Uniform boundedness and convergence of solutions to the systems with a single nonzero cross-diffusion

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Abstract

Uniform boundedness and convergence of global solutions are proved for quasilinear parabolic systems with a single nonzero cross-diffusion in population dynamics. Gagliardo–Nirenberg type inequalities are used in the estimates of solutions in order to establish $W^{1,2}$-bounds uniform in time. By using the uniform bound, convergence of solutions are established for systems with large diffusion coefficients in the weak competition case.

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1. Introduction

In an attempt to model spatial segregation phenomena between two competing species, Shigesada et al. [13] proposed the following quasilinear parabolic system in 1979:

\[
\begin{align*}
    u_t &= \Delta[(d_1 + \alpha_{11}u + \alpha_{12}v)u] + (a_1 - b_1u - c_1v)u \quad \text{in } \Omega \times (0, \infty), \\
    v_t &= \Delta[(d_2 + \alpha_{21}u + \alpha_{22}v)v] + (a_2 - b_2u - c_2v)v \quad \text{in } \Omega \times (0, \infty), \\
    \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} &= 0 \quad \text{on } \partial \Omega \times (0, \infty), \\
    u(x, 0) &= u_0(x) \geq 0, \quad v(x, 0) = v_0(x) \geq 0 \quad \text{in } \Omega,
\end{align*}
\]

(1.1)

where $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain and $\alpha_{ij} \geq 0$, $d_i$, $a_i$, $b_i$, $c_i$ are positive constants for $i, j = 1, 2$. In system (1.1) $u$ and $v$ are nonnegative functions which represent
the population densities of two competing species. \(d_1\) and \(d_2\) are the diffusion rates of the two species, respectively. \(a_1\) and \(a_2\) denote the intrinsic growth rates, \(b_1\) and \(c_2\) account for intra-specific competitions, \(b_2\) and \(c_1\) are the coefficients for inter-specific competitions. When \(a_{11} = a_{12} = a_{21} = a_{22} = 0\), \((1.1)\) reduces to the well-known Lotka–Volterra competition–diffusion system. \(a_{11}\) and \(a_{22}\) denote self-diffusion, and \(a_{12}, a_{21}\) are cross-diffusion pressures. By adopting the coefficients \(a_{ij}\) \((i, j = 1, 2)\), system \((1.1)\) takes into account the pressures created by mutually competing species. For more details on the backgrounds of this model we refer the reader to [10–13].

To describe results on system \((1.1)\) we use the following notation throughout this paper.

**Notation.** Let \(\Omega\) be a region in \(\mathbb{R}^n\). The norm in \(L^p(\Omega)\) is denoted by \(|·|_{L^p(\Omega)}, 1 \leq p \leq \infty\). The usual Sobolev spaces of real valued functions in \(\Omega\) with exponent \(k \geq 0\) are denoted by \(W^k_p(\Omega)\), \(1 \leq p \leq \infty\). And \(|·|_{W^k_p(\Omega)}\) represents the norm in the Sobolev space \(W^k_p(\Omega)\). For \(\Omega = [0, 1] \subset \mathbb{R}^1\) we shall use the simplified notation \(|·|_{k,p}\) for \(|·|_{W^k_p(\Omega)}\) and \(|·|_p\) for \(|·|_{L^p(\Omega)}\).

The local existence of solutions to \((1.1)\) was established by Amann [1–3]. According to his results system \((1.1)\) has a unique nonnegative solution \(u(·, t), v(·, t)\) in \(C([0, T), W^1_p(\Omega)) \cap C^\infty([0, T), C^\infty(\Omega))\), where \(T \in (0, \infty]\) is the maximal existence time for the solution \(u, v\). The following result is also due to Amann [2].

**Theorem 1.1.** Let \(u_0\) and \(v_0\) be in \(W^1_p(\Omega)\). System \((1.1)\) possesses a unique nonnegative maximal smooth solution \(u(x, t), v(x, t) \in C([0, T), W^1_p(\Omega)) \cap C^\infty([0, T), C^\infty(\Omega))\) for \(0 \leq t < T\), where \(T \in (0, \infty]\). If the solution satisfies the estimates

\[
\sup_{0 \leq t < T} \|u(·, t)\|_{W^1_p(\Omega)} < \infty, \quad \sup_{0 \leq t < T} \|v(·, t)\|_{W^1_p(\Omega)} < \infty,
\]

then \(T = +\infty\). If, in addition, \(u_0\) and \(v_0\) are in \(W^2_p(\Omega)\) then \(u(x, t), v(x, t) \in C([0, \infty), W^2_p(\Omega))\), and

\[
\sup_{0 \leq t < \infty} \|u(·, t)\|_{W^2_p(\Omega)} < \infty, \quad \sup_{0 \leq t < \infty} \|v(·, t)\|_{W^2_p(\Omega)} < \infty.
\]

System \((1.1)\) is a special case of the concrete example \((7), (8)\) in Introduction of [2], and the results stated in Theorem 1.1 are from the theorem in Introduction of [2].

So far the existence of nonnegative global solutions for system \((1.1)\) has been proved under very restrictive hypotheses only. Kim [7] proved the global existence of smooth nonnegative solutions for \(n = 1, d_1 = d_2,\) and \(a_{11} = a_{22} = 0\). Deuring [4] showed the global existence of classical positive solutions to system \((1.1)\) with \(a_{11} = a_{22} = 0\) and small coefficients depending on the initial values for \(n \geq 1\). In case \(n = 2\), Lou et al. [8] proved that system \((1.1)\) with \(a_{21} = 0\) has a unique smooth global solution. The arguments used in [8] still hold in the case \(n = 1\) with a minor modification. Yagi [14] established the global existence for system \((1.1)\) with \(n = 2\) under the condition either \(0 < a_{21} < 8\alpha_{11}, 0 < \alpha_{12} < 8\alpha_{22},\) or \(a_{21} = a_{22} = 0, \alpha_{11} > 0\). Kim [7] and Yagi [14] obtained estimates of Gronwall’s type depending on \(T\) to prove the global existence. And yet the qualitative
properties of those global solutions have not been proved for system (1.1) in the special cases mentioned above.

In this paper we study system (1.1) for $\Omega = [0, 1] \in \mathbb{R}^l$ with $\alpha_{21} = 0$ so that it is rewritten as follows:

$$
\begin{align*}
    u_t &= (d_1u + \alpha_{11}u^2 + \alpha_{12}uv)_x - \alpha_{22}uv + u(a_1 - b_1u - c_1v) & \text{in } [0, 1] \times (0, \infty), \\
    v_t &= (d_2v + \alpha_{22}v^2)_x + v(a_2 - b_2u - c_2v) & \text{in } [0, 1] \times (0, \infty), \\
    u_x(x, t) &= v_x(x, t) = 0 & \text{at } x = 0, 1, \\
    u(x, 0) &= u_0(x) \geq 0, & v(x, 0) &= v_0(x) \geq 0 & \text{in } [0, 1],
\end{align*}
$$

(1.2)

where $\alpha_{11}, \alpha_{12}, \alpha_{22}$ are nonnegative constants, and $d_1, a_1, b_1, c_1$ are positive constants for $i = 1, 2$. Throughout this paper we assume that the initial functions $u_0(x), v_0(x)$ are not identically zero and in the function space $W^1_{21}([0, 1]), W^2_{22}([0, 1])$, respectively. By applying the maximum principle (see [5, 12]) to system (1.2), we have for every $x \in [0, 1]$ and $t \geq 0$ that $0 \leq u(x, t) \leq M$ and $0 \leq v(x, t) \leq m_0$.

