Chaotic dynamics of a third-order Newton-type method

S. Amat\textsuperscript{a,*,1}, S. Busquier\textsuperscript{a,1}, S. Plaza\textsuperscript{b,2}

\textsuperscript{a} Departamento de Matemática Aplicada y Estadística, Universidad de Cartagena, Spain
\textsuperscript{b} Depto. de Matemáticas, Facultad de Ciencias, Universidad de Santiago de Chile, Casilla 307, Correo 2, Santiago, Chile

\textbf{A R T I C L E  I N F O}

Article history:
Received 8 September 2008
Submitted by Y. Huang

Keywords:
Iterative method
Dynamics
Real rational maps
Chaotic dynamical system

\textbf{A B S T R A C T}

The dynamics of a classical third-order Newton-type iterative method is studied when it is applied to degrees two and three polynomials. The method is free of second derivatives which is the main limitation of the classical third-order iterative schemes for systems. Moreover, each iteration consists only in two steps of Newton’s method having the same derivative. With these two properties the scheme becomes a real alternative to the classical Newton method. Affine conjugacy class of the method when is applied to a differentiable function is given. Chaotic dynamics have been investigated in several examples. Applying the root-finding method to a family of degree three polynomials, we have find a bifurcation diagram as those that appear in the bifurcation of the logistic map in the interval.

\textcopyright 2010 Elsevier Inc. All rights reserved.

1. Introduction

Let $f : \mathbb{R} \to \mathbb{R}$ be a $C^r$ function ($r \geq 1$). One of the most classical problems in numerical analysis is the solution of nonlinear equation $f(x) = 0$. To approximate the solution of these equations we can use iterative methods. An iterative method starts from an initial guess $x_0$, which is improved by means of an iteration, $x_{n+1} = \Phi (x_n)$. Conditions are imposed on $x_0$ and on the function $f$ to assure the convergence of the sequence $\{x_n\}_{n \geq 0}$ to a solution $x^*$ of the equation $f(x) = 0$ and to analyze the order of convergence.

Newton’s iterative method, $x_{n+1} = N_f(x_n) = x_n - \frac{f(x_n)}{f'(x_n)}$, and its variants are the most used and studied. We have that roots of $f$ are fixed points of $N_f$, and since $N_f'(x) = f(x)f''(x)/(f'(x))^2$ the simple roots, that is, roots of $f$ that are no critical points, are super-attracting fixed points of $N_f(x)$. This means that Newton’s method converges quadratically in a neighborhood of a simple root of $f$. Note that when $p$ is a polynomial, then its corresponding Newton’s iterative method is a rational map (quotient of two polynomials without common factors) on the real line.

The classical third-order methods require more computational cost than other simpler methods, which makes them disadvantageous to be used in general, only in some cases they should be considered [2,4–10]. However, in this paper we are interested in a third-order method free of second derivatives which is the main limitation of the classical third-order iterative schemes for systems. Moreover, each iteration consists in two steps of Newton’s method having the same derivative. In particular, if we consider a system of equations only one LU decomposition is necessary in each iteration. With these two properties the scheme can be considered a real alternative to the classical Newton method [1,3]. Our main interest...
This paper is the study of the dynamics of the discrete dynamical system defined by this method. In particular, we are interested to search chaos or sensitive dependence on initial conditions for this discrete dynamical system. In [12], Hurley and Martin showed that the classical Newton’s iterative method exhibits chaos for a large class of functions (see also [11,14,16]).

The paper is organized as follows. In Section 2 we introduce the scheme and present the associated Scaling Theorem. The Scaling Theorem allows up to suitable change of coordinates, to reduce the study of the dynamics of iterations of general maps, to the study of specific families of iterations of simpler maps. The dynamics for polynomials of degree two, having chaotic dynamics in some cases, are presented in Section 3. Finally, in Section 4 we study the dynamics for polynomials of degree three. In this case, we find a bifurcation diagram as those that appear in the bifurcation of the logistic map in the interval.

