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www.elsevier.com/locate/jalgebraCovering functors without groups [☆]José A. de la Peña ^a, María Julia Redondo ^{b,*}^a Instituto de Matemáticas, UNAM. Cd. Universitaria, México 04510 DF, Mexico^b Departamento de Matemática, Universidad Nacional del Sur, 8000, Bahía Blanca, Argentina

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ABSTRACT

Coverings in the representation theory of algebras were introduced for the Auslander–Reiten quiver of a representation-finite algebra in [Ch. Riedtmann, *Algebren, Darstellungsköcher, Überlagerungen und zurück*, *Comment. Math. Helv.* 55 (1980) 199–224] and later for finite-dimensional algebras in [K. Bongartz, P. Gabriel, *Covering spaces in representation theory*, *Invent. Math.* 65 (3) (1982) 331–378; P. Gabriel, *The universal cover of a representation-finite algebra*, in: *Proc. Representation Theory I, Puebla, 1980*, in: *Lecture Notes in Math.*, vol. 903, Springer, 1981, pp. 68–105; R. Martínez-Villa, J.A. de la Peña, *The universal cover of a quiver with relations*, *J. Pure Appl. Algebra* 30 (3) (1983) 277–292]. The best understood class of *covering functors* is that of *Galois covering functors* $F : A \rightarrow B$ determined by the action of a group of automorphisms of A . In this work we introduce the *balanced covering functors* which include the Galois class and for which classical Galois covering-type results still hold. For instance, if $F : A \rightarrow B$ is a balanced covering functor, where A and B are linear categories over an algebraically closed field, and B is tame, then A is tame.

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Introduction and notation

Let k be a field and A be a finite-dimensional (associative with 1) k -algebra. One of the main goals of the *representation theory of algebras* is the description of the category of finite-dimensional left modules ${}_A\text{mod}$. For that purpose it is important to determine the representation type of A . The finite representation type (that is, when A accepts only finitely many indecomposable objects in ${}_A\text{mod}$,

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up to isomorphism) is well understood. In that context, an important tool is the construction of Galois coverings $F : \tilde{A} \rightarrow A$ of A since \tilde{A} is a locally representation-finite category if and only if A is representation-finite [6,10]. For a tame algebra A and a Galois covering $F : \tilde{A} \rightarrow A$, the category \tilde{A} is also tame, but the converse does not hold [7,12].

Coverings were introduced in [13] for the Auslander–Reiten quiver of a representation-finite algebra. For algebras of the form $A = kQ/I$, where Q is a quiver and I an admissible ideal of the path algebra kQ , the notion of covering was introduced in [2,6,9]. Following [2], a functor $F : A \rightarrow B$, between two locally bounded k -categories A and B , is a *covering functor* if the following conditions are satisfied:

- (a) F is a k -linear functor which is onto on objects;
- (b) the induced morphisms

$$\bigoplus_{Fb'=j} A(a, b') \rightarrow B(Fa, j) \quad \text{and} \quad \bigoplus_{Fa'=i} A(a', b) \rightarrow B(i, Fb)$$

are bijective for all i, j in B and a, b in A .

We denote by $({}_b f_a^\bullet)_{b'} \mapsto f$ and $(\bullet_b f_{a'})_{a'} \mapsto f$ the corresponding bijections. We shall consider $F_\lambda : {}_A \text{mod} \rightarrow {}_B \text{mod}$ the left adjoint to the pull-up functor $F_\bullet : {}_B \text{mod} \rightarrow {}_A \text{mod}$, $M \mapsto MF$, where ${}_C \text{mod}$ denotes the category of left modules over the k -category C , consisting of covariant k -linear functors.

The best understood examples of covering functors are the *Galois covering functors* $A \rightarrow B$ given by the action of a group of automorphisms G of A acting freely on objects and where $F : A \rightarrow B = A/G$ is the quotient defined by the action. See [2,4,6,9,10] for results on Galois coverings. Examples of coverings which are not of Galois type will be exhibited in Section 1.

In this work we introduce *balanced coverings* as those coverings $F : A \rightarrow B$ where ${}_b f_a^\bullet = \bullet_b f_a$ for every $f \in B(Fa, Fb)$. Among many other examples, Galois coverings are balanced, see Section 2. We shall prove the following:

Theorem 0.1. *Let $F : A \rightarrow B$ be a balanced covering. Then every finitely generated A -module X is a direct summand of $F_\bullet F_\lambda X$.*

In fact, according to the notation in [1], we show that a balanced covering functor is a *cleaving* functor, see Section 3. This is essential for extending Galois covering-type results to more general situations. For instance, we show the following result.

