Nonlinearization of the Lax pairs for discrete Ablowitz–Ladik hierarchy

Xianguo Geng a,*, H.H. Dai b

a Department of Mathematics, Zhengzhou University, Zhengzhou, Henan 450052, People’s Republic of China
b Department of Mathematics, City University of Hong Kong, 83 Tat Chee Avenue, Kowloon, Hong Kong, People’s Republic of China

Received 23 December 2005
Available online 30 May 2006
Submitted by Steven G. Krantz

Abstract

The discrete Ablowitz–Ladik hierarchy with four potentials and the Hamiltonian structures are derived. Under a constraint between the potentials and eigenfunctions, the nonlinearization of the Lax pairs associated with the discrete Ablowitz–Ladik hierarchy leads to a new symplectic map and a class of finite-dimensional Hamiltonian systems. The generating function of the integrals of motion is presented, by which the symplectic map and these finite-dimensional Hamiltonian systems are further proved to be completely integrable in the Liouville sense. Each member in the discrete Ablowitz–Ladik hierarchy is decomposed into a Hamiltonian system of ordinary differential equations plus the discrete flow generated by the symplectic map.

© 2006 Elsevier Inc. All rights reserved.

Keywords: Discrete Ablowitz–Ladik hierarchy; Nonlinearization of the Lax pairs

1. Introduction

During the recent decades, the study of soliton equations is one of the most prominent events in the field of nonlinear sciences. Fairly satisfied understanding has been got for nonlinear integrable models. Quite a few methods of solution of such soliton equations are well established, such as the inverse scattering transformation, the bilinear transformation methods of Hirota, the...
dressing method, the Bäcklund and the Darboux transformation, the algebraic curve method, and so on [1–5].

A systemic approach, so-called nonlinearization of eigenvalue problems or Lax pairs, was developed in Refs. [6–8]. This approach offers an effective way to generate new finite-dimensional integrable Hamiltonian systems from soliton equations. It is always interesting to decompose a nonlinear partial differential equation into a pair of systems of ordinary differential equations both from theoretical point and practical point of view. The nonlinearization approach can decompose soliton equations into temporal and spatial finite-dimensional Hamiltonian systems, which also makes it very natural to compute solutions of soliton equations numerically, for instance, by the symplectic method. Some physically important solutions of soliton equations, including soliton solutions, periodic solutions and quasi-periodic solutions, may be engendered through solving the compatible system of ordinary differential equations [9–12]. Therefore it may give out a direct way to observe evolution behavior of nonlinear phenomena. Subsequently the nonlinearization approach has been applied to discrete soliton equations [13–18], for example, Toda lattice, Kac–van Moerbeke lattice, and others [9–22].

The main aim of the present paper is that the nonlinearization approach is developed and applied to the discrete Ablowitz–Ladik hierarchy with four potentials. We first derive the discrete Ablowitz–Ladik hierarchy with four potentials associated with a spectral problem and their Hamiltonian structures. Then we propose a constraint between the potentials and eigenfunctions. The nonlinearization of the Lax pairs for the discrete Ablowitz–Ladik hierarchy leads to a new integrable symplectic map and a class of finite-dimensional integrable Hamiltonian systems. As an application, solutions of the discrete Ablowitz–Ladik hierarchy are decomposed into solving compatible Hamiltonian systems of ordinary differential equations plus the discrete flow generated by the symplectic map.

The outline of the paper is as follows. In Section 2, with the aid of the decomposition of zero-curvature equation, we shall construct the discrete Ablowitz–Ladik hierarchy with four potentials. In Section 3, we derive the Lenard gradients and establish Hamiltonian structures of the discrete Ablowitz–Ladik hierarchy. In Section 4, we introduce the Bargmann constraint between the potentials and eigenfunctions, from which a new symplectic map and a class of finite-dimensional Hamiltonian systems are obtained. The generating function approach is used to calculate the involutivity of integrals, by which the symplectic map and these finite-dimensional Hamiltonian systems are further proved to be completely integrable in the Liouville sense. In Section 5, the discrete Ablowitz–Ladik hierarchy is decomposed into the integrable symplectic map and the compatible Hamiltonian system of ordinary differential equations.

2. The hierarchy of differential-difference equations

In this section, we shall derive the discrete Ablowitz–Ladik hierarchy associated with a discrete spectral problem with four potentials [23]

\[
\chi(n+1) = U_n \chi(n), \quad U_n = \frac{1}{\gamma_n \pi_n} \left( \begin{array}{cc}
\lambda + R_n S_n & Q_n + \lambda^{-1} S_n \\
R_n + \lambda T_n & \lambda^{-1} + Q_n T_n
\end{array} \right),
\]

where \(Q_n, R_n, S_n\) and \(T_n\) are four potentials, \(\lambda\) is a constant spectral parameter,

\[
\gamma_n = \sqrt{1 - Q_n R_n}, \quad \pi_n = \sqrt{1 - S_n T_n}, \quad \chi(n) = (\chi_1(n), \chi_2(n))^T.
\]

A number of research on the discrete Ablowitz–Ladik hierarchy has been conducted. For example, its inverse scattering transform, soliton solutions, the Hamiltonian structures, Darboux
transformation, and others have been discussed in Refs. [23–27]. In what follows, we first give solutions of the stationary discrete zero-curvature equation. Then we construct the discrete Ablowitz–Ladik hierarchy.

**Proposition 1.** Let two matrices \( V_n \) and \( \hat{V}_n \) satisfy

\[
\hat{V}_n U_n^{(1)} - U_n^{(1)} V_n = 0, \quad V_{n+1} U_n^{(2)} - U_n^{(2)} \hat{V}_n = 0,
\]

where \((z^2 = \lambda)\)

\[
U_n^{(1)} = \frac{1}{y_n} \begin{pmatrix} z & z^{-1} Q_n \\ z R_n & z^{-1} \end{pmatrix}, \quad U_n^{(2)} = \frac{1}{\pi_n} \begin{pmatrix} z & z^{-1} S_n \\ z T_n & z^{-1} \end{pmatrix}.
\]

Then \( V_n \) satisfies the discrete stationary zero-curvature equation

\[
V_{n+1} U_n - U_n V_n = 0,
\]

and \( \det V_n \) and \( \det \hat{V}_n \) are constants independent of \( n \).

**Proof.** Noticing \( U_n = U_n^{(2)} U_n^{(1)} \), we have

\[
V_{n+1} U_n - U_n V_n = \left( V_{n+1} U_n^{(2)} - U_n^{(2)} \hat{V}_n \right) U_n^{(1)} + U_n^{(2)} \left( \hat{V}_n U_n^{(1)} - U_n^{(1)} V_n \right) = 0.
\]

By (2.2), we arrive at \( \det V_{n+1} = \det \hat{V}_n \) and \( \det \hat{V}_n = \det V_n \), which imply that \( \det V_n \) and \( \det \hat{V}_n \) are constants independent of \( n \). The proof is completed. \( \square \)

Assume that solutions \( V_n \) and \( \hat{V}_n \) of (2.2) take form

\[
V_n = \begin{pmatrix} A_n & B_n \\ C_n & -A_n \end{pmatrix}, \quad \hat{V}_n = \begin{pmatrix} a_n & b_n \\ c_n & -a_n \end{pmatrix}.
\]

Then (2.2) can be written as

\[
\begin{align*}
& a_n - A_n + R_n b_n - \lambda^{-1} Q_n C_n = 0, \\
& b_n - \lambda B_n + Q_n (a_n + A_n) = 0, \\
& c_n - \lambda^{-1} C_n - R_n (a_n + A_n) = 0, \\
& A_n - a_n + Q_n c_n - \lambda R_n B_n = 0, \\
& A_{n+1} - a_n + T_n B_{n+1} - \lambda^{-1} S_n c_n = 0, \\
& B_{n+1} - \lambda b_n + S_n (a_n + A_{n+1}) = 0, \\
& C_{n+1} - \lambda^{-1} c_n - T_n (a_n + A_{n+1}) = 0, \\
& a_n - A_{n+1} - T_n b_n \lambda + S_n C_{n+1} = 0.
\end{align*}
\]

(2.6)

Let

\[
\begin{align*}
& A_n = \sum_{j=0}^{\infty} A_n^{(j)} \lambda^{-j}, \quad B_n = \sum_{j=0}^{\infty} B_n^{(j)} \lambda^{-j}, \quad C_n = \sum_{j=0}^{\infty} C_n^{(j)} \lambda^{-j}, \\
& a_n = \sum_{j=0}^{\infty} a_n^{(j)} \lambda^{-j}, \quad b_n = \sum_{j=0}^{\infty} b_n^{(j)} \lambda^{-j}, \quad c_n = \sum_{j=0}^{\infty} c_n^{(j)} \lambda^{-j}.
\end{align*}
\]

(2.8)
with the condition
\[ A_n^2 + B_nC_n = \alpha_0^2, \quad \alpha_0 = 2\alpha_0, \]  
\[ a_n^2 + b_n = \alpha_0^2, \]  
\[ (2.9) \]
where \( \alpha_0 \) is a constant. Substituting (2.8) into (2.6), (2.7) and (2.9) and comparing the coefficients of the same power for \( \lambda \), we obtain
\[ A_n^{(0)} = \alpha_0, \quad B_n^{(0)} = 0, \quad C_n^{(0)} = 2\alpha_0T_{n-1}, \]  
\[ a_n^{(0)} = \alpha_0, \quad b_n^{(0)} = 0, \quad c_n^{(0)} = 2\alpha_0R_n, \]  
\[ \alpha_n^{(j)} - \alpha_n^{(j)} + R_n\beta_n^{(j)} - Q_n\gamma_n^{(j-1)} = 0, \]  
\[ b_n^{(j)} - B_n^{(j+1)} + Q_n(a_n^{(j)} + \alpha_n^{(j)}) = 0, \]  
\[ c_n^{(j)} - C_n^{(j-1)} - R_n(a_n^{(j)} + \alpha_n^{(j)}) = 0, \]  
\[ A_n^{(j)} - \alpha_n^{(j)} + Q_n\beta_n^{(j)} - R_nB_n^{(j+1)} = 0, \]  
\[ A_n^{(j+1)} - a_n^{(j)} + T_nB_{n+1} - S_n\gamma_n^{(j-1)} = 0, \]  
\[ B_n^{(j+1)} - b_n^{(j+1)} + S_n(a_n^{(j)} + \alpha_n^{(j+1)}) = 0, \]  
\[ C_n^{(j+1)} - c_n^{(j+1)} - T_n(a_n^{(j)} + \alpha_n^{(j+1)}) = 0, \]  
\[ a_n^{(j)} - A_n^{(j)} - T_nB_n^{(j+1)} + S_nC_n^{(j+1)} = 0, \]  
\[ \sum_{k=0}^{j} A_n^{(k)} \alpha_n^{(j-k)} + \sum_{k=0}^{j} B_n^{(k)} C_n^{(j-k)} = 0, \]  
\[ \sum_{k=0}^{j} a_n^{(k)} \alpha_n^{(j-k)} + \sum_{k=0}^{j} b_n^{(k)} c_n^{(j-k)} = 0 \quad (j \geq 1). \]  
\[ (2.13) \]
Then \( A_n^{(j)}, B_n^{(j)}, C_n^{(j)}, a_n^{(j)}, b_n^{(j)}, c_n^{(j)} \) are uniquely determined by the recursion relations (2.10)–(2.13). It is easy to see that
\[ A_n^{(1)} = -2\alpha_0Q_nT_{n-1}, \quad B_n^{(1)} = 2\alpha_0Q_n, \]  
\[ C_n^{(1)} = 2\alpha_0R_{n-1} - 2\alpha_0T_{n-1}(Q_nT_{n-1} + R_{n-1}S_{n-1}), \]  
\[ A_n^{(2)} = -2\alpha_0(Q_nR_{n-1} + S_nT_{n-1}) + 2\alpha_0Q_nT_{n-1}(R_nS_n + Q_nT_{n-1} + R_{n-1}S_{n-1}), \]  
\[ B_n^{(2)} = 2\alpha_0S_n - 2\alpha_0Q_n(Q_nT_{n-1} + R_nS_n), \]  
\[ a_n^{(1)} = -2\alpha_0R_nS_n, \quad b_n^{(1)} = 2\alpha_0S_n, \]  
\[ c_n^{(1)} = 2\alpha_0T_{n-1} - 2\alpha_0R_n(Q_nT_{n-1} + R_nS_n), \]  
\[ a_n^{(2)} = -2\alpha_0(Q_{n+1}R_n + S_nT_{n-1}) + 2\alpha_0R_nS_n(Q_{n+1}T_n + Q_nT_{n-1} + R_nS_n), \]  
\[ b_n^{(2)} = 2\alpha_0Q_{n+1} - 2\alpha_0S_n(Q_{n+1}T_n + R_nS_n). \]  
\[ (2.14) \]

