J. Math. Anal. Appl. 366 (2010) 24–32

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Chaotic dynamics of a third-order Newton-type method

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ARTICLE INFO

Article history:
Received 8 September 2008
Submitted by Y. Huang

Keywords:
Iterative method
Dynamics
Real rational maps
Chaotic dynamical system

ABSTRACT

The dynamics of a classical third-order Newton-type iterative method is studied when it is applied to degrees two and three polynomials. The method is free of second derivatives which is the main limitation of the classical third-order iterative schemes for systems. Moreover, each iteration consists only in two steps of Newton's method having the same derivative. With these two properties the scheme becomes a real alternative to the classical Newton method. Affine conjugacy class of the method when is applied to a differentiable function is given. Chaotic dynamics have been investigated in several examples. Applying the root-finding method to a family of degree three polynomials, we have find a bifurcation diagram as those that appear in the bifurcation of the logistic map in the interval.

1. Introduction

Let $f:\mathbb{R}\to\mathbb{R}$ be a C^r function ($r \geqslant 1$). One of the most classical problems in numerical analysis is the solution of nonlinear equation $f(x)=0$. To approximate the solution of these equations we can use iterative methods. An iterative method starts from an initial guess x_0, which is improved by means of an iteration, $x_{n+1}=\Phi(x_n)$. Conditions are imposed on x_0 and on the function f to assure the convergence of the sequence $\{x_n\}_{n\geqslant 0}$ to a solution x^* of the equation $f(x)=0$ and to analyze the order of convergence.

Newton's iterative method, $x_{n+1}=N_f(x_n)=x_n-\frac{f(x_n)}{f'(x_n)}$, and its variants are the most used and studied. We have that roots of f are fixed points of N_f, and since $N_f'(x)=f(x)f''(x)/(f'(x))^2$ the simple roots, that is, roots of f that are no critical points, are super-attracting fixed points of $N_f(x)$. This means that Newton's method converges quadratically in a neighborhood of a simple root of f. Note that when p is a polynomial, then its corresponding Newton's iterative method is a rational map (quotient of two polynomials without common factors) on the real line.

The classical third-order methods require more computational cost than other simpler methods, which makes them disadvantageous to be used in general, only in some cases they should be considered [2,4–10]. However, in this paper we are interested in a third-order method free of second derivatives which is the main limitation of the classical third-order iterative schemes for systems. Moreover, each iteration consists in two steps of Newton's method having the same derivative. In particular, if we consider a system of equations only one LU decomposition is necessary in each iteration. With these two properties the scheme can be considered a real alternative to the classical Newton method [1,3]. Our main interest

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[1] Research supported in part by the Spanish grants MTM2007-62945 and 08662/PI/08.
[2] Part of this work was supported by Fondecyt Grant 1095025.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2010.01.047

in this paper is the study of the dynamics of the discrete dynamical system defined by this method. In particular, we are interested to search *chaos* or *sensitive dependence on initial conditions* for this discrete dynamical system. In [12], Hurley and Martin showed that the classical Newton's iterative method exhibits chaos for a large class of functions (see also [11,14,16]).

The paper is organized as follows. In Section 2 we introduce the scheme and present the associated *Scaling Theorem*. The Scaling Theorem allows up to suitable change of coordinates, to reduce the study of the dynamics of iterations of general maps, to the study of specific families of iterations of simpler maps. The dynamics for polynomials of degree two, having chaotic dynamics in some cases, are presented in Section 3. Finally, in Section 4 we study the dynamics for polynomials of degree three. In this case, we find a bifurcation diagram as those that appear in the bifurcation of the logistic map in the interval.

2. A third-order Newton-type iterative method

We are interested to analyze the dynamics of the following third-order iterative root-finding method [13,15,1,3]

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - \dfrac{f(y_n)}{f'(x_n)}. \end{cases} \tag{1}$$

This iterative root-finding method is defined by the following iterative function

$$M_f(x) = N_f(x) - \frac{f(N_f(x))}{f'(x)}, \tag{2}$$

where $N_f(x) = x - uf(x)$ and $u_f(x) = \frac{f(x)}{f'(x)}$. In other words, $x_{n+1} = M_f(x_n)$.

It is easy to see that the roots of f are fixed points of M_f. Recall that a root α is simple if $f'(\alpha) \neq 0$. We have that the simple roots of f are super-attracting fixed points of M_f, that is, $M'_f(\alpha) = 0$. In fact, we also have that $M''_f(\alpha) = 0$. If a root β of f is not simple, then β is an attracting, but not super-attracting fixed point of M_f, that is, $0 < |M'_f(\beta)| < 1$.

