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Some mixed finite element methods for biharmonic equation ☆

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Abstract

Some perturbed mixed finite element methods related to the reduced integration technique are considered for solving the biharmonic equation problem. On a rectangular mesh, a similar scheme was proposed in Malkus and Hughes (Comput. Methods Appl. Mech. Eng. 15 (1978) 63–81) and its convergence was analyzed in Johnson and Pitkäranta (Math. Comp. 38 (1982) 375–400). Here we modify the scheme proposed in Malkus and Hughes (1978) and prove the optimal order error estimate without the extra smoothness assumption on the solution made in Johnson and Pitkäranta (1982). On a triangular mesh, an analogous scheme is studied, and an order error estimate is proved. Some numerical results are given to show the convergence behavior of the numerical solutions.

MSC: 65N30

Keywords: Biharmonic equation; Mixed finite element method; Reduced integration; Error estimates

1. Introduction

We consider the numerical solution of the boundary value problem for the biharmonic equation

$$\begin{aligned} &\Delta^2 w = f \quad \text{in } \Omega, \\ &w = \frac{\partial w}{\partial n} = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

Here Ω is a domain in the d-dimensional space $\mathbb{R}^d$. The biharmonic operator Δ^2 is defined through

$$\Delta^2 w = \Delta(\Delta w)$$

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and

$$\Delta w = \sum_{i=1}^{d} \frac{\partial^2 w}{\partial x_i^2}.$$

The problem with $d = 2$ can be used to determine the deflection of a thin clamped plate under the action of a distributed load f.

Conforming finite element methods for solving (1.1) require C^1-elements; that is, the approximation of w involves the construction of basis functions which, together with their first partial derivatives, are continuous over $\bar{\Omega}$. Conforming elements are rarely used in practical computation, either because the dimension of the local interpolation space is large or because the structure of the local interpolation space is complicated. One way to avoid the difficulty of constructing C^1-elements is by using nonconforming finite elements. It is a very important and popular method to approximate high-order elliptic problems such as the biharmonic equation problem (1.1). For detail, see [7,8] and the references therein.

Another way to avoid C^1-elements is by using mixed finite element methods based on the following equivalent form for problem (1.1) [6,14]:

$$\begin{aligned} u + \Delta w &= 0 \quad \text{in } \Omega, \\ -\Delta u &= f \quad \text{in } \Omega, \\ w = \frac{\partial w}{\partial n} &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.2}$$

The literature on the mixed methods is vast, and we refer to monograph [6] and the survey paper [13] for detailed presentation and analysis of the methods in general, and their applications in solving the biharmonic problem in particular.

In this paper we discuss some perturbed mixed methods based on a penalty approximation combined with the reduced integration technique. It is easy to see that a variational formulation for problem (1.1) is

$$\inf_{v \in H_0^2(\Omega)} \left\{ \frac{1}{2} \int_\Omega |\Delta v|^2 \, \mathrm{d}x - \int_\Omega f v \, \mathrm{d}x \right\}. \tag{1.3}$$

We note that for $v \in H_0^2(\Omega)$,

$$\int_\Omega |\Delta v|^2 \, \mathrm{d}x = \int_\Omega \sum_{i=1}^{d} \sum_{j=1}^{d} \left(\frac{\partial^2 v}{\partial x_j \partial x_i} \right)^2 \mathrm{d}x.$$

Thus, introducing an auxiliary variable $\boldsymbol{\psi} = (\psi_1, \ldots, \psi_d)^{\mathrm{T}} = \nabla v$, we can rewrite (1.3) as

$$\inf_{\substack{(v,\boldsymbol{\psi}) \in V \times V^d \\ \boldsymbol{\psi} = \nabla v}} \left\{ \tfrac{1}{2} \, \|\nabla \boldsymbol{\psi}\|_0^2 - (f, v) \right\}, \tag{1.4}$$

where

$$V = H_0^1(\Omega),$$

$$\|\nabla \boldsymbol{\psi}\|_0^2 = \sum_{i=1}^{d} \|\nabla \psi_i\|_0^2,$$

$$(f,v)=\int_\Omega fv\,\mathrm{d}x.$$

Enforcing here the side condition $\boldsymbol{\psi}=\nabla v$ approximately via a penalty term, we are led to the following unconstrained minimization problem:

$$\inf_{(v,\boldsymbol{\psi})\in V\times V^d}\left\{\tfrac{1}{2}\,\|\nabla\boldsymbol{\psi}\|_0^2+\frac{1}{2\varepsilon}\,\|\boldsymbol{\psi}-\nabla v\|_0^2-(f,v)\right\}. \tag{1.5}$$

Actually, problem (1.5) with $d=2$ corresponds to a simplified version of the Reissner–Mindlin plate problem with thickness ε, taking shear deformations into account. Problem (1.4) can be viewed as the limiting problem of (1.5) as ε tends to zero. Below we shall only consider the case when ε is very small and compare the discrete solution of problem (1.5) with the exact solution of (1.4). It is well known that the standard discrete analogue of (1.5) fails to produce a good approximation, owing to the locking phenomenon. In [12], the following discrete approximation is proposed for problem (1.5) (for the case $d=2$):

Find $(w_h,\boldsymbol{\phi}_h)\in S_h\times[S_h]^2$, such that

$$J(w_h,\boldsymbol{\phi}_h)=\inf_{(v_h,\boldsymbol{\psi}_h)\in S_h\times[S_h]^2} J(v_h,\boldsymbol{\psi}_h), \tag{1.6}$$

where

$$J(v,\boldsymbol{\psi})=\frac{1}{2}\,\|\nabla\boldsymbol{\psi}\|_0^2+\frac{1}{2\varepsilon}\,\|\,\mathscr{P}_0(\boldsymbol{\psi}-\nabla v)\,\|_0^2-(f,v)$$

and the finite element space S_h consists of bilinear elements on a rectangular mesh, and $\mathscr{P}_0$ is the orthogonal projection from $[L^2(\Omega)]^2$ to the finite element space of piecewise constants. Scheme (1.6) can be viewed as one obtained from discretizing problem (1.5) when the term involving the small parameter ε is computed piecewisely by a one-point Gaussian quadrature (a reduced integration). An error analysis of method (1.6) is given in [10], again for the case $d=2$, where the following error estimate is proved for $0<\varepsilon\leqslant ch^2$

$$\|\,w-w_h\,\|_1+\|\nabla w-\boldsymbol{\phi}_h\|_1\leqslant Ch\,\|\,w\,\|_5\,. \tag{1.7}$$

The proof of the error estimate (1.7) is based on the use of a mesh-dependent norm and a super-approximating property. Note that in estimate (1.7), one has to assume the regularity $w\in H^5(\Omega)$, which is, in general, unrealistic. The most one can say is that $w\in H^s(\Omega)$ with $s\sim 4.73$ if $f\in H^1(\Omega)$, cf. [10].

Our purpose here is to further the investigation of numerical methods of the form (1.6) for solving problem (1.4). For a rectangular mesh, we use a new projection $\mathscr{P}_1$ to replace $\mathscr{P}_0$ in (1.6) and prove the optimal order error estimate for the modified method without the extra smoothness assumption on the solution. Our presentation here is for an arbitrary dimension d. Then for the planar problem, we propose and analyze an analogous scheme on a triangular mesh, and prove an order error estimate, again without the extra smoothness assumption on the solution. Finally, some numerical results are given to show the convergence behavior of the numerical solutions.

2. A modified scheme on rectangular mesh

In this section we assume the domain Ω is the union of a finite number of rectangular regions of the form $\prod_{i=1}^{d}[a_i,b_i]$ so that it can be partitioned into rectangular elements whose sides parallel the coordinate axes. Let $\Im_h$ be such a regular rectangular partition of the domain Ω into rectangular elements. Denote the meshsize parameter by h.

Define the multilinear finite element space

$$S_h = \{v \in H_0^1(\Omega)\colon v|_K \in Q_1(K)\ \forall K \in \Im_h\} \tag{2.1}$$

and an auxiliary space

$$\boldsymbol{Q}_h = \left\{\boldsymbol{\mu} \in [L^2(\Omega)]^d\colon \boldsymbol{\mu}|_K \in \prod_{i=1}^{d} Q_1^{(i)}(K)\ \forall K \in \Im_h\right\}. \tag{2.2}$$

Here $Q_1(K)$ denotes the space of multilinear functions on K, and for $1 \leqslant i \leqslant d$, $Q_1^{(i)}(K)$ is a subspace of $Q_1(K)$ consisting of functions constant in x_i. Then we propose a finite element method for solving problem (1.1):

Find $(w_h,\boldsymbol{\phi}_h) \in S_h \times [S_h]^d$ such that

$$(\nabla\boldsymbol{\phi}_h,\nabla\boldsymbol{\psi}_h) + \frac{1}{\varepsilon}(\mathscr{P}_1\boldsymbol{\phi}_h - \nabla w_h, \mathscr{P}_1\boldsymbol{\psi}_h - \nabla v_h) = (f,v_h) \quad \forall (v_h,\boldsymbol{\psi}_h) \in S_h \times [S_h]^d, \tag{2.3}$$

where $\mathscr{P}_1 : [L^2(\Omega)]^d \to \boldsymbol{Q}_h$ is the orthogonal projection onto $\boldsymbol{Q}_h$ in $[L^2(\Omega)]^d$. It is easy to see that problem (2.3) is equivalent to the minimization problem of finding $(w_h,\boldsymbol{\phi}_h) \in S_h \times [S_h]^d$ such that

$$J(w_h,\boldsymbol{\phi}_h) = \inf_{(v_h,\boldsymbol{\psi}_h)\in S_h\times[S_h]^d} J(v_h,\boldsymbol{\psi}_h),$$

where

$$J(v,\boldsymbol{\psi}) = \tfrac{1}{2}\|\nabla\boldsymbol{\psi}\|_0^2 + \frac{1}{2\varepsilon}\|\mathscr{P}_1(\boldsymbol{\psi} - \nabla v)\|_0^2 - (f,v).$$

Lemma 2.1. *There exists a unique solution* $(w_h,\boldsymbol{\phi}_h)$ *of problem* (2.3).