On the other hand, we consider the following assumption:
\[ A_n = \sum_{j=0}^{\infty} \hat{A}_n^{(j)} \lambda^j, \quad B_n = \sum_{j=0}^{\infty} \hat{B}_n^{(j)} \lambda^j, \quad C_n = \sum_{j=0}^{\infty} \hat{C}_n^{(j)} \lambda^j, \]  
\[ a_n = \sum_{j=0}^{\infty} \hat{a}_n^{(j)} \lambda^j, \quad b_n = \sum_{j=0}^{\infty} \hat{b}_n^{(j)} \lambda^j, \quad c_n = \sum_{j=0}^{\infty} \hat{c}_n^{(j)} \lambda^j \]  
\[ (2.15) \]
with the condition
\[ A_n^2 + B_n C_n = \beta_0^2, \quad a_n^2 + b_n c_n = \beta_0^2, \]  
where \( \beta_0 \) is a constant. Substituting (2.15) into (2.6), (2.7) and (2.16), we obtain
\[
\begin{align*}
\hat{A}_n^{(0)} &= \beta_0, \quad \hat{B}_n^{(0)} = -2\beta_0 S_{n-1}, \quad \hat{C}_n^{(0)} = 0, \\
\hat{a}_n^{(j)} - \hat{A}_n^{(j)} + R_n \hat{b}_n^{(j)} - Q_n \hat{C}_n^{(j+1)} &= 0, \\
\hat{b}_n^{(j)} - \hat{B}_n^{(j-1)} + Q_n (\hat{a}_n^{(j)} + \hat{A}_n^{(j)}) &= 0, \\
\hat{c}_n^{(j)} - \hat{C}_n^{(j+1)} - R_n (\hat{a}_n^{(j)} + \hat{A}_n^{(j)}) &= 0, \\
\hat{A}_n^{(j)} - \hat{a}_n^{(j)} + Q_n c_n^{(j)} - R_n \hat{b}_n^{(j-1)} &= 0, \\
\hat{A}_n^{(j)} - \hat{a}_n^{(j)} + T_n \hat{b}_n^{(j)} - S_n c_n^{(j+1)} &= 0, \\
\hat{b}_n^{(j)} - \hat{B}_n^{(j-1)} + S_n (\hat{a}_n^{(j)} + \hat{A}_n^{(j+1)}) &= 0, \\
\hat{c}_n^{(j)} - \hat{C}_n^{(j+1)} - T_n (\hat{a}_n^{(j)} + \hat{A}_n^{(j+1)}) &= 0, \\
\hat{a}_n^{(j)} - \hat{A}_n^{(j+1)} - T_n \hat{b}_n^{(j-1)} + S_n \hat{c}_n^{(j+1)} &= 0, \\
\sum_{k=0}^{j} \hat{A}_n^{(k)} \hat{A}_n^{(j-k)} + \sum_{k=0}^{j} \hat{b}_n^{(k)} \hat{C}_n^{(j-k)} &= 0, \\
\sum_{k=0}^{j} \hat{a}_n^{(k)} \hat{a}_n^{(j-k)} + \sum_{k=0}^{j} \hat{b}_n^{(k)} \hat{c}_n^{(j-k)} &= 0 \quad (j \geq 1).
\end{align*}
\]  
Then \( \hat{A}_n^{(j)}, \hat{B}_n^{(j)}, \hat{C}_n^{(j)}, \hat{a}_n^{(j)}, \hat{b}_n^{(j)}, \hat{c}_n^{(j)} \) are uniquely determined by the recursion relations (2.17)–(2.20), and
\[
\begin{align*}
\hat{A}_n^{(1)} &= -2\beta_0 R_n S_{n-1}, \quad \hat{B}_n^{(1)} = -2\beta_0 Q_n S_{n-1} + 2\beta_0 S_{n-1} R_n S_{n-1} + Q_{n-1} T_{n-1}, \\
\hat{C}_n^{(2)} &= -2\beta_0 T_n + 2\beta_0 R_n S_{n-1} + Q_n T_n, \\
\hat{A}_n^{(2)} &= -2\beta_0 (Q_n R_n + S_{n-1} T_n) + 2\beta_0 R_n S_{n-1} (Q_n T_n + R_n S_{n-1} + Q_{n-1} T_{n-1}), \\
\hat{a}_n^{(1)} &= -2\beta_0 Q_n T_n, \quad \hat{b}_n^{(1)} = -2\beta_0 S_{n-1} + 2\beta_0 Q_n (R_n S_{n-1} + Q_n T_n), \\
\hat{c}_n^{(1)} &= -2\beta_0 T_n, \quad \hat{c}_n^{(2)} = -2\beta_0 R_{n+1} + 2\beta_0 T_n (R_{n+1} S_n + Q_n T_n), \\
\hat{a}_n^{(2)} &= -2\beta_0 (Q_n R_{n+1} + S_{n-1} T_n) + 2\beta_0 Q_n T_n (R_{n+1} S_n + Q_n T_n + R_n S_{n-1}).
\end{align*}
\]  

**Proposition 2.** Let two matrices \( V_n^{(m)} \) and \( \hat{V}_n^{(m)} \) be solutions of the equations
\[
U_{ntm}^{(1)} = \hat{V}_n^{(m)} U_n^{(1)} - U_n^{(1)} V_n^{(m)}, \quad U_{ntm}^{(2)} = V_n^{(m+1)} U_n^{(2)} - U_n^{(2)} \hat{V}_n^{(m)},
\]
which imply that \( V_n^{(m)} \) satisfies the discrete zero-curvature equation
\[
U_{ntm} = V_n^{(m+1)} U_n - U_n V_n^{(m)}.
\]

Here \( U_n^{(1)} \) and \( U_n^{(2)} \) are defined by (2.3).
Proof. By using (2.2), a direct calculation shows that

\[ U_{ntm} - V_{ntn}^{(m)} U_n + U_n V_n^{(m)} = (U_{ntm}^{(2)} - V_{ntn}^{(m)} U_n + U_n^{(2)} \hat{V}_n^{(m)})U_n^{(1)} + U_n^{(2)} (U_{ntm}^{(1)} - \hat{V}_n^{(m)} U_n^{(1)} + U_n^{(1)} V_n^{(m)}) = 0. \]

The proof is completed. \( \square \)

Let \( V_n^{(m)} \) and \( \hat{V}_n^{(m)} \) of (2.22) take form

\[ V_n^{(m)} = \begin{pmatrix} \tilde{A}_n^{(m)} \\ \tilde{C}_n^{(m)} \end{pmatrix} = \begin{pmatrix} A_n^{(m)} \\ C_n^{(m)} \end{pmatrix}, \quad \hat{V}_n^{(m)} = \begin{pmatrix} b_n^{(m)} \\ \tilde{c}_n^{(m)} \end{pmatrix} = \begin{pmatrix} b_n^{(m)} \\ c_n^{(m)} \end{pmatrix}, \tag{2.24} \]

with

\[
\begin{align*}
\tilde{A}_n^{(m)} &= \sum_{j=0}^m A_n^{(j)} \lambda^{m-j} + \sum_{j=0}^m \hat{A}_n^{(j)} \lambda^{j-m} - \frac{1}{2} (\hat{A}_n^{(m)} + \tilde{A}_n^{(m)}), \\
\tilde{B}_n^{(m)} &= \sum_{j=0}^m B_n^{(j)} \lambda^{m-j} + \sum_{j=0}^{m-1} \hat{B}_n^{(j)} \lambda^{j-m}, \\
\tilde{C}_n^{(m)} &= \sum_{j=0}^{m-1} C_n^{(j)} \lambda^{m-j} + \sum_{j=0}^m \hat{C}_n^{(j)} \lambda^{j-m}, \quad \tag{2.25} \\
\tilde{a}_n^{(m)} &= \sum_{j=0}^m a_n^{(j)} \lambda^{m-j} + \sum_{j=0}^m \hat{a}_n^{(j)} \lambda^{j-m} - \frac{1}{2} (\hat{a}_n^{(m)} + \tilde{a}_n^{(m)}), \\
\tilde{b}_n^{(m)} &= \sum_{j=0}^m b_n^{(j)} \lambda^{m-j} + \sum_{j=0}^{m-1} \hat{b}_n^{(j)} \lambda^{j-m}, \\
\tilde{c}_n^{(m)} &= \sum_{j=0}^{m-1} c_n^{(j)} \lambda^{m-j} + \sum_{j=0}^m \hat{c}_n^{(j)} \lambda^{j-m}. \tag{2.26} \end{align*}
\]

Then (2.22) can be written as

\[
\begin{align*}
\gamma_n (\gamma_n^{-1})_{tm} &= \tilde{a}_n^{(m)} - \tilde{A}_n^{(m)} + R_n \tilde{b}_n^{(m)} - \lambda^{-1} Q_n \tilde{C}_n^{(m)}, \\
Q_n &+ Q_n \gamma_n (\gamma_n^{-1})_{tm} = \tilde{b}_n^{(m)} - \lambda \tilde{B}_n^{(m)} + Q_n (\tilde{a}_n^{(m)} + \tilde{A}_n^{(m)}), \\
R_n &+ R_n \gamma_n (\gamma_n^{-1})_{tm} = \tilde{c}_n^{(m)} - \lambda^{-1} \tilde{C}_n^{(m)} - R_n (\tilde{a}_n^{(m)} + \tilde{A}_n^{(m)}), \\
\gamma_n (\gamma_n^{-1})_{tm} &= \tilde{A}_n^{(m)} - \tilde{a}_n^{(m)} + Q_n \tilde{c}_n^{(m)} - \lambda R_n \tilde{B}_n^{(m)}, \tag{2.27} \\
\pi_n (\pi_n^{-1})_{tm} &= \tilde{a}_n^{(m)} - \tilde{a}_n^{(m)} + T_n \tilde{B}_n^{(m)} - \lambda^{-1} S_n \tilde{c}_n^{(m)}, \\
S_n &+ S_n \pi_n (\pi_n^{-1})_{tm} = \tilde{B}_n^{(m)} - \lambda \tilde{B}_n^{(m)} + S_n (\tilde{a}_n^{(m)} + \tilde{A}_n^{(m)}), \\
T_n &+ T_n \pi_n (\pi_n^{-1})_{tm} = \tilde{c}_n^{(m)} - \lambda^{-1} \tilde{c}_n^{(m)} - T_n (\tilde{a}_n^{(m)} + \tilde{A}_n^{(m)}), \\
\pi_n (\pi_n^{-1})_{tm} &= \tilde{a}_n^{(m)} - \tilde{a}_n^{(m)} - \lambda T_n \tilde{B}_n^{(m)} + S_n \tilde{c}_n^{(m)} + \tilde{c}_n^{(m)} - \lambda^{-1} \tilde{c}_n^{(m)} - T_n (\tilde{a}_n^{(m)} + \tilde{A}_n^{(m)}) \tag{2.28} \end{align*}
\]

that is
\[
\gamma_n (\gamma_n^{-1})_{tm} = \frac{1}{2} (a_n^{(m)} + \hat{a}_n^{(m)} - A_n^{(m)} - \hat{A}_n^{(m)}) + R_n b_n^{(m)} - Q_n C_n^{(m-1)},
\]
\[
Q_{ntm} + Q_n \gamma_n (\gamma_n^{-1})_{tm} = b_n^{(m)} - \hat{B}_n^{(m-1)} + \frac{1}{2} Q_n (a_n^{(m)} + \hat{a}_n^{(m)} + A_n^{(m)} + \hat{A}_n^{(m)}),
\]
\[
R_{ntm} + R_n \gamma_n (\gamma_n^{-1})_{tm} = \hat{c}_n^{(m)} - C_n^{(m-1)} - \frac{1}{2} R_n (a_n^{(m)} + \hat{a}_n^{(m)} + A_n^{(m)} + \hat{A}_n^{(m)}),
\]
\[
\gamma_n (\gamma_n^{-1})_{tm} = \frac{1}{2} (A_n^{(m)} + \hat{A}_n^{(m)} - a_n^{(m)} - \hat{a}_n^{(m)}) + Q_n \hat{c}_n^{(m)} - R_n \hat{B}_n^{(m-1)},
\]
\[
\tau_n (\tau_n^{-1})_{tm} = \frac{1}{2} (A_n^{(m)} + \hat{A}_n^{(m)} - a_n^{(m)} - \hat{a}_n^{(m)}) + T_n B_n^{(m)} - S_n \epsilon_n^{(m-1)},
\]
\[
S_{ntm} + S_n \tau_n (\tau_n^{-1})_{tm} = B_n^{(m)} - \hat{B}_n^{(m-1)} + \frac{1}{2} S_n (a_n^{(m)} + \hat{a}_n^{(m)} + A_n^{(m)} + \hat{A}_n^{(m)}),
\]
\[
T_{ntm} + T_n \tau_n (\tau_n^{-1})_{tm} = \hat{C}_n^{(m+1)} - \hat{C}_n^{(m-1)} - \frac{1}{2} T_n (a_n^{(m)} + \hat{a}_n^{(m)} + A_n^{(m+1)} + \hat{A}_n^{(m+1)}),
\]
\[
\tau_n (\tau_n^{-1})_{tm} = \frac{1}{2} (a_n^{(m)} + \hat{a}_n^{(m)} - A_n^{(m+1)} + \hat{A}_n^{(m+1)}) - T_n \hat{B}_n^{(m-1)} + S_n \hat{C}_n^{(m+1)}.
\]