We note that, M_f may have more fixed points than the roots of f. These fixed points are called *extraneous fixed points*.

Our main interest in this paper is the study of the dynamics of the discrete dynamical system defined M_f. In particular, we are interested to search *chaos* or *sensitive dependence on initial conditions* for this discrete dynamical system. In [12], Hurley and Martin showed that the classical Newton's iterative method $N_f(x) = x - u_f(x)$, exhibits chaos for a large class of functions. The key of this behavior is the existence of points x^* where $f'(x^*) = 0$ but $f(x^*) \neq 0$. In this paper, we show that chaos appears when the iterative method M_f is applied to an one-parameter family of cubic polynomials. In fact, we have a bifurcating diagram (period doubling bifurcation diagram) in the parameter space, similar to those that appear in the parameter space for the logistic map $q_\mu(x) = \mu x(1-x)$.

When we apply the iterative method M_f to a polynomial we may have some problems, since we obtain a rational map, say $M_p(x) = P(x)/Q(x)$, where P and Q, are polynomials, which we may suppose without common factors. The difficulty arises at those points where the evaluation of numerator is non-zero and the denominator is zero, such points correspond to the poles of the iterative method. In order to study the dynamics of M_f by means of graphical analysis we consider it as a map $\tilde{M}_f$ on $[0,1[$. For this, let $G:\mathbb{R} \to]0,1[$ be given by $G(x) = \frac{1}{\pi}\arctan(x) + \frac{1}{2}$. This map is a homeomorphism from $\mathbb{R}$ into $]0,1[$. Define the maps $\tilde{M}_f :]0,1[\to]0,1[$ by $\tilde{M}_f(x) = (G \circ M_f \circ G^{-1})(x)$, where M_f is the iterative root-finding method introduced above for a map $f:\mathbb{R} \to \mathbb{R}$. We may extend $\tilde{M}_f$ to maps from $[0,1]$ into itself. We use the same notation for this extension. Note that the extended function has fixed points at $x = 0$ and at $x = 1$, and that theses fixed points are repelling.

Now we have the following useful result.

Theorem 1 *(The Scaling Theorem). Let $f(x)$ be an analytic function, and let $T(x) = \alpha x + \beta$, with $\alpha \neq 0$, be an affine map. Let $g(x) = (f \circ T)(x)$. Then $T \circ M_g \circ T^{-1}(x) = M_f(x)$, that is, M_f and M_g are affine conjugated by T.*

Proof. We have

$$M_g\big(T^{-1}(x)\big) = T^{-1}(x) - u_g\big(T^{-1}(x)\big) - \frac{g(T^{-1}(x) - u_g(T^{-1}(x)))}{g'(T^{-1}(x))}.$$

On the other hand, since $g \circ T^{-1}(x) = f(x)$, and $(g \circ T^{-1})'(x) = \frac{1}{\alpha} g'(T^{-1}(x))$, then $g'(T^{-1}(x)) = \alpha (g \circ T^{-1})'(x) = \alpha f'(x)$, and by an easy induction process it follows that $g^{(k)}(T^{-1}(x)) = \alpha^k f^{(k)}(x)$. Hence, $u_g(T^{-1}(x)) = \frac{1}{\alpha} u_f(x)$.

Substituting these quantities in $M_g(T^{-1}(x))$ we obtain

$$\begin{aligned}
T \circ M_g \circ T^{-1}(x) &= T\big(M_g\big(T^{-1}(x)\big)\big) \\
&= \alpha M_g\big(T^{-1}(x)\big) + \beta \\
&= \alpha T^{-1}(x) - \alpha u_g\big(T^{-1}(x)\big) - \alpha \frac{g(T^{-1}(x) - u_g(T^{-1}(x)))}{g'(T^{-1}(x))} + \beta \\
&= x - u_f(x) - \frac{g(T^{-1}(x) - \frac{1}{\alpha} u_f(x))}{f'(x)}.
\end{aligned}$$

Finally, by a comparison of Taylor series expansions of f and g, we obtain

$$\begin{aligned}
g\left(T^{-1}(x) - \frac{1}{\alpha} u_f(x)\right) &= g\big(T^{-1}(x)\big) - g'\big(T^{-1}(x)\big)\frac{1}{\alpha} u_f(x) + \cdots \\
&= f(x) - \alpha f'(x) \frac{1}{\alpha} u_f(x) + \cdots \\
&= f\big(x - u_f(x)\big).
\end{aligned}$$

Therefore, $T \circ M_g \circ T^{-1}(x) = M_f(x)$.
This ends the proof. □

The theorem remains valid for $g(x) = c(f \circ T)(x)$, where c is a non-zero constant.