Theorem 0.2. *Assume that k is an algebraically closed field and let $F : A \rightarrow B$ be a covering functor. Then the following hold:*

- (a) *If F is induced from a map $f : (Q, I) \rightarrow (Q', I')$ of quivers with relations, where $A = kQ/I$ and $B = kQ'/I'$, then B is locally representation-finite if and only if so is A ;*
- (b) *If F is balanced and B is tame, then A is tame.*

More precise statements are shown in Section 4. For a discussion on the representation type of algebras we refer to [1,5,7,11,12].

1. Coverings: examples and basic properties

1.1. The pull-up and push-down functors

Following [2,6], consider a locally bounded k -category A , that is, A has a (possibly infinite) set of non-isomorphic objects A_0 such that

- (a) $A(a, b)$ is a k -vector space and the composition corresponds to linear maps $A(a, b) \otimes_k A(b, c) \rightarrow A(a, c)$ for every a, b, c objects in A_0 ;
- (b) $A(a, a)$ is a local ring for every a in A_0 ;
- (c) $\sum_b A(a, b)$ and $\sum_b A(b, a)$ are finite-dimensional for every a in A_0 .

For a locally bounded k -category A , we denote by ${}_A\text{Mod}$ (resp. Mod_A) the category of covariant (resp. contravariant) functors $A \rightarrow \text{Mod}_k$; by ${}_A\text{mod}$ (resp. mod_A) we denote the full subcategory of locally finite-dimensional functors $A \rightarrow \text{mod}_k$ of the category ${}_A\text{Mod}$ (resp. Mod_A). In case A_0 is finite, A can be identified with the finite-dimensional k -algebra $\bigoplus_{a,b \in A_0} A(a, b)$; in this case the category ${}_A\text{Mod}$ (resp. ${}_A\text{mod}$) is equivalent to the category of left A -modules (resp. finitely generated left A -modules).

According to [5], in case k is algebraically closed, there exist a quiver Q and an ideal I of the path category kQ , such that A is equivalent to the quotient kQ/I . Then any module $M \in {}_A\text{Mod}$ can be identified with a *representation* of the *quiver with relations* (Q, I) . Usually our examples will be presented by means of quivers with relations.

Let $F : A \rightarrow B$ be a k -linear functor between two locally bounded k -categories. The *pull-up* functor $F_\bullet : {}_B\text{Mod} \rightarrow {}_A\text{Mod}$, $M \mapsto MF$ admits a left adjoint $F_\lambda : {}_A\text{Mod} \rightarrow {}_B\text{Mod}$, called the *push-down* functor, which is uniquely defined (up to isomorphism) by the following requirements:

- (i) $F_\lambda A(a, -) = B(Fa, -)$;
- (ii) F_λ commutes with direct limits.

In particular, F_λ preserves projective modules. Denote by $F_\rho : {}_A\text{Mod} \rightarrow {}_B\text{Mod}$ the right adjoint to F_\bullet .

For covering functors $F : A \rightarrow B$ we get an explicit description of F_λ and F_ρ as follows:

Lemma 1.1. (See [2].) *Let $F : A \rightarrow B$ be a covering functor. Then:*

- (a) For any $X \in {}_A\text{mod}$ and $f \in B(i, j)$,

$$F_\lambda X(f) = (X({}_b f_a^\bullet)) : \bigoplus_{Fa=i} X(a) \rightarrow \bigoplus_{Fb=j} X(b), \quad \text{with } \sum_{Fb=j} F({}_b f_a^\bullet) = f.$$

In particular, $F_\bullet(a, -) : F_\lambda A(a, -) \rightarrow B(Fa, -)$ is the natural isomorphism given by $({}_b f_a^\bullet)_b \mapsto f$.

- (b) For any $X \in {}_A\text{mod}$ and $f \in B(i, j)$

$$F_\rho X(f) = (X({}_b^\bullet f_a)) : \prod_{Fa=i} X(a) \rightarrow \prod_{Fb=j} X(b), \quad \text{with } \sum_{Fa=i} F({}_b^\bullet f_a) = f.$$

In particular, $F_\bullet D(-, b) : F_\rho DA(-, b) \rightarrow DB(-, Fb)$ is the natural isomorphism induced by $({}_b^\bullet f_a)_a \mapsto f$.