2. A third-order Newton-type iterative method

We are interested to analyze the dynamics of the following third-order iterative root-finding method [13,15,13]

\[
\begin{align*}
&\left\{ \begin{array}{l}
y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\
x_{n+1} = y_n - \frac{f(y_n)}{f'(x_n)}.
\end{array} \right.
\end{align*}
\]

This iterative root-finding method is defined by the following iterative function

\[
M_f(x) = N_f(x) - \frac{f(N_f(x))}{f'(x)},
\]

where \(N_f(x) = x - u_f(x)\) and \(u_f(x) = \frac{f(x)}{f'(x)}\). In other words, \(x_{n+1} = M_f(x_n)\).

It is easy to see that the roots of \(f\) are fixed points of \(M_f\). Recall that a root \(\alpha\) is simple if \(f'(\alpha) \neq 0\). We have that the simple roots of \(f\) are super-attracting fixed points of \(M_f\), that is, \(M'_f(\alpha) = 0\). In fact, we also have that \(M''_f(\alpha) = 0\). If a root \(\beta\) of \(f\) is not simple, then \(\beta\) is an attracting, but not super-attracting fixed point of \(M_f\), that is, \(0 < |M'_f(\beta)| < 1\).

We note that, \(M_f\) may have more fixed points than the roots of \(f\). These fixed points are called extraneous fixed points.

Our main interest in this paper is the study of the dynamics of the discrete dynamical system defined \(M_f\). In particular, we are interested to search chaos or sensitive dependence on initial conditions for this discrete dynamical system. In [12], Hurley and Martin showed that the classical Newton's iterative method \(N_f(x) = x - u_f(x)\), exhibits chaos for a large class of functions. The key of this behavior is the existence of points \(x^*\) where \(f'(x^*) = 0\) but \(f(x^*) \neq 0\). In this paper, we show that chaos appears when the iterative method \(M_f\) is applied to an one-parameter family of cubic polynomials. In fact, we have a bifurcating diagram (period doubling bifurcation diagram) in the parameter space, similar to those that appear in the parameter space for the logistic map \(q(x) = \mu x(1 - x)\).

When we apply the iterative method \(M_f\) to a polynomial we may have some problems, since we obtain a rational map, say \(M_g(x) = P(x)/Q(x)\), where \(P\) and \(Q\), are polynomials, which we may suppose without common factors. The difficulty arises at those points where the evaluation of numerator is non-zero and the denominator is zero, such points correspond to the poles of the iterative method. In order to study the dynamics of \(M_f\) by means of graphical analysis we consider it as a map \(\tilde{M}_f\) on \([0, 1]\). For this, let \(G: \mathbb{R} \to ]0, 1]\) be given by \(G(x) = \frac{1}{2} \arctan(x) + \frac{1}{2}\). This map is a homeomorphism from \(\mathbb{R}\) into \([0, 1]\). Define the maps \(\tilde{M}_f: [0, 1] \to [0, 1]\) by \(\tilde{M}_f(x) = (G \circ M_f \circ G^{-1})(x)\), where \(M_f\) is the iterative root-finding method introduced above for a map \(f: \mathbb{R} \to \mathbb{R}\). We may extend \(\tilde{M}_f\) to maps from \([0, 1]\) into itself. We use the same notation for this extension. Note that the extended function has fixed points at \(x = 0\) and at \(x = 1\), and that these fixed points are repelling.

Now we have the following useful result.

**Theorem 1 (The Scaling Theorem).** Let \(f(x)\) be an analytic function, and let \(T(x) = \alpha x + \beta\), with \(\alpha \neq 0\), be an affine map. Let \(g(x) = (f \circ T)(x)\). Then \(T \circ M_g \circ T^{-1}(x) = M_f(x)\), that is, \(M_f\) and \(M_g\) are affine conjugated by \(T\).