A direct calculation shows that (2.29) and (2.30) are equivalent to the following difference-difference equations:
\[
Q_{ntm} = \gamma_n^2 b_n^{(m-1)} - \hat{B}_n^{(m-1)} + Q_n (A_n^{(m)} + \hat{A}_n^{(m)} + Q_n C_n^{(m-1)}),
\]
\[
R_{ntm} = \gamma_n^2 \epsilon_n^{(m-1)} - R_n (A_n^{(m)} + \hat{A}_n^{(m)} - R_n \hat{B}_n^{(m-1)}),
\]
\[
S_{ntm} = \pi_n^2 b_n^{(m+1)} - \hat{B}_n^{(m-1)} + S_n (a_n^{(m)} + \hat{a}_n^{(m)} + S_n \epsilon_n^{(m-1)}),
\]
\[
T_{ntm} = \pi_n^2 \hat{C}_n^{(m+1)} - \hat{C}_n^{(m-1)} - T_n (a_n^{(m)} + \hat{a}_n^{(m)} - T_n \hat{B}_n^{(m-1)}), \quad m \geq 0.
\]

The first three systems in (2.31) read
\[
Q_{nt0} = (\alpha_0 + \beta_0) Q_n, \quad R_{nt0} = -(\alpha_0 + \beta_0) R_n,
\]
\[
S_{nt0} = (\alpha_0 + \beta_0) S_n, \quad T_{nt0} = -(\alpha_0 + \beta_0) T_n,
\]
\[
Q_{nt1} = 2 (1 - Q_n R_n) (\alpha_0 S_n + \beta_0 S_{n-1}),
\]
\[
R_{nt1} = -2 (1 - Q_n R_n) (\beta_0 T_n + \alpha_0 T_{n-1}),
\]
\[
S_{nt1} = 2 (1 - S_n T_n) (\alpha_0 Q_{n+1} + \beta_0 Q_n),
\]
\[
T_{nt1} = -2 (1 - S_n T_n) (\beta_0 R_{n+1} + \alpha_0 R_n),
\]
\[
Q_{nt2} = 2 \alpha_0 (1 - Q_n R_n) [Q_{n+1} - S_n (Q_{n+1} T_n + R_n S_n + Q_n T_{n-1})]
+ 2 \beta_0 (1 - Q_n R_n) [Q_{n-1} - S_{n-1} (Q_n T_n + R_n S_{n-1} + Q_n T_{n-1})],
\]
\[
R_{nt2} = 2 \alpha_0 (1 - Q_n R_n) [-R_{n-1} + T_{n-1} (R_n S_n + Q_n T_{n-1} + R_{n-1} S_{n-1})]
+ 2 \beta_0 (1 - Q_n R_n) [-R_{n+1} + T_n (R_{n+1} S_n + Q_n T_n + R_n S_n)],
\]
\[
S_{nt2} = 2 \alpha_0 (1 - S_n T_n) [S_{n+1} - Q_{n+1} (Q_{n+1} T_n + R_{n+1} S_n + Q_{n+1} T_{n-1})]
+ 2 \beta_0 (1 - S_n T_n) [S_{n-1} - Q_n (R_{n+1} S_n + R_{n+1} S_{n-1} + Q_n T_n)],
\]
\[
T_{nt2} = 2 \alpha_0 (1 - S_n T_n) [-T_{n-1} + R_n (Q_{n+1} T_n + Q_n T_{n-1} + R_n S_n)]
+ 2 \beta_0 (1 - S_n T_n) [-T_{n+1} + R_{n+1} (Q_{n+1} T_{n+1} + R_{n+1} S_n + Q_n T_n)].
\]
In particular, letting \( R_n = \mp Q_n = u_n, \) \( S_n = \mp T_n = v_n, \) and \( \alpha_0 = -\beta_0 = \frac{1}{2}, \) (2.33) is reduced to the self-dual network \([23–28]\)

\[
u_{nt_1} = (1 \pm u_n^2) (\pm v_{n-1} \mp v_n), \quad v_{nt_1} = (1 \pm v_n^2) (\pm u_n \mp u_{n+1}). \tag{2.35}
\]

Similarly, we let \( R_n = \mp Q_n^*, \) \( S_n = \mp T_n^*, \) with \( \alpha_0 = \beta_0 = -\frac{1}{2} i, \) (2.33) is reduced to a discretized second order in time nonlinear Schrödinger equation \([23]\)

\[
R_{nt_1} = i (1 \pm R_n R_n^*) (\pm S_n^* \mp S_n^{*-1}), \quad S_{nt_1} = i (1 \pm S_n S_n^*) (\pm R_n^* \mp R_n^{*-1}). \tag{2.36}
\]

Alternatively, if \( Q_n = y_n, \) \( R_n = 0, \) \( S_n = 1 - x_n, \) \( T_n = 1, \) with \( \alpha_0 = -\beta_0 = \frac{1}{2}, \) we have from (2.33) that

\[
x_{nt_1} = x_n (y_n - y_{n+1}), \quad y_{nt_1} = x_{n-1} - x_n, \tag{2.37}
\]

which is the Toda lattice equation.

3. The Lenard gradients and the Hamiltonian structures

In this section, we shall derive the Lenard gradients and the Hamiltonian structures of differential-difference equations (2.31). To this end, we introduce a transformation

\[
\begin{align*}
\lambda^{-1} C_n + R_n A_n &= \gamma_n^2 S_n^{(1)}, \\
\lambda B_n - Q_n A_n &= \gamma_n^2 S_n^{(2)}, \\
\lambda^{-1} c_n + T_n a_n &= \pi_n^2 S_n^{(3)}, \\
\lambda b_n - S_n a_n &= \pi_n^2 S_n^{(4)}.
\end{align*}
\tag{3.1}
\]

Then (2.6) and (2.7) can be written as

\[
\begin{align*}
R_n (S_n a_n + \pi_n^2 S_n^{(4)}) &= \lambda (\gamma_n^2 Q_n g_n^{(1)} + \gamma_n^2 A_n - a_n), \\
S_n a_n + \pi_n^2 S_n^{(4)} &= \lambda (\gamma_n^2 S_n^{(2)} - Q_n a_n), \\
R_n a_n + \gamma_n^2 S_n^{(1)} &= \lambda (\pi_n^2 S_n^{(3)} - T_n a_n), \\
a_n - \gamma_n^2 A_n + \gamma_n^2 R_n S_n^{(2)} &= \lambda Q_n (\pi_n^2 S_n^{(3)} - T_n a_n), \\
T_n (Q_{n+1} A_{n+1} + \gamma_{n+1}^2 S_{n+1}^{(2)}) &= \lambda (\pi_n^2 a_n + \pi_n^2 S_n S_n^{(3)} - A_{n+1}), \\
Q_{n+1} A_{n+1} + \gamma_{n+1}^2 S_{n+1}^{(2)} &= \lambda (\pi_n^2 S_n - S_n A_{n+1}), \\
T_n A_{n+1} + \pi_n^2 S_n^{(3)} &= \lambda (\gamma_{n+1}^2 S_{n+1}^{(1)} - R_{n+1} A_{n+1}), \\
\pi_n^2 a_n - A_{n+1} - \pi_n^2 T_n S_n^{(4)} &= \lambda S_n (R_{n+1} A_{n+1} - \gamma_{n+1}^2 S_{n+1}^{(1)}).
\end{align*}
\tag{3.2}
\]

Proposition 3. Under the transformation (3.1), the matrices \( V_n \) and \( \hat{V}_n \) determined by (2.5) satisfy (2.2) if and if only the function \( g_\lambda(n) \) satisfies the discrete Lenard equation

\[
K_n g_\lambda(n) = \lambda J_n g_\lambda(n) \tag{3.4}
\]

where \( g_\lambda(n) = (g_n^{(1)}, g_n^{(2)}, g_n^{(3)}, g_n^{(4)}, a_n, A_n)^T \) and two matrix operators

\[
K_n = \begin{pmatrix}
0 & 0 & 0 & \pi_n^2 & S_n & 0 \\
\gamma_n^2 & 0 & 0 & 0 & R_n & 0 \\
0 & E \gamma_n^2 & 0 & 0 & 0 & E Q_n \\
0 & 0 & \pi_n^2 & 0 & 0 & T_n E \\
Q_n & -R_n & 0 & 0 & -1 & 1 \\
0 & 0 & S_n & -T_n & 1 & -E
\end{pmatrix}.
\]
Here the shift operator $E$ is defined as $Ef(n) = f(n + 1)$.

**Proof.** From (3.2) and (3.3), we have

\[
\begin{align*}
Q_n^2 s_n^{(1)} - R_n s_n^{(2)} - a_n + A_n &= 0, \\
S_n^2 s_n^{(3)} - T_n s_n^{(4)} + a_n - A_{n+1} &= 0,
\end{align*}
\tag{3.5}
\]

which, together with the second and third expressions of (3.2) and (3.3), imply (3.4). Contrarily, a direct calculation shows that (3.2) and (3.3) hold in view of (3.4). The proof is completed.

Substituting (2.8) and (2.15) into (3.1), respectively, we obtain two solutions of (3.4):

\[
\tilde{g}_\lambda(n) = \sum_{j=0}^{\infty} \tilde{g}_n^{(j)} \lambda^{-j}, \quad \hat{g}_\lambda(n) = \sum_{j=0}^{\infty} \hat{g}_n^{(j)} \lambda^j
\tag{3.6}
\]

with the Lenard gradient sequence

\[
\begin{align*}
\tilde{g}_n^{(j)} &= \begin{pmatrix}
\gamma_n^{-2} (C_n^{(j+1)} + R_n A_n^{(j)}) \\
\gamma_n^{-2} (B_n^{(j+1)} - Q_n A_n^{(j)}) \\
\pi_n^{-2} (C_n^{(j-1)} + T_n a_n^{(j)}) \\
\pi_n^{-2} (B_n^{(j+1)} - S_n a_n^{(j)}) \\
a_n^{(j)} \\
A_n^{(j)} 
\end{pmatrix}, \\
\hat{g}_n^{(j)} &= \begin{pmatrix}
\gamma_n^{-2} (\hat{C}_n^{(j+1)} + R_n \hat{A}_n^{(j)}) \\
\gamma_n^{-2} (\hat{B}_n^{(j-1)} - Q_n \hat{A}_n^{(j)}) \\
\pi_n^{-2} (\hat{C}_n^{(j+1)} + T_n \hat{a}_n^{(j)}) \\
\pi_n^{-2} (\hat{B}_n^{(j-1)} - S_n \hat{a}_n^{(j)}) \\
\hat{a}_n^{(j)} \\
\hat{A}_n^{(j)} 
\end{pmatrix},
\end{align*}
\tag{3.7}
\]

\[
\begin{align*}
\tilde{g}_n^{(0)} &= \alpha_0 \begin{pmatrix}
R_n \gamma_n^{-2} \\
Q_n \gamma_n^{-2} \\
T_n \pi_n^{-2} \\
S_n \pi_n^{-2} \\
1 \\
1
\end{pmatrix}, \\
\hat{g}_n^{(0)} &= -\beta_0 \begin{pmatrix}
R_n \gamma_n^{-2} \\
Q_n \gamma_n^{-2} \\
T_n \pi_n^{-2} \\
S_n \pi_n^{-2} \\
1 \\
1
\end{pmatrix}.
\tag{3.8}
\]

It is easy to calculate that

\[
\begin{align*}
\tilde{g}_n^{(1)} &= 2\alpha_0 \begin{pmatrix}
T_{n-1} \\
S_n \\
R_n \\
Q_{n+1} \\
-R_n S_n \\
-S_n T_{n-1}
\end{pmatrix}, \\
\hat{g}_n^{(1)} &= -2\beta_0 \begin{pmatrix}
T_n \\
S_{n-1} \\
R_{n+1} \\
Q_n \\
Q_n T_n \\
R_n S_{n-1}
\end{pmatrix}.
\tag{3.9}
\]
Utilizing the transformation (3.1), (2.5) can be put in the equivalent forms
\[ V_n (g_{\lambda}(n)) = \begin{pmatrix} A_n & \lambda^{-1}(\gamma_n^2 g_n^{(2)} + Q_n A_n) \\ \lambda(\gamma_n^2 g_n^{(1)} - R_n A_n) & -A_n \end{pmatrix}, \]
\[ \hat{V}_n (g_{\lambda}(n)) = \begin{pmatrix} a_n & \lambda^{-1}(\pi_n^2 g_n^{(4)} + S_n a_n) \\ \lambda(\pi_n^2 g_n^{(3)} - T_n a_n) & -a_n \end{pmatrix}. \]

Noting (2.9), (2.16) and (3.6), we arrive at the following fact.