1.2. The order of a covering

The following lemma allows us to introduce the notion of *order* of a covering.

Lemma 1.2. *Let $F : A \rightarrow B$ be a covering functor. Assume that B is connected and a fiber $F^{-1}(i)$ is finite, for some $i \in B_0$. Then the fibers have constant cardinality.*

Proof. Let $i \in B_0$ and $0 \neq f \in B(i, j)$. For $a \in F^{-1}(i)$, $\sum_{Fb=j} \dim_k A(a, b) = \dim_k B(i, j)$. Hence $|F^{-1}(i)| \dim_k B(i, j) = \sum_{Fa=i} \sum_{Fb=j} \dim_k A(a, b) = \sum_{Fb=j} \sum_{Fa=i} \dim_k A(a, b) = |F^{-1}(j)| \dim_k B(i, j)$ and $|F^{-1}(i)| = |F^{-1}(j)|$. Since B is connected, the claim follows. \square

In case $F : A \rightarrow B$ is a covering functor with B connected and A_0 is finite, we define the *order* of F as $\text{ord}(F) = |F^{-1}(i)|$ for any $i \in B_0$. Thus $\text{ord}(F) |B_0| = |A_0|$.

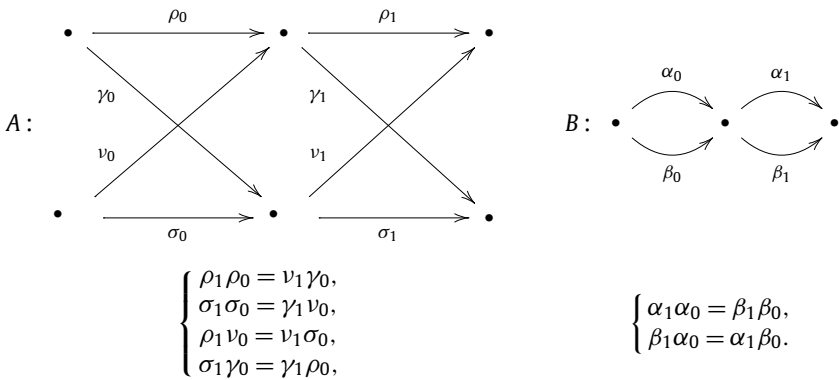
We recall from the Introduction that a covering functor $F : A \rightarrow B$ is *balanced* if ${}_b f_a^\bullet = {}_b f_a$ for every couple of objects a, b in A .

Lemma 1.3. *Let $F : A \rightarrow B$ be a balanced covering functor, then $F_\lambda = F_\rho$ as functors $A \text{ mod} \rightarrow B \text{ mod}$.*

1.3. Examples

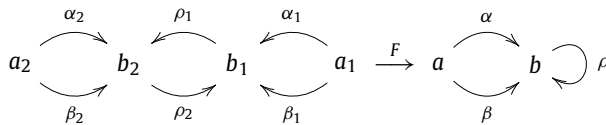
(a) Let A be a locally bounded k -category and let G be a group of k -linear automorphisms acting freely on A (that is, for $a \in A_0$ and $g \in G$ if $ga = a$, then $g = 1$). The quotient category A/G has as objects the G -orbits in the objects of A ; a morphism $f : i \rightarrow j$ in A/G is a family $f : ({}_b f_a) \in \prod_{a,b} A(a, b)$, where a (resp. b) ranges in i (resp. j) and $g \cdot {}_b f_a = {}_{gb} f_{ga}$ for all $g \in G$. The canonical projection $F : A \rightarrow A/G$ is called a *Galois covering* defined by the action of G .

A particular situation is illustrated by the following algebras (given as quivers with relations):



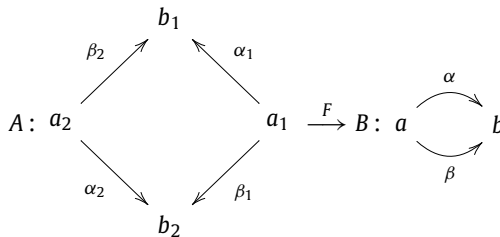
The algebra A is tame, but B is wild when $\text{char } k = 2$ [7]. The cyclic group C_2 acts freely on A and A/C_2 is isomorphic to B .