Proof. Problem (2.3) is a linear system with a square coefficient matrix. We only need to prove that $f=0$ implies $w_h=0$ and $\boldsymbol{\phi}_h=0$. Let $f=0$, then

$$(\nabla\boldsymbol{\phi}_h,\nabla\boldsymbol{\phi}_h)=0 \quad \text{and} \quad (\mathscr{P}_1\boldsymbol{\phi}_h - \nabla w_h, \mathscr{P}_1\boldsymbol{\phi}_h - \nabla w_h)=0.$$

It is then easy to see $\boldsymbol{\phi}_h=0$ and $w_h=0$ using the facts that $(w_h,\boldsymbol{\phi}_h) \in S_h \times [S_h]^d$ and $S_h \subset H_0^1(\Omega)$. □

Problem (2.3) can be written in a mixed formulation: Find $(w_h,\boldsymbol{\phi}_h,\boldsymbol{\lambda}_h) \in S_h \times [S_h]^d \times \boldsymbol{Q}_h$ such that

$$\begin{aligned}
&(\nabla\boldsymbol{\phi}_h,\nabla\boldsymbol{\psi}_h) + (\boldsymbol{\lambda}_h,\boldsymbol{\psi}_h) = 0 && \forall \boldsymbol{\psi}_h \in [S_h]^d,\\
&-(\boldsymbol{\lambda}_h,\nabla v_h) = (f,v_h) && \forall v_h \in S_h,\\
&\varepsilon(\boldsymbol{\lambda}_h,\boldsymbol{\mu}_h) - (\boldsymbol{\phi}_h - \nabla w_h,\boldsymbol{\mu}_h) = 0 && \forall \boldsymbol{\mu}_h \in \boldsymbol{Q}_h.
\end{aligned} \tag{2.4}$$

When $d=2$, system (2.4) is a discrete form of a simplified version of the Reissner–Mindlin plate model with thickness ε. Some works on the finite element methods for the Reissner–Mindlin plate can be found in [2,5]. We use (2.4) to approximate the solution of the biharmonic equation problem (1.1). The method studied here can be viewed as a version of the well-known MITC4 element mathematically analyzed in [3,4,11] for the Reissner–Mindlin plate model problem. The difference here is we use a perturbed numerical scheme to solve an original problem without a small parameter; also in this aspect, our theoretical result is different from those proved in the above-mentioned papers.

Let w be the solution of Eq. (1.1), and denote $\boldsymbol{\phi} = \Delta w$, $\boldsymbol{\lambda} = \Delta\boldsymbol{\phi}$. Then $(w, \boldsymbol{\phi}, \boldsymbol{\lambda}) \in H_0^1(\Omega) \times [H_0^1(\Omega)]^d \times [L^2(\Omega)]^d$ satisfies

$$\begin{aligned} &(\nabla\boldsymbol{\phi}, \nabla\boldsymbol{\psi}) + (\boldsymbol{\lambda}, \boldsymbol{\psi}) = 0 \quad && \forall \boldsymbol{\psi} \in [H_0^1(\Omega)]^d, \\ &-(\boldsymbol{\lambda}, \nabla v) = (f, v) && \forall v \in H_0^1(\Omega), \\ &(\boldsymbol{\phi} - \nabla w, \boldsymbol{\mu}) = 0 && \forall \boldsymbol{\mu} \in [L^2(\Omega)]^d. \end{aligned} \tag{2.5}$$

We give an error analysis for method (2.4). First we present two lemmas.

Lemma 2.2. *Assume $K = \prod_{i=1}^d [a_i, b_i]$ and $v \in H^3(K)$. Denote $x_i^K = (a_i + b_i)/2$, $1 \leqslant i \leqslant d$ and $\delta = \max\{b_i - a_i,\ 1 \leqslant i \leqslant d\}$. Let $\tilde{v}$ be the multilinear function interpolating v at the vertices of the element K. For $1 \leqslant i \leqslant d$, denote*

$$L_{i,K}^{(i)}(v) = \int_K \frac{\partial}{\partial x_i}(v - \tilde{v})\,\mathrm{d}x, \tag{2.6}$$

$$L_{i,K}^{(j)}(v) = \int_K (x_j - x_j^K)\frac{\partial}{\partial x_i}(v - \tilde{v})\,\mathrm{d}x, \quad j \neq i. \tag{2.7}$$

Then we have the estimates

$$|L_{i,K}^{(i)}(v)| \leqslant C\delta^{d/2+2}|v|_{3,K}, \qquad |L_{i,K}^{(j)}(v)| \leqslant C\delta^{d/2+3}|v|_{3,K}, \quad j \neq i \tag{2.8}$$

for some constant C independent of the element K.

If we denote

$$\sigma = \min\{(b_i - a_i)/\delta,\ 1 \leqslant i \leqslant d\},$$

then there is a constant depending only on σ, such that

$$\left| \int_K \nabla(v - \tilde{v}) \cdot \boldsymbol{\mu}\,\mathrm{d}x \right| \leqslant c\delta^2 |v|_{3,K} \|\boldsymbol{\mu}\|_{0,K} \quad \forall \boldsymbol{\mu} \in \prod_{i=1}^d Q_1^{(i)}(K). \tag{2.9}$$

Proof. If $v \in Q_1(K)$, then $\tilde{v} = v$ and so $L_{i,K}^{(j)}(v) = 0$, $1 \leqslant i,\ j \leqslant d$.

For any $1 \leqslant l \leqslant d$, let $v = (x_l - a_l)(x_l - b_l)$, then $\tilde{v} = 0$ and it can be verified that $L_{i,K}^{(j)}(v) = 0$, $1 \leqslant i,\ j \leqslant d$.

Therefore,

$$L_{i,K}^{(j)}(v) = 0 \quad \forall v \in P_2(K),\ 1 \leqslant i,\ j \leqslant d,$$

where $P_2(K)$ is the space of polynomials of degree less than or equal to 2 on K. By the Bramble–Hilbert Lemma and a standard scaling argument (cf. [7]), we get the super-approximating property (2.8).

Now let $\boldsymbol{\mu} = (\mu_1, \dots, \mu_d)^{\mathrm{T}} \in \prod_{i=1}^{d} Q_1^{(i)}(K)$, and write

$$\mu_i(x) = c_i^{(i)} + \sum_{j \neq i} c_j^{(i)}(x_j - x_j^K).$$

Then using the estimates (2.8), we get

$$\begin{aligned}
&\left| \int_K \nabla(v - \tilde{v}) \cdot \boldsymbol{\mu} \, \mathrm{d}x \right| \\
&\quad = \left| \int_K \sum_{i=1}^{d} \frac{\partial}{\partial x_i}(v - \tilde{v}) \left[c_i^{(i)} + \sum_{j \neq i} c_j^{(i)}(x_j - x_j^K) \right] \mathrm{d}x \right| \\
&\quad \leqslant \sum_{i=1}^{d} \left[\left| \int_K \frac{\partial}{\partial x_i}(v - \tilde{v}) \, \mathrm{d}x \right| |c_i^{(i)}| + \sum_{j \neq i} \left| \int_K (x_j - x_j^K) \frac{\partial}{\partial x_i}(v - \tilde{v}) \, \mathrm{d}x \right| |c_j^{(i)}| \right] \\
&\quad \leqslant c\delta^{d/2+2} \sum_{i=1}^{d} \left(|c_i^{(i)}| + \sum_{j \neq i} |c_j^{(i)}| \right) |v|_{3,K}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\|\boldsymbol{\mu}\|_{0,K}^2 &= \sum_{i=1}^{d} \int_K |\mu_i|^2 \, \mathrm{d}x \\
&= \sum_{i=1}^{d} \int_K \left[c_i^{(i)} + \sum_{j \neq i} c_j^{(i)}(x_j - x_j^K) \right]^2 \mathrm{d}x \\
&= \sum_{i=1}^{d} \int_K \left[|c_i^{(i)}|^2 + \sum_{j \neq i} |c_j^{(i)}|^2 |(x_j - x_j^K)|^2 \right] \mathrm{d}x \\
&\geqslant c\delta^d \sum_{i=1}^{d} \left(|c_i^{(i)}|^2 + \sum_{j \neq i} |c_j^{(i)}|^2 \delta^2 \right) \\
&\geqslant c\delta^d \left[\sum_{i=1}^{d} \left(|c_i^{(i)}| + \sum_{j \neq i} |c_j^{(i)}| \delta \right) \right]^2.
\end{aligned}$$

Hence, estimate (2.9) holds. □

Lemma 2.3. *Assume* $\boldsymbol{\lambda} \in [H^1(\Omega)]^d$, *then*

$$\|\boldsymbol{\lambda} - \mathscr{P}_1 \boldsymbol{\lambda}\|_0 \leqslant Ch|\boldsymbol{\lambda}|_1, \tag{2.10}$$

and

$$\|\mathscr{P}_1 \boldsymbol{\lambda}\|_0 \leqslant \|\boldsymbol{\lambda}\|_0, \tag{2.11}$$

where C *is a constant independent of* h.