**Proposition 4.**

\[ \text{det} V_n (\tilde{g}_{\lambda}(n)) = \text{det} \hat{V}_n (\tilde{g}_{\lambda}(n)) = -\alpha_0^2, \]
\[ \text{det} V_n (\hat{g}_{\lambda}(n)) = \text{det} \hat{V}_n (\hat{g}_{\lambda}(n)) = -\beta_0^2. \]

For the sake of convenience, we introduce an inverse operator by
\[ \tilde{J}_n = \begin{pmatrix} 0 & \gamma_n^2 & 0 & 0 & -Q_n & 0 \\ 0 & 0 & \pi_n^2 & 0 & -T_n & 0 \\ 0 & 0 & 0 & \pi_n^2 & 0 & S_n E \\ E \gamma_n^2 & 0 & 0 & 0 & 0 & -E R_n \\ Q_n & -R_n & 0 & 0 & -1 & 1 \\ 0 & 0 & S_n & -T_n & 1 & -E \end{pmatrix}. \]

With the help of (3.4) and (3.6), we arrive at
\[ K_n \tilde{g}_{n}^{(j)} = J_n \tilde{g}_{n}^{(j+1)}, \quad J_n \tilde{g}_{n}^{(0)} = 0, \]
\[ K_n \hat{g}_{n}^{(j)} = J_n \hat{g}_{n}^{(j+1)}, \quad K_n \hat{g}_{n}^{(0)} = 0, \]
which are equivalent to
\[ K_n \tilde{g}_{n}^{(j)} = \tilde{J}_n \tilde{g}_{n}^{(j+1)}, \quad \tilde{J}_n \tilde{g}_{n}^{(0)} = 0, \]
\[ K_n \hat{g}_{n}^{(j+1)} = \tilde{J}_n \hat{g}_{n}^{(j)}, \quad K_n \hat{g}_{n}^{(0)} = 0. \]

Utilizing (2.11) and (2.12), we have
\[ \gamma_n^2 b_n^{(m)} = B_n^{(m+1)} - 2Q_n A_n^{(m)} - Q_n^2 C_n^{(m-1)}, \]
\[ \gamma_n^2 c_n^{(m)} = \hat{c}_n^{(m+1)} + 2R_n \hat{A}_n^{(m)} - R_n^2 \hat{B}_n^{(m-1)}, \]
\[ \pi_n^2 b_n^{(m+1)} = b_n^{(m+1)} - 2S_n a_n^{(m)} - S_n^2 c_n^{(m-1)}, \]
\[ \pi_n^2 c_n^{(m+1)} = \hat{c}_n^{(m+1)} + 2T_n \hat{a}_n^{(m)} - T_n^2 \hat{b}_n^{(m-1)}. \]

Resorting to (3.7), (3.15) and (3.16), Eqs. (2.31) can be written as
\[ (Q_{ntm}, R_{ntm}, S_{ntm}, T_{ntm})^T = X_m(n) = J_n \mathcal{L}(\tilde{g}_n^{(m)} - \hat{g}_n^{(m)}), \quad m \geq 0, \]
where $\mathcal{L}$ is the projective map $f = (f^{(1)}, f^{(2)}, f^{(3)}, f^{(4)}, f^{(5)}, f^{(6)})^T \rightarrow (f^{(1)}, f^{(2)}, f^{(3)}, f^{(4)})^T$, and $J_n$ is a skew-symmetric operator.
\[ J_n = \begin{pmatrix} 0 & \gamma_n^2 & 0 & 0 \\ -\gamma_n^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \pi_n^2 \\ 0 & 0 & -\pi_n^2 & 0 \end{pmatrix} \]

To establish Hamiltonian structures of the discrete Ablowitz–Ladik hierarchy (2.31), we need the following quantities, which are easy to calculate (\( V_n = V_n U_n^{-1} \))

\[ V_n = \frac{1}{\gamma_n \pi_n} \left( (\lambda^{-1} + Q_n T_n) A_n - (\lambda T_n + R_n) B_n \right) \left( (\lambda T_n + R_n) A_n + (\lambda^{-1} + Q_n T_n) C_n - (Q_n + \lambda^{-1} S_n) A_n \right), \]

\[ \text{tr} \left( V_n \frac{\partial U_n}{\partial \lambda} \right) = \frac{1}{\gamma_n^2} \left( 2\lambda^{-1} A_n - R_n B_n + \lambda^{-2} Q_n C_n \right), \tag{3.19} \]

\[ \text{tr} \left( V_n \frac{\partial U_n}{\partial Q_n} \right) = \frac{1}{\gamma_n^2} \frac{\lambda}{2} \left( \lambda R_n + \lambda T_n \right) B_n + \lambda^{-1} \left( \lambda^{-1} + Q_n T_n \right) C_n \right), \]

\[ \text{tr} \left( V_n \frac{\partial U_n}{\partial R_n} \right) = \frac{1}{\gamma_n^2} \left\{ 2\lambda R_n + S_n \left( 1 + Q_n R_n \right) \right\} A_n \]

\[ - Q_n \left( Q_n + \lambda^{-1} S_n \right) C_n + \lambda \left( \lambda + R_n S_n \right) B_n \right\}. \tag{3.20} \]

From (2.6), we have

\[ a_n \gamma_n^2 = (1 + Q_n R_n) A_n - \lambda R_n B_n + \lambda^{-1} Q_n C_n, \]

\[ c_n \gamma_n^2 = \lambda^{-1} C_n - \lambda R_n^2 C_n + 2 R_n A_n, \]

\[ b_n \gamma_n^2 = \lambda B_n - \lambda^{-1} Q_n^2 C_n - 2 Q_n A_n, \]

\[ 2\lambda^{-1} A_n - R_n B_n + \lambda^{-2} Q_n C_n = \lambda^{-1} \left( 2\gamma_n^2 A_n - 2 Q_n R_n a_n - R_n b_n + Q_n c_n \right), \tag{3.22} \]

which, together with (3.21), (3.19), implies

\[ \text{tr} \left( V_n \frac{\partial U_n}{\partial S_n} \right) = \frac{1}{\pi_n^2} \left( \lambda^{-1} c_n + T_n a_n \right), \quad \text{tr} \left( V_n \frac{\partial U_n}{\partial T_n} \right) = \frac{1}{\pi_n^2} \left( \lambda b_n - S_n a_n \right), \]

\[ \text{tr} \left( V_n \frac{\partial U_n}{\partial \lambda} \right) = \frac{1}{\gamma_n^2} \left( 2\gamma_n^2 A_n - 2 Q_n R_n a_n - R_n b_n + Q_n c_n \right). \tag{3.23} \]

Using (3.1), (2.8), (2.15), (3.6) and the trace identity [29], we arrive at

\[ (\delta / \delta Q_n, \delta / \delta R_n, \delta / \delta S_n, \delta / \delta T_n)^T \hat{H}_j = \mathcal{L}_{\mathcal{G}_n}^{\gamma_n(j)}, \]

\[ (\delta / \delta Q_n, \delta / \delta R_n, \delta / \delta S_n, \delta / \delta T_n)^T \hat{H}_j = \mathcal{L}_{\hat{\mathcal{G}}_n}^{\gamma_n(j)}, \quad j \geq 1, \tag{3.24} \]

where

\[ \hat{H}_j = \frac{1}{J \gamma_n^2} \left( 2 Q_n R_n a_n^{(j)} + R_n b_n^{(j)} - Q_n c_n^{(j)} - 2 \gamma_n^2 A_n^{(j)} \right), \]

\[ \hat{H}_j = \frac{1}{J \gamma_n^2} \left( 2 \gamma_n^2 A_n^{(j)} - 2 Q_n R_n \hat{a}_n^{(j)} - R_n \hat{b}_n^{(j)} + Q_n \hat{c}_n^{(j)} \right). \tag{3.25} \]
Therefore, we deduce that the desired Hamiltonian form of the Ablowitz–Ladik hierarchy (3.18)
\[(Q_{ntm}, R_{ntm}, S_{ntm}, T_{ntm})^T = J_n(\delta/\delta Q_n, \delta/\delta R_n, \delta/\delta S_n, \delta/\delta T_n)^T H_m, \quad m \geq 1,\] (3.26)
with \(H_j = \tilde{H}_j - \hat{H}_j\). Specially, (2.33) can be written as the Hamiltonian form
\[(Q_{nt1}, R_{nt1}, S_{nt1}, T_{nt1})^T = J_n(\delta/\delta Q_n, \delta/\delta R_n, \delta/\delta S_n, \delta/\delta T_n)^T H_1\] (3.27)
with the Hamiltonian
\[H_1 = 2\alpha_0(Q_nT_n - 1 + R_nS_n) + 2\beta_0(Q_nT_n + R_nS_n - 1).\]

4. A new integrable symplectic map

For simplicity, in what follows we write \(f(n) = f, f(n+k) = E_k f, \) and so on. Assume that \(\lambda_j\) and \(\chi = (p_j, q_j)^T (1 \leq j \leq N)\) are distinct eigenparameters and the associated eigenfunctions for the spectral problem (2.1). Then \(N\) copies of the discrete spectral problem (2.1) can be written as
\[
\begin{align*}
\pi E p_j &= z_j \hat{p}_j + z_j^{-1} S \hat{q}_j, & \pi E q_j &= z_j T \hat{p}_j + z_j^{-1} \hat{q}_j, \\
\gamma \hat{p}_j &= z_j p_j + z_j^{-1} Q q_j, & \gamma \hat{q}_j &= z_j R p_j + z_j^{-1} q_j, & 1 \leq j \leq N.
\end{align*}
\]
(4.1)

It is easy to calculate that
\[
\begin{pmatrix}
\delta \lambda_j / \delta Q \\
\delta \lambda_j / \delta R \\
\delta \lambda_j / \delta S \\
\delta \lambda_j / \delta T
\end{pmatrix} =
\begin{pmatrix}
-\gamma^{-2}(\lambda_j^{-1} q_j^2 + R p_j q_j) \\
\gamma^{-2}(\lambda_j p_j^2 + Q p_j q_j) \\
-\pi^{-2}(\lambda_j^{-1} \hat{q}_j^2 + T \hat{p}_j \hat{q}_j) \\
\pi^{-2}(\lambda_j \hat{p}_j^2 + S \hat{p}_j \hat{q}_j)
\end{pmatrix}
\] (4.2)
up to a constant factor, which is extended to
\[
\nabla \lambda_j =
\begin{pmatrix}
-\gamma^{-2}(\lambda_j^{-1} q_j^2 + R p_j q_j) \\
\gamma^{-2}(\lambda_j p_j^2 + Q p_j q_j) \\
-\pi^{-2}(\lambda_j^{-1} \hat{q}_j^2 + T \hat{p}_j \hat{q}_j) \\
\pi^{-2}(\lambda_j \hat{p}_j^2 + S \hat{p}_j \hat{q}_j) \\
-\hat{p}_j \hat{q}_j \\
-\hat{p}_j \hat{q}_j
\end{pmatrix}.
\] (4.3)

Noticing the identities
\[
\begin{align*}
Q \frac{\delta \lambda_j}{\delta Q} - R \frac{\delta \lambda_j}{\delta R} + \hat{p}_j \hat{q}_j - p_j q_j &= 0, \\
S \frac{\delta \lambda_j}{\delta S} - T \frac{\delta \lambda_j}{\delta T} - \hat{p}_j \hat{q}_j + E p_j q_j &= 0.
\end{align*}
\] (4.4)
it is not difficult to verify that
\[K \nabla \lambda_j = \lambda_j \tilde{J} \nabla \lambda_j.\] (4.5)