(b) Consider the algebras given by quivers with relations and the functor F as follows:



both algebras with $\text{rad}^2 = 0$ and $F\alpha_1 = \alpha, F\alpha_2 = \alpha + \beta, F\beta_1 = \beta, F\rho_i = \rho, i = 1, 2$. It is a simple exercise to check that F is a balanced covering, but obviously it is not of Galois type.

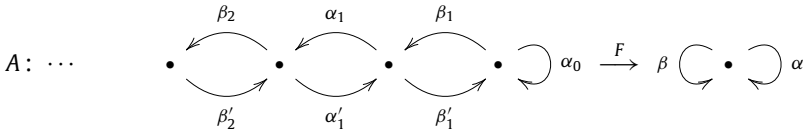
(c) Consider the functor



where $F\alpha_i = \alpha$, $i = 1, 2$, $F\beta_1 = \beta$, $F\beta_2 = \alpha + \beta$. Since $F(\beta_2 - \alpha_2) = \beta$ and $F(\beta_1) = \beta$, then ${}_{b_2}\beta_{a_2}^\bullet = -\alpha_2$ and ${}_{b_2}\beta_{a_2} = 0$. Hence F is a non-balanced covering functor.

For the two-dimensional indecomposable A -module X given by $X(a_2) = k$, $X(b_2) = k$, $X(\alpha_2) = \text{id}$ and zero otherwise, it follows that $F_\bullet F_\lambda X$ is indecomposable and hence X is not a direct summand of $F_\bullet F_\lambda X$.

(d) As a further example, consider the infinite category A and the balanced covering functor defined in the obvious way:



where both categories A and B have $\text{rad}^2 = 0$.

1.4. Coverings of schurian categories

We say that a locally bounded k -category B is *schurian* if for every $i, j \in B_0$, $\dim_k B(i, j) \leq 1$.

Lemma 1.4. *Let $F : A \rightarrow B$ be a covering functor and assume that B is schurian, then F is balanced.*

Proof. Let $0 \neq f \in B(i, j)$ and $Fa = i$, $Fb = j$. Since B is schurian, there is a unique $0 \neq {}_{b'}f_a^\bullet \in A(a, b')$ with $Fb' = j$ and a unique $0 \neq {}_b f_{a'}^\bullet \in A(a', b)$ with $Fa' = i$ satisfying $F_{b'}f_a^\bullet = f = F_b f_{a'}^\bullet$. In case $b = b'$, then $a = a'$ and ${}_b f_a^\bullet = {}_b f_a$. Else $b \neq b'$ and ${}_b f_a^\bullet = 0$. In this situation $a \neq a'$ and ${}_{b'}f_a = 0$. \square

Proposition 1.5. *Let $F : A \rightarrow B$ be a covering functor with finite order and B schurian. Then for every $M \in {}_B \text{mod}$, $F_\lambda F_\bullet M \cong M^{\text{ord}(F)}$.*

Proof. For any $0 \neq f \in B(i, j)$ we get

$$\begin{array}{ccc} F_\lambda F_\bullet M(i) = \bigoplus_{Fa=i} M(i) & \xrightarrow{\sim} & M^{\text{ord}(F)}(i) \\ \downarrow (M(F_b f_a^\bullet)) & & \downarrow \text{diag}(M(f), \dots, M(f)) \\ F_\lambda F_\bullet M(j) = \bigoplus_{Fb=j} M(j) & \xrightarrow{\sim} & M^{\text{ord}(F)}(j) \end{array}$$

Since for each a there is a unique b with ${}_b f_a^\bullet \neq 0$ such that $F_b f_a^\bullet = f$, then the square commutes. \square

Remark 1.6. If B is not schurian the result may not hold as shown in [7, (3.1)] for a Galois covering $F : B \rightarrow C$ with B as in example 1.3(a).

1.5. Coverings induced from a map of quivers

Let $q : Q' \rightarrow Q$ be a *covering map of quivers*, that is, q is an onto morphism of oriented graphs inducing bijections $i^+ \rightarrow q(i)^+$ and $i^- \rightarrow q(i)^-$ for every vertex i in Q' , where x^+ (resp. x^-) denotes those arrows $x \rightarrow y$ (resp. $y \rightarrow x$). For the concept of covering and equitable partitions in graphs, see [8].