Proof. Inequality (2.11) follows immediately from the definition of the orthogonal projection $\mathscr{P}_1$. Estimate (2.10) follows from the standard interpolation theory, cf. [7]. □

Now we state and prove the main result of the section.

Theorem 2.4. *Let* $(w, \boldsymbol{\phi}, \boldsymbol{\lambda})$ *be the solution of problem* (2.5), $(w_h, \boldsymbol{\phi}_h, \boldsymbol{\lambda}_h)$ *the solution of problem* (2.4). *Assume* $w \in H^4(\Omega)$ *and* $\varepsilon = \alpha h^2$ *for some constant* $\alpha > 0$, *then*

$$\|\nabla(\boldsymbol{\phi} - \boldsymbol{\phi}_h)\|_0 + \|\nabla(w - w_h)\|_0 \leqslant Ch\|w\|_4, \tag{2.12}$$

where C *is a constant independent of* h.

Proof. From (2.5) and (2.4), we get the following error relations:

$$(\nabla(\boldsymbol{\phi} - \boldsymbol{\phi}_h), \nabla\boldsymbol{\psi}_h) + (\boldsymbol{\lambda} - \boldsymbol{\lambda}_h, \boldsymbol{\psi}_h) = 0 \quad \forall \boldsymbol{\psi}_h \in [S_h]^d, \tag{2.13}$$

$$(\boldsymbol{\lambda} - \boldsymbol{\lambda}_h, \nabla v_h) = 0 \quad \forall v_h \in S_h, \tag{2.14}$$

$$(\boldsymbol{\phi} - \boldsymbol{\phi}_h, \boldsymbol{\mu}_h) = (\nabla(w - w_h), \boldsymbol{\mu}_h) - \varepsilon(\boldsymbol{\lambda}_h, \boldsymbol{\mu}_h) \quad \forall \boldsymbol{\mu}_h \in \boldsymbol{Q}_h. \tag{2.15}$$

For any $\boldsymbol{\psi}_h \in [S_h]^d$, we have

$$\begin{aligned}\|\nabla(\boldsymbol{\phi} - \boldsymbol{\psi}_h)\|_0^2 &= \|\nabla(\boldsymbol{\phi} - \boldsymbol{\phi}_h)\|_0^2 + \|\nabla(\boldsymbol{\phi}_h - \boldsymbol{\psi}_h)\|_0^2 + 2(\nabla(\boldsymbol{\phi} - \boldsymbol{\phi}_h), \nabla(\boldsymbol{\phi}_h - \boldsymbol{\psi}_h)) \\ &\geqslant \|\nabla(\boldsymbol{\phi} - \boldsymbol{\phi}_h)\|_0^2 + 2(\nabla(\boldsymbol{\phi} - \boldsymbol{\phi}_h), \nabla(\boldsymbol{\phi}_h - \boldsymbol{\psi}_h)).\end{aligned}$$

By (2.13), we obtain

$$\|\nabla(\boldsymbol{\phi} - \boldsymbol{\phi}_h)\|_0^2 \leqslant \|\nabla(\boldsymbol{\phi} - \boldsymbol{\psi}_h)\|_0^2 + 2(\boldsymbol{\lambda} - \boldsymbol{\lambda}_h, \boldsymbol{\phi}_h - \boldsymbol{\psi}_h). \tag{2.16}$$

For the second term of the right-hand side, we write

$$(\boldsymbol{\lambda} - \boldsymbol{\lambda}_h, \boldsymbol{\phi}_h - \boldsymbol{\psi}_h) = (\boldsymbol{\lambda} - \mathscr{P}_1\boldsymbol{\lambda}, \boldsymbol{\phi}_h - \boldsymbol{\psi}_h) + (\mathscr{P}_1\boldsymbol{\lambda} - \boldsymbol{\lambda}_h, \boldsymbol{\phi}_h - \boldsymbol{\phi}) + (\mathscr{P}_1\boldsymbol{\lambda} - \boldsymbol{\lambda}_h, \boldsymbol{\phi} - \boldsymbol{\psi}_h). \tag{2.17}$$

Using (2.15) with $\boldsymbol{\mu}_h = \mathscr{P}_1\boldsymbol{\lambda} - \boldsymbol{\lambda}_h \in \boldsymbol{Q}_h$, we have

$$(\mathscr{P}_1\boldsymbol{\lambda} - \boldsymbol{\lambda}_h, \boldsymbol{\phi}_h - \boldsymbol{\phi}) = \varepsilon(\boldsymbol{\lambda}_h, \mathscr{P}_1\boldsymbol{\lambda} - \boldsymbol{\lambda}_h) - (\nabla(w - w_h), \mathscr{P}_1\boldsymbol{\lambda} - \boldsymbol{\lambda}_h),$$

which can be rewritten as

$$(\mathscr{P}_1\boldsymbol{\lambda} - \boldsymbol{\lambda}_h, \boldsymbol{\phi}_h - \boldsymbol{\phi}) = -\varepsilon\|\mathscr{P}_1\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_0^2 + \varepsilon(\mathscr{P}_1\boldsymbol{\lambda}, \mathscr{P}_1\boldsymbol{\lambda} - \boldsymbol{\lambda}_h) - (\nabla(w - w_h), \mathscr{P}_1\boldsymbol{\lambda} - \boldsymbol{\lambda}_h). \tag{2.18}$$

Using (2.14), we have, for any $v_h \in S_h$,

$$(\nabla v_h, \mathscr{P}_1\boldsymbol{\lambda} - \boldsymbol{\lambda}_h) = (\nabla v_h, \boldsymbol{\lambda} - \boldsymbol{\lambda}_h) = 0.$$

Hence for the last term of (2.18),

$$(\nabla(w - w_h), \mathscr{P}_1\boldsymbol{\lambda} - \boldsymbol{\lambda}_h) = (\nabla(w - \tilde{w}), \mathscr{P}_1\boldsymbol{\lambda} - \boldsymbol{\lambda}_h), \tag{2.19}$$

where $\tilde{w} \in S_h$ is the piecewise bilinear interpolant of w. By the above relations (2.16)–(2.19) we deduce that

$$\begin{aligned}&\|\nabla(\boldsymbol{\phi} - \boldsymbol{\phi}_h)\|_0^2 + 2\varepsilon\|\mathscr{P}_1\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_0^2 \\ &\quad \leqslant \|\nabla(\boldsymbol{\phi} - \boldsymbol{\psi}_h)\|_0^2 + 2(\boldsymbol{\lambda} - \mathscr{P}_1\boldsymbol{\lambda}, \boldsymbol{\phi}_h - \boldsymbol{\psi}_h) + 2\varepsilon(\mathscr{P}_1\boldsymbol{\lambda}, \mathscr{P}_1\boldsymbol{\lambda} - \boldsymbol{\lambda}_h) \\ &\qquad - 2(\nabla(w - \tilde{w}), \mathscr{P}_1\boldsymbol{\lambda} - \boldsymbol{\lambda}_h) + 2(\mathscr{P}_1\boldsymbol{\lambda} - \boldsymbol{\lambda}_h, \boldsymbol{\phi} - \boldsymbol{\psi}_h).\end{aligned}$$

Since the partition is regular, we use (2.9) to get

$$|(\nabla(w-\tilde{w}),\mathscr{P}_1\boldsymbol{\lambda}-\boldsymbol{\lambda}_h)| \leqslant Ch^2|w|_3\cdot \|\mathscr{P}_1\boldsymbol{\lambda}-\boldsymbol{\lambda}_h\|_0 .$$

Using the assumption $\varepsilon=\alpha h^2$, we then obtain

$$\|\nabla(\boldsymbol{\phi}-\boldsymbol{\phi}_h)\|_0^2+h^2\|\mathscr{P}_1\boldsymbol{\lambda}-\boldsymbol{\lambda}_h\|_0^2 \leqslant C\{\|\nabla(\boldsymbol{\phi}-\boldsymbol{\psi}_h)\|_0^2+h^{-2}\|\boldsymbol{\phi}-\boldsymbol{\psi}_h\|_0^2+\|\boldsymbol{\lambda}-\mathscr{P}_1\boldsymbol{\lambda}\|_0^2+h^2(\|\mathscr{P}_1\boldsymbol{\lambda}\|_0^2+|w|_3^2)\}. \tag{2.20}$$

The above relation holds for any $\boldsymbol{\psi}_h\in[S_h]^d$. In particular, let us choose $\boldsymbol{\psi}_h$ to be the interpolant of $\boldsymbol{\phi}$ in $[S_h]^d$. Then from (2.20) and Lemma 2.3, we have

$$\|\nabla(\boldsymbol{\phi}-\boldsymbol{\phi}_h)\|_0+h\|\mathscr{P}_1\boldsymbol{\lambda}-\boldsymbol{\lambda}_h\|_0 \leqslant Ch\|w\|_4 . \tag{2.21}$$

Finally we estimate the error for the displacement. Recall that $\tilde{w}\in S_h$ is the bilinear element interpolant of w.

