13]. Each operator was required to be r-fold degenerate. For any N, we have been able to compute $\Delta_{\vec{\beta}}$ and $\{h_q\}$ for two distinct doublets ($\Phi^{\pm 1}$ and $\Phi^{\pm 2}$ in the Q notation) and the disorder operator R. This approach settles point (1) above, and provides valuable partial answers to points (2) and (3). The calculations also reveal at which levels degeneracy has to be imposed (see below). Moreover, they strongly corroborate the assumed Coulomb gas formulae.

Second, we have used the technique of Weyl reflections. In a way analogous to the BRST structure of the (Virasoro algebra based) minimal models [15], the reflections in the hyperplanes which border the physical domain (11) put in correspondence the operators outside the physical domain with the degenerate combinations of descendent fields inside the modules of physical operators (i.e., operators positioned within the physical domain). The exact correspondence is furnished by the *simple reflections* $s_{\vec{e}_a} \equiv s_a$ which act on the weight lattice as the generators of the Weyl group:

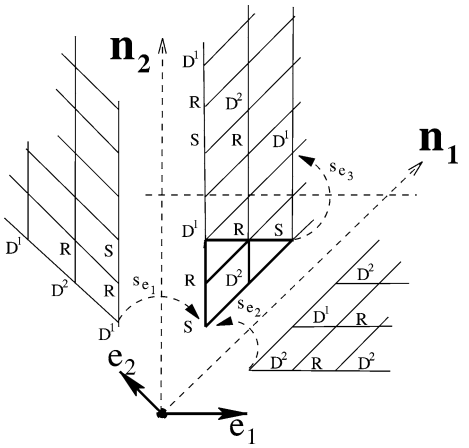

Fig. 2. The Weyl reflection technique illustrated for $N = 5$.

$$s_a \vec{\beta}_{(1,\ldots,1)(n'_1,\ldots,n'_r)} = \vec{\beta}_{(1,\ldots,1)(n'_1,\ldots,n'_r)} - n'_a \alpha_- \vec{e}_a. \tag{12}$$

Here $\{\vec{e}_a\}$, with $a = 1, 2, \ldots, r$, are the simple roots of the given Lie algebra. In the case of unitary theories, there is an extra simple reflection based on the affine simple root $\vec{e}_{r+1}$.

Since a given simple reflection connects a ghost operator and a degenerate (or singular) state inside the module of a physical operator, the difference of conformal dimensions of the ghost operator and the corresponding physical operator should be compatible with the levels available in the module, as given by δ_k^q. For the difference of dimensions one obtains, from Eq. (7),

$$\Delta_{\vec{\beta}} - \Delta_{s_a\vec{\beta}} = \Delta^{(0)}_{\vec{\beta}} - \Delta^{(0)}_{s_a\vec{\beta}} + B_{\vec{\beta}} - B_{s_a\vec{\beta}}. \tag{13}$$

Given the position, sector label and boundary term of some operator, the reflection technique allows, in general, to provide the same information for all operators in the Weyl orbit of that operator; see Fig. 2. Ignoring some sporadic non-regular possibilities for large N, it allows for a unique identification of the operators in the elementary cell.

We can now summarize our results. The physical domain of the unitary theory (3) has the Z_2 symmetry

$$n'_1 \to p+2-\Sigma\big(\{n'_a\}\big), \qquad n_1 \to p-\Sigma\big(\{n_a\}\big). \tag{14}$$

For even N there is an additional Z_2 symmetry:

$$n'_{r-1} \leftrightarrow n'_r, \qquad n_{r-1} \leftrightarrow n_r. \tag{15}$$

With $p = N-2$ these are also the symmetries of the elementary cell. The assignment of sector labels (singlet S^Q, doublet D^Q or disorder R) to its independent operators (writing only the α_- indices) is:

$$\Phi_{(1,1,\ldots,1,1)} = I = S^0, \qquad \Phi_{(1,1,\ldots,2,\ldots,1,1)} = D^Q \tag{16}$$

for $Q = 1, 2, \ldots, r-2$ (only $n'_Q = 2$). Further, for $N = 2r+1$ odd:

$$\Phi_{(1,\ldots,2,1)} = D^{r-1}, \qquad \Phi_{(1,\ldots,1,3)} = D^r,$$
$$\Phi_{(1,\ldots,1,2)} = R;$$

and for $N = 2r$ even:

$$\Phi_{(1,\ldots,2,2)} = D^{r-1}, \qquad \Phi_{(1,\ldots,3,1)} = S^r,$$
$$\Phi_{(1,\ldots,2,1)} = R.$$