Assume that Q is a finite quiver. Let I be an *admissible ideal* of the path algebra kQ , that is, $J^n \subset I \subset J^2$ for J the ideal of kQ generated by the arrows of Q . We say that I is *admissible with respect to* q if there is an ideal I' of the path category kQ' such that the induced map $qk : kQ \rightarrow kQ'$ restricts to

isomorphisms $\bigoplus_{q(a)=i} l'(a, b) \rightarrow I(i, j)$ for $q(b) = j$ and $\bigoplus_{q(b)=j} l'(a, b) \rightarrow I(i, j)$ for $q(a) = i$. Observe that most examples in Section 1.3 (not example (c)) are built according to the following proposition:

Proposition 1.7. *Let $q: Q' \rightarrow Q$ be a covering map of quivers, I an admissible ideal of kQ and l' an admissible ideal of kQ' making I admissible with respect to q as in the above definition. Then the induced functor $F: kQ'/l' \rightarrow kQ/I$ is a balanced covering functor.*

Proof. Since q is a covering of quivers, it has the unique lifting property of paths. Hence for any pair of vertices i in Q and a in Q' with $q(a) = i$, we have that

$$\begin{array}{ccc} \bigoplus_{q(b)=j} kQ'(a, b) & \xrightarrow{k(q)} & kQ(i, j) \\ \downarrow & & \downarrow \\ \bigoplus_{F(b)=j} kQ'/l'(a, b) & \xrightarrow{F} & kQ/I(i, j) \end{array}$$

is a commutative diagram with F an isomorphism. This shows that F is a covering functor.

For any arrow $i \xrightarrow{\alpha} j$ in Q and $q(a) = i$, there is a unique b in Q' and an arrow $a \xrightarrow{\alpha'} b$ with $q(\alpha') = \alpha$. Hence the class ${}_b f_a^*$ of α' in $kQ'/l'(a, b)$ satisfies that $F({}_b f_a^*)$ is the class $f = \bar{\alpha}$ of α in $kQ/I(i, j)$. By symmetry, ${}_b f_a^* = {}_b^* f_a$. For arbitrary $f \in kQ/I(i, j)$, f is the linear combination $\sum \lambda_i f_i$, where f_i is the product of classes of arrows in Q . Observe that for arrows $i \xrightarrow{\alpha} j \xrightarrow{\beta} m$ we have ${}_c(\bar{\beta}\bar{\alpha})_a^* = ({}_c\bar{\beta}_j^*)({}_b^*\bar{\alpha}_a) = ({}_c\bar{\beta}_j^*)({}_b\bar{\alpha}_a^*)$. It follows that F is balanced. \square

In the above situation we shall say that the functor F is induced from a map $q: (Q', l') \rightarrow (Q, I)$ of quivers with relations.

2. On Galois coverings

2.1. Galois coverings are balanced

Proposition 2.1. *Let $F: A \rightarrow B$ be a Galois covering, then F is balanced.*

Proof. Assume F is determined by the action of a group G of automorphisms of A , acting freely on the objects A_0 . Let i, j be objects of B and $f \in B(i, j)$. Consider a, b in A with $Fa = i, Fb = j$ and $({}_{b'} f_a^*)_{b'} \in \bigoplus_{Fb'=j} A(a, b')$ with $\sum_{Fb'=j} F({}_{b'} f_a^*) = f$.

For each object b' with $Fb' = j$, there is a unique $g_{b'} \in G$ with $g_{b'}(b') = b$. Then $(g_{b'}({}_{b'} f_a^*))_{b'} \in \bigoplus_{b'} A(g_{b'}(a), b) = \bigoplus_{Fa'=i} A(a', b)$ with $\sum_{b'} F(g_{b'}({}_{b'} f_a^*)) = \sum_{b'} F({}_{b'} f_a^*) = f$. Hence $g_{b'}({}_{b'} f_a^*) = {}_b^* f_{g_{b'}(a)}$ for every $Fb' = j$. In particular, for $g_b = 1$ we get ${}_b f_a^* = {}_b^* f_a$. \square

2.2. The smash-product

We say that a k -category B is G -graded with respect to the group G if for each pair of objects i, j there is a vector space decomposition $B(i, j) = \bigoplus_{g \in G} B^g(i, j)$ such that the composition induces linear maps

$$B^g(i, j) \otimes B^h(j, m) \rightarrow B^{gh}(i, m).$$

Then the smash product $B \# G$ is the k -category with objects $B_0 \times G$, and for pairs $(i, g), (j, h) \in B_0 \times G$, the set of morphisms is

$$(B \# G)((i, g), (j, h)) = B^{g^{-1}h}(i, j)$$

with compositions induced in natural way.