**Proof.** We have

\[
M_g(T^{-1}(x)) = T^{-1}(x) - u_g(T^{-1}(x)) - \frac{g(T^{-1}(x) - u_g(T^{-1}(x)))}{g'(T^{-1}(x))}.
\]

On the other hand, since \(g \circ T^{-1}(x) = f(x)\), and \((g \circ T^{-1})'(x) = \frac{1}{a} g'(T^{-1}(x))\), then \(g'(T^{-1}(x)) = \alpha (g \circ T^{-1})'(x) = \alpha f'(x)\), and by an easy induction process it follows that \(g^{(k)}(T^{-1}(x)) = \alpha^k f^{(k)}(x)\). Hence, \(u_g(T^{-1}(x)) = \frac{1}{a} u_f(x)\).
Substituting these quantities in $M_g(T^{-1}(x))$ we obtain

$$T \circ M_g \circ T^{-1}(x) = T(M_g(T^{-1}(x)))$$

$$= \alpha M_g(T^{-1}(x)) + \beta$$

$$= \alpha T^{-1}(x) - \alpha u_g(T^{-1}(x)) - \frac{\alpha g(T^{-1}(x) - u_g(T^{-1}(x)))}{g'(T^{-1}(x))} + \beta$$

$$= x - u_f(x) - \frac{g(T^{-1}(x) - \frac{1}{\alpha} u_f(x))}{f'(x)}.$$ 

Finally, by a comparison of Taylor series expansions of $f$ and $g$, we obtain

$$g \left( T^{-1}(x) - \frac{1}{\alpha} u_f(x) \right) = g(T^{-1}(x)) - g'(T^{-1}(x)) \frac{1}{\alpha} u_f(x) + \cdots$$

$$= f(x) - \alpha f'(x) \frac{1}{\alpha} u_f(x) + \cdots$$

$$= f(x - u_f(x)).$$

Therefore, $T \circ M_g \circ T^{-1}(x) = M_f(x)$.

This ends the proof. □

The theorem remains valid for $g(x) = c(f \circ T)(x)$, where $c$ is a non-zero constant.

The Scaling Theorem allows up to suitable change of coordinates, to reduce the study of the dynamics of iterations $M_f$, to the study of specific families of iterations of simpler maps. For example, each quadratic polynomial $f(x) = ax^2 + bx + c$, with $a \neq 0$, which we may suppose that is monic, by an affine change of variables reduces to one of the following polynomials $f_{-}(x) = x^2 - 1$, $f_0(x) = x^2$ or $f_{+}(x) = x^2 + 1$ if $f(x) = 0$ has two, one (double) or not real roots. Therefore, the study of the dynamics of $M_f$, when it is applied to a quadratic polynomial, reduces to the study of the dynamics of $M_f_{s}$, where $s = +, 0, −$, up to an affine change of variables. As we indicated above, the study of the dynamics of the iterative methods $M_f$ reduces to the study of the dynamics of the interval map $M_f_{s}$. Similarly, any cubic polynomial reduces to one of the simplest cubic polynomials $f_{-}(x) = x^3$, $f_0(x) = x^3 + x$, $f_{+}(x) = x^3 - x$ or to a member of the one-parameter family of cubic maps $f_{f'}(x) = x^3 + \gamma x + 1$. This is nothing but an appropriate rescaling that puts $M_f$ inside the conjugacy class.

3. Quadratic polynomials

Let $f(x)$ be a quadratic polynomial. By an affine change of variables the polynomial $f$ reduces to one of the following polynomials $f_{-}(x) = x^2 - 1$, $f_0(x) = x^2$ or $f_{+}(x) = x^2 + 1$ if $f$ has two, one (double) or not real roots. Therefore, the study the dynamics of $M_f$, reduces to the study of the dynamics of $M_f_{s}$, where $s = +, 0, −$, up to an affine change of variables. As we indicated, the study of the dynamics of the iterative method $M_f$ reduces to the study of the dynamics of the interval map $M_f_{s}$.

3.1. Case $f_0(x) = x^2$

In this case $M_{f_0}(x) = \frac{1}{2}x$. This function is a linear contraction. Therefore, its dynamics is trivial, the unique fixed point is $x = 0$, which is a global attractor, but not super-attracting.