In order to prove that $u(x, t)$ is also uniformly bounded independent of $t$, we consider the following three cases for system (1.2):

(i) $\alpha_{21} = 0, \alpha_{11} > 0$,
(ii) $\alpha_{21} = \alpha_{22} = 0$,
(iii) $\alpha_{21} = \alpha_{11} = 0, \alpha_{22} > 0$, and $d_1d_2 > C(a_1/b_1)^2\alpha_{12}^2$, where $C$ is the positive constant from the calculus inequality (3.3).

Applying Gagliardo–Nirenberg type inequalities, we establish the uniform $W^1_{21}$-bound of the solutions from $\Omega$, the maximal existence time, for the solutions obtained in Theorem 1.1. Thus we have the global existence and the uniform $L_\infty$-bound of the solutions from Theorem 1.1 and the Sobolev embedding theorems. Using the uniform boundedness of $u(x, t)$ and $v(x, t)$, we obtain convergence results on the solution in the weak competition case for large $d_1, d_2$.

Here we state the main theorems of this paper.

**Theorem 1.2.** Let $(u(x, t), v(x, t))$ be the maximal solution to system (1.2) as stated in Theorem 1.1. In each case (i), (ii), and (iii), there exist positive constants $t_0, M' = M'(m_0, d_i, \alpha_{11}, \alpha_{12}, \alpha_{22}, a_i, b_i, c_i, i = 1, 2)$, and $M = M(m_0, d_i, \alpha_{11}, \alpha_{12}, \alpha_{22}, a_i, b_i, c_i, i = 1, 2)$ such that

$$
\max\left\{\|u(\cdot, t)\|_{1, 2}, \|v(\cdot, t)\|_{2, 2} : t \in (t_0, T)\right\} \leq M',
\max\{u(x, t), v(x, t) : (x, t) \in [0, 1] \times (t_0, T)\} \leq M,
$$

and $T = +\infty$.

**Theorem 1.3.** Suppose for system (1.2) that $a_1/a_2 < b_1/b_2$, and $u_0, v_0 \in W^2_{22}([0, 1])$. In each case (i), (ii), and (iii), if $d_1, d_2$ satisfy that

$$
b_2\alpha_{12}^2\bar{m}m_0^2 < 4c_1\bar{v}d_1d_2,
$$

(1.4)
where
\[(\bar{u}, \bar{v}) = \left( \frac{a_1c_2 - a_2c_1}{b_1c_2 - b_2c_1}, \frac{b_1a_2 - b_2a_1}{b_1c_2 - b_2c_1} \right), \]
then the solution \((u(t), v(t))\) converges uniformly in \([0, 1]\) as \(t \to \infty\), and \((\bar{u}, \bar{v})\) is globally asymptotically stable.

**Remark.** The condition \(c_1/c_2 < a_1/a_2 < b_1/b_2\) is called the weak competition condition throughout this paper.

The proof of Theorem 1.3 is given in Section 2. In Section 3 we collect calculus inequalities which are necessary for the proof of Theorem 1.2. Theorem 1.2 is proved in Section 4. In Section 4 each of cases (i), (ii), (iii) are proved in Sections 4.1, 4.2, 4.3, respectively.

### 2. Convergence in the weak competition case

**Proof of Theorem 1.3.** In this section we consider the weak competition case for system (1.2), that is, \(c_1/c_2 < a_1/a_2 < b_1/b_2\). Using the functional \(H(u, v)\) defined in the following, we observe the convergence of global solutions of system (1.2) in cases (i), (ii), (iii) in which we have the uniform boundedness of solutions from Theorem 1.2:

\[
H(u, v) = \int_0^1 \left\{ \frac{b_2}{u} \left( u - \bar{u} - \bar{u} \log \frac{u}{\bar{u}} \right) + c_1 \left( v - \bar{v} - \bar{v} \log \frac{v}{\bar{v}} \right) \right\} \, dx,
\]

where
\[
(\bar{u}, \bar{v}) = \left( \frac{a_1c_2 - a_2c_1}{b_1c_2 - b_2c_1}, \frac{b_1a_2 - b_2a_1}{b_1c_2 - b_2c_1} \right)
\]
is the stable constant steady-state of system (1.2) in the weak competition case. In order to prove the convergence theorems, first we observe the time derivative of \(H(u(t), v(t))\) for the solution of system (1.2):

\[
\frac{dH(u(t), v(t))}{dt} = \int_0^1 \left\{ \frac{b_2}{u} \left( 1 - \bar{u} \right) u_t + c_1 \left( 1 - \bar{v} \right) v_t \right\} \, dx
\]

\[
= \int_0^1 \left\{ \frac{b_2}{u} \left( 1 - \bar{u} \right) (d_1u + \alpha_1 u^2 + \alpha_2 uv)_{xx} \right. \\
+ c_1 \left( 1 - \bar{v} \right) (d_2v + \alpha_2 u v + \alpha_2 v^2)_{xx} \right\} \, dx
\]

\[
+ \int_0^1 \left\{ b_2(u - \bar{u}) f + c_1(v - \bar{v}) g \right\} \, dx
\]
From the weak competition condition for every \( \delta \) such that
\[
0 < \delta < \frac{b_2c_1(b_1c_2 - b_2c_1)}{b_1b_2 + c_1c_2},
\]
we have
\[
b_1b_2(u - \bar{u})^2 + 2b_2c_1(u - \bar{u})(v - \bar{v}) + c_1c_2(v - \bar{v})^2 \geq \delta \{(u - \bar{u})^2 + (v - \bar{v})^2\}.
\]
(2.1)

Now using the uniform boundness results in (1.3) and Theorem 1.2, it is given that
\[
v(x, t) \leq M_0, \quad u(x, t) \leq M
\]
(2.2)
for every \((x, t) \in [0, 1] \times [\tau_0, \infty)\). Using (2.2) and condition (1.4) for every constant \( \gamma \) such that
\[
0 < \gamma < \frac{b_2\bar{u}(4c_1\bar{v}d_1d_2 - b_2\alpha_1^2\bar{u}m_0^2)}{4[b_2\bar{u}(d_1 + 2\alpha_1M + \alpha_1m_0)m_0^2 + c_1\bar{v}(d_2 + 2\alpha_2m_0)M^2]},
\]
we have the following inequality:
\[
b_2\bar{u}(d_1 + 2\alpha_1u + \alpha_1v)u_x^2 + \frac{b_2\alpha_1\bar{u}}{u}u_t v_x + \frac{c_1\bar{v}}{v^2}(d_2 + 2\alpha_2v)v_x^2 \geq \gamma \{u_x^2 + v_x^2\},
\]
(2.3)
since
\[
\left( \frac{b_2\alpha_1\bar{u}}{u} \right)^2 - 4 \left\{ \frac{b_2\bar{u}}{u} (d_1 + 2\alpha_1u + \alpha_1v) - \gamma \left\{ \frac{c_1\bar{v}}{v^2}(d_2 + 2\alpha_2v) - \gamma \right\} \right\} \leq \frac{b_2\alpha_1^2\bar{u}^2}{u^2} - \frac{b_2\alpha_1\bar{u}d_1d_2}{u^2v} + 4\gamma \left\{ \frac{b_2\bar{u}}{u^2}(d_1 + 2\alpha_1M + \alpha_1m_0) + \frac{c_1\bar{v}}{v^2}(d_2 + 2\alpha_2m_0) \right\} \leq \frac{1}{u^2v^2} \left[ b_2\alpha_1^2\bar{u}^2m_0^2 - 4b_2\alpha_1\bar{u}\bar{v}d_1d_2 + 4\gamma \{b_2\bar{u}(d_1 + 2\alpha_1M + \alpha_1m_0)m_0^2 + c_1\bar{v}(d_2 + 2\alpha_2m_0)M^2 \} \right] < 0.
\]