$$\begin{aligned}\|\nabla(\tilde{w}-w_h)\|_0 &= \sup_{\boldsymbol{\mu}\in[L^2(\Omega)]^d}\frac{(\nabla(\tilde{w}-w_h),\boldsymbol{\mu})}{\|\boldsymbol{\mu}\|_0}\\ &= \sup_{\boldsymbol{\mu}\in[L^2(\Omega)]^d}\frac{(\nabla(\tilde{w}-w_h),\mathscr{P}_1\boldsymbol{\mu})}{\|\boldsymbol{\mu}\|_0}\\ &\leqslant \sup_{\boldsymbol{\mu}\in[L^2(\Omega)]^d}\frac{(\nabla(w-\tilde{w}),\mathscr{P}_1\boldsymbol{\mu})}{\|\boldsymbol{\mu}\|_0}+\sup_{\boldsymbol{\mu}\in[L^2(\Omega)]^d}\frac{(\nabla(w-w_h),\mathscr{P}_1\boldsymbol{\mu})}{\|\boldsymbol{\mu}\|_0}\\ &\leqslant \|\nabla(w-\tilde{w})\|_0+\sup_{\boldsymbol{\mu}\in[L^2(\Omega)]^d}\frac{(\boldsymbol{\phi}-\boldsymbol{\phi}_h,\mathscr{P}_1\boldsymbol{\mu})+\varepsilon(\boldsymbol{\lambda}_h,\mathscr{P}_1\boldsymbol{\mu})}{\|\boldsymbol{\mu}\|_0}\\ &\leqslant Ch|w|_2+\|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_0+\varepsilon\|\boldsymbol{\lambda}_h\|_0\\ &\leqslant Ch|w|_2+\|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_0+\varepsilon\|\mathscr{P}_1\boldsymbol{\lambda}-\boldsymbol{\lambda}_h\|_0+\varepsilon\|\boldsymbol{\lambda}\|_0,\end{aligned}$$

where the error relation (2.15) was used. Thus, using (2.21),

$$\|\nabla(\tilde{w}-w_h)\|_0 \leqslant Ch\|w\|_4 . \tag{2.22}$$

Then

$$\|\nabla(w-w_h)\|_0 \leqslant \|\nabla(w-\tilde{w})\|_0+\|\nabla(\tilde{w}-w_h)\|_0 \leqslant Ch\|w\|_4 \tag{2.23}$$

and the proof is completed. □

From Theorem 2.4, we see that the error estimate

$$\|w-w_h\|_1+\|\nabla w-\boldsymbol{\phi}_h\|_1 \leqslant C_\alpha h\|w\|_4 \tag{2.24}$$

holds for $\varepsilon=\alpha h^2$, where w and (w_h,ϕ_h) are the solutions of (1.1) and (2.3), respectively. Our result only requires the achievable smoothness assumption of the solution, namely, $w\in H^4(\Omega)$, in contrast to estimate (1.7). Numerical experiments show that (2.24) holds even when we only assume $\varepsilon=\mathrm{O}(h^2)$.

3. A scheme on triangular mesh

In this section we extend the previous error analysis to cover the case of a triangular mesh for planar domains. There is no super-approximating property like Lemma 2.2 on a triangular mesh, and we will split problem (1.1) into four equations. As usual, we use P_1 and P_0 to denote the spaces of linear functions and constants.

Let $\Im_h$ be a regular triangulation of Ω, where as usual h stands for the mesh size. Define the linear finite element space

$$T_h = \{v \in H_0^1(\Omega)\colon\ v|_K \in P_1(K)\ \forall K \in \Im_h\}, \tag{3.1}$$

and an auxiliary space

$$\boldsymbol{\Gamma}_h = \{\boldsymbol{\mu} \in [L^2(\Omega)]^2\colon\ \boldsymbol{\mu}|_K \in [P_0(K)]^2\ \forall K \in \Im_h\}. \tag{3.2}$$

Then we use the following scheme to solve problem (1.1).

Find $(w_h, \boldsymbol{\phi}_h) \in T_h \times [T_h]^2$ such that

$$(\nabla\boldsymbol{\phi}_h, \nabla\boldsymbol{\psi}_h) + \frac{1}{\varepsilon}(\mathscr{P}_0\boldsymbol{\phi}_h - \nabla w_h, \mathscr{P}_0\boldsymbol{\psi}_h - \nabla v_h) = (f, v_h) \quad \forall (v_h, \boldsymbol{\psi}_h) \in T_h \times [T_h]^2, \tag{3.3}$$

where $\mathscr{P}_0 : [L^2(\Omega)]^2 \to \boldsymbol{\Gamma}_h$ is the orthogonal projection. It is easy to see that Eq. (3.3) is equivalent to the following minimization problem:

$$J(w_h, \boldsymbol{\phi}_h) = \inf_{(v_h, \boldsymbol{\psi}_h) \in T_h \times [T_h]^2} J(v_h, \boldsymbol{\psi}_h),$$

$$J(v, \boldsymbol{\psi}) = \tfrac{1}{2}\ \|\nabla\boldsymbol{\psi}\|_0^2 + \frac{1}{2\varepsilon}\,\|\mathscr{P}_0(\boldsymbol{\psi} - \nabla v)\|_0^2 - (f, v).$$

Helmholtz Theorem states that any L^2 vector field can be decomposed uniquely into the sum of the gradient of a function $r \in H_0^1$ and the curl of a function $p \in \hat{H}^1$; moreover, the two summands are orthogonal in L^2. Here

$$\hat{H}^1(\Omega) = \left\{ v \in H^1(\Omega)\colon\ \int_\Omega v\,\mathrm{d}x = 0 \right\}.$$

When the vector field is piecewise constant, it is not true that r and p must be continuous piecewise linear functions. A discrete version of the Helmholtz theorem is proved in [2] by using a nonconforming element. Here, we prove another orthogonal decomposition which will be used in error analysis of method (3.3).

Define a differential operator $\boldsymbol{curl}_h$ element-wise by

$$\boldsymbol{curl}_h|_K = \boldsymbol{curl}|_K, \quad \boldsymbol{curl}\, v = \left(\frac{\partial v}{\partial x_2}, -\frac{\partial v}{\partial x_1}\right)^{\mathrm{T}}.$$

Let

$$\hat{M}_h = \Big\{ v \in L^2(\Omega)\colon\ v|_K \in P_1(K),\ \forall K \in \Im_h,\ v \text{ is continuous at midpoints of element edges},\ \int_\Omega v\,\mathrm{d}x = 0 \Big\}. \tag{3.4}$$

Lemma 3.1. *The following* $L^2(\Omega)$-*orthogonal decomposition holds*:

$$\boldsymbol{\Gamma}_h = \nabla T_h \oplus \boldsymbol{curl}_h \hat{M}_h.$$

Proof. Let $r \in T_h$ and $p \in \hat{M}_h$. Obviously, $\nabla r, \boldsymbol{curl}_h p \in \boldsymbol{\Gamma}_h$. Let us show that ∇r and $\boldsymbol{curl}_h p$ are orthogonal in $L^2(\Omega)$. We have

$$(\nabla r, \boldsymbol{curl}_h p) = \sum_T \int_T \nabla r \cdot \boldsymbol{curl}\, p \,\mathrm{d}x = -\sum_T \int_{\partial T} p \frac{\partial r}{\partial \boldsymbol{\tau}_T} \,\mathrm{d}s.$$

Here $\boldsymbol{\tau}_T$ is the unit tangential vector on ∂T, positive with respect to T. Let e be any interior edge of the triangulation, say $e = T_+ \cap T_-$. Let $r_+ = r|_{T_+}$ and $r_- = r|_{T_-}$. Since r is a piecewise linear function, the derivatives $\partial r_+/\partial \boldsymbol{\tau}_{T_+}$ and $\partial r_-/\partial \boldsymbol{\tau}_{T_-}$ are constant on e, and since r is continuous, $\partial r_+/\partial \boldsymbol{\tau}_{T_+}|_e = -\partial r_-/\partial \boldsymbol{\tau}_{T_-}|_e$. Since $p \in \hat{M}_h$, $p_+ - p_-$ is a linear function on e vanishing at the midpoint. It follows that

$$\int_e p_+ \frac{\partial r_+}{\partial \boldsymbol{\tau}_{T_+}} \,\mathrm{d}s + \int_e p_- \frac{\partial r_-}{\partial \boldsymbol{\tau}_{T_-}} \,\mathrm{d}s = 0.$$

If e lies on the boundary $\partial\Omega$, then $\partial r/\partial \boldsymbol{\tau}_T = 0$, since $r \in T_h$. Adding over all element edges, we get

$$(\nabla r, \boldsymbol{curl}_h p) = 0.$$

Now, we check the dimensions of the spaces involved in the orthogonal decomposition. Let N_{IS}, N_{BS}, N_{T}, N_{IV} and N_{BV} denote the number of interior sides, boundary sides, triangles, interior vertices and boundary vertices, respectively. Obviously $N_{\mathrm{BS}} = N_{\mathrm{BV}}$ and

$$\dim(\nabla T_h) = N_{\mathrm{IV}},$$

$$\dim(\boldsymbol{curl}_h \hat{M}_h) = N_{\mathrm{IS}} + N_{\mathrm{BS}} - 1,$$

$$\dim(\boldsymbol{\Gamma}_h) = 2N_{\mathrm{T}}.$$

We then use Euler's relation on $\mathfrak{T}_h$, namely

$$N_{\mathrm{T}} + (N_{\mathrm{IV}} + N_{\mathrm{BV}}) - (N_{\mathrm{IS}} + N_{\mathrm{BS}}) = 1$$

and

$$3N_{\mathrm{T}} = N_{\mathrm{BS}} + 2N_{\mathrm{IS}}.$$

Evidently, $2N_{\mathrm{T}} = N_{\mathrm{IS}} + N_{\mathrm{IV}} + N_{\mathrm{BS}} - 1$, and this implies that

$$\dim(\boldsymbol{\Gamma}_h) = \dim(\nabla T_h) + \dim(\boldsymbol{curl}_h \hat{M}_h).$$