3.2. Case $f_{-}(x) = x^2 - 1$

For $f_{-}(x) = x^2 - 1$, we have that $M_{f_{-}}(x) = \frac{3x^3 + 6x - 1}{8x^3}$. Now, the fixed points of $M_{1.f_{-}}$ are $x_{1,2} = \pm 1$ (the roots of $f_{-}$) which are super-attracting, and $x_{3,4} = \pm \frac{1}{2}\sqrt{5}$, which are extraneous fixed points. Since $M_{f_{-}}'(x) = \frac{3}{8}x^2 - 2x^2 + 1$, we have $M_{f_{-}}'(\pm \frac{1}{2}\sqrt{5}) = 6$, thus the two extraneous fixed points of $M_{f_{-}}$ are repelling. Note that $M_{f_{-}}(x)$ has a vertical asymptote at $x = 0$. 

Note: This is a partial transcription. The full context and mathematical details are essential for a complete understanding of the document.
Let \( I = [p_1, p_2] \) be the interval determined by the repelling fixed points \( p_1 = G(x_3) \) and \( p_2 = G(x_4) \) of \( \tilde{M}_f^- \). The following picture shows the restriction of \( \tilde{M}_f^- \) to the interval \( I \).

It is clear that \( \tilde{M}_f^- \) restricted to the interval \( I \) have an invariant Cantor set \( \Lambda_- \) of zero Lebesgue measure of non-escaping points, that is, \( \Lambda_- \) is the set of points whose orbits remain in the interval \( I \) under iteration by \( \tilde{M}_f^- \), in other words, these points are not attracted to one of the super-attracting fixed points of \( \tilde{M}_f^- \).

From the analysis of above we have that the orbit of any point \( x \in [0, 1] - (\Lambda_- \cup \{p_1, p_2\}) \) is attracted to one of the super-attracting fixed points of \( \tilde{M}_f^- \). Consequently, the orbit of any point in \( \mathbb{R} - (G^{-1}(\Lambda_-) \cup [x_3, x_4]) \) is attracted to one of the fixed points of \( M_f^- \).

### 3.3. Case \( f_+(x) = x^2 + 1 \)

In this case, the equation \( f_+ \) has not real roots, thus \( M_{f_+}(x) \) has not real fixed points and its dynamics is chaotic.
4. Cubic polynomials

We now consider cubic polynomials $f : \mathbb{R} \to \mathbb{R}$. By an affine change of coordinates $\tau(x) = \alpha x + \beta$, polynomial $f$ reduces to one of the simplest polynomials $f_*(x) = x^3$, $f_+(x) = x^3 + x$, $f_-(x) = x^3 - x$ or to a member of the one-parameter family of cubic polynomials $f_\gamma(x) = x^3 + \gamma x + 1$. Therefore, we analyze the dynamics of these simpler cubic polynomials, in the last case depending on the parameter $\gamma$.

4.1. Case $f_*(x) = x^3$

For $f_*(x) = x^3$, we have $M_{f_*}(x) = \frac{46}{27}x$, which is a linear contraction. Therefore it is dynamically trivial, since $x = 0$ is the unique fixed point, which is a global attractor, but not super-attractor.

4.2. Case $f_+(x) = x^3 + x$

In this case, $f_+$ has only one real root at $x = 0$.

We have

$$M_{f_+}(x) = \frac{2x^5(23x^4 + 18x^2 + 3)}{(3x^2 + 1)^2}.$$  

Since $f'_+(x) = 3x^2 + 1 > 0$ for all $x$, it follows that $f_+$ has no critical points, hence $M_{f_+}(x)$ is a global homeomorphism from $\mathbb{R}$ into itself.

Also we can see that there are not extraneous fixed points. In this case the dynamics of $M_{f_+}(x)$ is trivial, since the unique fixed point is $x = 0$ which is a global super-attracting fixed point.