The Scaling Theorem allows up to suitable change of coordinates, to reduce the study of the dynamics of iterations M_f, to the study of specific families of iterations of simpler maps. For example, each quadratic polynomial $f(x) = ax^2 + bx + c$, with $a \neq 0$, which we may suppose that is monic, by an affine change of variables reduces to one of the following polynomials $f_-(x) = x^2 - 1$, $f_0(x) = x^2$ or $f_+(x) = x^2 + 1$ if $f(x) = 0$ has two, one (double) or not real roots. Therefore, the study of the dynamics of M_f, when it is applied to a quadratic polynomial, reduces to the study of the dynamics of M_{f_s}, where $s = +, 0, -$, up to an affine change of variables. As we indicated above, the study of the dynamics of the iterative methods M_f reduces to the study of the dynamics of the interval map $\tilde{M}_{f_s}$. Similarly, any cubic polynomial reduces to one of the simplest cubic polynomials $f_\bullet(x) = x^3$, $f_+(x) = x^3 + x$, $f_-(x) = x^3 - x$ or to a member of the one-parameter family of cubic maps $f_\gamma(x) = x^3 + \gamma x + 1$. This is nothing but an appropriate rescaling that puts M_f inside the conjugacy class.

3. Quadratic polynomials

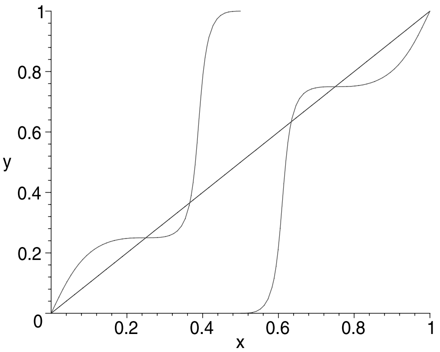
Let $f(x)$ be a quadratic polynomial. By an affine change of variables the polynomial f reduces to one of the following polynomials $f_-(x) = x^2 - 1$, $f_0(x) = x^2$ or $f_+(x) = x^2 + 1$ if f has two, one (double) or not real roots. Therefore, the study the dynamics of M_f, reduces to the study of the dynamics of M_{f_s}, where $s = +, 0, -$, up to an affine change of variables. As we indicated, the study of the dynamics of the iterative method M_f reduces to the study of the dynamics of the interval map $\tilde{M}_f$.

3.1. *Case $f_0(x) = x^2$*

In this case $M_{f_0}(x) = \frac{3}{8}x$. This function is a linear contraction. Therefore, its dynamics is trivial, the unique fixed point is $x = 0$, which is a global attractor, but not super-attracting.

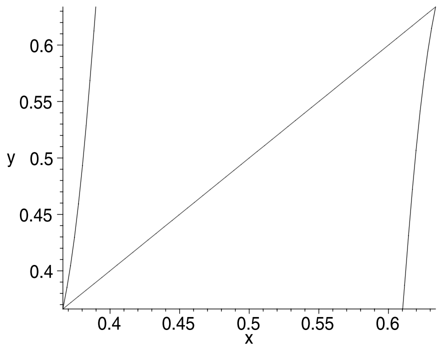
3.2. *Case $f_-(x) = x^2 - 1$*

For $f_-(x) = x^2 - 1$, we have that $M_{f_-}(x) = \frac{3x^4+6x-1}{8x^3}$. Now, the fixed points of M_{1,f_-} are $x_{1,2} = \pm 1$ (the roots of f_-) which are super-attracting, and $x_{3,4} = \pm\frac{1}{5}\sqrt{5}$, which are extraneous fixed points. Since $M'_{f_-}(x) = \frac{3}{8}\frac{x^4-2x^2+1}{x^4}$, we have $M'_{f_-}(\pm\frac{1}{5}\sqrt{5}) = 6$, thus the two extraneous fixed points of M_{f_-} are repelling. Note that $M_{f_-}(x)$ has a vertical asymptote at $x = 0$.



$\tilde{M}_{f_-}$ as a circle map

Let $I=[p_1,p_2]$ be the interval determined by the repelling fixed points $p_1=G(x_3)$ and $p_2=G(x_4)$ of $\tilde{M}_{f_-}$. The following picture shows the restriction of $\tilde{M}_{f_-}$ to the interval I.