Consider the Bargmann constraint [6]
\[L(\tilde{g}^{(0)} + \hat{g}^{(0)}) = \sum_{j=1}^{N} L \nabla \lambda_j,\] (4.6)
which is equivalent to
\[ Q = \alpha_1 \langle \Lambda p, p \rangle, \quad R = \beta_1 \langle \Lambda^{-1} q, q \rangle, \]
\[ S = \hat{\alpha}_1 \langle \Lambda \hat{p}, \hat{p} \rangle, \quad T = \hat{\beta}_1 \langle \Lambda^{-1} \hat{q}, \hat{q} \rangle \quad (4.7) \]
with
\[ \alpha_1 = \frac{1}{\alpha_0 - \beta_0 - \langle p, q \rangle}, \quad \beta_1 = \frac{1}{\beta_0 - \alpha_0 - \langle p, q \rangle}, \]
\[ \hat{\alpha}_1 = \frac{1}{\alpha_0 - \beta_0 - \langle \hat{p}, \hat{q} \rangle}, \quad \hat{\beta}_1 = \frac{1}{\beta_0 - \alpha_0 - \langle \hat{p}, \hat{q} \rangle}. \]

where \( \langle \cdot, \cdot \rangle \) stands for the canonical inner product in \( \mathbb{R}^N \), \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_N) \), \( p = (p_1, \ldots, p_N)^T \), \( q = (q_1, \ldots, q_N)^T \), \( \hat{p} = (\hat{p}_1, \ldots, \hat{p}_N)^T \), \( \hat{q} = (\hat{q}_1, \ldots, \hat{q}_N)^T \). Substituting (4.7) into (4.1) yields
\[ E_p = \frac{Z \hat{p} + \hat{\alpha}_1 \langle \Lambda \hat{p}, \hat{p} \rangle Z^{-1} \hat{q}}{\sqrt{1 - \hat{\alpha}_1 \hat{\beta}_1 \langle \Lambda \hat{p}, \hat{p} \rangle \langle \Lambda^{-1} \hat{q}, \hat{q} \rangle}}, \quad E_q = \frac{\hat{\beta}_1 \langle \Lambda^{-1} \hat{q}, \hat{q} \rangle Z \hat{p} + Z^{-1} \hat{q}}{\sqrt{1 - \hat{\alpha}_1 \hat{\beta}_1 \langle \Lambda \hat{p}, \hat{p} \rangle \langle \Lambda^{-1} \hat{q}, \hat{q} \rangle}}, \]
\[ \hat{p} = \frac{Z p + \alpha_1 \langle \Lambda p, p \rangle Z^{-1} q}{\sqrt{1 - \alpha_1 \beta_1 \langle \Lambda p, p \rangle \langle \Lambda^{-1} q, q \rangle}}, \quad \hat{q} = \frac{\beta_1 \langle \Lambda^{-1} q, q \rangle Z p + Z^{-1} q}{\sqrt{1 - \alpha_1 \beta_1 \langle \Lambda p, p \rangle \langle \Lambda^{-1} q, q \rangle}} \quad (4.10) \]
with \( Z = \text{diag}(z_1, \ldots, z_N) \). Equation (4.10) is equivalently written as follows
\[ E \left( \begin{array}{c} p \\ q \end{array} \right) = S \left( \begin{array}{c} \hat{p} \\ \hat{q} \end{array} \right), \quad \left( \begin{array}{c} \hat{p} \\ \hat{q} \end{array} \right) = S^{-1} \left( \begin{array}{c} p \\ q \end{array} \right), \quad (4.11) \]
where
\[ S: \left( \begin{array}{c} \xi \\ \eta \end{array} \right) \rightarrow \left( \begin{array}{c} \xi' \\ \eta' \end{array} \right) = \frac{1}{\sqrt{1 - \hat{\alpha}_1 \hat{\beta}_1 \langle \Lambda \xi, \xi \rangle \langle \Lambda^{-1} \eta, \eta \rangle}} \left( \begin{array}{c} Z \xi + \hat{\alpha}_1 \langle \Lambda \xi, \xi \rangle Z^{-1} \eta \\ \hat{\beta}_1 \langle \Lambda^{-1} \eta, \eta \rangle Z \xi + Z^{-1} \eta \end{array} \right) \quad (4.12) \]
with
\[ \tilde{\alpha}_1 = \frac{1}{\alpha_0 - \beta_0 - \langle \xi, \eta \rangle}, \quad \hat{\beta}_1 = \frac{1}{\beta_0 - \alpha_0 - \langle \xi, \eta \rangle}. \]

**Proposition 5.** \( S \) determined by (4.12) is a symplectic map in \( (\mathbb{R}^{2N}, dp \wedge dq) \).

**Proof.** Through tedious calculations, we obtain that \( d\xi' \wedge d\eta' = d\xi \wedge d\eta. \) \( \square \)

**Proposition 6.** Let \( p \) and \( q \) be solutions of (4.10). Then \( \langle p, q \rangle \) and \( \langle \hat{p}, \hat{q} \rangle \) are conserved integrals of the \( n \)-flow, and \( \langle p, q \rangle = \langle \hat{p}, \hat{q} \rangle \). This implies that \( \hat{\alpha}_1 = \alpha_1, \hat{\beta}_1 = \beta_1 \).

**Proof.** Using (4.10) and (4.7), we arrive at
\[ \gamma^2 \langle \hat{p}, \hat{q} \rangle = (\alpha_1 + \beta_1 + \alpha_1 \beta_1 \langle p, q \rangle) \langle \Lambda p, p \rangle \langle \Lambda^{-1} q, q \rangle + \langle p, q \rangle \]
\[ = (\alpha_1^{-1} + \beta_1^{-1} + \langle p, q \rangle) QR + \langle p, q \rangle = \gamma^2 \langle p, q \rangle, \]
which gives rise to \( \langle p, q \rangle = \langle \hat{p}, \hat{q} \rangle, \hat{\alpha}_1 = \alpha_1, \hat{\beta}_1 = \beta_1 \). The latter two expressions of (4.7) can be written as
\[ S = \frac{\alpha_1(\langle A^2, p, p \rangle + 2\alpha_1\langle Ap, p \rangle\langle Ap, q \rangle + \alpha_1^2\langle Ap, p \rangle^2\langle q, q \rangle)}{1 - \alpha_1\beta_1\langle Ap, p \rangle\langle \Lambda^{-1}, q, q \rangle}, \]
\[ T = \frac{\beta_1(\langle \Lambda^{-2}, q, q \rangle + 2\beta_1\langle \Lambda^{-1}, q, q \rangle\langle \Lambda^{-1}, p, q \rangle + \beta_1^2\langle \Lambda^{-1}, q, q \rangle^2\langle p, p \rangle)}{1 - \alpha_1\beta_1\langle Ap, p \rangle\langle \Lambda^{-1}, q, q \rangle}. \tag{4.13} \]

Therefore, we have
\[ \gamma^2\pi^2(E - 1)\langle p, q \rangle \]
\[ = T\left(\langle A^2, p, p \rangle + 2\alpha_1\langle Ap, p \rangle\langle Ap, q \rangle + \alpha_1^2\langle Ap, p \rangle^2\langle q, q \rangle\right) \]
\[ + S\left(\langle \Lambda^{-2}, q, q \rangle + 2\beta_1\langle \Lambda^{-1}, q, q \rangle\langle \Lambda^{-1}, p, q \rangle + \beta_1^2\langle \Lambda^{-1}, q, q \rangle^2\langle p, p \rangle\right) \]
\[ + R(1 + ST)\langle Ap, p \rangle + Q(1 + ST)\langle \Lambda^{-1}, q, q \rangle + 2(QR + ST)\langle p, q \rangle \]
\[ = \gamma^2ST(\alpha^{-1} + \beta^{-1}) + QR(1 + ST)(\alpha^{-1} + \beta^{-1}) + 2(QR + ST)\langle p, q \rangle = 0. \tag{4.14} \]

The proof is finished. \[ \square \]

The first two expressions of (4.7) and Eq. (4.13) can be denoted as
\[ \begin{pmatrix} Q \\ R \\ S \\ T \end{pmatrix} = f_S(p, q). \tag{4.15} \]

In order to prove the integrability of the symplectic map \( S \), we introduce two functions
\[ G_\lambda = \begin{pmatrix} [-2\alpha_1P_\lambda(q, q) + R(1 - 2\alpha_1P_\lambda(Ap, q))]/(2\alpha_1\gamma^2) \\ [2\alpha_1P_\lambda(A^2, p, p) + Q(1 + 2\alpha_1P_\lambda(Ap, q))]/(2\alpha_1\gamma^2) \\ [-2\hat{\alpha}_1P_\lambda(q, \hat{q}) + T(1 - 2\hat{\alpha}_1P_\lambda(Ap, q))]/(2\hat{\alpha}_1\pi^2) \\ [2\hat{\alpha}_1P_\lambda(A^2, \hat{p}, \hat{q}) + S(1 + 2\hat{\alpha}_1P_\lambda(Ap, \hat{q}))]/(2\hat{\alpha}_1\pi^2) \\ [1 - 2\hat{\alpha}_1P_\lambda(\Lambda\hat{p}, \hat{q})]/(2\hat{\alpha}_1) \\ [1 - 2\alpha_1P_\lambda(Ap, q)])//(2\alpha_1) \end{pmatrix}, \tag{4.16} \]
\[ \hat{G}_\lambda = \begin{pmatrix} [-2\beta_1P_\lambda(\Lambda^{-2}, q, q) - R(1 + 2\beta_1P_\lambda(\Lambda^{-1}, p, q))]/(2\beta_1\gamma^2) \\ [2\beta_1P_\lambda(p, p) - Q(1 - 2\beta_1P_\lambda(A^{-1}, p, q))]/(2\beta_1\gamma^2) \\ [-2\hat{\beta}_1P_\lambda(\Lambda^{-2}, \hat{q}, \hat{q}) - T(1 + 2\hat{\beta}_1P_\lambda(\Lambda^{-1}, \hat{p}, \hat{q}))]/(2\hat{\beta}_1\pi^2) \\ [2\hat{\beta}_1P_\lambda(\hat{p}, \hat{q}) - S(1 + 2\hat{\beta}_1P_\lambda(A^{-1}, \hat{q}))]/(2\hat{\beta}_1\pi^2) \\ [1 - 2\hat{\beta}_1P_\lambda(\Lambda^{-1}, \hat{p}, \hat{q})]/(2\hat{\beta}_1) \\ [1 - 2\beta_1P_\lambda(\Lambda^{-1}, p, q)])//(2\beta_1) \end{pmatrix}, \tag{4.17} \]

where bilinear functions \( P_\lambda(\xi, \eta) \) and \( P_\lambda(A^{-1}, p, q) \) on \( \mathbb{R}^N \) and their partial-fraction expansions and Laurent expansions are defined as
\[ P_\lambda(\xi, \eta) = \langle(\lambda - A)^{-1}, \xi, \eta \rangle = \sum_{j=1}^{N} \frac{\xi j \eta j}{\lambda - \lambda_j} = \sum_{m=0}^{\infty} \lambda^{-m-1}\langle A^m, \xi, \eta \rangle, \]
\[ P_\lambda(A^{-1}, p, q) = \langle(\lambda^{-1} - A^{-1})^{-1}, \xi, \eta \rangle = \sum_{j=1}^{N} \frac{\xi j \eta j}{\lambda^{-1} - \lambda_j^{-1}} = \sum_{m=0}^{\infty} \lambda^{m+1}\langle A^{-m}, \xi, \eta \rangle. \tag{4.18} \]
The first expansion and second one in (4.18) are convergent absolutely outside $|\lambda| > \max\{|\lambda_1|, \ldots, |\lambda_N|\}$ and inside $|\lambda| < \min\{|\lambda_1|, \ldots, |\lambda_N|\}$, respectively. Now we introduce two matrices $V_\lambda$ and $V_{\lambda_1}$ by

$$V_\lambda = \begin{pmatrix} \frac{1}{2\alpha_1} - P_\lambda(\Lambda p, q) & P_\lambda(\Lambda p, p) \\ -\lambda P_\lambda(q, q) & -\frac{1}{2\alpha_1} + P_\lambda(\Lambda p, q) \end{pmatrix},$$

$$V_{\lambda_1} = \begin{pmatrix} \frac{1}{2\beta_1} - P_{\lambda_1}(\Lambda^{-1} p, q) & \lambda^{-1} P_{\lambda_1}(p, p) \\ -P_{\lambda_1}(\Lambda^{-1} q, q) & -\frac{1}{2\beta_1} + P_{\lambda_1}(\Lambda^{-1} p, q) \end{pmatrix}.$$  (4.19)

(4.20)

**Proposition 7.** The matrices $V_\lambda$ and $V_{\lambda_1}$ given by (4.19) and (4.20) satisfy Eq. (2.2), respectively.