4.3. Case $f_-(x) = x^3 - x$

In this case $M_{f_-}$ is given by

$$M_{f_-}(x) = \frac{46x^3 - 36x^7 + 6x^3}{(3x^2 - 1)^4}.$$  

The fixed points of $M_{f_-}$ are $x_1 = -1$, $x_0 = 0$, $x_2 = 1$ (the roots of $f$) which are super-attracting, and the extraneous fixed points $x_3 \approx -0.7941044878$, $x_4 \approx -0.4759631495$, $x_5 \approx -0.4472135955$, $x_6 \approx 0.4472125955$, $x_7 \approx 0.4759631455$ and $x_8 \approx 0.7941044878$. Evaluating $M'_{f_-}(x)$ at the points $x_j$ for $j = 3, \ldots, 8$ we find that $|M'_{f_-}(x_j)| > 1$. Therefore, the extraneous fixed points of $M_{f_-}$ are repelling. Now $M_{f_-}(x)$ has asymptotes at the points $a_1 = -\frac{\sqrt{3}}{2}$ and $a_2 = \frac{\sqrt{3}}{2}$, and we have $\lim_{a \to a_1^+} M_{f_-}(x) = +\infty$ and $\lim_{a \to a_2^+} M_{f_-}(x) = -\infty$.

In this case, the dynamics of $M_{f_-}$ is trivial.

4.4. Case $f_\gamma(x) = x^3 + \gamma x + 1$

For the sake of simplicity in the notation, we denote $M_\gamma$ for $M_{f_\gamma}$. Now, we have

$$M_\gamma(x) = \frac{46x^9 + 36\gamma x^7 - 42x^6 + 6\gamma^2 x^5 - 45\gamma x^4 - 6x^3 - 12\gamma^2 x^2 - \gamma^3 + 1}{(3x^2 + \gamma)^4}.$$  

We analyze the dynamics of the family of rational iterative root-finding methods $M_\gamma$ depending on the parameter $\gamma$. Now, we have

$$M'_\gamma(x) = \frac{6x(23x^3 + 51\gamma x^2 + 33\gamma^2 x^3 + 42x^6 + 48\gamma x^4 + 5\gamma^3 x^3 + 6\gamma^2 x^2 + 15x^3 - 3\gamma x - 4)}{(3x^2 + \gamma)^5}$$  

and

$$M''_\gamma(x) = -12(213\gamma^2 x^6 - 57\gamma^2 x^4 - 66\gamma x^2 + 3\gamma^2 x + 189x^5 + 135x^5 - 54x^2$$  

$$+ 38\gamma x^5 - 6\gamma^2 x^3 - 54\gamma^3 x^4 - 10\gamma^3 x^3 - 9\gamma^3 x^2 - 2\gamma + (3x^2 + \gamma)^6.$$

For $\gamma \neq 0$, let $m_\gamma = M_\gamma(0) = \frac{1 - \gamma^2}{\gamma^2}$. We have that $x = 0$ is a critical point with $m_\gamma$ its corresponding critical value of $M_\gamma$, which is a local maximum of $M_\gamma$ for $\gamma > 0$, since $M''_\gamma(0) = -\frac{24}{\gamma^2}$.
If the parameter $\gamma > 0$ decreases to zero, then the local maximum $m_\gamma$ is negative and increases. It is easy to see that there exists a parameter value $\gamma_{sn} \approx 1.00768$, such that for each $\gamma > \gamma_{sn}$, the iterative map $M_\gamma$ has a unique fixed point, $x_{sa,\gamma}$, corresponding to the unique real root of $f_\gamma$. Also we see that the iterates of any point of $\mathbb{R}$ under $M_\gamma$ converge to $x_{sa,\gamma}$, and therefore its dynamics is trivial in this case. Note that the function $\gamma \mapsto x_{sa,\gamma}$ is decreasing when $\gamma$ decreases to zero. The next picture shows a generic graph of $M_\gamma$ for $\gamma > \gamma_{sn}$. This configuration is the same for all parameters $\gamma > \gamma_{sn}$.