Hence the result of the lemma holds. □

We apply Lemma 3.1 to split problem (3.3) into subproblems. Let

$$\frac{1}{\varepsilon}(\nabla w_h - \mathscr{P}_0 \boldsymbol{\phi}_h) = \nabla r_h + \boldsymbol{curl}_h p_h$$

for some $r_h \in T_h$ and $p_h \in \hat{M}_h$. It can be seen that $(r_h, \boldsymbol{\phi}_h, p_h, w_h) \in T_h \times [T_h]^2 \times \hat{M}_h \times T_h$ satisfies

$$\begin{aligned}
&(\nabla r_h, \nabla \mu_h) = (f, \mu_h) && \forall \mu_h \in T_h,\\
&(\nabla \boldsymbol{\phi}_h, \nabla \boldsymbol{\psi}_h) - (\boldsymbol{curl}_h p_h, \boldsymbol{\psi}_h) = (\nabla r_h, \boldsymbol{\psi}_h) && \forall \boldsymbol{\psi}_h \in [T_h]^2,\\
&\varepsilon(\boldsymbol{curl}_h p_h, \boldsymbol{curl}_h q_h) + (\boldsymbol{\phi}_h, \boldsymbol{curl}_h q_h) = 0 && \forall q_h \in \hat{M}_h,\\
&(\nabla w_h, \nabla s_h) = (\boldsymbol{\phi}_h + \varepsilon \nabla r_h, \nabla s_h) && \forall s_h \in T_h.
\end{aligned} \tag{3.5}$$

To obtain error estimates for method (3.3) (or equivalently, (3.5)), we introduce three new variables to split problem (1.1) into a new mixed formulation consisting of four equations:

Find $(r,\boldsymbol{\phi},p,w)\in H_0^1(\Omega)\times[H_0^1(\Omega)]^2\times\hat{H}^1(\Omega)\times H_0^1(\Omega)$ such that

$$
\begin{aligned}
&(\nabla r,\nabla\mu)=(f,\mu) && \forall\mu\in H_0^1(\Omega),\\
&(\nabla\boldsymbol{\phi},\nabla\boldsymbol{\psi})-(\mathbf{curl}\,p,\boldsymbol{\psi})=(\nabla r,\boldsymbol{\psi}) && \forall\boldsymbol{\psi}\in[H_0^1(\Omega)]^2,\\
&(\boldsymbol{\phi},\mathbf{curl}\,q)=0 && \forall q\in\hat{H}^1(\Omega),\\
&(\nabla w,\nabla s)=(\boldsymbol{\phi},\nabla s) && \forall s\in H_0^1(\Omega).
\end{aligned}
\tag{3.6}
$$

We observe that system (3.6) is just two Poisson equations plus a Stokes equation problem subject to the change of variables $(\phi_1,\phi_2)\mapsto(\phi_2,-\phi_1)$.

Theorem 3.2. *There exists a unique solution* $(r,\boldsymbol{\phi},p,w)\in H_0^1(\Omega)\times[H_0^1(\Omega)]^2\times\hat{H}^1(\Omega)\times H_0^1(\Omega)$ *of problem* (3.6). *Problem* (1.1) *and problem* (3.6) *are equivalent in the sense that if w is a solution of problem* (1.1), *then there exists* $(r,\boldsymbol{\phi},p)\in H_0^1(\Omega)\times[H_0^1(\Omega)]^2\times\hat{H}^1(\Omega)$ *such that* $(r,\boldsymbol{\phi},p,w)$ *solves problem* (3.6); *conversely, for a solution* $(r,\boldsymbol{\phi},p,w)$ *of problem* (3.6), *the function w is a solution of problem* (1.1). *Moreover, if* $f\in L^2(\Omega)$, *we have the regularity estimate*

$$\|r\|_2+\|\boldsymbol{\phi}\|_3+\|p\|_2+\|w\|_2\leqslant C\,\|f\|_0\,. \tag{3.7}$$

Proof. The existence and uniqueness of problem (3.6) are well known. To prove the equivalence of the two problems, we only need to show that if w is a solution of problem (1.1), then there exists $(r,\boldsymbol{\phi},p)\in H_0^1(\Omega)\times[H_0^1(\Omega)]^2\times\hat{H}^1(\Omega)$ such that $(r,\boldsymbol{\phi},p,w)$ solves problem (3.6). Let w be the solution of problem (1.1). Define $\boldsymbol{\phi}=\nabla w$ and $\boldsymbol{\lambda}=\Delta\boldsymbol{\phi}$. We have

$$\operatorname{div}\boldsymbol{\lambda}=\Delta\operatorname{div}\Delta w=\Delta^2 w=f.$$

By Helmholtz Theorem, there exist $r\in H_0^1(\Omega)$ and $p\in\hat{H}^1(\Omega)$ such that

$$\boldsymbol{\lambda}=\nabla r+\mathbf{curl}\,p.$$

It is readily verified that $(r,\boldsymbol{\phi},p,w)$ defined in this way solves problem (3.6).

To prove the regularity estimate (3.7), we notice that the first and fourth equations of (3.6) are Poisson equations. So

$$\|r\|_2\leqslant C\,\|f\|_0,\quad \|w\|_2\leqslant C\,\|\operatorname{div}\boldsymbol{\phi}\|_0\,.$$

The second and third equations form a Stokes-like system, then from [9]

$$\|\boldsymbol{\phi}\|_3+\|p\|_2\leqslant C\,\|\Delta r\|_1\,.$$

Combining the above relations, we get the regularity estimate (3.7). □

Now, we present two lemmas to be used in proving error estimates.

Lemma 3.3. *Let* $q_h\in\hat{M}_h$, $e=\bar{K}_1\cap\bar{K}_2$ *an internal edge of* $\Im_h$. *Then*

$$\int_e (q_h|_{K_1}-q_h|_{K_2})^2\,\mathrm{d}s\leqslant Ch\,\|\mathbf{curl}_h\,q_h\|_{0,K_1\cup K_2}^2,$$

where C is a constant independent of h.

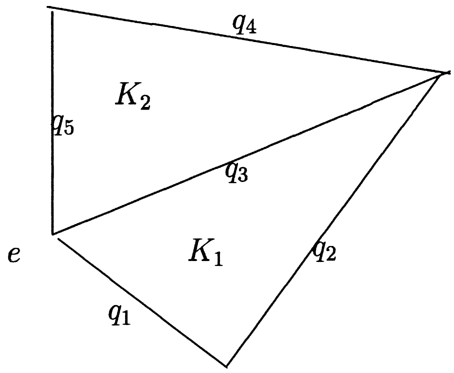

Fig. 1.

Proof. Let q_h on $K_1 \cup K_2$ be defined as in Fig. 1.

Recall that Simpson's rule is exact for cubics. Hence,

$$\begin{aligned}\int_e (q_h|_{K_1} - q_h|_{K_2})^2 \, ds &= \frac{|e|}{6}[(q_1 - q_2 + q_4 - q_5)^2 + (q_2 - q_1 + q_5 - q_4)^2] \\ &\leqslant Ch[(q_1 - q_2)^2 + (q_4 - q_5)^2].\end{aligned}$$

By a simple computation in the standard reference element K_0 with nodes $(0,0),(1,0)$ and $(0,1)$ [7,15], we find

$$\| \boldsymbol{curl}_h \, q_h \|_{0,K_1}^2 \geqslant C[(q_2 - q_3)^2 + (q_2 - q_1)^2],$$

$$\| \boldsymbol{curl}_h \, q_h \|_{0,K_2}^2 \geqslant C[(q_4 - q_3)^2 + (q_4 - q_5)^2].$$

Then the result follows. □

Lemma 3.4. *The following inequalities hold*:

$$|(\boldsymbol{curl}_h \, q_h, \boldsymbol{\phi})| \leqslant Ch \, \| \boldsymbol{\phi} \|_2 \| \boldsymbol{curl}_h \, q_h \|_0 \quad \forall \boldsymbol{\phi} \in [H^2(\Omega) \cap H_0^1(\Omega)]^2, \; q_h \in \hat{M}_h, \tag{3.8}$$

$$|(\boldsymbol{curl}_h \, q_h, \boldsymbol{\phi})| \leqslant Ch \, \| \boldsymbol{\phi} \|_1 \| \boldsymbol{curl}_h \, q_h \|_0 \quad \forall \boldsymbol{\phi} \in [T_h]^2, \; q_h \in \hat{M}_h. \tag{3.9}$$