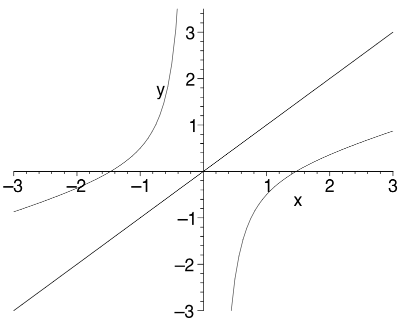
$\tilde{M}_{f_-}$ restricted to the interval $[p_1,p_2]$

It is clear that $\tilde{M}_{f_-}$ restricted to the interval I have an invariant Cantor set Λ_- of zero Lebesgue measure of non-escaping points, that is, Λ_- is the set of points whose orbits remain in the interval I under iteration by $\tilde{M}_{f_-}$, in other words, these points are not attracted to one of the super-attracting fixed points of $\tilde{M}_{f_-}$.

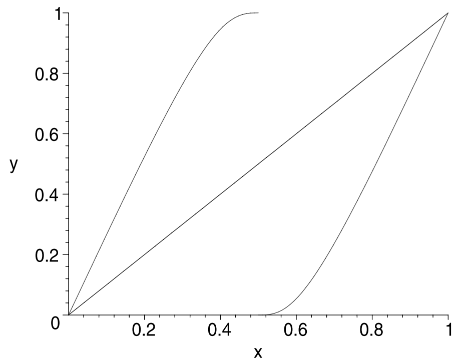
From the analysis of above we have that the orbit of any point $x\in[0,1]-(\Lambda_-\cup\{p_1,p_2\})$ is attracted to one of the super-attracting fixed points of $\tilde{M}_{f_-}$. Consequently, the orbit of any point in $\mathbb{R}-(G^{-1}(\Lambda_-)\cup\{x_3,x_4\})$ is attracted to one of the fixed points of M_{f_-}.

3.3. Case $f_+(x)=x^2+1$

In this case, the equation f_+ has not real roots, thus $M_{f_+}(x)$ has not real fixed points and its dynamics is chaotic.



M_f



$\tilde{M}_f$ as a circle map

4. Cubic polynomials

We now consider cubic polynomials $f:\mathbb{R}\to\mathbb{R}$. By an affine change of coordinates $\tau(x)=\alpha x+\beta$, polynomial f reduces to one of the simplest polynomials $f_\bullet(x)=x^3$, $f_+(x)=x^3+x$, $f_-(x)=x^3-x$ or to a member of the one-parameter family of cubic polynomials $f_\gamma(x)=x^3+\gamma x+1$. Therefore, we analyze the dynamics of these simpler cubic polynomials, in the last case depending on the parameter γ.

4.1. Case $f_\bullet(x)=x^3$

For $f_\bullet(x)=x^3$, we have $M_{f_\bullet}(x)=\frac{46}{81}x$, which is a linear contraction. Therefore it is dynamically trivial, since $x=0$ is the unique fixed point, which is a global attractor, but not super-attractor.

4.2. Case $f_+(x)=x^3+x$

In this case, f_+ has only one real root at $x=0$.

We have

$$M_{f_+}(x)=\frac{2x^5(23x^4+18x^2+3)}{(3x^2+1)^2}.$$

Since $f_+'(x)=3x^2+1>0$ for all x, it follows that f_+ has no critical points, hence $M_{f_+}(x)$ is a global homeomorphism form $\mathbb{R}$ into itself.

Also we can see that there are not extraneous fixed points. In this case the dynamics of $M_{f_+}(x)$ is trivial, since the unique fixed point is $x=0$ which is a global super-attracting fixed point.

4.3. Case $f_-(x)=x^3-x$

In this case M_{f_-} is given by

$$M_{f_-}(x)=\frac{46x^9-36x^7+6x^5}{(3x^2-1)^4}.$$

The fixed points of M_{f_-} are $x_1=-1$, $x_0=0$, $x_2=1$ (the roots of f) which are super-attracting, and the extraneous fixed points $x_3\approx-0.7941044878$, $x_4\approx-0.4759631495$, $x_5\approx-0.4472135955$, $x_6\approx0.4472125955$, $x_7\approx0.4759631455$ and $x_8\approx0.7941044878$. Evaluating $M_{f_-}'(x)$ at the points x_j for $j=3,\dots,8$ we find that $|M_{f_-}'(x_j)|>1$. Therefore, the extraneous fixed points of M_{f_-} are repelling. Now $M_{f_-}(x)$ has asymptotes at the points $a_1=-\frac{\sqrt{3}}{3}$ and $a_2=\frac{\sqrt{3}}{3}$, and we have $\lim_{a\to a_1^\pm}M_{f_-}(x)=+\infty$ and $\lim_{a\to a_2^\pm}M_{f_-}(x)=-\infty$.

