JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 93, 15–26 (1983)

On Some Logical Connectives for Fuzzy Sets Theory

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Submitted by L. Zadeh

INTRODUCTION

It was proved by Bellman and Giertz [2] that, under reasonable hypotheses (especially distributivity), the only truth-functional logical connectives for fuzzy sets are the usual min and max. The following easy argument proves that distributivity, monotonicity and boundary conditions are essential assumptions:

$$x = F(x, 1) = F(x, G(1, 1)) = G(F(x, 1), F(x, 1)) = G(x, x),$$

$$\begin{aligned}\max(x, y) &= G(\max(x, y), \max(x, y)) \geqslant G(x, y)\\ &\geqslant \max(G(x, 0), G(0, y)) = \max(x, y),\end{aligned}$$

i.e., $G(x, y) = \max(x, y)$. Here F and G are, respectively, functions from $[0, 1] \times [0, 1]$ into $[0, 1]$ generating the "meet" and the "join" by means of

$$(A \cap B)(x) = F(A(x), B(x)), \qquad (A \cup B)(x) = G(A(x), B(x)),$$

for every $x \in X$ and $A, B \in \mathbf{P}(X) = [0, 1]^X$.

Zadeh [18], founder of the theory, is the first to introduce nondistributive and dual connectives by considering the couples

$$G_m(x, y) = \min\{x + y, 1\}, \qquad F_m(x, y) = \max\{x + y - 1, 0\};$$

$$G_p(x, y) = x + y - xy, \qquad G_p(x, y) = xy.$$

Since then some authors (for instance, Hamacher [7]) have considered some classes of nondistributive connectives.

Obviously, the loss of global idempotency causes the loss of the lattice structure in $\mathbf{P}(X)$ when $F \neq \min$ or $G \neq \max$. For arbitrary connectives F and G, the only sublattice is the Boolean algebra of characteristic functions (classical subsets). It is also interesting to observe that the idempotents for the couples G_m, F_m and G_p, F_p are 0 and 1, and that the former verifies universally the excluded middle and noncontradiction laws. On the other

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hand, these laws are satisfied by the couple G_p, F_p only in the case of classical sets.

In this article it is given a general functional form for logical connectives "or" and "and" (nondistributive but associative) using additive generators. We also study the Kleene's character of the obtained logic and DeMorgan laws (with adequate strong negation functions, if they exist). We give further characterizations of the min–max pair.

1. Preliminaries on Fuzzy Connectives

Let F and G be two binary operations on $[0, 1]$. In Bellman–Giertz [2] it was shown that $F = \min$, $G = \max$ are the unique solutions of these 8 conditions:

(1) associativity, $F(x, F(y, z)) = F(F(x, y), z)$; $G(x, G(y, z)) = G(G(x, y), z)$;

(2) commutativity, $F(x, y) = F(y, x)$, $G(x, y) = G(y, x)$;

(3) nondecreasing, $F(x, y) \leqslant F(x', y')$, $G(x, y) \leqslant G(x', y')$, if $x \leqslant x'$, $y \leqslant y'$;

(4) $F(x, x) < F(x', x')$, $G(x, x) < G(x', x')$, if $x < x'$;

(5) $F(1, 1) = 1$, $G(0, 0) = 0$;

(6) $F(x, y) \leqslant \min(x, y)$; $G(x, y) \geqslant \max(x, y)$;

(7) F, G are continuous;

(8) distributivity, $F(x, G(y, z)) = G(F(x, y), F(x, z))$, $G(x, F(y, z)) = F(G(x, y), G(x, z))$.

Hamacher [7] proved that conditions (1), (3), (4), (6), and (8) are enough in order to conclude that min–max are the unique solutions. Thus it is clear that if we want to study fuzzy connectives, different from the classical min–max, it would be necessary to avoid some of the given requirements. Several authors have studied this problem (see, e.g., Dubois–Prade [4]).