![Graph of $M_\gamma$ for $\gamma > \gamma_{sn}$](image1.png)

For all parameter value $\gamma = \gamma_{sn}$, appears other fixed point of $M_\gamma$. This new fixed point, denoted by $x_{sn,\gamma}$, is a saddle-node fixed point, that is, $M_\gamma'(x_{sn}) = 1$. Thus at the parameter $\gamma = \gamma_{sn}$ the iterative map $M_\gamma$ has two fixed points, one super-attracting and one saddle-node. The super-attracting one, $x_{sa,\gamma}$, corresponds to the unique root of $f_\gamma$ and the saddle-node $x_{sn,\gamma}$ is an extraneous fixed point. See the next picture.

![Graph of $M_{\gamma_{sn}}$](image2.png)

Let $p_{sn,\gamma} = G(x_{sn,\gamma})$ and $q_{sn,\gamma} = G(-x_{sn,\gamma})$. Define the interval $J_{sn,\gamma} = [p_{sn,\gamma}, q_{sn,\gamma}]$. Then, the restriction of $\tilde{M}_{\gamma_{sn},\gamma}$ to the interval $J_{sn,\gamma}$, has a unique fixed point at $p_{sn,\gamma}$ and $\tilde{M}_{\gamma_{sn},\gamma}'(p_{sn,\gamma}) = 1$. We have the following picture.

![Graph of restricted $\tilde{M}_{\gamma_{sn},\gamma}$](image3.png)
Let $I_{m,\gamma} = G^{-1}(J_{m,\gamma})$. We have that, for all $x \in \mathbb{R} - \bigcup_{n \geq 0} M_{\gamma}^{-n}(I_{m,\gamma})$, the iterates under $M_{\gamma}$ converge to the super-attracting fixed point $x_{sa,\gamma}$.

If we continue decreasing the values of $\gamma$, then saddle-node fixed point $x_{sn}$ bifurcates into two fixed points, one repelling and one attracting, we denote by $x_{r,\gamma}$ the repelling one and by $x_{a,\gamma}$ the attracting one.

Let $p_{r,\gamma} = G(x_{r,\gamma})$, $q_{r,\gamma} = G(-x_{r,\gamma})$ and $p_{a,\gamma} = G(x_{a,\gamma})$. Then the interval $J_{\gamma} = \{p_{r,\gamma}, q_{r,\gamma}\}$ contains the attracting fixed point $p_{a,\gamma}$ and $\tilde{M}_{\gamma\gamma}$ has a repelling fixed point at $p_{r,\gamma}$ and attracting fixed point at $p_{a,\gamma}$.

For $\gamma = 1$, the critical point $m_{\gamma}$ of $M_{\gamma}$ becomes a fixed point, in fact a super-attracting fixed point. Thus, for $\gamma < 1$, but near 1, the attracting fixed point $x_{a,\gamma}$ is such that $M'_{\gamma}(x_{a,\gamma}) < 0$, with $|M'_{\gamma}(x_{a,\gamma})| < 1$. If $\gamma < 1$ decreases then the derivative $M'_{\gamma}$ decreases also, and there exists a parameter value $\gamma = \gamma_{pd}$, such that $M'_{\gamma}(x_{a,\gamma}) = -1$, that is, we have a period doubling bifurcation.

We see from the above analysis that a period doubling bifurcating has begin to appear, which is shown in the next picture.

As above, let $p_{r,\gamma} = G(x_{r,\gamma})$ and $q_{r,\gamma} = G(-x_{r,\gamma})$. Now we have, varying the parameter $\gamma$, that there exists a set $\Gamma$ of positive Lebesgue measure such that for $\gamma \in \Gamma$ the map $M_{\gamma}$ admits a probability invariant measure, which is absolutely continuous with respect to the Lebesgue measure and also has positive Lyapunov exponents. This implies, in particular, that for these parameter values, $M_{\gamma}$ is chaotic.