In this case, the dynamics of M_{f_-} is trivial.

4.4. Case $f_\gamma(x)=x^3+\gamma x+1$

For the sake of simplicity in the notation, we denote M_γ for M_{f_γ}. Now, we have

$$M_\gamma(x)=\frac{46x^9+36\gamma x^7-42x^6+6\gamma^2x^5-45\gamma x^4-6x^3-12\gamma^2x^2-\gamma^3+1}{(3x^2+\gamma)^4}.$$

We analyze the dynamics of the family of rational iterative root-finding methods M_γ depending on the parameter γ. Now, we have

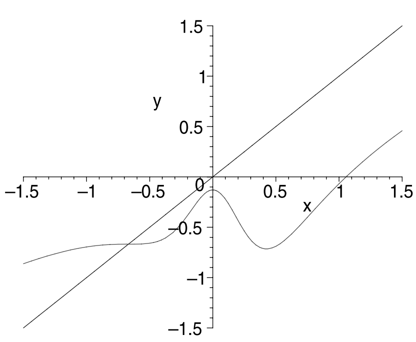
$$M_\gamma'(x)=\frac{6x(23x^9+51\gamma x^7+33\gamma^2x^5+42x^6+48\gamma x^4+5\gamma^3x^3+6\gamma^2x^2+15x^3-3\gamma x-4)}{(3x^2+\gamma)^5}$$

and

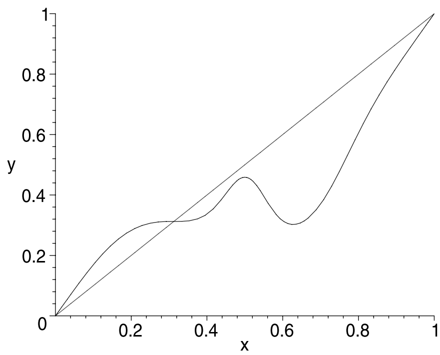
$$\begin{aligned}M_\gamma''(x)=-12\big(&213\gamma x^6-57\gamma^2x^4-66\gamma x^3+3\gamma^2x+189x^8+135x^5-54x^2\\&+38\gamma x^9-6\gamma^2x^7-54\gamma^3x^5-10\gamma^4x^3-9\gamma^3x^2+2\gamma\big)/\big(3x^2+\gamma\big)^6.\end{aligned}$$

For $\gamma\neq0$, let $m_\gamma=M_\gamma(0)=\frac{1-\gamma^3}{\gamma^4}$. We have that $x=0$ is a critical point with m_γ its corresponding critical value of M_γ, which is a local maximum of M_γ for $\gamma>0$, since $M_\gamma''(0)=-\frac{24}{\gamma^5}$.

If the parameter $\gamma > 0$ decreases to zero, then the local maximum m_γ is negative and increases. It is easy to see that there exists a parameter value $\gamma_{sn} \approx 1.00768$, such that for each $\gamma > \gamma_{sn}$, the iterative map M_γ has a unique fixed point, $x_{sa,\gamma}$, corresponding to the unique real root of f_γ. Also we see that the iterates of any point of $\mathbb{R}$ under M_γ converge to $x_{sa,\gamma}$, and therefore its dynamics is trivial in this case. Note that the function $\gamma \mapsto x_{sa,\gamma}$ is decreasing when γ decreases to zero. The next picture shows a generic graph of M_γ for $\gamma > \gamma_{sn}$. This configuration is the same for all parameters $\gamma > \gamma_{sn}$.