We first note that if condition (5) is replaced by

(5′) (a) $F(x, 1) = F(1, x) = x$, (b) $G(0, x) = G(x, 0) = x$, for all $x \in [0, 1]$,

then (3) and (5′) imply (6) and (3), (5′) together with (8), admit min–max as the only solutions. Thus we shall exclude (6) and (8) and we can also avoid (2) and (4). More precisely, in this article we shall restrict our attention to the following classes of connectives:

$$\mathscr{F}_{ab} = \{F\colon [a, b]^2 \to [a, b] / F \text{ satisfies (1), (3), (5′(a)) and (7)}\},$$

$$\mathscr{G}_{ab} = \{G\colon [a, b]^2 \to [a, b] / G \text{ satisfies (1), (3), (5′(b)) and (7)}\}$$

and

$$\mathscr{F}^A_{ab} = \{F \in \mathscr{F}_{ab}/F \text{ is Archimedean, i.e., } F(x,x) < x, \text{ for all } x \in (a,b)\},$$

$$\mathscr{G}^A_{ab} = \{G \in \mathscr{G}_{ab}/G \text{ is Archimedean, i.e., } G(x,x) > x, \text{ for all } x \in (a,b)\}.$$

In the sequel we shall write $\mathscr{F} = \mathscr{F}_{01}$, $\mathscr{G} = \mathscr{G}_{01}$, $\mathscr{F}_A = \mathscr{F}^A_{01}$, and $\mathscr{G}_A = \mathscr{G}^A_{01}$.

These functional sets have a long distinguished history in the field of functional equations (Aczél [1]) because they are related to the classical problem of the associativity equation. Such sets have been analyzed with detail and have become a basic tool in the theory of probabilistic metric spaces (Schweizer–Sklar [13]) and in the theory of information (Kampé de Fériet [8]).

From Aczél [1] and Ling [9] we have the following important characterization of the operations in $\mathscr{F}^A_{ab}$ and $\mathscr{G}^A_{ab}$:

THEOREM 1.1. *Let $H \in \mathscr{F}_{ab}$ (resp. $H \in \mathscr{G}_{ab}$). Then $H \in \mathscr{F}^A_{ab}$ (resp. $H \in \mathscr{G}^A_{ab}$) if and only if there exists a continuous and strictly decreasing function h (resp. continuous and strictly increasing) from $[a,b]$ into $[0,+\infty]$, such that H is representable in the form*

$$H(x,y) = h^{(-1)}(h(x) + h(y)),$$

where $h^{(-1)}$ is the pseudoinverse of h, that is,

$$\begin{aligned} h^{(-1)}(x) &= b, && \text{if } 0 \leqslant x \leqslant h(b) && (\text{resp. } a, \text{ if } 0 \leqslant x \leqslant h(a)),\\ &= h^{-1}(x), && \text{if } h(b) \leqslant x \leqslant h(a) && (\text{resp. } h^{-1}(x), \text{ if } h(a) \leqslant x \leqslant h(b)),\\ &= a, && \text{if } h(a) \leqslant x && (\text{resp. } b, \text{ if } h(b) \leqslant x), \end{aligned}$$

where h^{-1} is the usual inverse of h on $[h(b), h(a)]$ (resp. $[h(a), h(b)]$).

The function h is called *an additive generator* of the Archimedean operation H and it is unique up to a positive constant, i.e., kh $(k > 0)$ also generates H. We remark that if f generates $F \in \mathscr{F}_A$, then $f(1) = 0$, and if g generates $G \in \mathscr{G}_A$, then $g(0) = 0$.