If we continue decreasing $\gamma$, there exists a parameter value $\gamma_{T}$, such that $m_{\gamma} = q_{r,\gamma}$. For $\gamma_{T}$ we have $\tilde{M}_{\gamma_{T}}(p_{a,\gamma_{T}}) = p_{r,\gamma_{T}}$ and $\tilde{M}_{\gamma_{T}J_{\gamma_{T}}}$ has the same shape as the one-parameter map $q_{\mu}(x) = \mu x(1-x)$, for $\mu = 4$.

Now, for $0 < \gamma < \gamma_{T}$, an invariant Cantor set of non-escaping points appears. It is not difficult to show that these invariant Cantor set has zero Lebesgue measure. This invariant Cantor set of non-escaping points exists for all parameter values $\gamma$, with $0 < \gamma < \gamma_{T}$. 

\[ \tilde{M}_{\gamma_{T}} \text{ as a circle map restricted to } J_{\gamma_{T}} \]
Now, for \( \gamma = 0 \), \( f_0(x) = x^3 + 1 \). In this case, \( M_0 \) is given by
\[
M_0(x) = \frac{46x^9 - 42x^6 - 6x^3 + 1}{81x^8}.
\]
This map has one super-attracting fixed point at \( x_1 = -1 \). Moreover, there are other two fixed points which are extraneous repelling fixed points.

For \( \gamma < 0 \), let \( c_1 = \frac{(-3\gamma)^{1/2}}{3} \) be the positive critical point of \( f_\gamma \) and let \( \gamma_c \) be the parameter values \( \gamma_c = -\frac{12^{2/3}3^{1/3}}{4} \). This value of the parameter \( \gamma \) is the solution of the equation
\[
\frac{(-3\gamma)^{3/2}}{27} + \frac{\gamma(-3\gamma)^{1/2}}{3} + 1 = 0.
\]
Note that \( f_{\gamma_c} \) has a double root at \( c_1 \).
For $\gamma < 0$ the iterative method $M_\gamma(x)$ has two vertical asymptotes at $a_{1,\gamma} = -\sqrt{-\frac{3\gamma}{3}}$ and at $a_{2,\gamma} = \sqrt{-\frac{3\gamma}{3}}$, with $\lim_{x \to a_{1,\gamma}} M_\gamma(x) = \lim_{x \to a_{2,\gamma}} M_\gamma(x) = M_{\gamma}(x) = +\infty$. Also, note that $M_{\gamma}(0) = -\frac{1}{\sqrt{3}} + \frac{1}{\gamma^2}$ is the minimum of $M_{\gamma}(x)$ in the interval $[a_{1,\gamma}, a_{2,\gamma}]$. On the other hand, the fixed points of $M_{\gamma}(x)$ are $x_{a_{1,\gamma}} < 0$ which correspond to the roots of $f_\gamma$, and then super-attracting. Also there exist others two fixed points $x_{2,\gamma}$ and $x_{3,\gamma}$, with $x_{1,\gamma} < x_{2,\gamma} < a_{1,\gamma}$ and $a_{2,\gamma} < x_{3,\gamma}$. The fixed points $x_{2,\gamma}$ and $x_{3,\gamma}$ are extraneous fixed points, and they are repelling. This configuration for $M_{\gamma}(x)$ is the same for all parameter values $\gamma = \frac{2^{1/3} + 3^{1/3}}{2} < 0$.

For the parameter values $\gamma < 0$ ($f_\gamma$ has only one real root) $\gamma = \gamma_c$ ($f_\gamma$ has two real root, one of them, the positive one, is a double root) and for $\gamma < \gamma_c$ ($f_\gamma$ has three real roots).

Finally, for $\gamma < \gamma_c$ the iterative method $M_{\gamma}(x)$ has three super-attracting fixed point, three repelling fixed points and two vertical asymptotes.

This configuration remains valid for all $\gamma < \gamma_c$.

5. Conclusion

In this work we have studied the dynamics of a third-order method for approximating roots of nonlinear equations, and have shown that, as with Newton’s method, when applied to a one-parameter family of cubic polynomials, bifurcations and chaos appear. From the numerical point of view, this represents a great difficulty in order to determine the regions of convergence of a method to the roots of a nonlinear equation.

References