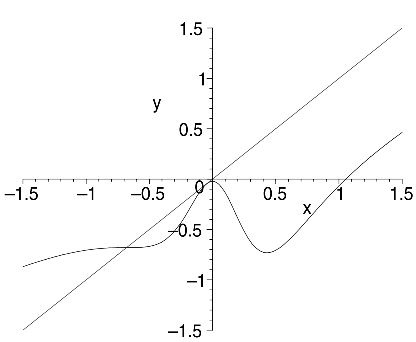


M_γ

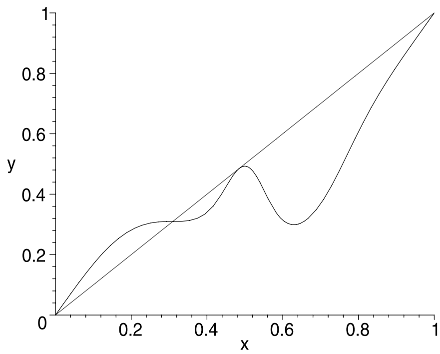


$\tilde{M}_\gamma$ as a circle map

For all parameter value $\gamma = \gamma_{sn}$, appears other fixed point of M_γ. This new fixed point, denoted by $x_{sn,\gamma}$, is a saddle-node fixed point, that is, $M'_\gamma(x_{sn}) = 1$. Thus at the parameter $\gamma = \gamma_{sn}$ the iterative map M_γ has two fixed points, one super-attracting and one saddle-node. The super-attracting one, $x_{sa,\gamma}$, corresponds to the unique root of f_γ and the saddle-node $x_{sn,\gamma}$ is an extraneous fixed point. See the next picture.

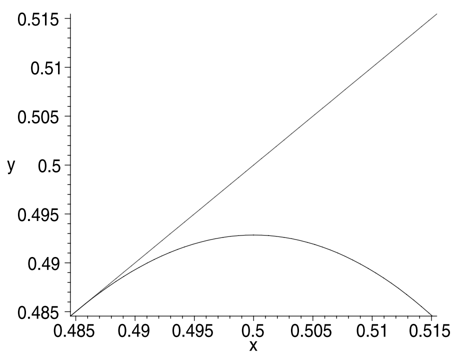


$M_{\gamma_{sn}}$



$\tilde{M}_{\gamma_{sn}}$ as a circle map

Let $p_{sn,\gamma} = G(x_{sn,\gamma})$ and $q_{sn,\gamma} = G(-x_{sn,\gamma})$. Define the interval $J_{sn,\gamma} = [p_{sn,\gamma}, q_{sn,\gamma}]$. Then, the restriction of $\tilde{M}_{\gamma_{sn,\gamma}}$ to the interval $J_{sn,\gamma}$, has a unique fixed point at $p_{sn,\gamma}$ and $\tilde{M}'_{\gamma_{sn,\gamma}}(p_{sn,\gamma}) = 1$. We have the following picture.



$\tilde{M}_{\gamma_{sn,\gamma}}$ restricted to the interval $J_{sn,\gamma}$

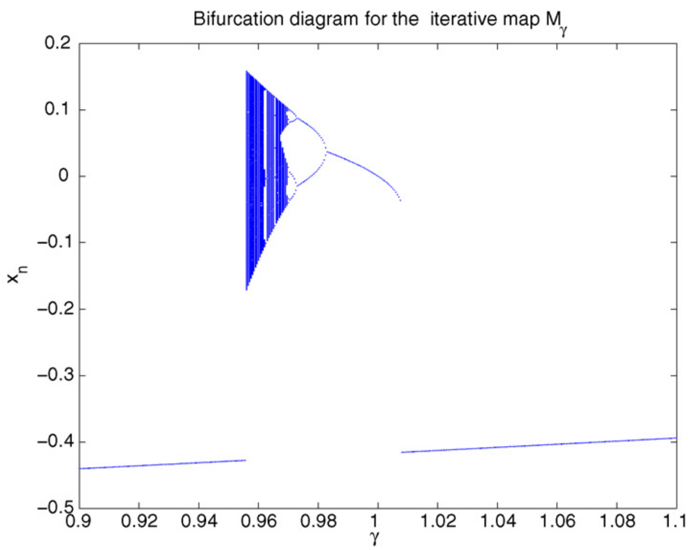
Let $I_{sn,\gamma} = G^{-1}(J_{sn,\gamma})$. We have that, for all $x \in \mathbb{R} - \bigcup_{n\geqslant 0} M_\gamma^{-n}(I_{sn,\gamma})$, the iterates under M_γ converge to the super-attracting fixed point $x_{sa,\gamma}$.

If we continue to decreasing the values of γ, then saddle-node fixed point x_{sn} bifurcates into two fixed points, one repelling and one attracting, we denote by $x_{r,\gamma}$ the repelling one and by $x_{a,\gamma}$ the attracting one.

Let $p_{r,\gamma} = G(x_{r,\gamma})$, $q_{r,\gamma} = G(-x_{r,\gamma})$ and $p_{a,\gamma} = G(x_{a,\gamma})$. Then the interval $J_\gamma = [p_{r,\gamma}, q_{r,\gamma}]$ contains the attracting fixed point $p_{a,\gamma}$ and $\tilde{M}_{\gamma|J_\gamma}$ has a repelling fixed point at $p_{r,\gamma}$ and attracting fixed point at $p_{a,\gamma}$.