**Proof.** With the aid of (4.1) and the identities

$$E\langle p, p \rangle + \beta_1^{-1} S = 0, \quad E\langle q, q \rangle + \alpha_1^{-1} T = 0,$$

$$\langle \hat{p}, \hat{p} \rangle + \beta_1^{-1} Q = 0, \quad \langle \hat{q}, \hat{q} \rangle + \alpha_1^{-1} R = 0,$$

$$P_\lambda(\Lambda^{j+1} q, q) = \lambda P_\lambda(\Lambda^j q, q) - \langle \Lambda^j p, q \rangle,$$

$$P_{\lambda_1}(\Lambda^{-j} p, q) = \lambda^{-1} P_{\lambda_1}(\Lambda^{-j-1} p, q) - \langle \Lambda^{-j} q, q \rangle,$$  (4.21)

a direct calculation shows that the desired result holds. The proof is completed. □

Therefore, the generating functions $F_\lambda = \det V_\lambda$ and $F_{\lambda_1} = \det V_{\lambda_1}$ are independent constants of $n$-flow. This means that $F_\lambda$ and $F_{\lambda_1}$ are invariant under the action of the symplectic map $S$. Then we have

$$F_\lambda = \lambda \left| \begin{array}{cc} P_\lambda(\Lambda p, p) & P_\lambda(\Lambda p, p) \\ P_\lambda(\Lambda p, q) & P_\lambda(\Lambda p, q) \end{array} \right| + (\alpha_0 - \beta_0) P_\lambda(\Lambda p, q) - \frac{1}{4\alpha_1^2}$$

$$= \sum_{m=-1}^{\infty} \lambda^{-m-1} F_m,$$  (4.23)

$$F_{\lambda_1} = \lambda^{-1} \left| \begin{array}{cc} P_{\lambda_1}(p, p) & P_{\lambda_1}(\Lambda^{-1} p, q) \\ P_{\lambda_1}(p, q) & P_{\lambda_1}(\Lambda^{-1} q, q) \end{array} \right| + (\beta_0 - \alpha_0) P_{\lambda_1}(\Lambda^{-1} p, q) - \frac{1}{4\beta_1^2}$$

$$= \sum_{m=-1}^{\infty} \lambda^{m+1} F_m,$$  (4.24)

where

$$F_{-1} = -\frac{1}{4}(\alpha_0 - \beta_0 - \langle p, q \rangle)^2,$$

$$F_m = \sum_{j=0}^{m} \left| \begin{array}{cc} \langle \Lambda^{j+1} p, p \rangle & \langle \Lambda^{m-j} p, q \rangle \\ \langle \Lambda^{j+1} p, q \rangle & \langle \Lambda^{m-j} q, q \rangle \end{array} \right| + (\alpha_0 - \beta_0) \langle \Lambda^{m+1} p, p \rangle, \quad m \geq 0,$$  (4.25)

$$F_{-1} = -\frac{1}{4}(\beta_0 - \alpha_0 - \langle p, q \rangle)^2,$$

$$F_m = \sum_{j=0}^{m} \left| \begin{array}{cc} \langle \Lambda^{-j} p, p \rangle & \langle \Lambda^{-m+j-1} p, q \rangle \\ \langle \Lambda^{-j} p, q \rangle & \langle \Lambda^{-m+j-1} q, q \rangle \end{array} \right| + (\beta_0 - \alpha_0) \langle \Lambda^{-m-1} p, p \rangle, \quad m \geq 0.$$  (4.26)
In order to prove the involutivity of \{F_m\} and \{\mathcal{F}_m\}, we introduce the generating function method, which is convenient in a series of later calculations. Denote the variables of \(F_\lambda\)-flow by \(\tau_\lambda\) and \(\tilde{\tau}_\lambda\)-flow by \(\tilde{\tau}_\lambda\), respectively. Let us consider \(F_\lambda\) and \(\mathcal{F}_\lambda\) as two Hamiltonians in the symplectic space \((\mathbb{R}^{2N}, dp \wedge dq)\). A direct calculation gives the canonical equations of the \(F_\lambda\)-flow and \(\mathcal{F}_\lambda\)-flow:

\[
\frac{d}{d\tau_\lambda} \left( \begin{array}{c} p_k \\
q_k \end{array} \right) = I \nabla_k F_\lambda = \left( \begin{array}{c} -\frac{\partial F_\lambda}{\partial q_k} \\
\frac{\partial F_\lambda}{\partial p_k} \end{array} \right) = W(\lambda, \lambda_k) \left( \begin{array}{c} p_k \\
q_k \end{array} \right),
\]

and

\[
\frac{d}{d\tilde{\tau}_\lambda} \left( \begin{array}{c} p_k \\
q_k \end{array} \right) = I \nabla_k \mathcal{F}_\lambda = \left( \begin{array}{c} -\frac{\partial \mathcal{F}_\lambda}{\partial q_k} \\
\frac{\partial \mathcal{F}_\lambda}{\partial p_k} \end{array} \right) = W(\lambda, \lambda_k) \left( \begin{array}{c} p_k \\
q_k \end{array} \right),
\]

where

\[
W(\lambda, \mu) = \frac{1}{\lambda - \mu} \begin{pmatrix}
(\lambda + \mu)[P_\lambda(\Lambda p, q) - \frac{1}{2\alpha_1}] & -2\lambda P_\lambda(\Lambda p, p) \\
2\lambda \mu P_\lambda(q, q) & -(\lambda + \mu)[P_\lambda(\Lambda p, q) - \frac{1}{2\alpha_1}]\end{pmatrix},
\]

\[
W(\lambda, \mu) = \frac{1}{\lambda^{-1} - \mu^{-1}} \begin{pmatrix}
(\lambda^{-1} + \mu^{-1})[P_\lambda(A^{-1} p, q) - \frac{1}{2\beta_1}] & -2\lambda^{-1} \mu^{-1} P_\lambda(p, p) \\
2\lambda^{-1} \mu^{-1} P_\lambda(A^{-1} q, q) & -(\lambda^{-1} + \mu^{-1})[P_\lambda(A^{-1} p, q) - \frac{1}{2\beta_1}]\end{pmatrix}.
\]

The Poisson bracket of two smooth functions \(h_1\) and \(h_2\) in the symplectic space \((\mathbb{R}^{2N}, dp \wedge dq)\) is defined as

\[
\{h_1, h_2\} = \sum_{j=1}^{N} \left( \frac{\partial h_1}{\partial q_j} \frac{\partial h_2}{\partial p_j} - \frac{\partial h_1}{\partial p_j} \frac{\partial h_2}{\partial q_j} \right).
\]

**Theorem 8.** The matrices \(V_\mu\) and \(V_\mu\) satisfy the Lax equations alone the \(\tau_\lambda\)-flow and \(\tilde{\tau}_\lambda\)-flow:

\[
\frac{d}{d\tau_\lambda} V_\mu = [W(\lambda, \mu), V_\mu],
\]

\[
\frac{d}{d\tilde{\tau}_\lambda} V_\mu = [W(\lambda, \mu), V_\mu],
\]

which imply

\[
\{F_\mu, F_\lambda\} = 0, \quad \{\mathcal{F}_\mu, \mathcal{F}_\lambda\} = 0, \quad \forall \lambda, \mu \in \mathbb{C},
\]

\[
\{F_j, F_k\} = 0, \quad \{\mathcal{F}_j, \mathcal{F}_k\} = 0, \quad \forall j, k \geq -1.
\]

**Proof.** Using (4.27) and (4.28), we have

\[
\frac{d}{d\tau_\lambda} (p, q) = 0, \quad \frac{d}{d\tilde{\tau}_\lambda} (p, q) = 0.
\]

Notice the identities (4.22) and
\[
\langle (\mu - \Lambda)^{-1} (\lambda - \Lambda)^{-1} A^l \xi, \eta \rangle = \frac{1}{\mu - \lambda} \left( P_\lambda (A^l \xi, \eta) - P_\mu (A^l \xi, \eta) \right),
\]
\[
\langle (\mu^{-1} - \Lambda^{-1})^{-1} (\lambda^{-1} - \Lambda^{-1})^{-1} A^l \xi, \eta \rangle = \frac{1}{\mu^{-1} - \lambda^{-1}} \left( P_\lambda (A^l \xi, \eta) - P_\mu (A^l \xi, \eta) \right),
\]
\[(4.36)\]
a direct calculation shows that (4.31) and (4.32) hold. As a consequence of the Lax equations (4.31) and (4.32) we obtain
\[
0 = \frac{d}{d\tau_\lambda} \det V_\mu = \frac{dF_\mu}{d\tau_\lambda} = \{F_\mu, F_\lambda\},
\]
\[
0 = \frac{d}{d\tilde{\tau}_\lambda} \det V_\mu = \frac{dF_\mu}{d\tilde{\tau}_\lambda} = \{F_\mu, F_\lambda\}.
\]
Substituting the expansions (4.23) and (4.24) into (4.33) gives (4.34) by comparing the same power of \(\lambda, \mu\). \(\Box\)

From (4.23), it is easy to see that
\[
F_\lambda = -\left( V_{\lambda}^{11} \right)^2 - V_{\lambda}^{12} V_{\lambda}^{21} = \sum_{j=1}^{N} \frac{\Gamma_j}{\lambda - \lambda_j} - \frac{1}{4\alpha_1^2} \frac{w(\lambda)}{u(\lambda)} = -\frac{r(\lambda)}{4\alpha_1^2 u^2(\lambda)},
\]
\[(4.37)\]
\[
V_{\lambda}^{12} = P_\lambda (A\mu, p) = \frac{n(\lambda)}{u(\lambda)},
\]
\[(4.38)\]
where
\[
\Gamma_j = \lambda_j \left[ (\alpha_0 - \beta_0 - \langle p, q \rangle) p_j q_j + \langle q, q \rangle p_j^2 + \sum_{k=1, k \neq j}^{N} \frac{\lambda_k (p_j q_k - p_k q_j)^2}{\lambda_j - \lambda_k} \right],
\]
\[(4.39)\]
\[
u(\lambda) = \prod_{j=1}^{N} (\lambda - \lambda_j), \quad w(\lambda) = \prod_{j=1}^{N} (\lambda - \lambda_{j+N}),
\]
\[
u(\lambda) = \prod_{j=1}^{N} (\lambda - \lambda_j), \quad w(\lambda) = \prod_{j=1}^{N} (\lambda - \lambda_{j+N}),
\]
\[
u(\lambda) = \prod_{j=1}^{N} (\lambda - \lambda_j), \quad w(\lambda) = \prod_{j=1}^{N} (\lambda - \lambda_{j+N}).
\]
\[(4.40)\]
Here the roots \(\{\mu_j\}\) are defined as elliptic variables.