From (2.1) and (2.3) we have
\[
\frac{dH(u(t), v(t))}{dt} \leq -\gamma \int_0^1 (u_x^2 + v_x^2) \, dx - \delta \int_0^1 \{(u - \bar{u})^2 + (v - \bar{v})^2\} \, dx \leq 0.
\]
We notice that \( \frac{dH(u(t), v(t))}{dt} = 0 \) only if \( u(x, t) \equiv \bar{u} \) and \( v(x, t) \equiv \bar{v} \). Thus it is shown that \( H(u(x, t), v(x, t)) \downarrow 0 \) as \( t \to 0 \). We obtain the \( L^2 \) convergences \( |u(t) - \bar{u}|_2 \to 0 \), \( |v(t) - \bar{v}|_2 \to 0 \) as \( t \to \infty \) by using the uniform boundedness of \((u(x, t), v(x, t))\) in \([0, 1]\). From Theorem 1.1, \( \sup_{0 \leq t < \infty} |uxx(t)|_2^2 < \infty \) and \( \sup_{0 \leq t < \infty} |vxx(t)|_2^2 < \infty \). Applying the calculus inequality (3.4) to the functions \( u(x, t) - \bar{u} \) and \( v(x, t) - \bar{v} \), we obtain the convergence \( (u(t), v(t)) \to (\bar{u}, \bar{v}) \) as \( t \to \infty \) in \( W^{1, 2}_1([0, 1]) \). By using the Sobolev embedding theorem we show that \( (u(t), v(t)) \) converges to \( (\bar{u}, \bar{v}) \) uniformly in \([0, 1]\) as \( t \to \infty \). We also obtain that \((\bar{u}, \bar{v})\) is locally asymptotically stable in \( C([0, 1]) \) by using the fact that \( H(u(t), v(t)) \) is decreasing for \( t \geq 0 \). Thus we conclude that \((\bar{u}, \bar{v})\) is globally asymptotically stable. 

3. Calculus inequalities

**Theorem 3.1.** Let \( \Omega \in \mathbb{R}^n \) be a bounded domain with \( \partial \Omega \) in \( C^m \). For every function \( u \) in \( W^{m, r}(\Omega) \), \( 1 \leq q, r \leq \infty \), the derivative \( D^j u \), \( 0 \leq j < m \), satisfies the inequality

\[
|D^j u|_p \leq C \left( |D^m u|_r^a |u|_1^{1-a} + |u|_q \right),
\]

where

\[
\frac{1}{p} = \frac{j}{n} + a \left( \frac{1}{r} - \frac{m}{n} \right) + (1 - a) \frac{1}{q}
\]

for all \( a \) in the interval \( j/m \leq a < 1 \), provided one of the following three conditions:

(i) \( r \leq q \),
(ii) \( 0 < n(r - q)/(mrq) < 1 \), or
(iii) \( n(r - q)/(mrq) = 1 \) and \( m - n/q \) is not a nonnegative integer.

(The positive constant \( C \) depends only on \( n, m, j, q, r, a \).)

**Proof.** We refer the reader to Theorem 10.1 in Part 1 of Friedman [5] or Nirenberg [9] for the proof of this well-known calculus inequality. 

**Corollary 3.1.** There exist positive constants \( \tilde{C} \) and \( C \) such that for every function \( u \) in \( W^2_2([0, 1]) \):

\[
|u|_2 \leq \tilde{C} \left( |u_{xx}|^{1/3}_2 |u|^{2/3}_1 + |u|_1 \right),
\]

\[
|u|_4 \leq C \left( |u_{xx}|^{1/2}_2 |u|_1^{1/2} + |u|_1 \right).
\]

**Proof.** \( m = 1, r = 2, q = 1 \) satisfy condition (ii) in Theorem 3.1. 

**Lemma 3.1.** For every function \( u \) in \( W^2_2([0, 1]) \) with \( u_x(0) = u_x(1) = 0 \),

\[
|u_x|_2 \leq |u_{xx}|^{1/2}_2 |u|^{1/2}_1,
\]
and for every function \( u \) in \( W_2^3([0,1]) \) with \( u_x(0) = u_x(1) = 0 \),
\[
|u_{xx}|_2 \leq |u_{xxx}|_2^{2/3} |u|_2^{1/3}.
\tag{3.5}
\]

**Proof.** Using the given boundary conditions and Hölder’s inequality
\[
\int_0^1 u_x^2 \, dx = - \int_0^1 uu_{xx} \, dx \leq |u_{xx}|_2 |u|_2,
\]
and thus inequality (3.4) holds. From inequality (3.4) we have
\[
\int_0^1 u_{xx}^2 \, dx = - \int_0^1 u_x u_{xxx} \, dx \leq |u_x|_2 |u_{xxx}|_2 \leq |u|_2^{1/2} |u_{xx}|_2^{1/2} |u_{xxx}|_2.
\]
Thus \( |u_{xx}|_2^{3/2} \leq |u_{xxx}|_2 |u|_2^{1/2} \) and (3.5) is proved. \( \square \)

**Lemma 3.2.** If a function \( f \) is in \( W_2^1([0,1]) \) then there exists a constant \( C > 0 \) such that
\[
|f|_\infty \leq C \left( \left( 1 + \frac{1}{\epsilon} \right) |f|_2^2 + \epsilon |f_{x}|_2^2 \right)
\tag{3.6}
\]
for every \( 0 < \epsilon < 1 \).

**Proof.** Suppose first \( f \in C^1[0,1] \). By Lemma 5.2 of [5], there exists a function \( F \) in \( C_0^1(\mathbb{R}) \) such that \( F = f \) in the interval \([0,1]\) and \( \|F\|_{W_2^j(\mathbb{R})} \leq C \|f\|_{j,2}, j = 0, 1 \). For the function \( F \) we have the inequalities
\[
|F^2|_{L_\infty(\mathbb{R})} \leq \int_{\mathbb{R}^1} |(F^2)_x| \, dx = 2 \int_{\mathbb{R}^1} |F F_x| \, dx \\
\leq \int_{\mathbb{R}^1} \left( \epsilon |F_x|^2 + \frac{1}{\epsilon} |F|^2 \right) \, dx = \epsilon |F_x|_{L_2(\mathbb{R})}^2 + \frac{1}{\epsilon} |F|_{L_2(\mathbb{R})}^2.
\]
Thus now for \( f \) we have
\[
|f^2|_\infty \leq |F^2|_{L_\infty(\mathbb{R})} \leq \epsilon |F_x|_{L_2(\mathbb{R})}^2 + \frac{1}{\epsilon} |F|_{L_2(\mathbb{R})}^2 \leq \epsilon \|f\|_{1,2}^2 + \frac{C}{\epsilon} |f|_2^2
\tag{3.7}
\]
for every \( \epsilon > 0 \). Suppose now that \( f \in W_2^1([0,1]) \). There exists a sequence \( \{f_i\} \) in \( C^1[0,1] \) such that \( \|f_i - f\|_{1,2} \to 0 \), \( \|f_i - f\|_{0,2} \to 0 \), \( |f_i - f|_{\infty} \to 0 \) as \( i \to \infty \). Hence by passing limits in inequality (3.7) for \( f_i \), we obtain inequality (3.7) for \( f \in W_2^1([0,1]) \) and thus inequality (3.6) for every \( 0 < \epsilon < 1 \). \( \square \)