We also have from Paalman de Miranda [12] a characterization for the operations $F \in \mathscr{F}_{ab} - \mathscr{F}^A_{ab}$ and $G \in \mathscr{G}_{ab} - \mathscr{G}^A_{ab}$. If H is one such operation, let

$$E(H) = \{x \in [a,b]/H(x,x) = x\},$$

then $[a,b] - E(H) = \bigcup_{i \in j} (a_i, b_i)$, where $\{(a_i, b_i)/i \in j\}$ is a finite or countable collection of disjoint open intervals. Let H_i be the restriction of H to $[a_i, b_i]^2$ given by $H_i(x,y) = a_i + (b_i - a_i) H_i((x - a_i)/(b_i - a_i), (y - a_i)/(b_i - a_i))$. Then $([a_i, b_i], H_i)$ is an Archimedean semigroup and we have the representation (*ordinal sum*):

$$\begin{aligned} H(x,y) &= H_i(x,y), && \text{if } (x,y) \in [a_i, b_i]^2 \text{ for some } i \in J, \\ &= \min(x,y), && \text{if } (x,y) \notin \bigcup_{i \in J} [a_i, b_i]^2 \text{ and } H \in \mathscr{F}_{ab} - \mathscr{F}^A_{ab}, \\ &= \max(x,y), && \text{if } (x,y) \notin \bigcup_{i \in J} [a_i, b_i]^2 \text{ and } H \in \mathscr{G}_{ab} - \mathscr{G}^A_{ab}; \end{aligned}$$

i.e., H consists of Archimedean "blocks" along the diagonal of $[a, b]^2$ and $H = \min$ (resp. max) outside of these blocks.

In the literature about fuzzy connectives other couples of operations different from the classical couple min–max have been considered. We summarize some of these operations, together with their additive generators:

$$\begin{aligned} F_m(x,y) &= \max(x+y-1, 0), & f_m(x) &= 1-x, \\ G_m(x,y) &= \min(x+y, 1), & g_m(x) &= x; \end{aligned} \tag{1.1}$$

$$\begin{aligned} F_p(x,y) &= xy, & f_p(x) &= -\ln x, \\ G_p(x,y) &= x+y-xy, & g_p(x) &= -\ln(1-x); \end{aligned} \tag{1.2}$$

$$\begin{aligned} F_0(x,y) &= \frac{xy}{x+y-xy}, & f_0(x) &= \frac{1-x}{x}, \\ G_{-1}(x,y) &= \frac{x+y-2xy}{1-xy}, & g_{-1}(x) &= \frac{x}{1-x}; \end{aligned} \tag{1.3}$$

$$\begin{aligned} F_a(x,y) &= \frac{xy}{a+(1-a)(x+y-xy)}, & f_a(x) &= \log\frac{a-(a-1)x}{x}, \quad (a>0), \\ G_b(x,y) &= \frac{(b-1)\,xy+x+y}{1+bxy}, & g_b(x) &= \log\frac{bx-1}{1-x}, \quad (b>-1); \end{aligned} \tag{1.4}$$

$$\begin{aligned} F_\lambda(x,y) &= 1-\min\{1, [(1-x)^\lambda + (1-y)^\lambda]^{1/\lambda}\}, & f_\lambda(x) &= (1-x)^\lambda, \\ G_\lambda(x,y) &= \min\{1, (x^\lambda + y^\lambda)^{1/\lambda}\}, & g_\lambda(x) &= x^\lambda, \quad (\lambda > 1). \end{aligned} \tag{1.5}$$

Examples (1.1) and (1.2) were used by Zadeh [18], the operations (1.3) and (1.4) were considered by Hamacher [7], and Yager [17] introduced the pair (1.5). Another interesting couple introduced by Frank [5] is given by

$$\begin{aligned} F_s(x,y) &= \log_s\left(\frac{(s^x-1)(s^y-1)}{s-1}+1\right); & f_s(x) &= \log_s\frac{s-1}{s^x-1}, \quad (s>0), \\ G_s(x,y) &= 1-F_s(1-x, 1-y); & g_s(x) &= 1-f_s(1-x), \quad (s>0). \end{aligned} \tag{1.6}$$

In a remarkable article (Frank [5]) *it has been proved that* (1.1), (1.2), *and* (1.6) *are the only possible Archimedean solutions of the functional equation* $F(x, y) + G(x, y) = x + y$.