In [3] it was shown that $B \# G$ accepts a free action of G such that

$$(B \# G)/G \xrightarrow{\sim} B.$$

Moreover, if $B = A/G$ is a quotient, then B is a G -graded k -category and

$$(A/G) \# G \xrightarrow{\sim} A.$$

Proposition 2.2. *Let $F : A \rightarrow B$ be a covering functor and assume that B is a G -graded k -category. Then*

- (i) *Assume A accepts a G -grading compatible with F , that is, $F(A^g(a, b)) \subseteq B^g(Fa, Fb)$, for every pair $a, b \in A_0$ and $g \in G$. Then there is a covering functor $F \# G : A \# G \rightarrow B \# G$ completing a commutative square*

$$\begin{array}{ccc} A \# G & \xrightarrow{F \# G} & B \# G \\ \downarrow & & \downarrow \\ A & \xrightarrow{F} & B \end{array}$$

where the vertical functors are the natural quotients. Moreover F is balanced if and only if $F \# G$ is balanced.

- (ii) *In case B is a schurian algebra, then A accepts a G -grading compatible with F .*

Proof. (i) For each $a, b \in A_0$, consider the decomposition $A(a, b) = \bigoplus_{g \in G} A^g(a, b)$ and $B(Fa, Fb) = \bigoplus_{g \in G} B^g(Fa, Fb)$. Since these decompositions are compatible with F , then $A^g(a, b) = F^{-1}(B^g(Fa, Fb))$, for every $g \in G$.

For $\alpha \in (A \# G)((a, g), (b, h)) = A^{g^{-1}h}(a, b) = F^{-1}(B^{g^{-1}h}(Fa, Fb))$, we have

$$(F \# G)(\alpha) = F\alpha \in B^{g^{-1}h}(Fa, Fb) = (B \# G)((Fa, g), (Fb, h)).$$

(ii) Assume B is schurian and take $a, b \in A_0$ and $g \in G$. Either $B^g(Fa, Fb) = B(Fa, Fb) \neq 0$, if $A(a, b) \neq 0$ or $B^g(Fa, Fb) = 0$, correspondingly we set $A^g(a, b) = A(a, b)$ or $A^g(a, b) = 0$. Observe that the composition induces linear maps $A^g(a, b) \otimes A^h(b, c) \rightarrow A^{gh}(a, c)$, hence A accepts a G -grading compatible with F . \square

Remark. In the situation above, the fact that A and $B \# G$ are connected categories does not guaranty that $A \# G$ is connected. For instance, if $B = A/G$, then $A \# G = A \times G$.

The following result is a generalization of Proposition 2.2(ii).

Proposition 2.3. *Let $F : A \rightarrow B$ be a (balanced) covering functor induced from a map of quivers with relations. Let $F' : B' \rightarrow B$ be a Galois covering functor induced from a map of quivers with relations defined by the action of a group G . Assume moreover that B' is schurian. Then A accepts a G -grading compatible with F making the following diagram commutative.*

$$\begin{array}{ccc} A \# G & \xrightarrow{F \# G} & B' \\ \downarrow & & \downarrow F' \\ A & \xrightarrow{F} & B \end{array}$$

Proof. Let $A = k\Delta/J$, $B = kQ/I$ and $B' = kQ'/I'$ be the corresponding presentations as quivers with relations, F induced from the map $\delta: \Delta \rightarrow Q$, while F' induced from the map $q: Q' \rightarrow Q$. For each vertex a in Δ fix a vertex a' in Q' such that $F'a' = Fa$.

Consider an arrow $a \xrightarrow{\alpha} b$ in Δ and $\bar{\alpha}$ the corresponding element of A . We claim that there exists an element $g_\alpha \in G$ such that $F(\bar{\alpha}) \in B^{g_\alpha}(Fa, Fb)$. Indeed, we get $F(\bar{\alpha}) = \bar{\beta} = F'(\bar{\beta}')$ for arrows $Fa \xrightarrow{\beta} Fb$ and $a' \xrightarrow{\beta'} g_\alpha b'$ for a unique $g_\alpha \in G$. Therefore $F(\bar{\alpha}) \in B^{g_\alpha}(Fa, Fb)$. We shall define $A^{g_\alpha}(a, b)$ as containing the space $k\bar{\alpha}$. For this purpose, consider $g \in G$ and any vertices a, b in Δ , then $A^g(a, b)$ is the space generated by the classes \bar{u} of the paths $u: a \rightarrow b$ such that $F(\bar{u}) \in B^g(Fa, Fb)$. Since the classes of the arrows in Δ generate A , then $A(a, b) = \bigoplus_{g \in G} A^g(a, b)$. We shall prove that there are linear maps