**4. Uniform boundedness**

By taking integration of the first equation in (1.2) over the domain \([0,1]\), we have
The above inequalities show that
\[
\int_0^1 u(t) \, dx \leq M_0 \quad \text{for every } t \in (\tau_0, \infty),
\] (4.1)
where $M_0 = \frac{a_1}{b_1} + \delta$ and $\tau_0, \delta$ are positive constants. Now multiplying the second equation in (1.2) by $v$ and integrating over $[0, 1]$, we have
\[
\frac{1}{2} \frac{d}{dt} \int_0^1 v^2 \, dx = \int_0^1 v(d_2v + \alpha_{22}v^2)_{xx} \, dx + \int_0^1 v^2(a_2 - b_2u - c_2v) \, dx
\]
\[
= -\int_0^1 v_x(d_2v_x + 2\alpha_{22}vv_x) \, dx + \int_0^1 v^2(a_2 - b_2u - c_2v) \, dx
\]
\[
\leq a_2 \int_0^1 v^2 \, dx - c_2 \int_0^1 v^3 \, dx \leq a_2 \int_0^1 v^2 \, dx - c_2 \left( \int_0^1 v^2 \, dx \right)^{3/2},
\]
and thus there exists a positive constant $\tilde{M}_1$ depending only on $a_2, c_2$ such that
\[
\int_0^1 v^2(t) \, dx \leq \tilde{M}_1 \quad \text{for every } t \in (\tau_0, \infty) \quad (4.2)
\]
for a positive constant $\tau_{0.3}$.

4.1. Proof of Theorem 1.2 in case (i)

**Proof.** Step 1. We multiply the first equation in (1.2) by $u$ and integrate over $[0, 1]$ to have
\[
\frac{1}{2} \frac{d}{dt} \int_0^1 u^2 \, dx = \int_0^1 u(d_1u + \alpha_{11}u^2 + \alpha_{12}uv)_{xx} \, dx + \int_0^1 u^2(a_1 - b_1u - c_1v) \, dx
\]
\[
= -\int_0^1 (d_1 + 2\alpha_{11}u + \alpha_{12}v)u_x^2 \, dx
\]
\[-\int_{0}^{1} \alpha_{12} uu_{x} v_{x} \, dx + \int_{0}^{1} u^{2}(a_{1} - b_{1} u - c_{1} v) \, dx \]
\[\leq -d_{1} \int_{0}^{1} u_{x}^{2} \, dx - 2\alpha_{13} \int_{0}^{1} uu_{x}^{2} \, dx \]
\[\quad - \alpha_{12} \int_{0}^{1} uu_{x} v_{x} \, dx + a_{1} \int_{0}^{1} u^{2} \, dx. \]  
(4.1.1)

The mixed term of $u$ and $v$ in the last line of (4.1.1) is estimated as follows:
\[
\left| \int_{0}^{1} uu_{x} v_{x} \, dx \right| \leq \frac{\epsilon}{2} \int_{0}^{1} uu_{x}^{2} \, dx + \frac{1}{2\epsilon} \int_{0}^{1} v_{x}^{2} \, dx \\
\leq \frac{\epsilon}{2} \int_{0}^{1} uu_{x}^{2} \, dx + \frac{1}{2\epsilon} M_{0} |v_{x}|_{\infty}^{2} \\
\leq \frac{\epsilon}{2} \int_{0}^{1} uu_{x}^{2} \, dx + C \int_{0}^{1} v_{x}^{2} \, dx + \epsilon \int_{0}^{1} v_{x x}^{2} \, dx
\]
for any small $\epsilon > 0$ and for some positive constant $C$ by inequality (3.6). Thus we have
\[
\frac{1}{2} \frac{d}{dt} \int_{0}^{1} u^{2} \, dx \leq -d_{1} \int_{0}^{1} u_{x}^{2} \, dx - \left(2\alpha_{11} - \frac{\epsilon}{2} \alpha_{12}\right) \int_{0}^{1} uu_{x}^{2} \, dx \\
+ C \int_{0}^{1} v_{x}^{2} \, dx + \epsilon \alpha_{12} \int_{0}^{1} v_{x x}^{2} \, dx \\
\leq -d_{1} \int_{0}^{1} u_{x}^{2} \, dx + C \int_{0}^{1} v_{x}^{2} \, dx + \epsilon \alpha_{12} \int_{0}^{1} v_{x x}^{2} \, dx. \]  
(4.1.2)

In order to deal with the $v$-terms in the last line of (4.1.2), multiply the second equation in (1.2) by $-v_{x x}$ and integrate over $[0, 1]$ so that we have
\[
\frac{1}{2} \frac{d}{dt} \int_{0}^{1} v_{x}^{2} \, dx = -\int_{0}^{1} v_{x x} (d_{2} v + 2\alpha_{22} v^{2}) \, dx - \int_{0}^{1} v_{x x} v(a_{2} - b_{2} u - c_{2} v) \, dx
\]
\[
\begin{align*}
= & - \int_0^1 v_{xx} (d_2 v_{xx} + 2 \alpha_{22} v^2_x + 2 \alpha_{22} v v_{xx}) \, dx \\
& - a_2 \int_0^1 v v_{xx} \, dx + b_2 \int_0^1 u v v_{xx} \, dx + c_2 \int_0^1 v^2 v_{xx} \, dx \\
\leq & - d_2 \int_0^1 v^2_{xx} \, dx - 2 \alpha_{22} \int_0^1 v_x^2 v_{xx} \, dx \\
& + a_2 \int_0^1 v_x^2 \, dx + b_2 \int_0^1 u v v_{xx} \, dx + c_2 \int_0^1 v^2 v_{xx} \, dx. 
\end{align*}
\]

(4.1.3)

For the terms in the last line of (4.1.3) we make the following observations:

\[
\int_0^1 v_x^2 v_{xx} \, dx = 0. 
\]

(4.1.4)

From estimate (1.3) we have

\[
\begin{align*}
\int_0^1 u v v_{xx} \, dx &= - \int_0^1 u_x v_{xx} \, dx - \int_0^1 u v_x^2 \, dx \\
& \leq - \int_0^1 u_x v_{xx} \, dx \leq \epsilon \int_0^1 u_x^2 \, dx + C \int_0^1 v_x^2 \, dx, 
\end{align*}
\]

(4.1.5)

\[
\int_0^1 v^2 v_{xx} \, dx \leq C + \epsilon \int_0^1 v_x^2 \, dx 
\]

(4.1.6)

for any small \( \epsilon > 0 \) and some \( C > 0 \). Substituting estimates (4.1.4), (4.1.5), and (4.1.6) into (4.1.3), we obtain

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \int_0^1 v_x^2 \, dx \leq & -(d_2 - \epsilon \alpha_{22}) \int_0^1 v_{xx} \, dx + (a_2 + C b_2) \int_0^1 v_x^2 \, dx + \epsilon b_2 \int_0^1 u_x^2 \, dx + C.
\end{align*}
\]

(4.1.7)

Now adding (4.1.2) and (4.1.7), we have
\[ \frac{1}{2} \frac{d}{dt} \int_0^1 (u^2 + v_x^2) \, dx \leq -(d_1 - \varepsilon b_2) \int_0^1 u_x^2 \, dx - (d_2 - \varepsilon \alpha_{12} - \varepsilon c_2) \int_0^1 v_{xx}^2 \, dx + a_1 \int_0^1 u^2 \, dx + (a_2 + Cb_2 + C) \int_0^1 v_{xx}^2 \, dx + C. \] (4.1.8)

Let us denote \( \tilde{C}_2 = \min\{d_1 - \varepsilon b_2, d_2 - \varepsilon \alpha_{12} - \varepsilon c_2\} \) for the simplicity of notation. Here we can choose small enough \( \varepsilon \) so that \( \tilde{C}_2 > 0 \). Then finally for \( t > \tau_1' > 0 \) we have

\[ \frac{1}{2} \frac{d}{dt} \int_0^1 (u^2 + v_x^2) \, dx \leq -\tilde{C}_2 \left( \int_0^1 u_x^2 \, dx + \int_0^1 v_{xx}^2 \, dx \right) + C_1 \int_0^1 (u^2 + v_x^2) \, dx + C_0 \]

\[ \leq -C_2 \left( \int_0^1 u^2 \, dx \right)^3 + C_1 \int_0^1 (u^2 + v_x^2) \, dx + C_0 \] (4.1.9)

by using inequalities (3.2), (3.4), and the uniform boundedness of \( |u|_1 \) and \( |v|_2 \). Therefore we conclude that there exists a positive constant \( M_1 \) depending only on \( m_0, d_i, \alpha_{1i}, \alpha_{12}, \alpha_{22}, a_i, b_i, c_i, i = 1, 2 \), such that

\[ \int_0^1 u_x^2(t) \, dx \leq M_1 \] and \[ \int_0^1 v_{xx}^2(t) \, dx \leq M_1 \] for \( t \in (\tau_1, \infty) \), where \( \tau_1 \) is a positive constant.