**Proposition 9.** The elliptic coordinates satisfy the evolution equation along the \(\tau_\lambda\) flow:
\[
\frac{\alpha_1}{2\sqrt{r(\mu_j)}} \frac{d\mu_j}{d\tau_\lambda} = -\frac{\lambda n(\lambda)}{u(\lambda)(\lambda - \mu_j)n'(\mu_j)}, \quad 1 \leq j \leq N - 1.
\]
\[(4.41)\]
Moreover,
\[
\sum_{j=1}^{N-1} \frac{\alpha_1 \mu_j^{N-1-l} d\mu_j}{2\sqrt{r(\mu_j)} d\tau_\lambda} = -\frac{\lambda^{N-l}}{u(\lambda)}, \quad 1 \leq l \leq N - 1.
\]
\[(4.42)\]
**Proof.** Substituting \(\lambda = \mu_j\) in (4.37), we have
\[
V_{\mu_j}^{11} = \frac{\sqrt{r(\mu_j)}}{2\alpha_1 u(\mu_j)}.
\]
In the second component of (4.31),
\[
\frac{d}{d\tau\lambda} V_{12}^{\mu} = \frac{2}{\lambda - \mu} [2\lambda V_{11}^{11} V_{12}^{\mu} - (\lambda + \mu) V_{11}^{11} V_{12}^{\mu}],
\]
let \( \mu = \mu_j \). After some calculation, we obtain (4.41). By means of the interpolation formula for polynomials, it is easy to see that (4.42) holds. \( \Box \)

For fixed \( \lambda_0 \), we introduce the quasi-Abel–Jacobi coordinates:
\[
\rho_l = \sum_{j=1}^{N-1} \int_{\lambda_0}^{\mu_j} \frac{\alpha_j \mu^{N-l} d\mu}{2\sqrt{r(\mu)}}, \quad 1 \leq l \leq N - 1.
\]
(4.43)

Then we have
\[
\frac{d\rho_l}{d\tau\lambda} = -\frac{\lambda^{N-l}}{u(\lambda)}.
\]
(4.44)

It is not difficult to verify that the coefficient of the expansion
\[
\frac{1}{(1 - \lambda_1 \lambda^{-1}) \cdots (1 - \lambda_N \lambda^{-1})} = \sum_{k=0}^{\infty} \omega_k \lambda^{-k}
\]
(4.45)
can be determined recursively by
\[
\omega_0 = 1, \quad \omega_1 = s_1, \quad \omega_k = \frac{1}{k} \left( s_k + \sum_{i+j=k, i,j \geq 1} s_i \omega_j \right),
\]
(4.46)
where \( s_k = \lambda_1^k + \cdots + \lambda_N^k \). Denote the variable of \( F_k \)-flow by \( \tau_k \) \( (k \geq -1) \). Noticing the definition of the Poisson bracket and comparing the coefficients of \( \lambda^{-k-1} \) in the expansion of (4.44) give
\[
\frac{d\rho_l}{d\tau_k} = \{ \rho_l, F_k \} = -\omega_{k+1-l}, \quad 1 \leq l \leq N - 1,
\]
(4.47)
with supplementary definition \( \omega_{-j} = 0 \) \( (j = 1, 2, \ldots) \). Thus we have
\[
\frac{d\rho}{d\tau_{-1}} = 0,
\]
(4.48)
\[
\left( \frac{d\rho}{d\tau_0}, \frac{d\rho}{d\tau_1}, \ldots, \frac{d\rho}{d\tau_{N-2}} \right) = -
\begin{pmatrix}
1 & \omega_1 & \cdots & \omega_{N-2} \\
& 1 & \omega_1 & \cdots & \omega_{N-3} \\
& & \ddots & \ddots & \vdots \\
& & & \ddots & \omega_1 \\
& & & & 1
\end{pmatrix},
\]
where \( \rho = (\rho_1, \ldots, \rho_{N-1})^T \).

**Proposition 10.** The conserved integrals \( \{ F_{-1}, F_0, \ldots, F_{N-2} \} \) and \( \{ F_{-1}, F_0, \ldots, F_{N-2} \} \) are functionally independent, respectively.

**Proof.** We need only prove the linear independence of the differentials \( dF_{-1}, dF_0, \ldots, dF_{N-2} \).
Suppose \( \sum_{k=-1}^{N-2} \kappa_k dF_k = 0 \). We have
0 = \sum_{j=-1}^{N-2} \kappa_j \alpha_j^2 (Id F_j, Id \rho_l) = \sum_{j=-1}^{N-2} \kappa_j \rho_l, \quad F_j = \sum_{j=0}^{N-2} \kappa_j \frac{d \rho_l}{d \tau_j}, \quad l = 1, \ldots, N - 1.

Thus \( \kappa_0 = \cdots = \kappa_{N-2} = 0 \) because of (4.48). Thus \( \kappa_{-1} d F_{-1} = 0 \), which implies that \( \kappa_{-1} = 0 \) since

\[ d F_{-1} = \frac{1}{2} (\alpha_0 - \beta_0 - \langle p, q \rangle) \sum_{j=1}^{N} (q_j dp_j + p_j dq_j) \neq 0 \]

as long as \( \alpha_0 - \beta_0 - \langle p, q \rangle \neq 0 \). Similarly, we can prove the linear independence of the differentials \( d F_{-1}, d F_0, \ldots, d F_{N-2} \).

The invariance of \( F_\lambda \) and \( \tilde{F}_\lambda \) under the symplectic map \( S \) implies the invariance of \( F_m \) and \( \tilde{F}_m \). The integrals \{\( F_k \)\}_{k=-1}^{N-2} or \{\( \tilde{F}_k \)\}_{k=-1}^{N-2} are in involution in pair and functionally independent. Therefore, we have the following assertion [30,31].

**Theorem 11.** The symplectic map \( S \) defined by (4.12) and the finite-dimensional Hamiltonian systems

\[
\frac{d}{d \tau_m} \left( \begin{array}{c} p \\ q \end{array} \right) = \left( \begin{array}{c} \frac{\partial F_m}{\partial p} \\ \frac{\partial F_m}{\partial q} \end{array} \right), \quad 0 \leq m \leq N - 2, \quad (4.49)
\]

and

\[
\frac{d}{d \hat{\tau}_m} \left( \begin{array}{c} p \\ q \end{array} \right) = \left( \begin{array}{c} \frac{\partial \tilde{F}_m}{\partial p} \\ \frac{\partial \tilde{F}_m}{\partial q} \end{array} \right), \quad 0 \leq m \leq N - 2, \quad (4.50)
\]

are completely integrable in the Liouville sense.

**5. Decomposition of the discrete Ablowitz–Ladik hierarchy**

In this section, we shall give a decomposition of the discrete Ablowitz–Ladik hierarchy. With the aid of Proposition 6 and (4.6), we have

\[
\sum_{j=1}^{N} \nabla \lambda_j = \tilde{g}^{(0)} + \hat{g}^{(0)} + \varepsilon_0 \sigma_0, \quad (5.1)
\]

where \( \varepsilon_0 = -\alpha_0 - \beta_0 - \langle p, q \rangle \), \( \sigma_0 = (0, 0, 0, 0, 1, 1)^T \). It is easy to see that

\[
\ker K = \{ \hat{\ell}_0 \tilde{g}^{(0)} |_{\rho_0=1} : \forall \hat{\ell}_0 \}, \quad \ker \tilde{J} = \{ \ell_0 \tilde{g}^{(0)} |_{\rho_0=1} : \forall \ell_0 \},
\]

\[
\tilde{J}^{-1} K \sigma_0 = \frac{1}{2 \alpha_0} \tilde{g}^{(1)} + \ker \tilde{J}, \quad K^{-1} \tilde{J} \sigma_0 = \frac{1}{2 \beta_0} \hat{g}^{(1)} + \ker K.
\]

Acting with \( \tilde{J}^{-1} K \) and \( K^{-1} \tilde{J} \) upon (5.1) \( k \) times, respectively, we obtain

\[
\sum_{j=1}^{N} \lambda_j \nabla \lambda_j = \left( 1 + \frac{\varepsilon_0}{2 \alpha_0} \right) \tilde{g}^{(k)} + \hat{\ell}_1 \hat{g}^{(k-1)} + \cdots + \hat{\ell}_{k-1} \hat{g}^{(1)} + \hat{\ell}_k \hat{g}^{(0)}, \quad k \geq 1, \quad (5.2)
\]

\[
\sum_{j=1}^{N} \lambda_j^{-k} \nabla \lambda_j = \left( 1 + \frac{\varepsilon_0}{2 \beta_0} \right) \tilde{g}^{(k)} + \hat{\ell}_1 \hat{g}^{(k-1)} + \cdots + \hat{\ell}_{k-1} \hat{g}^{(1)} + \hat{\ell}_k \hat{g}^{(0)}, \quad k \geq 1, \quad (5.3)
\]
in view of (3.15), (3.16) and (4.5), where \( \ell_j \) and \( \hat{\ell}_j \) are constants. Using (4.16), (4.3) and noting Proposition 6, we have

\[
G_\lambda = \frac{1}{2\alpha_0}\tilde{g}(0) + \sum_{j=1}^N \frac{\lambda_j \nabla \lambda_j}{\lambda - \lambda_j} = \left(1 + \frac{\varepsilon_0}{2\alpha_0}\right)\tilde{g}(0) + \sum_{k=1}^\infty \lambda^{-k} \sum_{j=1}^N \lambda^k_j \nabla \lambda_j
\]

\[
= \left(1 + \frac{\varepsilon_0}{2\alpha_0} + \sum_{k=1}^\infty \left(1 + \frac{\varepsilon_0}{2\alpha_0}\right)\tilde{g}(k) + \ell_1 \hat{\tilde{g}}(k-1) + \cdots + \ell_{k-1} \hat{\tilde{g}}(1) + \ell_k \hat{\tilde{g}}(0)\right)
\]

\[
= \left(1 + \frac{\varepsilon_0}{2\alpha_0} + \sum_{k=1}^\infty \hat{\ell}_k \hat{\lambda}^{-k}\right) \sum_{k=0}^\infty \lambda^{-k}\hat{\tilde{g}}(k) = \ell_\lambda \hat{\tilde{g}}_\lambda
\]

(5.4)

with

\[
\ell_\lambda = 1 + \frac{\varepsilon_0}{2\alpha_0} + \sum_{k=1}^\infty \hat{\ell}_k \hat{\lambda}^{-k}.
\]

(5.5)

In a similar way, we obtain from (4.17), (4.3) and (5.3) that

\[
\hat{G}_\lambda = \hat{\ell}_\lambda \hat{\tilde{g}}_\lambda, \quad \hat{\ell}_\lambda = 1 + \frac{\varepsilon_0}{2\beta_0} + \sum_{k=1}^\infty \hat{\ell}_k \hat{\beta}^k.
\]

(5.6)

According to Proposition 4 and noticing (5.5), (5.6), we arrive at

\[
F_\lambda = -\alpha_0^2 \ell_\lambda^2, \quad \mathcal{F}_\lambda = -\beta_0^2 \hat{\ell}_\lambda^2.
\]

(5.7)

Let us introduce two sets of integrals \( \{H_k\}, \{\mathcal{H}_k\} \) by

\[
H_0 = -\frac{1}{2}\alpha_0(p, q), \quad H_k = \alpha_0^2 \ell_k, \quad k \geq 1,
\]

(5.8)

and

\[
\mathcal{H}_0 = -\frac{1}{2}\beta_0(p, q), \quad \mathcal{H}_k = \beta_0^2 \hat{\ell}_k, \quad k \geq 1,
\]

(5.9)

which are put in the equivalent forms

\[
2\alpha_0 \ell_\lambda = \alpha_0 - \beta_0 + \frac{2}{\alpha_0} H_\lambda,
\]

(5.10)

\[
2\beta_0 \hat{\ell}_\lambda = \beta_0 - \alpha_0 + \frac{2}{\beta_0} \mathcal{H}_\lambda
\]

(5.11)

with the aid of the generating functions

\[
H_\lambda = \sum_{k=0}^\infty H_k \lambda^{-k}, \quad \mathcal{H}_\lambda = \sum_{k=0}^\infty \mathcal{H}_k \lambda^k.
\]

(5.12)

Using (5.7), (5.10) and (5.11), we have

\[
-F_\lambda = \frac{1}{4} \left(\alpha_0 - \beta_0 + \frac{2}{\alpha_0} H_\lambda\right)^2, \quad -\mathcal{F}_\lambda = \frac{1}{4} \left(\beta_0 - \alpha_0 + \frac{2}{\beta_0} \mathcal{H}_\lambda\right)^2.
\]

(5.13)

which imply
\[ H_0 = -\frac{1}{2} \alpha_0 (p, q), \quad H_1 = -\alpha_0 \alpha_1 F_0, \]

\[ H_{k+1} = -\alpha_0 \alpha_1 F_k - \alpha_1 \alpha_0^{-1} \sum_{j=1}^{k} H_j H_{k+1-j}, \quad k \geq 1, \quad (5.14) \]

and

\[ H_0 = -\frac{1}{2} \beta_0 (p, q), \quad H_1 = -\beta_0 \beta_1 F_0, \]

\[ H_{k+1} = -\beta_0 \beta_1 F_k - \beta_1 \beta_0^{-1} \sum_{j=1}^{k} H_j H_{k+1-j}, \quad k \geq 1, \quad (5.15) \]

in view of (4.23) and (4.24). The involutivity of \( \{ H_k \} \) and \( \{ H_k \} \) is respectively based on the equalities

\[ \{ H_\mu, H_\lambda \} = \frac{\alpha_0^2}{4 \sqrt{F_\mu F_\mu}} \{ F_\mu, F_\lambda \} = 0, \]

\[ \{ H_\mu, H_\lambda \} = \frac{\beta_0^2}{4 \sqrt{F_\mu F_\mu}} \{ F_\mu, F_\lambda \} = 0. \quad (5.16) \]

Denote the variables of the \( H_\lambda \)-flow and \( H_\lambda \)-flow by \( \tilde{t}_\lambda \) and \( \hat{t}_\lambda \), respectively. By the Leibniz rule of the Poisson bracket we get

\[ \{ h, F_\lambda \} = -2 \ell_\lambda \{ h, H_\lambda \}, \quad \{ h, F_\lambda \} = -2 \hat{\ell}_\lambda \{ h, H_\lambda \} \quad (5.17) \]

for any smooth function \( h \). Therefore, we have

\[ \frac{d}{d t_\lambda} = -\frac{1}{2 \ell_\lambda} \frac{d}{d \tau_\lambda}, \quad \frac{d}{d \hat{t}_\lambda} = -\frac{1}{2 \hat{\ell}_\lambda} \frac{d}{d \hat{\tau}_\lambda}. \quad (5.18) \]