Proof. Let $\mathscr{E} = \{e\}$ denote the set of the internal edges of the triangulation. For each internal edge e, the two associated neighboring triangles are denoted by $K_e^{(1)}$ and $K_e^{(2)}$. We use $\boldsymbol{\tau}_e$ for the unit tangential vector along e, positive with respect to the element $K_e^{(1)}$. Then we have

$$\begin{aligned}(\boldsymbol{curl}_h \, q_h, \boldsymbol{\phi}) &= -\sum_{K \in \Im_h} \int_{\partial K} \boldsymbol{\phi} \cdot \boldsymbol{\tau} q_h \, ds \\ &= -\sum_{K \in \Im_h} \int_{\partial K} (\boldsymbol{\phi} - \boldsymbol{s}) \cdot \boldsymbol{\tau} q_h \, ds \\ &= \sum_{e \in \mathscr{E}} \int_e (\boldsymbol{\phi} - \boldsymbol{s}) \cdot \boldsymbol{\tau}_e (q_h|_{K_e^{(2)}} - q_h|_{K_e^{(1)}}) \, ds,\end{aligned}$$

where $\boldsymbol{s}$ denotes an arbitrary piecewise constant function defined on the set $\mathscr{E}$; that is, $\boldsymbol{s}|_e$ is a constant for any $e \in \mathscr{E}$. We then obtain, for any piecewise constant function $\boldsymbol{s}$ on $\mathscr{E}$,

$$|(\boldsymbol{curl}_h\, q_h, \boldsymbol{\phi})| \leqslant \sum_{e\in\mathscr{E}} \left[\int_e |\boldsymbol{\phi} - \boldsymbol{s}|^2 \,\mathrm{d}s \right]^{1/2} \left[\int_e (q_h|_{K_e^{(2)}} - q_h|_{K_e^{(1)}})^2 \,\mathrm{d}s \right]^{1/2}. \tag{3.10}$$

Let us estimate the term

$$\inf_{\boldsymbol{s}\ \text{constant}} \int_e |\boldsymbol{\phi} - \boldsymbol{s}|^2 \,\mathrm{d}s.$$

Let K_0 be the standard reference triangle, where coordinate variables are denoted by $\xi = (\xi_1, \xi_2)$. We view e as one edge of the element $K_e^{(1)}$. Let $x = Q_{K_e^{(1)}}(\xi)$ be a linear mapping function from K_0 to $K_e^{(1)}$. Denote e_0 the edge of ∂K_0 obtained under the mapping function. For simplicity, we will use the same letter $\boldsymbol{\phi}$ to denote the function in both the x-coordinates and the ξ-coordinates, and use $\mathrm{d}s_0$ for the infinitesimal line element on ∂K_0. Then applying the standard reference element technique [7], we have

$$\inf_{\boldsymbol{s}\ \text{constant}} \int_e |\boldsymbol{\phi} - \boldsymbol{s}|^2 \,\mathrm{d}s \leqslant h \inf_{\boldsymbol{s}\ \text{constant}} \int_{e_0} |\boldsymbol{\phi} - \boldsymbol{s}|^2 \,\mathrm{d}s_0 \leqslant Ch \int_{e_0} \left| \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{\tau}_\xi} \right|^2 \mathrm{d}s_0,$$

where $\boldsymbol{\tau}_\xi$ is the unit tangential vector on e_0. We apply a trace theorem on K_0 to obtain

$$\int_{e_0} \left| \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{\tau}_\xi} \right|^2 \mathrm{d}s_0 \leqslant C \int_{K_0} (|\nabla_\xi \boldsymbol{\phi}|^2 + |\nabla_\xi^2 \boldsymbol{\phi}|^2) \,\mathrm{d}\xi.$$

Returning to the original element $K_e^{(1)}$, we get

$$\int_{e_0} \left| \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{\tau}_\xi} \right|^2 \mathrm{d}s_0 \leqslant C \int_{K_e^{(1)}} (|\nabla_x \boldsymbol{\phi}|^2 + h^2 |\nabla_x^2 \boldsymbol{\phi}|^2) \,\mathrm{d}x \leqslant C \parallel \boldsymbol{\phi} \parallel_{2,K_e^{(1)}}^2 .$$

Therefore,

$$\inf_{\boldsymbol{s}\ \text{constant}} \int_e |\boldsymbol{\phi} - \boldsymbol{s}|^2 \,\mathrm{d}s \leqslant Ch \parallel \boldsymbol{\phi} \parallel_{2,K_e^{(1)}}^2 . \tag{3.11}$$

By Lemma 3.3, we have

$$\int_e (q_h|_{K_e^{(2)}} - q_h|_{K_e^{(1)}})^2 \,\mathrm{d}s \leqslant Ch \parallel \boldsymbol{curl}_h\, q_h \parallel_{0,K_e^{(1)}\cup K_e^{(2)}}^2 . \tag{3.12}$$

Combining (3.10)–(3.12), we obtain estimate (3.8). The above argument also reveals estimate (3.9). □

We are now ready to prove the following error estimate.

Theorem 3.5. *There exists a unique solution* $(r_h, \boldsymbol{\phi}_h, p_h, w_h) \in T_h \times [T_h]^2 \times \hat{M}_h \times T_h$ *to the discrete problem* (3.5). *If* $f \in L^2(\Omega)$ *and* $\varepsilon = \alpha h^{2\gamma}$ *for some constants* $\alpha > 0$ *and* $\gamma \in (0, \frac{1}{2}]$, *then*

$$\|\nabla(\boldsymbol{\phi} - \boldsymbol{\phi}_h) \|_0 + h^\gamma \| \boldsymbol{curl}_h(p - p_h) \|_0 \leqslant Ch^{1-\gamma} \| f \|_0,$$

where w *is the solution of* (1.1), $\boldsymbol{\phi} = \nabla w$, *and* C *is a constant independent of* h *and* w.

Proof. The unique solvability of problem (3.5) is standard, and we only need to prove the error estimate.

Let us start with the error $r - r_h$, r_h being the usual conforming linear finite element solution of the problem

$$-\Delta r = f \text{ in } \Omega, \quad r = 0 \text{ on } \partial\Omega.$$

It is well known that (see [7])

$$\| r - r_h \|_0 \leqslant Ch^2 \| f \|_0, \quad \| r - r_h \|_1 \leqslant Ch \| f \|_0 . \tag{3.13}$$

We now derive estimates for the errors $\boldsymbol{\phi} - \boldsymbol{\phi}_h$ and $p - p_h$. Let $\boldsymbol{\phi}^I \in [T_h]^2$ and $p^I \in \hat{M}_h$ be the interpolants of $\boldsymbol{\phi}$ and p, respectively. From the standard interpolation theory [7], we have

$$\begin{aligned} &\| \boldsymbol{\phi} - \boldsymbol{\phi}^I \|_1 \leqslant Ch \| \boldsymbol{\phi} \|_2, \\ &\| p - p^I \|_0 \leqslant Ch \| p \|_1, \quad \| \boldsymbol{curl}_h(p - p^I) \|_0 \leqslant Ch \| p \|_2 . \end{aligned} \tag{3.14}$$

From the second equations of (3.5) and (3.6), we have

$$(\nabla(\boldsymbol{\phi} - \boldsymbol{\phi}_h), \nabla \boldsymbol{\psi}_h) - (\boldsymbol{curl}_h(p - p_h), \boldsymbol{\psi}_h) = (\nabla(r - r_h), \boldsymbol{\psi}_h) \quad \forall \boldsymbol{\psi}_h \in [T_h]^2. \tag{3.15}$$

In particular,

$$(\nabla(\boldsymbol{\phi} - \boldsymbol{\phi}_h), \nabla(\boldsymbol{\phi}^I - \boldsymbol{\phi}_h)) - (\boldsymbol{curl}_h(p - p_h), \boldsymbol{\phi}^I - \boldsymbol{\phi}_h) = (\nabla(r - r_h), \boldsymbol{\phi}^I - \boldsymbol{\phi}_h).$$

Hence,

$$\begin{aligned} \|\nabla(\boldsymbol{\phi} - \boldsymbol{\phi}_h)\|_0^2 &= (\nabla(\boldsymbol{\phi} - \boldsymbol{\phi}_h), \nabla(\boldsymbol{\phi} - \boldsymbol{\phi}^I)) + (\nabla(\boldsymbol{\phi} - \boldsymbol{\phi}_h), \nabla(\boldsymbol{\phi}^I - \boldsymbol{\phi}_h)) \\ &= (\nabla(\boldsymbol{\phi} - \boldsymbol{\phi}_h), \nabla(\boldsymbol{\phi} - \boldsymbol{\phi}^I)) + (\boldsymbol{curl}_h(p - p_h), \boldsymbol{\phi}^I - \boldsymbol{\phi}_h) + (\nabla(r - r_h), \boldsymbol{\phi}^I - \boldsymbol{\phi}_h) \\ &= (\nabla(\boldsymbol{\phi} - \boldsymbol{\phi}_h), \nabla(\boldsymbol{\phi} - \boldsymbol{\phi}^I)) + (\boldsymbol{curl}_h(p - p_h), \boldsymbol{\phi}^I - \boldsymbol{\phi}) + (\boldsymbol{curl}_h(p - p^I), \boldsymbol{\phi} - \boldsymbol{\phi}_h) \\ &\quad + (\boldsymbol{curl}_h(p^I - p_h), \boldsymbol{\phi} - \boldsymbol{\phi}_h) + (\nabla(r - r_h), \boldsymbol{\phi}^I - \boldsymbol{\phi}_h). \end{aligned}$$