**Step 2.** Multiplying the first equation in (1.2) by \(-u_{xx}\) and integrating over \([0, 1]\), we obtain

\[ \frac{1}{2} \frac{d}{dt} \int_0^1 u_x^2 \, dx = -\int_0^1 u_{xx}(d_1u + \alpha_{11}u^2 + \alpha_{12}uv)_{xx} \, dx \]

\[ -\int_0^1 u_{xx}u(a_1 - b_1u - c_1v) \, dx \]

\[ \leq -d_1 \int_0^1 (u_{xx})^2 \, dx - 2\alpha_{12} \int_0^1 u_xv_xu_{xx} \, dx - \alpha_{12} \int_0^1 uu_{xx}v_x \, dx \]

\[ + a_1 \int_0^1 u_x^2 \, dx + b_1 \int_0^1 u^2u_{xx} \, dx + c_1 \int_0^1 uvu_{xx} \, dx, \] (4.1.10)

by noticing that \( \int_0^1 u_x^2u_{xx} \, dx = 0 \). For the terms in the last line of (4.1.10), we make the following observations:
\[ \left| \int_0^1 u_x v_x u_{xx} \, dx \right| \leq |u_x|_\infty |v_x|_2 |u_{xx}|_2 \leq C |u_x|_\infty |u_{xx}|_2 \]
\[ \leq C |u_x|_\infty^2 + \frac{1}{2} \epsilon |u_{xx}|_2^2 \leq C |u_x|_2^2 + \epsilon |u_{xx}|_2^2. \]  
\hfill (4.1.11)\\
\[ \left| \int_0^1 u u_{xx} v_{xx} \, dx \right| \leq |v_{xx}|_\infty |u|_2 |u_{xx}|_2 \leq C |v_{xx}|_\infty |u_{xx}|_2 \]
\[ \leq C |v_{xx}|_\infty^2 + \epsilon |u_{xx}|_2^2 \leq C |v_{xx}|_2^2 + \epsilon |v_{xxx}|_2^2 + \epsilon |u_{xx}|_2. \]  
\hfill (4.1.12)\\
\[ \left| \int_0^1 u^2 u_{xx} \, dx \right| \leq C \int_0^1 u^4 \, dx + \epsilon \int_0^1 u_{xx}^2 \, dx \leq C \int_0^1 u^2 \, dx + \epsilon \int_0^1 u_{xx}^2 \, dx, \]  
\hfill (4.1.13)\\
\[ \left| \int_0^1 u v u_{xx} \, dx \right| \leq C + \epsilon \int_0^1 u_{xx}^2 \, dx \]  
\hfill (4.1.14)\\
for any \(0 < \epsilon < 1\) and some \(C > 0\), by using (1.3), (3.6), and the result in Step 1. Substituting (4.1.11), (4.1.12), (4.1.13), and (4.1.14) into (4.1.10), we have
\[ \frac{1}{2} \frac{d}{dt} \int_0^1 u_x^2 \, dx \leq -(d_1 - 4\epsilon) \int_0^1 (u_{xx})^2 \, dx + (a_1 + C) \int_0^1 u_x^2 \, dx \]
\[ + C \int_0^1 v_{xx}^2 \, dx + \epsilon \int_0^1 v_{xxx}^2 \, dx + C. \]  
\hfill (4.1.15)\\
Now we take the second derivative of the second equation of (1.2) with respect to \(x\), multiply by \(v_{xx}\), and integrate over \([0, 1]\) to obtain
\[ \frac{1}{2} \frac{d}{dt} \int_0^1 v_{xx}^2 \, dx = -d_2 \int_0^1 v_{xxx}^2 \, dx - 6a_2 \int_0^1 v_x v_{xx} v_{xxx} \, dx + \int_0^1 G_{xx} v_{xx} \, dx, \]  
\hfill (4.1.16)\\
where \(G = v(a_2 - b_2 u - c_2 v)\), by noticing that
\[ \int_0^1 (v_{xxx})^2 v_{xx} \, dx = \int_0^1 (2v v_{xx})_{xxx} v_{xx} \, dx \]
\[ = 2 \int_0^1 (v_{xx} v_{xx})_{xx} v_{xx} \, dx = -2 \int_0^1 (3v_x v_{xx} + v v_{xxx}) v_{xx} \, dx, \]
since $v_x = v_{xxx} = 0$ at $x = 0, 1$ because of system (1.2). In the following we estimate the terms on the right-hand side of (4.1.16),

$$\left| \int_0^1 v_x v_{xx} v_{xxx} \, dx \right| \leq |v_{xx}|_\infty |v_x|_2 |v_{xxx}|_2 \leq C |v_{xx}|_\infty |v_{xxx}|_2$$

$$\leq C |v_{xx}|_\infty^2 + \frac{\epsilon}{2} |v_{xxx}|_2^2 \leq C |v_{xx}|_2^2 + \epsilon |v_{xxx}|_2^2. \quad (4.1.17)$$

$$\int_0^1 G_{xx} v_{xx} \, dx = - \int_0^1 G_x v_{xxx} \, dx \leq C \int_0^1 G_x^2 \, dx + \epsilon \int_0^1 v_{xxx}^2 \, dx$$

$$\leq C \int_0^1 (u_x^2 + v_x^2) \, dx + \epsilon \int_0^1 u_{xx}^2 \, dx + \epsilon \int_0^1 v_{xx}^2 \, dx + \epsilon \int_0^1 v_{xxx}^2 \, dx,$$

since

$$\int_0^1 G_x^2 \, dx \leq C \int_0^1 (u_x^2 + v_x^2) \, dx + C \int_0^1 (u_x^2 + v_x^2)(u_x^2 + v_x^2) \, dx$$

$$\leq C \int_0^1 (u_x^2 + v_x^2) \, dx + C (|u_x|_\infty^2 + |v_x|_\infty^2)$$

$$\leq C \int_0^1 (u_x^2 + v_x^2) \, dx + \epsilon (|u_{xx}|_2^2 + |v_{xx}|_2^2).$$