2. On Strong Negations and DeMorgan's Laws

The classical equality $\aleph_{\bar{A}} = 1 - \aleph_A$ has its natural generalization in the realm of fuzzy sets theory $\bar{A}(x) = 1 - A(x)$, where A denotes any fuzzy set. This last equality can be written as $\bar{A} = N \circ A$, where $N(x) = 1 - x$ is the standard negation function on $[0, 1]$. In Bellman–Giertz [2] it was pointed out that it would be interesting to study the functions n from $[0, 1]$ into $[0, 1]$ with $\bar{A}(x) = n(A(x))$, satisfying only some of the classical DeMorgan's laws. We shall pay attention to this problem.

To begin with we remark that the condition $\bar{\bar{A}} = A$ whenever $\bar{A}(x) = n(A(x))$ yields that the function n from $[0, 1]$ into $[0, 1]$ must satisfy the condition $n \circ n = j$ (where $j(x) = x$ denotes the usual identity function). Thus we shall restrict our attention to the set $S([0, 1])$ of strong negations, i.e.,

$$S([0, 1]) = \{n\colon [0, 1] \to [0, 1] / n \text{ is continuous and strictly decreasing, } n(0) = 1, n(1) = 0 \text{ and } n \circ n = j\}.$$

A functional characterization of strong negations was given by Trillas [15].

Theorem 2.1. *If n is any map from $[0, 1]$ into $[0, 1]$, then $n \in S([0, 1])$ if and only if there exists a function t from $[0, 1]$ into $[0, +\infty)$ such that t is increasing, $t(0) = 0$ and n admits the representation*

$$n(x) = t^{-1}(t(1) - t(x)).$$

Such a function is called a generator of the negation n. Note that kt ($k > 0$) also generates n. For example:

(i) the *classical negation* $N(x) = 1 - x$ is generated by $t(x) = x$,

(ii) the *round negation* $n_c(x) = \sqrt{1 - x^2}$ is generated by $t_c(x) = x^2$,

(iii) *Sugeno's negation* $n_s(x) = (1 - x)/(1 + sx)$ $(s > -1)$ is generated by $t_s(x) = (1/s) \log(1 + sx)$. Note that n_s is the unique rational negation of the form $(ax + b)/(cx + d)$.

In many uses, if some "nice" additional conditions to that of strong negation are required, then $N(x) = 1 - x$ is the only possible one, e.g., *N is the unique element in $S([0, 1])$ satisfying the Lipschitz inequality* $|n(x) - n(y)| \leqslant k\,|x - y|$ $(0 < k < 1)$, for all $x, y \in [0, 1]$.

We now turn our attention to the standard DeMorgan's law $N \circ (A \cup B) = (N \circ A) \cap (N \circ B)$.

DEFINITION 2.1. Let $F \in \mathscr{F}$, $G \in \mathscr{G}$, and $n \in S([0, 1])$. Then F and G will be called *n-duals* if $F \circ (n \times n) = n \circ G$, i.e.,

$$F(n(x), n(y)) = n(G(x, y)) \qquad \text{for all} \quad x, y \in [0, 1].$$

It is easily seen that the connectives min–max, as well as the examples (1.1)–(1.3), (1.5), and (1.6), are examples of N-duality. In order to build more collections of n-dual connectives we note

THEOREM 2.2. *If f is an additive generator of $F \in \mathscr{F}_A$ and $n \in S([0, 1])$, then $g' = f \circ n$ generates additively a function $G \in \mathscr{G}_A$ which is n-dual of F. Analogously, if $G \in \mathscr{G}_A$ is generated by g and $n \in S([0, 1])$, then $f' = g \circ n$ generates an n-dual operation $F \in \mathscr{F}_A$.*

We shall now give a functional characterization of the n-duality in terms of the additive generators of Archimedean connectives. If $p > 0$, let $l_p(x) = px$.