Therefore,

$$\begin{aligned} \|\nabla(\boldsymbol{\phi} - \boldsymbol{\phi}_h)\|_0^2 &= (\nabla(\boldsymbol{\phi} - \boldsymbol{\phi}_h), \nabla(\boldsymbol{\phi} - \boldsymbol{\phi}^I)) + (\boldsymbol{curl}_h(p - p_h), \boldsymbol{\phi}^I - \boldsymbol{\phi}) \\ &\quad + (\boldsymbol{curl}_h(p - p^I), \boldsymbol{\phi} - \boldsymbol{\phi}_h) + (\boldsymbol{curl}_h(p^I - p_h), \boldsymbol{\phi}) \\ &\quad - (\boldsymbol{curl}_h(p^I - p_h), \boldsymbol{\phi}_h) + (\nabla(r - r_h), \boldsymbol{\phi}^I - \boldsymbol{\phi}_h). \end{aligned} \tag{3.16}$$

By Lemma 3.4, we have

$$(\boldsymbol{curl}_h(p^I - p_h), \boldsymbol{\phi}) \leqslant Ch \| \boldsymbol{\phi} \|_2 \| \boldsymbol{curl}_h(p^I - p_h) \|_0 . \tag{3.17}$$

For the term $-(\boldsymbol{curl}_h(p^I - p_h), \boldsymbol{\phi}_h)$, we have, from the third equation in (3.5),

$$-(\boldsymbol{curl}_h(p^I - p_h), \boldsymbol{\phi}_h) = \varepsilon(\boldsymbol{curl}_h p_h, \boldsymbol{curl}_h(p^I - p_h))$$

which is rewritten as

$$-(\boldsymbol{curl}_h(p^I - p_h), \boldsymbol{\phi}_h) = -\varepsilon(\boldsymbol{curl}_h(p^I - p_h), \boldsymbol{curl}_h(p^I - p_h)) + \varepsilon(\boldsymbol{curl}_h p^I, \boldsymbol{curl}_h(p^I - p_h)). \tag{3.18}$$

We can derive the following inequality from (3.16)–(3.18):

$$\begin{aligned}
&\|\nabla(\boldsymbol{\phi}-\boldsymbol{\phi}_h)\|_0^2+\varepsilon\,\|\boldsymbol{curl}_h(p^I-p_h)\|_0^2\\
&\quad\leqslant \|\nabla(\boldsymbol{\phi}-\boldsymbol{\phi}_h)\|_0\|\nabla(\boldsymbol{\phi}-\boldsymbol{\phi}^I)\|_0\\
&\qquad+\|\boldsymbol{curl}_h(p-p_h)\|_0\|\boldsymbol{\phi}^I-\boldsymbol{\phi}\|_0+\|\boldsymbol{curl}_h(p-p^I)\|_0\|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_0\\
&\qquad+Ch\,\|\boldsymbol{\phi}\|_2\|\boldsymbol{curl}_h(p^I-p_h)\|_0+\varepsilon(\boldsymbol{curl}_h\,p^I,\boldsymbol{curl}_h(p^I-p_h))\\
&\qquad+\|\nabla(r-r_h)\|_0\|\boldsymbol{\phi}^I-\boldsymbol{\phi}_h\|_0\,.
\end{aligned}$$

Using estimates (3.14), an algebraic manipulation of the above inequality yields, with $\varepsilon=\alpha h^{2\gamma}$,

$$\|\nabla(\boldsymbol{\phi}-\boldsymbol{\phi}_h)\|_0+h^\gamma\|\boldsymbol{curl}_h(p^I-p_h)\|_0\leqslant Ch^{1-\gamma}\,\|f\|_0\,.$$

Then the error estimate follows. □

Theorem 3.6. *Assume $\varepsilon=\alpha h$. Then*

$$\|w-w_h\|_1\leqslant ch\,\|f\|_0\,. \tag{3.19}$$

Proof. From the fourth equations of (3.5) and (3.6), we obtain

$$\|\nabla(w-w_h)\|_0\leqslant C\{\|\nabla(w-s_h)\|_0+\|\nabla\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_0+\varepsilon\,\|\nabla(r-r_h)\|_0\}.$$

Thus, it is enough to prove

$$\|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_0\leqslant ch\,\|f\|_0\,. \tag{3.20}$$

We define a Stokes-like dual problem: find $(\boldsymbol{\phi}^*,p^*)\in[H_0^1(\Omega)]^2\times\hat{H}^1(\Omega)$, such that

$$\begin{aligned}
&(\nabla\boldsymbol{\phi}^*,\nabla\boldsymbol{\psi})+(\boldsymbol{curl}\,p^*,\boldsymbol{\psi})=(\boldsymbol{g},\boldsymbol{\psi})\quad\forall\boldsymbol{\psi}\in[H_0^1(\Omega)]^2,\\
&(\boldsymbol{curl}\,q,\boldsymbol{\phi}^*)=0\quad\forall q\in\hat{H}^1(\Omega).
\end{aligned}\tag{3.21}$$

The problem has a unique solution and we have the regularity estimate

$$\|\boldsymbol{\phi}^*\|_2+\|p^*\|_1\leqslant c\,\|\boldsymbol{g}\|_0\,. \tag{3.22}$$

Now

$$\|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_0=\sup_{\boldsymbol{g}\in[L^2(\Omega)]^2}\frac{(\boldsymbol{\phi}-\boldsymbol{\phi}_h,\boldsymbol{g})}{\|\boldsymbol{g}\|_0}. \tag{3.23}$$

Let $\boldsymbol{\phi}^{*I}$ be the orthogonal projection of $\boldsymbol{\phi}^*$ to $[T_h]^2$ in $[H_0^1(\Omega)]^2$, and p^{*I} be the orthogonal projection of p^* to $\hat{M}_h$ in $H_0^1(\Omega)$. Then from (3.21),

$$\begin{aligned}
&(\boldsymbol{\phi}-\boldsymbol{\phi}_h,\boldsymbol{g})\\
&\quad=(\nabla\boldsymbol{\phi}^*,\nabla(\boldsymbol{\phi}-\boldsymbol{\phi}_h))+(\boldsymbol{curl}\,p^*,\boldsymbol{\phi}-\boldsymbol{\phi}_h)\\
&\quad=(\nabla(\boldsymbol{\phi}^*-\boldsymbol{\phi}^{*I}),\nabla(\boldsymbol{\phi}-\boldsymbol{\phi}_h))+(\nabla\boldsymbol{\phi}^{*I},\nabla(\boldsymbol{\phi}-\boldsymbol{\phi}_h))-(p^*,\mathrm{rot}(\boldsymbol{\phi}-\boldsymbol{\phi}_h))\\
&\quad=(\nabla(\boldsymbol{\phi}^*-\boldsymbol{\phi}^{*I}),\nabla(\boldsymbol{\phi}-\boldsymbol{\phi}_h))+(\nabla\boldsymbol{\phi}^{*I},\nabla(\boldsymbol{\phi}-\boldsymbol{\phi}_h))\\
&\qquad-(p^*-p^{*I},\mathrm{rot}(\boldsymbol{\phi}-\boldsymbol{\phi}_h))-(p^{*I},\mathrm{rot}(\boldsymbol{\phi}-\boldsymbol{\phi}_h))\\
&\quad\leqslant ch\,\|\boldsymbol{\phi}^*\|_2\|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_1+ch\,\|p^*\|_1\|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_1-I_3+I_4,
\end{aligned}$$

where

$$I_3 = (p^{*I}, \mathrm{rot}(\boldsymbol{\phi} - \boldsymbol{\phi}_h)),$$
$$I_4 = (\nabla \boldsymbol{\phi}^{*I}, \nabla(\boldsymbol{\phi} - \boldsymbol{\phi}_h)).$$

We have

$$\begin{aligned} I_3 &= (p^{*I}, -\mathrm{rot}\, \boldsymbol{\phi}_h) \\ &= (\boldsymbol{curl}_h p^{*I}, \boldsymbol{\phi}_h) - \sum_K \int_{\partial K} p^{*I} \boldsymbol{\phi}_h \cdot \boldsymbol{\tau}\, \mathrm{d}s \\ &= -\varepsilon(\boldsymbol{curl}_h p_h, \boldsymbol{curl}_h p^{*I}) - \sum_K \int_{\partial K} p^{*I} \boldsymbol{\phi}_h \cdot \boldsymbol{\tau}\, \mathrm{d}s. \end{aligned}$$

By Lemma 3.4, we have

$$\left| \sum_K \int_{\partial K} p^{*I} \boldsymbol{\phi}_h \cdot \boldsymbol{\tau}\, \mathrm{d}s \right| \leqslant ch \, \| \boldsymbol{\phi}_h \|_1 \| \boldsymbol{curl}_h p^{*I} \|_0 \, .$$

For I_4, we have using (3.15),

$$\begin{aligned} I_4 &= (\boldsymbol{curl}_h(p - p_h), \boldsymbol{\phi}^{*I}) + (\nabla(r - r_h), \boldsymbol{\phi}^{*I}) \\ &= (\boldsymbol{curl}_h(p - p_h), \boldsymbol{\phi}^{*}) + (\boldsymbol{curl}_h(p - p_h), \boldsymbol{\phi}^{*I} - \boldsymbol{\phi}^{*}) + (\nabla(r - r_h), \boldsymbol{\phi}^{*I}). \end{aligned}$$

For the second term on the right-hand side, we have

$$(\boldsymbol{curl}_h(p - p_h), \boldsymbol{\phi}^{*I} - \boldsymbol{\phi}^{*}) \leqslant ch^2 \| \boldsymbol{\phi}^{*} \|_2 \| \boldsymbol{curl}_h(p - p_h) \|_0 \, .$$