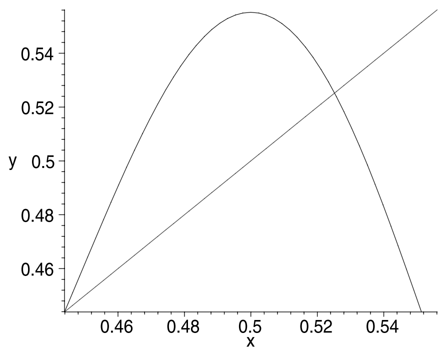
For $\gamma = 1$, the critical point m_γ of M_γ becomes a fixed point, in fact a super-attracting fixed point. Thus, for $\gamma < 1$, but near 1, the attracting fixed point $x_{a,\gamma}$ is such that $M'_\gamma(x_{a,\gamma}) < 0$, with $|M'_\gamma(x_{a,\gamma})| < 1$. If $\gamma < 1$ decreases then the derivative M'_γ decreases also, and there exists a parameter value $\gamma = \gamma_{pd}$, such that $M'_{\gamma_{pd}}(x_{a,\gamma_{pd}}) = -1$, that is, we have a period doubling bifurcation.

We see from the above analysis that a period doubling bifurcating has begin to appear, which is shown in the next picture.



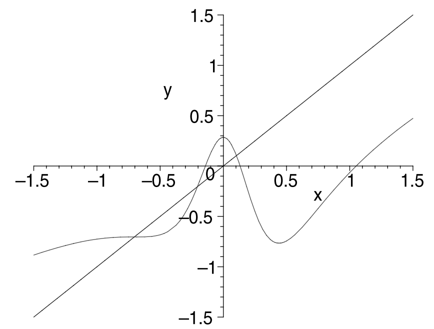
As above, let $p_{r,\gamma} = G(x_{r,\gamma})$ and $q_{r,\gamma} = G(-x_{r,\gamma})$. Now we have, varying the parameter γ, that there exists a set Γ of positive Lebesgue measure such that for $\gamma \in \Gamma$ the map M_γ admits a probability invariant measure, which is absolutely continuous with respect to the Lebesgue measure and also has positive Lyapunov exponents. This implies, in particular, that for these parameter values, M_γ is chaotic.

If we continue decreasing γ, there exists a parameter value γ_T, such that $m_{\gamma_T} = q_{r,\gamma}$. For γ_T we have $\tilde{M}_{\gamma_T}(p_{a,\gamma_T}) = p_{r,\gamma_T}$ and $\tilde{M}_{\gamma_T\, J_{\gamma_T}}$ has the same shape as the one-parameter map $q_\mu(x) = \mu x(1-x)$, for $\mu = 4$.

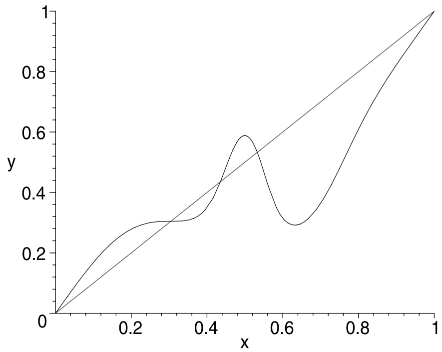


$\tilde{M}_{\gamma_T}$ as a circle map resctricted to J_{γ_T}

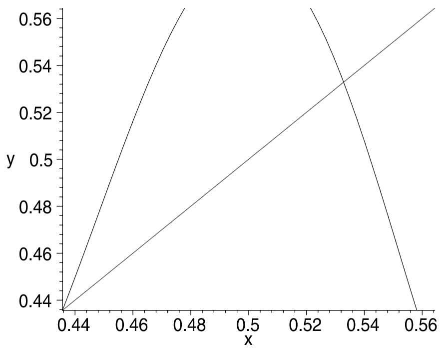
Now, for $0 < \gamma < \gamma_T$, an invariant Cantor set of non-escaping points appears. It is not difficult to show that these invariant Cantor set has zero Lebesgue measure. This invariant Cantor set of non-escaping points exists for all parameter values γ, with $0 < \gamma < \gamma_T$.