THEOREM 2.3. *Let $F \in \mathscr{F}_A$ and $G \in \mathscr{G}_A$ be generated by f and g, respectively. Then F and G are n-duals, if and only if there exists a positive constant p such that*

$$f^{(-1)} \circ l_p \circ g = g^{(-1)} \circ l_{1/p} \circ f, \tag{$*$}$$

and "a fortiori" n is given by the left-hand side (or by the right-hand side) of the equality $(*)$.

Proof. The condition $n \circ F \circ (n \times n) = G$, can be written in terms of the additive generators as follows:

$$(n \circ f^{(-1)})(f(n(x)) + f(n(y))) = g^{(-1)}(g(x) + g(y)),$$

but this equation states that $f \circ n$ generates G as well as g. Consequently, $f \circ n$ and g must differ only in a positive constant p, i.e., $f \circ n = l_p \circ g$, and $n = f^{(-1)} \circ l_p \circ g = g^{(-1)} \circ l_{1/p} \circ f$.

Remark. We note in view of $(*)$ that if $f(0) < +\infty$ and $g(1) < +\infty$, then $(*)$ yields $p = f(0)/g(1)$, i.e., the constant p is unique. If $f(0)$ (resp. $g(1)$) is finite and $g(1)$ (resp. $f(0)$) is infinite, then there exists *no* constant satisfying $(*)$ i.e., the corresponding connectives F and G generated by f and g, respectively, never will be n-duals. But if $f(0) = g(1) = +\infty$, then several possibilities are available, e.g., p can be unique and there will exist a unique negation n satisfying the n-duality or p is not fixed and there exists a large collection of negations establishing the n-duality. Examples 2.1 and 2.2 will clarify this remark.

EXAMPLE 2.1. We consider the connectives F_a and G_b given by (1.4). Then we have, considering the generators $\hat{f}_a(x) = tf_a(x)$ and $\hat{g}_b(x) = t'g_b(x)$,

$$(\hat{f}_a^{-1} \circ l_p \circ \hat{g}_b)(x)$$
$$= a(1-x)^{pt'/t}/((bx-1)^{pt'/t} + (a-1)(1-x)^{pt'/t})$$

and

$$(\hat{g}_b^{-1} \circ l_{1/p} \circ \hat{f}_a)(x)$$
$$= ((a-(a-1)x)^{t/pt'} - x^{t/pt'})/((a-(a-1)x)^{t/pt'} + bx^{t/pt'}),$$

and $(*)$ holds for $p = t/t'$. In this case the operations F_a and G_b are n_s-duals, being n_s Sugeno's negation $(1-x)/(1+sx)$, where $s = (b-a+1)/a$.

EXAMPLE 2.2. Consider F_0 and G_{-1} given by (1.3). Then

$$(f_0^{-1} \circ l_p \circ g_{-1})(x) = (g_{-1}^{-1} \circ l_{1/p} \circ f_0)(x) = (1-x)/(1+(p-1)x),$$

i.e., $(*)$ holds for any $p > 0$ and the n-duality is given by anyone of Sugeno's negations $n_p(x) = (1-x)/(1+(p-1)x)$, $p > 0$.

We now study the relation between n-duality and ordinal sums. Note first that *if* $F \in \mathscr{F} - \mathscr{F}_A$ *and* $G \in \mathscr{G} - \mathscr{G}_A$ *are n-duals, then necessarily* $n(E(F)) = E(G)$.

THEOREM 2.4. *Let* $F \in \mathscr{F} - \mathscr{F}_A$ *be an ordinal sum determined by a family of Archimedean semigroups* $\{([a_i, b_i], F_i)/i \in J\}$ *and let* $n \in s([0,1])$. *Then the function* $G(x, y) = n(F(n(x), n(y))$ *is a n-dual operation of F which is an ordinal sum of the Archimedean semigroups* $\{([n(b_i), n(a_i)], G_i)/i \in j\}$, *where* $G_i = n \circ F_i \circ (n \times n)$.