For the third term, noting that

$$(\nabla(r - r_h), \boldsymbol{\phi}^{*I}) = (\nabla(r - r_h), \boldsymbol{\phi}^{*I} - \nabla \mu_h) \quad \forall \mu_h \in T_h,$$

we get

$$(\nabla(r - r_h), \boldsymbol{\phi}^{*I}) \leqslant ch^2 \| r \|_2 \| \boldsymbol{\phi}^{*I} \|_1 \, .$$

For the first term, we write

$$(\boldsymbol{curl}_h(p - p_h), \boldsymbol{\phi}^{*}) = (\boldsymbol{curl}_h(p - p^I), \boldsymbol{\phi}^{*}) + (\boldsymbol{curl}_h(p^I - p_h), \boldsymbol{\phi}^{*}).$$

Now,

$$(\boldsymbol{curl}_h(p - p^I), \boldsymbol{\phi}^{*}) \leqslant ch \, \| p \|_2 \| \boldsymbol{\phi}^{*} \|_0,$$

and

$$(\boldsymbol{curl}_h(p^I - p_h), \boldsymbol{\phi}^{*}) = \sum_K \int_{\partial K} (p^I - p_h) \boldsymbol{\phi}^{*} \cdot \boldsymbol{\tau}\, \mathrm{d}s$$

by Lemma 3.4. Thus,

$$\begin{aligned} &(\boldsymbol{\phi} - \boldsymbol{\phi}_h, \boldsymbol{g}) \\ &\quad \leqslant ch(\| \boldsymbol{\phi}^{*} \|_2 + \| p^{*} \|_1) \| \boldsymbol{\phi} - \boldsymbol{\phi}_h \|_1 + \varepsilon(\boldsymbol{curl}_h p_h, \boldsymbol{curl}_h p^{*I}) \\ &\qquad + ch \, \| \boldsymbol{\phi}_h \|_1 \| \boldsymbol{curl}_h p^{*I} \|_0 + ch \, \| p \|_2 \| \boldsymbol{\phi}^{*} \|_0 \\ &\qquad + ch \, \| \boldsymbol{curl}_h(p^I - p_h) \|_0 \| \boldsymbol{\phi}^{*} \|_2 + ch^2 \| \boldsymbol{\phi}^{*} \|_2 \| \boldsymbol{curl}_h(p - p_h) \|_0 + ch^2 \| r \|_2 \| \boldsymbol{\phi}^{*I} \|_1 \end{aligned}$$

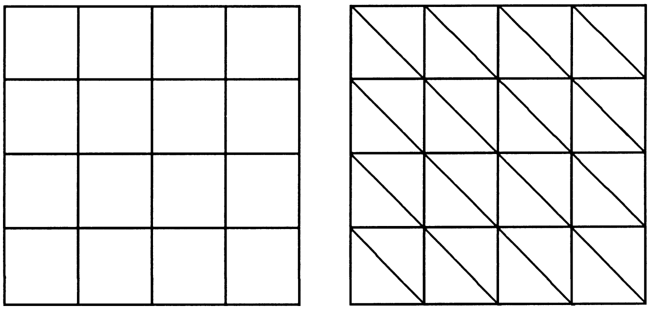

Fig. 2. Rectangular and triangular meshes.

$$\leqslant ch \,\|\boldsymbol{g}\|_0 \|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_1 + (\varepsilon \,\|\boldsymbol{curl}_h p_h\|_0 + ch\,\|\boldsymbol{\phi}_h\|_1)\,\|\boldsymbol{curl}_h p^{*I}\|_0 + ch\,\|p\|_2\|\boldsymbol{g}\|_0$$
$$+ ch\,\|\boldsymbol{g}\|_0\|\boldsymbol{curl}_h(p^I - p_h)\|_0 + ch^2\|\boldsymbol{g}\|_0\|\boldsymbol{curl}_h(p-p_h)\|_0 + ch^2\|r\|_2\|\boldsymbol{\phi}^{*I}\|_1 .$$

We have

$$\|\boldsymbol{\phi}^{*I}\|_1 \leqslant \|\boldsymbol{\phi}^*\|_1 + \|\boldsymbol{\phi}^* - \boldsymbol{\phi}^{*I}\|_1 \leqslant c\,\|\boldsymbol{\phi}^*\|_2 \leqslant c\,\|\boldsymbol{g}\|_0$$

and

$$\|\boldsymbol{curl}_h p^{*I}\|_0 \leqslant \|\nabla p^*\|_0 \leqslant c\,\|\boldsymbol{g}\|_0 .$$

Thus we have shown that

$$(\boldsymbol{\phi}-\boldsymbol{\phi}_h, \boldsymbol{g}) \leqslant c[h\,\|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_1 + \varepsilon\,\|\boldsymbol{curl}_h p_h\|_0 + h\,\|\boldsymbol{\phi}_h\|_1 + h\,\|p\|_2$$
$$+ h\,\|\boldsymbol{curl}_h(p^I - p_h)\|_0 + h^2\|\boldsymbol{curl}_h(p-p_h)\|_0 + h^2\|r\|_2\,]\|\boldsymbol{g}\|_0 .$$

Applying the estimates obtained in Theorem 3.5, we then have

$$\|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_0 = \sup_{g\in[L^2(\Omega)]^2} \frac{(\boldsymbol{\phi}-\boldsymbol{\phi}_h,\boldsymbol{g})}{\|\boldsymbol{g}\|_0} \leqslant ch\,\|f\|_0 .$$

So a proof of the theorem is now completed. □

4. Numerical experiments

We consider two examples for deflections of the thin clamped unit square plate. In Example A, the load is uniform and we take $f(x,y)=1$. In Example B, we choose $f(x,y)=\delta(x-\frac{1}{2}, y-\frac{1}{2})$, δ being the Delta function. So in the second example, the plate is under the action of a concentrated central load. For comparison, we use the same regularization parameter $\varepsilon=\alpha h^2$ for both the rectangular mesh solutions and the triangular mesh solutions.

Using the symmetry of the displacement and antisymmetry of the rotation for Examples A and B, we only need to solve the problems in a quarter domain $[0,\frac{1}{2}]\times[0,\frac{1}{2}]$. We divide the unit interval $[0,1]$ into N equal parts and set $h=1/N$. The rectangular mesh and triangular mesh on $[0,\frac{1}{2}]\times[0,\frac{1}{2}]$ for $h=\frac{1}{8}$ are given in Fig. 2 below.

Table 1
The percentage relative error at point $(0.5, 0.5)$, $\varepsilon = h^2$

	$h = 1/8$	$h = 1/16$	$h = 1/32$	$h = 1/64$
Method (2.3), Example A	93.62	24.54	6.45	1.68
Method (1.6), Example A	95.09	24.66	6.46	1.68
Method (2.3), Example B	159.04	48.26	14.30	4.24
Method (1.6), Example B	241.51	76.40	23.25	6.95

Table 2
The percentage relative error at point $(0.5, 0.5)$, $h = 1/32$

	$\varepsilon = h^2$	$\varepsilon = 0.1h^2$	$\varepsilon = 0.05h^2$	$\varepsilon = 0.01h^2$
Method (2.3), Example A	6.45	0.58	0.24	0.03
Method (1.6), Example A	6.46	0.58	0.25	0.03
Method (2.3), Example B	14.30	1.31	0.58	0.01
Method (1.6), Example B	23.25	2.20	1.02	0.08

Table 3
The percentage relative error at point $(0.5, 0.5)$, $\varepsilon = h^2$

	$h = 1/8$	$h = 1/16$	$h = 1/32$	$h = 1/64$
Method (3.3), Example A	81.39	20.53	4.37	1.01
Method (3.3), Example B	119.34	34.06	7.09	1.51

Table 4
The percentage relative error at point $(0.5, 0.5)$, $h = 1/32$

	$\varepsilon = 0.40h^2$	$\varepsilon = 0.55h^2$	$\varepsilon = 0.65h^2$	$\varepsilon = 0.85h^2$	$\varepsilon = h^2$
Method (3.3), Example A	2.17	0.07	1.22	3.12	4.37
Method (3.3), Example B	6.34	1.93	0.44	4.45	7.09

We list the percentage relative error at the point $(\frac{1}{2}, \frac{1}{2})$ where the maximal deflection occurs,

$$\text{relative error } (\%) = 100 \times \frac{|w_h(\frac{1}{2}, \frac{1}{2}) - w_{\text{exact}}(\frac{1}{2}, \frac{1}{2})|}{w_{\text{exact}}(\frac{1}{2}, \frac{1}{2})}.$$

From [1], $w_{\text{exact}}(\frac{1}{2}, \frac{1}{2}) \sim 0.001265$ for Example A and $w_{\text{exact}}(\frac{1}{2}, \frac{1}{2}) \sim 0.0056$ for Example B. See Tables 1–4 for some numerical results.

Since the H^1-norm error for the rectangular element solutions of w is $O(h)$, we expect the pointwise errors will behave almost like $O(h^2)$, when the exact solution is smooth enough. From Table 1, we observe that with the rectangular meshes, for Example A, the pointwise error at the point $(0.5, 0.5)$ converges to 0 quadratically, while the convergence order at the same point for

Example B is less than 2, due to the less solution regularity caused by the singular load force. With the triangular meshes, however, the numerical results in Table 3 seem to suggest a quadratic convergence order also for Example B. It is an open problem to show the first-order convergence in the H^1-norm for the solution computed from scheme (3.3) with $\varepsilon = \alpha h^2$. For rectangular meshes, the smaller the parameter α in the relation $\varepsilon = \alpha h^2$, the more accurate the numerical approximations.

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