From (4.27) and (4.28), we arrive at

\[ \frac{d}{d \tau_\lambda} \left( \frac{Q}{R} \right) = 2 \left( \frac{\alpha_1 (A p, q_\tau_\lambda)}{\beta_1 (A^{-1} q, q_\tau_\lambda)} \right) = -2 \left( \frac{P_\lambda (A^2 p, p) + Q (1/2a_1) + P_\lambda (A p, q))}{P_\lambda (q, q) - R (1/2a_1) - P_\lambda (A p, q))} \right), \quad (5.19) \]

\[ \frac{d}{d \hat{\tau}_\lambda} \left( \frac{Q}{R} \right) = 2 \left( \frac{\alpha_1 (A p, q_\hat{\tau}_\lambda)}{\beta_1 (A^{-1} q, q_\hat{\tau}_\lambda)} \right) = 2 \left( \frac{\mathcal{P}_\lambda (p, p) - Q (1/2a_1) - \mathcal{P}_\lambda (A^{-1} p, q))}{\mathcal{P}_\lambda (A^{-2} q, q) + R (1/2a_1) + \mathcal{P}_\lambda (A^{-1} p, q))} \right). \quad (5.20) \]

Resorting to the second expression of (4.10) and (4.27), we have

\[ \gamma q = Z \hat{q} - R Z \hat{p}, \quad \gamma p = Z^{-1} \hat{p} - Q Z^{-1} \hat{q}, \quad (5.21) \]

\[ \gamma Z p_\tau_\lambda = \hat{V}_\lambda^{11} (\lambda - A)^{-1} (\lambda + A) (\hat{p} - Q \hat{q}) - 2 \lambda \hat{V}_\lambda^{12} (\lambda - A)^{-1} A (\hat{q} - R \hat{p}), \]

\[ \gamma Z q_\tau_\lambda = 2 \lambda \hat{V}_\lambda^{21} (\lambda - A)^{-1} A (\hat{p} - Q \hat{q}) - \hat{V}_\lambda^{11} (\lambda - A)^{-1} A (\hat{q} - R \hat{p}), \quad (5.22) \]

where

\[ \hat{V}_\lambda^{11} = P_\lambda (A p, q) - \frac{1}{2 \alpha_1}, \quad \hat{V}_\lambda^{12} = P_\lambda (A p, q), \quad \hat{V}_\lambda^{21} = P_\lambda (q, q). \]

Then we obtain by (4.10) that
Substituting (5.26) into (5.24) and (5.25) yields
\[
\hat{p}_{t\lambda} = \frac{1}{\gamma} \left[ Zp_{t\lambda} + Q_{t\lambda} Z^{-1} q + Q Z^{-1} q_{t\lambda} + \frac{1}{2\gamma} (QR)_{t\lambda} \hat{p} \right],
\]
\[
\hat{q}_{t\lambda} = \frac{1}{\gamma} \left[ R_{t\lambda} Zp + R Z p_{t\lambda} + Z^{-1} q_{t\lambda} + \frac{1}{2\gamma} (QR)_{t\lambda} \hat{q} \right].
\]
Using (5.19) and (5.21)–(5.23), a direct calculation gives that
\[
\frac{\gamma^2}{2\alpha_1} S_{t\lambda} = \gamma^2 \langle A^2 \hat{p}, \hat{p}_{t\lambda} \rangle = 2 \left[ (1 + QR) \hat{V}_{\lambda}^{11} + Q \hat{V}_{\lambda}^{21} + \lambda R \hat{V}_{\lambda}^{12} \right] P_\lambda (A^2 \hat{p}, \hat{p})
- 2\lambda \left[ 2Q \hat{V}_{\lambda}^{11} + \lambda \hat{V}_{\lambda}^{12} + Q^2 \hat{V}_{\lambda}^{21} \right] P_\lambda (A \hat{p}, \hat{q})
+ \left[ (1 + QR) \hat{V}_{\lambda}^{11} + 2Q \hat{V}_{\lambda}^{21} + \frac{1}{2} QR_{t\lambda} - \frac{1}{2} R Q_{t\lambda} \right] \langle A \hat{p}, \hat{q} \rangle
+ \left[ 2Q \hat{V}_{\lambda}^{11} + 2\lambda \hat{V}_{\lambda}^{12} + Q_{t\lambda} \right] \langle A \hat{p}, \hat{q} \rangle,
\]
\[
\frac{\gamma^2}{2\beta_1} T_{t\lambda} = \gamma^2 \langle A^{-1} \hat{q}, \hat{q}_{t\lambda} \rangle = -2 \left[ (1 + QR) \hat{V}_{\lambda}^{11} + Q \hat{V}_{\lambda}^{21} + \lambda R \hat{V}_{\lambda}^{12} \right] P_\lambda (\hat{q}, \hat{q})
+ 2 \left[ 2R \hat{V}_{\lambda}^{11} + \lambda R^2 \hat{V}_{\lambda}^{12} + \hat{V}_{\lambda}^{21} \right] P_\lambda (\hat{q}, \hat{q})
- \left[ (1 + QR) \hat{V}_{\lambda}^{11} + 2Q \hat{V}_{\lambda}^{21} + \frac{1}{2} QR_{t\lambda} - \frac{1}{2} R Q_{t\lambda} \right] \langle A^{-1} \hat{q}, \hat{q} \rangle
+ \left[ 2R \hat{V}_{\lambda}^{11} + 2 \hat{V}_{\lambda}^{21} + R_{t\lambda} \right] \langle A^{-1} \hat{p}, \hat{q} \rangle.
\]
Noticing (4.22) and (5.19), it is easy to calculate that
\[
(1 + QR) \hat{V}_{\lambda}^{11} + Q \hat{V}_{\lambda}^{21} + \lambda R \hat{V}_{\lambda}^{12} = \gamma^2 \left[ P_\lambda (A \hat{p}, \hat{q}) - \frac{1}{2\alpha_1} \right],
\]
\[
2Q \hat{V}_{\lambda}^{11} + \lambda \hat{V}_{\lambda}^{12} + Q^2 \hat{V}_{\lambda}^{21} = \gamma^2 P_\lambda (A \hat{p}, \hat{p}),
\]
\[
(1 + QR) \hat{V}_{\lambda}^{11} + 2Q \hat{V}_{\lambda}^{21} + \frac{1}{2} QR_{t\lambda} - \frac{1}{2} R Q_{t\lambda} = \gamma^2 \left[ P_\lambda (A \hat{p}, \hat{q}) - \frac{1}{2\alpha_1} \right],
\]
\[
2Q \hat{V}_{\lambda}^{11} + 2\lambda \hat{V}_{\lambda}^{12} + Q_{t\lambda} = 0,
\]
\[
2R \hat{V}_{\lambda}^{11} + \hat{V}_{\lambda}^{21} + \lambda R^2 \hat{V}_{\lambda}^{12} = \gamma^2 P_\lambda (\hat{q}, \hat{q}),
\]
\[
2R \hat{V}_{\lambda}^{11} + 2 \hat{V}_{\lambda}^{21} + R_{t\lambda} = 0.
\]
Substituting (5.26) into (5.24) and (5.25) yields
\[
\frac{d}{dt} \left( \begin{array}{c} S \\ T \end{array} \right) = -2 \left( \begin{array}{c} P_\lambda (A^2 \hat{p}, \hat{p}) + S(\frac{1}{2\alpha_1} + P_\lambda (A \hat{p}, \hat{q})) \\ P_\lambda (\hat{q}, \hat{q}) - T(\frac{1}{2\beta_1} - P_\lambda (A \hat{p}, \hat{q})) \end{array} \right),
\]
where \( \alpha_1 = \hat{\alpha}_1 \) and \( \beta_1 = \hat{\beta}_1 \) are used. Similarly, we obtain
\[
\gamma Z p_{\tilde{t} \lambda} = \hat{V}_{\lambda}^{11} (\lambda^{-1} - A^{-1})^{-1} (\lambda^{-1} + A^{-1}) (\hat{p} - Q \hat{q})
- 2\lambda^{-1} \hat{V}_{\lambda}^{12} (\lambda^{-1} - A^{-1})^{-1} (\hat{q} - R \hat{p}),
\]
\[
\gamma Z q_{\tilde{t} \lambda} = 2\lambda^{-1} \hat{V}_{\lambda}^{21} (\lambda^{-1} - A^{-1})^{-1} A^{-1} (\hat{p} - Q \hat{q})
- \hat{V}_{\lambda}^{11} (\lambda^{-1} - A^{-1})^{-1} (\lambda^{-1} + A^{-1}) (\hat{q} - R \hat{p})
\]
with
From (5.19), (5.20), (5.27) and (5.32), we have

Substituting (5.4) and (5.6) into (5.33) yields

Therefore, we arrive at

and

Therefore, we arrive at

From (5.19), (5.20), (5.27) and (5.32), we have

Substituting (5.4) and (5.6) into (5.33) yields

\[
\frac{d}{dt} \begin{pmatrix} Q \\ R \\ S \\ T \end{pmatrix} = \mathcal{J} \mathcal{L} \tilde{g}_\lambda = \mathcal{J} \mathcal{L} \sum_{j=0}^{\infty} \tilde{g}^{(j)}_\lambda \Lambda^{-j},
\]
\[
\frac{d}{d\tilde{t}_j} \begin{pmatrix} Q \\ R \\ S \\ T \end{pmatrix} = -JL \hat{g}_j = -JL \sum_{j=0}^{\infty} \hat{g}^{(j)} \lambda^j
\]  
(5.34)

in view of (5.18) and (3.6).

Assume that \(\tilde{t}_j\) and \(\hat{t}_j\) are the variables of \(H_j\)-flow and \(\mathcal{H}_j\)-flow, respectively. Therefore we get from (5.34) that

\[
\frac{d}{d\tilde{t}_j} \begin{pmatrix} Q \\ R \\ S \\ T \end{pmatrix} = JL \hat{g}_j, \quad \frac{d}{d\hat{t}_j} \begin{pmatrix} Q \\ R \\ S \\ T \end{pmatrix} = -JL \hat{g}_j, \quad j \geq 0.
\]  
(5.35)

**Theorem 12.** Let \((p(n, t_m), q(n, t_m))^T\) be a compatible solution of the discrete \(S^2\)-flow (4.10) and \(\bar{H}_m\)-flow

\[
\frac{d}{dt_m} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} -\partial \bar{H}_m/\partial p \\ \partial \bar{H}_m/\partial q \end{pmatrix}, \quad m \geq 0,
\]  
(5.36)

where \(\bar{H}_m = H_m + \mathcal{H}_m\), and \(t_m\) stands for the variable of \(\bar{H}_m\)-flow. Then

\[
\begin{pmatrix} Q \\ R \\ S \\ T \end{pmatrix} = f_S(p(n, t_m), q(n, t_m))
\]  
(5.37)

determined by (4.15) solves the \(m\)th discrete Ablowitz–Ladik equation (3.18)

\[
\frac{\partial}{\partial t_m} \begin{pmatrix} Q \\ R \\ S \\ T \end{pmatrix} = X_m, \quad m \geq 0.
\]  
(5.38)

According to Theorem 12, let \((p^0(t_m), q^0(t_m))^T\) be the solution of the initial-value problem

\[
\begin{cases} 
\frac{d}{dt_m} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} -\partial \bar{H}_m/\partial p \\ \partial \bar{H}_m/\partial q \end{pmatrix}, & m \geq 0, \\
\begin{pmatrix} p \\ q \end{pmatrix} |_{t_m=0} = \begin{pmatrix} p^0 \\ q^0 \end{pmatrix}.
\end{cases}
\]  
(5.39)

Then the algorithm

\[
\begin{pmatrix} p^0(t_m) \\ q^0(t_m) \end{pmatrix} \xrightarrow{S} \begin{pmatrix} p(n, t_m) \\ q(n, t_m) \end{pmatrix} \xrightarrow{f_S} \begin{pmatrix} Q_n(t_m) \\ R_n(t_m) \\ S_n(t_m) \\ T_n(t_m) \end{pmatrix}
\]  

yields a solution to the \(m\)th discrete Ablowitz–Ladik equation (3.18).

**Acknowledgments**

The work described in this paper was supported by a grant from City University of Hong Kong (Project No. 7001645). One of the authors (X.G.G.) acknowledges the support by National Natural Science Foundation of China (Project 10471132) and the Special Funds for Chinese Major State Basic Research Project “Nonlinear Science.”
References