Let $F \in \mathscr{F} - \mathscr{F}_A$ and $G \in \mathscr{G} - \mathscr{G}_A$ be ordinal sums determined by the Archimedean semigroups $\{([a_i, b_i], F_i)/i \in J\}$, and $\{([c_i, d_i], G_i)/i \in J\}$, respectively (both collections with identical index sets). Let f_i and g_i $(i \in J)$ be additive generators of F_i and G_i, respectively. Assume the additional condition

$$a_i < b_i \leqslant a_{i+1} < b_{i+1} \qquad \text{and} \qquad c_i < d_i \leqslant c_{i-1} < d_{i-1}. \qquad (**)$$

If F and G are n-duals for some $n \in S([0,1])$, then $n(E(F)) = E(G)$,

$n(a_i) = d$ and $n(b_i) = c_i$, for all $i \in J$ and there exists a sequence $(p_i)_{i \in J}$ in $(0, +\infty)$ such that, for all $i \in J$

$$n(x) = c_i + (d_i - c_i)\, g_i^{-1}[\, p_i f_i((x_i - a_i)/(b_i - a_i))] \qquad \text{for all} \quad x \in [a_i, b_i].$$

Note that in this case n is determined only in $\bigcup_{i \in J} [a_i, b_i]$.

3. Some Logical Properties of Nondistributive Connectives

As we pointed out in the introduction if $F \neq \min$ and $G \neq \max$, then $(\mathbf{P}(X), F, G)$ is not a lattice, but other properties for $\mathbf{P}(X)$ (e.g., Kleene's algebra (Cignoli [3]) can be considered for nonclassical connectives.

THEOREM 3.1. *If $F \in \mathscr{F}$, $G \in \mathscr{G}$, and $n \in S([0, 1])$, then Kleene's inequality $F(x, n(x)) \leqslant G(y, n(y))$ holds for all $x, y \in [0, 1]$.*

Proof. Let x_n be the fixed point of n. Then we have for all $x, y \in [0, 1]$,

$$F(x, n(x)) \leqslant \min(x, n(x)) \leqslant x_n \leqslant \max(y, n(y)) \leqslant G(y, n(y)).$$

In view of this inequality we can consider the parameter

$$K(F; n) = \inf\{n(F(x, n(x)) - F(x, n(x))/x \in [0, 1]\},$$

for given $F \in \mathscr{F}$ and $n \in S([0, 1])$. Then it is easy to prove:

(a) $0 \leqslant K(F; n) \leqslant 1$.

(b) $K(F; n) = 0$ if and only if $F(x_n, x_n) = x_n$, i.e., the fixed point x_n of n is an idempotent element of F.

(c) If $F \in \mathscr{F}_A$, then $K(F; n) > 0$.

(d) If $F \in \mathscr{F}_A$ and F is strictly increasing, then $K(F; n) < 1$.

Let $G \in \mathscr{G}_A$ and $F \in \mathscr{F}_A$ with respective additive generators g and f such that $g(1) < +\infty$ and $f(0) < +\infty$. Then G and F induce the strong negations n_G and n_F given by

$$n_G(x) = g^{(-1)}(g(1) - g(x)), \qquad n_F(x) = f^{(-1)}(f(0) - f(x)).$$

For example, the couple (1.1) induces $n_{F_m} = n_{G_m} = N$ and the couple (1.5) yields $n_{G_\lambda}(x) = (1 - x^\lambda)^{1/\lambda}$ and $n_{F_\lambda}(x) = 1 - (1 - (1 - x)^\lambda)^{1/\lambda}$.