M_γ for $0 < \gamma < \gamma_T$, but γ not so small



$\tilde{M}_\gamma$ as a circle map

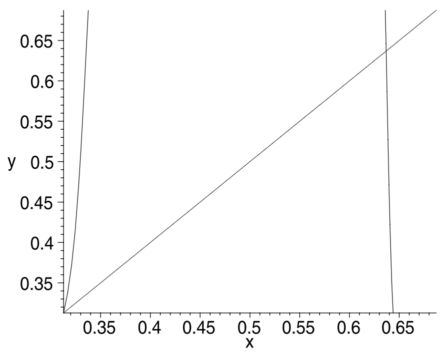


$\tilde{M}_\gamma$ as a circle map resctricted to J_γ

Now, for $\gamma = 0$, $f_0(x) = x^3 + 1$. In this case, M_0 is given by

$$M_0(x) = \frac{46x^9 - 42x^6 - 6x^3 + 1}{81x^8}.$$

This map has one super-attracting fixed point at $x_1 = -1$. Moreover, there are other two fixed points which are extraneous repelling fixed points.



$\tilde{M}_\gamma$ as a circle map resctricted to J_γ

has an invariant Cantor set of zero Lebesgue measure

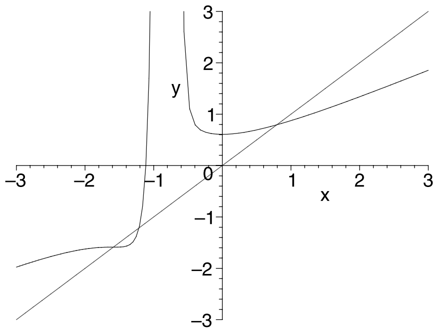
of points which do not converge to the super–attracting

fixed point of $\tilde{M}_\gamma$

For $\gamma < 0$, let $c_1 = \frac{(-3\gamma)^{1/2}}{3}$ be the positive critical point of f_γ and let γ_c be the parameter values $\gamma_c = -\frac{(12)^{2/3}3^{1/3}}{4}$. This value of the parameter γ is the solution of the equation

$$\frac{(-3\gamma)^{3/2}}{27} + \frac{\gamma(-3\gamma)^{1/2}}{3} + 1 = 0.$$

Note that f_{γ_c} has a double root at c_1.



M_{γ_c}

For $\gamma_c < \gamma < 0$ the iterative method $M_\gamma(x)$ has two vertical asymptotes at $a_{1,\gamma} = -\frac{\sqrt{-3\gamma}}{3}$ and at $a_{2,\gamma} = \frac{\sqrt{-3\gamma}}{3}$, with $\lim_{x\to a_{1,\gamma}^-} M_\gamma(x) = \lim_{x\to a_{1,\gamma}^+} M_\gamma(x) = \lim_{x\to a_{2,\gamma}^-} M_\gamma(x) = \lim_{x\to a_{2,\gamma}^+} M_\gamma(x) = +\infty$. Also, note that $M_\gamma(0) = -\frac{1}{\gamma} + \frac{1}{\gamma^4}$ is the minimum of $M_\gamma(x)$ in the interval $]a_{1,\gamma}, a_{2,\gamma}[$. On the other hand, the fixed points of $M_\gamma(x)$ are $x_{sa,\gamma} < 0$ which correspond to the roots of f_γ, and then super-attracting. Also there exist others two fixed points $x_{2,\gamma}$ and $x_{3,\gamma}$, with $x_{1,\gamma} < x_{2,\gamma} < a_{1,\gamma}$ and $a_{2,\gamma} < x_{3,\gamma}$. The fixed points $x_{2,\gamma}$ and $x_{3,\gamma}$ are extraneous fixed points, and they are repelling. This configuration for $M_\gamma(x)$ is the same for all parameter values $\gamma_c = -\frac{(12)^{2/3}3^{1/3}}{4} < \gamma < 0$.

For the parameter values $\gamma_c < \gamma < 0$ (f_γ has only one real root) $\gamma = \gamma_c$ (f_γ has two real root, one of them, the positive one, is a double root) and for $\gamma < \gamma_c$ (f_γ has three real roots).

Finally, for $\gamma < \gamma_c$ the iterative method $M_\gamma(x)$ has three super-attracting fixed point, three repelling fixed points and two vertical asymptotes.

This configuration remains valid for all $\gamma < \gamma_c$.

5. Conclusion

In this work we have studied the dynamics of a third-order method for approximating roots of nonlinear equations, and have shown that, as with Newton's method, when applied to a one-parameter family of cubic polynomials, bifurcations and chaos appear. From the numerical point of view, this represents a great difficulty in order to determine the regions of convergence of a method to the roots of a nonlinear equation.

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