Substituting (4.1.17) and (4.1.18) into (4.1.16), we obtain

$$\frac{1}{2} \frac{d}{dt} \int_0^1 v_{xx}^2 \, dx \leq -(d_2 - 2\epsilon) \int_0^1 v_{xxx}^2 \, dx + \epsilon \int_0^1 u_{xx}^2 \, dx$$

$$+ C \int_0^1 v_{xxx}^2 \, dx + C \int_0^1 u_{xx}^2 \, dx + C. \quad (4.1.19)$$

Adding (4.1.15) and (4.1.19), we have for $t > \tau'_2 > 0$

$$\frac{1}{2} \frac{d}{dt} \int_0^1 (u_x^2 + v_{xx}^2) \, dx$$

$$\leq -(d_1 - 5\epsilon) \int_0^1 (u_{xx})^2 \, dx - (d_2 - 3\epsilon) \int_0^1 v_{xxx}^2 \, dx$$
\[ + C \int_0^1 u_x^2 \, dx + C \int_0^1 v_{xx}^2 \, dx + C \]
\[ \leq -C_2 \left( \int_0^1 u_x^2 \, dx \right)^2 - C_2 \left( \int_0^1 v_{xx}^2 \, dx \right)^{3/2} + C \int_0^1 (u_x^2 + v_{xx}^2) \, dx + C, \]

where \( C_2 > 0 \), by using the inequalities (3.4), (3.5), and the uniform boundedness of \( |u|_2 \) and \( |v|_2 \). Therefore we conclude that there exists a positive constant \( M_2 \) depending only on \( m_0, d_i, \alpha_{11}, \alpha_{12}, \alpha_{22}, b_i, c_i, i = 1, 2 \) such that

\[ \int_0^1 u_x^2 x(t) \, dx \leq M_2 \]
\[ \int_0^1 v_{xx}^2 x(t) \, dx \leq M_2 \]
for \( t \in (\tau_2, \infty) \), where \( \tau_2 \) is a positive constant.

From the results of Step 1 and Step 2 we have a constant \( M \) independent of \( T \) and depending only on \( m_0, d_i, \alpha_{11}, \alpha_{12}, \alpha_{22}, b_i, c_i, i = 1, 2 \), such that

\[ \max \{ \| u(\cdot, t) \|_1, \| v(\cdot, t) \|_2 : t \in (t_0, T) \} \leq M \]
for \((u, v)\), the maximal solution to system (1.2). We also conclude that \( T = +\infty \) from Theorem 1.1.

\[ \blacksquare \]

4.2. Proof of Theorem 1.2 in case (ii)

**Proof.** In the case (ii) the first equation and the second of system (1.2) are rewritten as follows:

\[ \begin{align*}
  u_t &= (d_1 u + \alpha_{11} u^2 + \alpha_{12} u v)_{xx} + u(a_1 - b_1 u - c_1 v), \\
  v_t &= d_2 v_{xx} + v(a_2 - b_2 u - c_2 v).
\end{align*} \]

(4.2.1)

**Step 1.** In order to find the bound of the \( L_2 \)-norm of \( u \), we first estimate \( |v_x|_\infty \). Let \( A \) denote the operator defined by

\[ A \phi := -d_2 \phi_{xx} + \phi \]

(4.2.2)

with the domain of definition

\[ D(A) = \{ \phi \in W^2_2([0, 1]): \phi_x = 0 \text{ at } x = 0, 1 \}. \]

(4.2.3)

It is well know that \(-A\) generates an analytic semigroup \( \{ e^{-At} : t \geq 0 \} \) in \( L_2([0, 1]) \) and that, for any \( \alpha \geq 0 \), the fractional power \( A^\alpha \) of \( A \) is well defined (see [11, Section 2.6]). Moreover, for any \( t > 0 \), \( A^\alpha e^{-At} \) acts continuously from \( L_2 \) to \( L_2 \) with norm bounded by

\[ \| A^\alpha e^{-At} \|_{L_2 \to L_2} \leq C_\alpha t^{-\alpha} e^{-\delta t} \]

(4.2.4)

for \( t > 0 \), \( \alpha > 0 \), where \( C_\alpha, \delta \) are positive constants independent of \( t \). The \( \alpha \)-norm of \( \phi \in D(A^\alpha) \) is defined by \( \| \phi \|_\alpha := \| A^\alpha \phi \|_2 \) in this proof. We will make use of the interpolation inequality

\[ \| \phi \|_\beta \leq C_{\alpha, \beta} \| \phi \|_{0}^{1-\beta/\alpha} \| \phi \|_{\alpha}^{\beta/\alpha} \]

(4.2.5)

for any \( \phi \in D(A^\alpha) \). The following continuous embeddings (see [6, Section 1.6]) are also being used:
\[ D(A^\alpha) \subset W^{1,2}_2([0, 1]) \quad \text{for } 1/2 < \alpha < 1, \quad (4.2.6) \]
\[ D(A^\beta) \subset C^1([0, 1]) \quad \text{for } 3/4 < \beta < 1, \quad (4.2.7) \]
\[ W^{1,2}_2([0, 1]) \subset D(A^\gamma) \quad \text{for } \gamma < 1/2, \quad (4.2.8) \]
where \( W^{1,2}_{2, \beta}([0, 1]) = \{ \phi \in W^{1,2}_2([0, 1]) : \phi \geq 0 \text{ in } [0, 1] \text{ and } \phi_x = 0 \text{ at } x = 0, 1 \} \).

Now we consider the abstract integral version of the second equation of system (4.2.1) in \( L^2_{2}([0, 1]) \):
\[ v(t) = e^{-A t} v_0 + \int_0^t e^{-A(t-s)} G(s) \, ds \quad \text{for } t > 0, \quad (4.2.9) \]
where \( G(s) := v(s)(a_2 - b_2 u(s) - c_2 v(s)) + v(s) \) for \( s \geq 0 \). Taking \( \alpha \)-norm in (4.2.9) gives
\[ \| v \|_\alpha \leq \| e^{-A t} v_0 \|_\alpha + \int_0^t C_\alpha(t - s)^{-\alpha} e^{-\delta(t - s)} \| G(s) \|_2 \, ds. \quad (4.2.10) \]
From (4.2.8) and (4.2.4) we have
\[ \| e^{-A t} v_0 \|_\alpha \leq \| A^{\alpha - \gamma} e^{-A t} \|_{L^2_{2}, L^2_{2}} \| v_0 \|_\gamma \leq C \| A^{\alpha - \gamma} e^{-A t} \|_{L^2_{2}, L^2_{2}} \| v_0 \|_{1, 2} \leq t^{\gamma - \alpha} w_1(t) \quad (4.2.11) \]
for \( \gamma < 1/2 \) and \( 3/4 < \alpha < 1 \), where \( w_1 \) is a continuous nonnegative function and \( w_1(t) \to 0 \) as \( t \to \infty \). From (1.3) we have
\[ \| G(s) \|_2 \leq C(1 + \| u(s) \|_2). \quad (4.2.12) \]
Substituting (4.2.11) and (4.2.12) into (4.2.10) leads to
\[ \| v \|_\alpha \leq t^{\gamma - \alpha} w_1(t) + C + C \int_0^t (t - s)^{-\alpha} e^{-\delta(t - s)} \| u(s) \|_2 \, ds. \quad (4.2.13) \]
Here we use (1.3), (4.2.7), and (4.2.5) to estimate \( |v_x|\infty \):
\[ |v_x|\infty \leq C \| v \|_\beta \leq C \| v \|_0^{1 - \beta/\alpha} \| v \|^{\beta/\alpha}_\alpha \leq C + t^{(\gamma - \alpha)\beta/\alpha} w(t) + C \left( \int_0^t (t - s)^{-\alpha} e^{-\delta(t - s)} \| u(s) \|_2 \, ds \right)^{\beta/\alpha}. \quad (4.2.14) \]
for \( 3/4 < \beta < \alpha < 1 \), where \( w \) is a nonnegative continuous function and \( w(t) \to 0 \) as \( t \to \infty \).