THEOREM 3.2. *Let* $n \in S([0, 1])$, *let* $G \in \mathscr{G}_A$ *be such that* $g(1) < +\infty$ *and let* $F \in \mathscr{F}_A$ *be such that* $f(0) < +\infty$. *Then*:

(i) *G satisfies the* excluded middle *respect to n, i.e.,* $G(x, n(x)) = 1$ *for all x in* $[0, 1]$, *if and only if* $n \geqslant n_G$.

(ii) *F satisfies the* noncontradiction law, *i.e.,* $F(x, n(x)) = 0$ *for all x in* $[0, 1]$, *if and only if* $n \leqslant n_F$.

(iii) n_G *(resp.* n_F*) is the minimal negation of* $S([0, 1])$ *(resp. maximal) such that* $F'(x, y) = n_G(G(n_G(x), n_G(y)))$ *(resp.* $G'(x, y) = n_F(F(n_F(x), n_F(y)))$*) is* n_G*-dual of G and both G and F' satisfy* (i) *and* (ii) *(resp. is* n_F*-dual of F and both F and G' satisfy* (i) *and* (ii).

EXAMPLE 3.1. If G_λ is given by (1.5), then $n_{G_\lambda}(x) = (1 - x^\lambda)^{1/\lambda} > N(x)$, for all $\lambda > 1$, and $F'(x, y) = n_{G_\lambda}(G_\lambda(n_{G_\lambda}(x), n_{G_\lambda}(y))) = \max\{[(x^\lambda + y^\lambda) - 1]^{1/\lambda}, 0\}$ while F_λ and G_λ as given by (1.5) are N-duals. This example also shows that (i) will be satisfied only for negations of n such that $n \geqslant n_{G_\lambda} > N$.

In the particular case that F and G are n-duals then $n_G \circ n = n \circ n_F$, and $(G - n)$ satisfies (i) if and only if $(F - n)$ satisfies (ii). If $n_F = n_G$, then F and G can be n-duals only for $n = n_F = n_G$. Finally, we remark that $K(F; n) = 1$ if and only if $n \leqslant n_F$, i.e., n and F satisfy the "noncontradiction law."

Theorem 3.2 has some applications in the study of nonclassical implications, e.g., $I(x, y) = G(n(x), y)$ satisfies $I(x, x) = 1$, if and only if $n \geqslant n_G$ (when such n_G exists). (See Trillas *et al.* [16].)

4. FURTHER CHARACTERIZATION OF THE CLASSICAL CONNECTIVES

It is well known that min and max are n-duals for all $n \in S([0, 1])$. We shall show first that this fact characterizes these classical connectives:

THEOREM 4.1. *If* $F \in \mathscr{F}$ *and* $G \in \mathscr{G}$ *are n-duals for all* $n \in S([0, 1])$, *then necessarily* $F = \min$ *and* $G = \max$.

Proof. It is sufficient to show that $G(x_0, x_0) = x_0$ for all $x_0 \in (0, 1)$. Let $x_0 \in (0, 1)$ and consider two negations $n_1, n_2 \in S([0, 1])$ such that $n_1(x_0) = n_2(x_0) = x_0$ and $n_1(x) < n_2(x)$ for all $x \in [0, 1] - \{0, 1, x_0\}$. Then the duality conditions yields $n_1(G(x_0, x_0)) = F(n_1(x_0), n_1(x_0)) = F(x_0, x_0) = F(n_2(x_0), n_2(x_0)) = n_2(G(x_0, x_0))$, i.e., $G(x_0, x_0) \in \{0, 1, x_0\}$. But $G(x_0, x_0) \geqslant x_0 > 0$, i.e., $G(x_0, x_0)$ is either 1 or x_0. If $G(x_0, x_0) = 1$, we would have, taking the sequence of strong negations,

$$\begin{aligned} n_k(x) &= (x/(1-k)) + 1, && \text{if } 0 \leqslant x \leqslant 1 - (1/k), \\ &= (1-k)(x-1), && \text{if } 1 - (1/k) \leqslant x \leqslant 1, \end{aligned}$$

for all $k \in N$, $0 = n_k(1) = n_k G(x_0, x_0) = F(n_k(x_0), n_k(x_0))$, consequently,

$$\begin{aligned} 0 &= \lim_{k\to\infty} F(n_k(x_0), n_k(x_0)) \\ &= F(\lim_{k\to\infty} n_k(x_0), \lim_{k\to\infty} n_k(x_0)) = F(1, 1) = 1. \end{aligned}$$

This contradiction implies that $G(x_0, x_0) = x_0$ and "a fortiori" $G = \max$ and $F = \min$.