The boundary terms for the singlet/doublet operator of charge $Q = 0, 1, \ldots, r$, and for the disorder operator, read for all N

$$B_{(Q)} = \frac{Q(N-2Q)}{4N}, \qquad B_R = \frac{1}{16}\left\lfloor \frac{N-1}{2} \right\rfloor. \tag{17}$$

It remains to assign sector labels to *all* the sites of the weight lattice. It can be argued that the result should only depend on $\tilde{n}_a \equiv |n_a - n'_a|$ [11]; it suffices therefore to treat the case $\{n_a = 1\}$. As already discussed, the reflection method determines the ghost environment of the fundamental operators, cf. Fig. 2. This can also be applied to operators identified via the symmetries (14), (15) of the elementary cell. Finally, the labels of elementary cell operators and the surrounding ghosts are spread over the lattice by using fusions with the singlet ($Q = 0$) operators. As in Ref. [11] we assume that the principal channel amplitudes are non-vanishing in all fusions of singlets with other operators.

This method assigns sector labels to all $\{n_a = 1\}$ operators. The end result can be stated quite simply [13]. Once sector labels have been assigned to the operators of the elementary cell, the assignment of the rest of the $\{n_a = 1\}$ operators is obtained by repeatedly reflecting the elementary cell in all its faces, filling progressively in this way the whole lattice.

Note that these reflections (technically, they are shifted Weyl reflections) have no bearing on the structure of modules of primary operators. Their only significance is with respect to the sector assignment. Such reflections also appear in a general analysis of the distribution of boundary terms in coset-based CFTs [16,17]. Since there are degeneracies in the boundary terms (17) for even N, our method is more complete than Refs. [16,17], and suggests that the shifted Weyl reflections can actually be used to distribute the sector labels over the weight lattice.

Algebraically, the sector assignment reads as follows.

Define $x_a = \tilde{n}_a$ for $a = 1, 2, \ldots, r-2$. For N odd we further set $x_{r-1} = \tilde{n}_{r-1}$ and $x_r = \tilde{n}_r/2$; and for N even we set $x_{r-1} = \tilde{n}_r$ and $x_r = (\tilde{n}_{r-1} - \tilde{n}_r)/2$. If x_r is non-integer, we have a disorder operator R. Otherwise, the doublet charge Q associated with the position $\vec{\beta}_{(n_1,\ldots,n_r)(n'_1,\ldots,n'_r)}$ is given by

$$Q(x_1, x_2, \ldots, x_r) = \sum_{a=1}^{r}\left[\left(\sum_{b=a}^{r} x_b\right) \bmod 2\right]. \tag{18}$$

Alternatively, choose an orthonormal basis such that: $\vec{\omega}_a = (1, \ldots, 1, 0, \ldots, 0)$ (with a 1's) for $a = 1, 2, \ldots, r-2$, and $\vec{\omega}_r = (1/2, \ldots, 1/2)$. Further, $\vec{\omega}_{r-1} = (1, \ldots, 1, 0)$ for N odd, and $\vec{\omega}_{r-1} = (1/2, \ldots, 1/2, -1/2)$ for N even. Let y_a be the coordinates of $[\vec{\beta}_{(1,\ldots,1)(n'_1,\ldots,n'_r)} - 2\vec{\alpha}_0]/\alpha_-$ with respect to this basis (hypercubic lattice). Then $Q = 2\sum_{a=1}^{r}(y_a \bmod 1)$ for both $N = 2r$ and $N = 2r+1$.

The CFT that we have constructed is based on the same weight lattices as the WB_r and WD_r theories [3]. The crucial difference is that the coset (3) has got another "shift" (2 instead of 1), and this makes the elementary cell bigger, cf. Eq. (11). This makes room for more sectors than in the W theories (WD_r has one sector, and WB_r two sectors, for any r).

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