Now we estimate the \( L^2_{2} \)-norm of \( u \) using the estimate (4.2.14) of \( v \). By multiplying the first equation in (4.2.1) by \( u \) and integrating over \([0, 1] \), for \( t > t_{1, 1} \) we have
\[ \frac{1}{2} \frac{d}{dt} \int_0^1 u^2 \, dx = \int_0^1 u(d_1 u + \alpha_1 u^2 + \alpha_1 2 u v) \, dx + \int_0^1 u^2(a_1 - b_1 u - c_1 v) \, dx \]
from (4.2.16) we have
\[ 0 \leq \frac{\alpha_2}{4d_1} + \frac{\alpha_1}{d_1} \]
For the simplicity of notation let us denote \( \tau \)
here we let \( \int_{-\infty}^{t} u^2 dx \) where
Supposing contrarily, let \( \tilde{s} \) be such that \( k(t) < 2k(0) \) for every \( t \in (0, \tau] \). For \( t > \tau \) from (4.2.16) we have
\[ k(t) \leq k(t) \left[ C - Ck(t) + Ct^{2(\gamma - \alpha)} \alpha \right] \]
where \( K(t) = \sup \{ k(s): 0 \leq s \leq t \} \). Noticing that \( \beta / \alpha < 1 \) denote by \( \kappa \) the unique positive solution to the equation \( C - C_3 + C_3 \beta / \alpha = 0 \). Then we claim that
\[ k(t) \leq \max \{ 2k(0), \kappa \} \quad \text{for all } t \geq 0. \] (4.2.18)
Supposing contrarily, let \( \tilde{t} > 0 \) be such that \( k(\tilde{t}) = R + \epsilon \) and \( k(s) < R + \epsilon \) for every \( 0 \leq s < \tilde{t} \), where \( \epsilon > 0 \) and \( R = \max \{ 2k(0), \kappa \} \). Note that \( \tilde{t} > \tau \) and \( k'(\tilde{t}) \geq 0 \), but inequality (4.2.17) gives that \( k'(\tilde{t}) < k(\tilde{t})[C_1 + C_2(R + \epsilon) + C_3(R + \epsilon) \beta / \alpha] < 0 \), since \( R + \epsilon > \kappa \). This contradiction shows that there is no such \( t \). And since \( \epsilon > 0 \) was arbitrary, (4.2.18) is proved.
Using the \( L_2 \)-boundedness of \( u \) now we estimate \( \int_{0}^{1} u^2 \) by \( -v_{xx} \) and integrate over \( [0, 1] \) so that we have
\[ \frac{1}{2} \frac{d}{dt} \int_{0}^{1} u_x^2 dx = -d_2 \int_{0}^{1} u_x^2 dx + a_2 \int_{0}^{1} u_x^2 dx + b_2 \int_{0}^{1} u v_{xx} dx + c_2 \int_{0}^{1} u_{xx}^2 dx. \] (4.2.19)
For the terms in (4.2.19) we make the following observations for \( t > \tau_{1.2} \):
\[ \int_{0}^{1} u v_{xx} dx \leq C \int_{0}^{1} |u v_{xx}| dx \leq C \int_{0}^{1} u^2 dx + \epsilon \int_{0}^{1} u_{xx}^2 dx \leq C + \epsilon \int_{0}^{1} u_{xx}^2 dx. \] (4.2.20)
\[ \int_0^1 v^2 v_{xx} \, dx \leq C + \epsilon \int_0^1 v_x^2 \, dx \quad (4.2.21) \]

for any small \( \epsilon > 0 \) and some \( C > 0 \) by using estimates (1.3) and (4.2.18). Substituting estimates (4.2.20) and (4.2.21) into (4.2.19), we obtain from the calculus inequality (3.4) that for \( t > t_1 \)

\[ \frac{1}{2} \frac{d}{dt} \int_0^1 v_x^2 \, dx \leq -(d_2 - \epsilon(b_2 + c_2)) \int_0^1 v_x^2 \, dx + a \int_0^1 v_x^2 \, dx + C \]

\[ \leq -\tilde{C}_2 \int_0^1 v_x^2 \, dx + a \int_0^1 v_x^2 \, dx + C \]

\[ \leq -C_2 \left( \int_0^1 v_x^2 \, dx \right) + a \int_0^1 v_x^2 \, dx + C, \quad (4.2.22) \]

where \( \epsilon \) is chosen small enough and \( \tau_1, \tilde{C}_2, C_2 \) are positive constants.

Therefore we conclude from (4.2.18) and (4.2.22) that there exists a positive constant \( M_1 \) depending only on \( m_0, d_i, a_{11}, a_{12}, a_{22}, b_i, c_i, i = 1, 2 \), such that \( \int_0^1 u_x^2(t) \, dx \leq M_1 \) and \( \int_0^1 v_x^2(t) \, dx \leq M_1 \) for \( t \in (t_1, \infty) \), where \( t_1 \) is a positive constant.

Step 2. The uniform boundedness of \( \int_0^1 u_x^2(t) \, dx \) and \( \int_0^1 v_x^2(t) \, dx \) are obtained similarly as in the proof of Theorem 1.2 in case (i).

From the results of Step 1 and Step 2, we have a constant \( M \) independent of \( T \) and depending only on \( m_0, d_i, a_{11}, a_{12}, a_{22}, b_i, c_i, i = 1, 2 \), such that \( \max\{\|u(\cdot, t)\|_{1,2}, \|v(\cdot, t)\|_{2,2}; t \in (t_0, T)\} \leq M \) for \((u, v)\), the maximal solution to system (1.2). We also conclude that \( T = +\infty \) from Theorem 1.1.

4.3. Proof of Theorem 1.2 in case (iii)

**Proof.** We rewrite the first equation and the second of system (1.2) by using the conditions in (i):

\[
\begin{align*}
    u_t &= (d_1 u + a_{12} u v)_{xx} + u(a_1 - b_1 u - c_1 v), \\
    v_t &= (d_2 v + a_{22} v^2)_{xx} + v(a_2 - b_2 u - c_2 v). 
\end{align*}
\quad (4.3.1)
\]

Step 1. By multiplying the first equation in (4.3.1) by \( u \) and integrating over \([0, 1]\), we have

\[
\frac{1}{2} \frac{d}{dt} \int_0^1 u^2 \, dx = \int_0^1 u(d_1 u + a_{12} u v)_{xx} \, dx + \int_0^1 u^2(a_1 - b_1 u - c_1 v) \, dx 
\]

\[
= -\int_0^1 u_x(d_1 u_x + a_{12} u_x v + a_{12} u v_x) \, dx + \int_0^1 u_x^2(a_1 - b_1 u - c_1 v) \, dx 
\]

\[
\leq -(d_1 u_x^2 + a_{12} u_x^2 v + a_{12} u v_x^2) \, dx + \int_0^1 u_x^2(a_1 - b_1 u - c_1 v) \, dx 
\]

\[
= -(d_1 u_x^2 + a_{12} u_x^2 v + a_{12} u v_x^2) \, dx + \int_0^1 u_x^2(a_1 - b_1 u - c_1 v) \, dx 
\]

\[
\leq -(d_1 u_x^2 + a_{12} u_x^2 v + a_{12} u v_x^2) \, dx + \int_0^1 u_x^2(a_1 - b_1 u - c_1 v) \, dx 
\]

where \( \epsilon \) is chosen small enough and \( \tau_1, \tilde{C}_2, C_2 \) are positive constants.
\[ \leq -d_1 \int_0^1 u_x^2 \, dx - \alpha_{12} \int_0^1 uu_x v_x \, dx + a_1 \int_0^1 u^2 \, dx. \]  
(4.3.2)