Now we shall give an equation which characterizes at the same time min, max, and $N = 1 - j$.

THEOREM 4.2. *Let $F \in \mathscr{F}$ and $n \in S([0, 1])$. Assume that $G = n \circ F \circ (n \times n)$, i.e., G is the dual operation of F via n. Then*

$$F(x, y) + G(x, y) - F(x, y)\, G(x, y) = 1 - n(x)\, n(y), \tag{4.1}$$

for all $x, y \in [0, 1]$, if and only if $F = \min$, $G = \max$, and $n = N$.

Proof. Substituting $y = 0$ in (4.1), we obtain $n(x) = 1 - x$. Thus (4.1) can be written as

$$F(x, y) + G(x, y) + xy = x + y + F(x, y)\, G(x, y),$$

which is equivalent (via $G(x, y) = 1 - F(1 - x, 1 - y)$) to

$$1 + xy - F(1 - x, 1 - y) = x + y - F(x, y)\, F(1 - x, 1 - y). \tag{4.2}$$

Choose any $t \in (0, 1)$. The substitution $x = y = t$ on (4.2) yields

$$1 + t^2 - F(1 - t, 1 - t) = 2t - F(t, t)\, F(1 - t, 1 - t), \tag{4.3}$$

and the substitution $x = y = 1 - t$ on (4.2) gives

$$1 + (1 - t)^2 - F(t, t) = 2 - 2t - F(1 - t, 1 - t)\, F(t, t). \tag{4.4}$$

Thus by (4.3), we have

$$F(1 - t, 1 - t) = (1 + t^2 - 2t)/(1 - F(t, t)), \tag{4.5}$$

and (4.5) together with (4.4) yield:

$$F(t, t)^2 - 2tF(t, t) + t^2 = 0,$$

whence $F(t, t) = t$ and $F = \min$. Note that this proof does not require either continuity assumptions or associative hypotheses.

Using a similar argument to that of Theorem 4.2 it is easy to prove the following characterization:

THEOREM 4.3. *Let* $F \in \mathscr{F}$ *and let* $G \in \mathscr{G}$ *be its N-dual,* $G(x, y) = 1 - F(1-x, 1-y)$. *Then*

$$F(x, y)\, G(x, y) = xy \qquad \text{for all} \quad x, y \in [0, 1],$$

if and only if $F = \min$ *and* $G = \max$.

Another characterization of the classical couple min–max, which is independent of the hypotheses on n-duality is

THEOREM 4.4. *Let* $F \in \mathscr{F}$ *and* $G \in \mathscr{G}$. *Then*

$$F(x, y) + G(x, y) = x + y, \qquad F(x, y)\, G(x, y) = xy$$
$$\text{for all} \quad x \in [0, 1],$$

if and only if $F = \min$ *and* $G = \max$.

Proof. Using the result of Frank [5] this theorem follows as a trivial corollary. We shall give a short argument. Take $x = y = t$. Then by the hypotheses, we have

$$F(t, t) + G(t, t) = 2t, \qquad F(t, t)\, G(t, t) = t^2,$$

whence (using the fact that $F(t, t) > 0$, for all $t > 0$) we have

$$F(t, t) + (t^2/F(t, t)) = 2t,$$

and it follows that $F(t, t) = t$, i.e., necessarily $F = \min$ and "a fortiori" $G = \max$.

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