The mixed term of \( u \) and \( v \) in the last line of (4.3.2) is estimated as follows:

\[
\left| \int_0^1 uu_x v_x \, dx \right| = -\frac{1}{2} \int_0^1 u^2 v_{xx} \, dx \leq \frac{1}{2} \left( \frac{1}{2\epsilon_1} u^4 + \frac{\epsilon_1}{2} v_{xx}^2 \right) \, dx,
\]

for any small \( \epsilon_1 > 0 \). And by using (4.1), for some positive constant \( \tau_{2,1} \) we choose a positive constant \( M_0 > a_1/b_1 \) such that

\[ \int_0^1 u \, dx \leq M_0 \text{ for every } t > \tau_{2,1}, \]  
and

\[ d_1 d_2 > C M_0^2 \alpha_{12} > C \left( \frac{a_1}{b_1} \right)^2 \alpha_{12}. \]  
(4.3.3)

Thus for \( t > \tau_{2,1} \) we have

\[
\frac{1}{2} \frac{d}{dt} \int_0^1 u^2 \, dx \leq -d_1 \int_0^1 u_x^2 \, dx + \frac{\alpha_{12}}{4\epsilon_1} \int_0^1 u^4 \, dx + \frac{\alpha_{12}}{4} \epsilon_1 \int_0^1 v_{xx}^2 \, dx + a_1 \int_0^1 u^2 \, dx
\]

\[
\leq -\left( d_1 - \frac{2C_4 M_0^2 \alpha_{12}}{\epsilon_1} \right) \int_0^1 u_x^2 \, dx
\]

\[
+ \frac{\alpha_{12}}{4} \int_0^1 v_{xx}^2 \, dx + a_1 \int_0^1 u^2 \, dx + \frac{2C_4 M_0^4 \alpha_{12}}{\epsilon_1},
\]  
(4.3.4)

by noticing that from inequality (3.3)

\[ \int_0^1 u^4 \, dx \leq 8C^4 M_0^4 \int_0^1 u_x^2 \, dx + 8C^4 M_0^4. \]

In order to deal with the term \( \int_0^1 v_{xx}^2 \, dx \) in the last line of (4.3.4), multiply the second equation in (4.3.1) by \(-v_{xx} \) and integrate over \([0, 1]\) so that we have

\[
\frac{1}{2} \frac{d}{dt} \int_0^1 v_x^2 \, dx = -\int_0^1 v_{xx} (d_2 v + 2\alpha_{22} v_x^2) \, dx - \int_0^1 v_{xx} v(a_2 - b_2 u - c_2 v) \, dx
\]

\[
= -\int_0^1 v_{xx} (d_2 v_{xx} + 2\alpha_{22} v_x^2 + 2\alpha_{22} v v_{xx}) \, dx
\]

\[
- a_2 \int_0^1 v v_{xx} \, dx + b_2 \int_0^1 u v v_{xx} \, dx + c_2 \int_0^1 v^2 v_{xx} \, dx
\]
\[
\begin{align*}
&\leq -d_2 \int_0^1 v_x^2 \, dx - 2\alpha_{22} \int_0^1 v_x^2 v_{xx} \, dx \\
&\quad + a_2 \int_0^1 v_x^2 \, dx + b_2 \int_0^1 u v v_{xx} \, dx + c_2 \int_0^1 v^2 v_{xx} \, dx.
\end{align*}
\] (4.3.5)

For the terms in the last line of (4.3.5) we make the following observations for \( t > \tau_{2,2} \):

\[
\begin{align*}
&\int_0^1 v_x^2 v_{xx} \, dx = 0, \quad (4.3.6) \\
&\int_0^1 u v v_{xx} \, dx = - \int_0^1 u_x v v_x \, dx - \int_0^1 u v_x^2 \, dx \\
&\quad \leq - \int_0^1 u_x v v_x \, dx \leq \epsilon \int_0^1 u_x^2 \, dx + C \int_0^1 v_x^2 \, dx, \quad (4.3.7) \\
&\int_0^1 v^2 v_{xx} \, dx \leq C + \epsilon \int_0^1 v_{xx}^2 \, dx, \quad (4.3.8)
\end{align*}
\]

for any small \( \epsilon > 0 \) and some \( C > 0 \), by using (1.3). Substituting the estimates (4.3.6), (4.3.7), and (4.3.8) into (4.3.5), we obtain

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \int_0^1 v_x^2 \, dx &\leq -(d_2 - \epsilon c_2) \int_0^1 v_{xx}^2 \, dx \\
&\quad + (a_2 + C b_2) \int_0^1 v_x^2 \, dx + \epsilon_2 b_2 \int_0^1 u_x^2 \, dx + C c_2. \quad (4.3.9)
\end{align*}
\]

Now adding (4.3.4) and (4.3.9), we have

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \int_0^1 (u_x^2 + v_x^2) \, dx &\leq - \left( d_1 - \frac{2\tilde{C}^4 M_0^2 \alpha_{12}}{\epsilon_1} - \epsilon b_2 \right) \int_0^1 u_x^2 \, dx - \left( d_2 - \frac{\alpha_{12}}{4} \epsilon_1 - \epsilon c_2 \right) \int_0^1 v_{xx}^2 \, dx \\
&\quad + a_1 \int_0^1 u_x^2 \, dx + (a_2 + C b_2) \int_0^1 v_x^2 \, dx + \frac{2\tilde{C}^4 M_0^4 \alpha_{12}}{\epsilon_1} + C c_2. \quad (4.3.10)
\end{align*}
\]
In order to observe the coefficients of the terms \( \int_0^1 u_2^2 \, dx \) and \( \int_0^1 v_2^2 \, dx \) in (4.3.10), let us define two functions of \( \epsilon > 0 \),

\[
h_1(\epsilon) = \frac{2C_4 M_\alpha^2_0 \eta_1}{\epsilon}, \\
h_2(\epsilon) = \frac{\alpha_1}{4} \epsilon.
\]

Since \( h_2(4d_2/\eta_1) = d_2 \), \( h_1(4d_2/\eta_1) = C_4 M_\alpha^2_0 \eta_1^2/(2d_2) \), and \( d_1 > h_1(4d_2/\eta_1) \) by (4.3.3), we can find a positive number \( \epsilon_1 < 4d_2/\eta_1 \) such that \( d_1 > h_1(\epsilon_1) = C_4 M_\alpha^2_0 \eta_1^2/(2d_2) \). Let us denote \( C_2 \) for the simplicity of notation. Here we can choose small enough \( \epsilon \) so that \( C_2 > 0 \). Then finally for \( t > \tau_{2,3} \) we have

\[
\frac{1}{2} \frac{d}{dt} \int_0^1 (u_2^2 + v_2^2) \, dx \leq -C_2 \left( \int_0^1 u_2^2 \, dx + \int_0^1 v_2^2 \, dx \right) + C_1 \int_0^1 (u^2 + v_2^2) \, dx + C_0,
\]

and similarly as in the proof of Theorem 1.2, we conclude the uniform boundedness of \( \int_0^1 u_2^2(t) \, dx \) and \( \int_0^1 v_2^2(t) \, dx \) in time.

Step 2. The uniform boundedness of \( \int_0^1 u_2^2(t) \, dx \) and \( \int_0^1 v_2^2(x) \, dx \) are obtained similarly as in the proof of Theorem 1.2 in case (i).

From the results of Step 1 and Step 2, we have a constant \( M \) independent of \( T \) and depending only on \( m_0, d_1, \eta_1, \eta_2, a_i, b_i, c_i, i = 1, 2 \), such that \( \max \{ \| u(\cdot, t) \|_{L^2}, \| v(\cdot, t) \|_{L^2} \} \leq M \) for \( (u, v) \), the maximal solution to system (1.2). We also conclude that \( T = +\infty \) from Theorem 1.1. \( \square \)

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References