$$A^g(a, b) \otimes A^h(b, c) \rightarrow A^{gh}(a, c).$$

Indeed, if $\bar{u} \in A^g(a, b)$ and $\bar{v} \in A^h(b, c)$ for paths $u: a \rightarrow b$ and $v: b \rightarrow c$ in Δ , let $F(\bar{u}) = F'(\bar{u}')$ and $F(\bar{v}) = F'(\bar{v}')$ for paths $u': a' \rightarrow gb'$ and $v': b' \rightarrow hc'$ in Q' . Since B' is schurian then the class of the lifting of $F(\bar{v}\bar{u})$ to B' is $\overline{(gv')u'}$. Therefore

$$F(\bar{v})F(\bar{u}) = F'(\overline{(gv')u'}) \in B^{gh}(Fa, Fb).$$

By definition, the G -grading of A is compatible with F . We get the commutativity of the diagram from Proposition 2.2. \square

2.3. Universal Galois covering

Let $B = kQ/I$ be a finite-dimensional k -algebra. According to [9] there is a k -category $\tilde{B} = k\tilde{Q}/\tilde{I}$ and a Galois covering functor $\tilde{F}: \tilde{B} \rightarrow B$ defined by the action of the fundamental group $\pi_1(Q, I)$ which is universal among all the Galois coverings of B , that is, for any Galois covering $F: A \rightarrow B$ there is a covering functor $F': \tilde{B} \rightarrow A$ such that $F = FF'$. In fact, the following more general result is implicitly shown in [9]:

Proposition 2.4. (See [9].) *The universal Galois covering $\tilde{F}: \tilde{B} \rightarrow B$ is universal among all (balanced) covering functors $F: A \rightarrow B$ induced from a map $q: (Q', I') \rightarrow (Q, I)$ of quivers with relations, where $A = kQ'/I'$.*

3. Cleaving functors

3.1. Balanced coverings are cleaving functors

Consider the k -linear functor $F: A \rightarrow B$ and the natural transformation $F(a, b): A(a, b) \rightarrow B(Fa, Fb)$ in two variables. The following is the main observation of this work.

Theorem 3.1. *Assume $F: A \rightarrow B$ is a balanced covering, then the natural transformation $F(a, b): A(a, b) \rightarrow B(Fa, Fb)$ admits a retraction $E(a, b): B(Fa, Fb) \rightarrow A(a, b)$ of functors in two variables a, b such that $E(a, b)F(a, b) = \mathbf{1}_{A(a, b)}$ for all $a, b \in A_0$.*

Proof. Set $E(a, b): B(Fa, Fb) \rightarrow A(a, b)$, $f \mapsto \bullet_b f_a$ which is a well-defined map. For any $\alpha \in A(a, a')$, $\beta \in A(b, b')$, we shall prove the commutativity of the diagrams:

$$\begin{array}{ccc} B(Fa, Fb) & \xrightarrow{E(a,b)} & A(a, b) \\ \downarrow B(Fa, F\beta) & & \downarrow A(a, \beta) \\ B(Fa, Fb') & \xrightarrow{E(a,b')} & A(a, b') \end{array} \qquad \begin{array}{ccc} B(Fa', Fb) & \xrightarrow{E(a',b)} & A(a', b) \\ \downarrow B(F\alpha, Fb) & & \downarrow A(\alpha, b) \\ B(Fa, Fb) & \xrightarrow{E(a,b)} & A(a, b) \end{array}$$

For the sake of clarity, let us denote by \circ the composition of maps. Indeed, let $f \in B(Fa, Fb)$ and calculate $\sum_{Fa'=Fa} F(\beta \circ_b^* f_a') = F\beta \circ f$, hence

$$A(a, \beta) \circ E(a, b)(f) = \beta \circ_b^* f_a = {}_b^*(F\beta \circ f)_a = E(a, b') \circ B(Fa, F\beta)(f),$$

and the first square commutes. Moreover, let $h \in B(Fa', Fb)$ and calculate $\sum_{Fb'=Fb} F({}_b^* h_a' \circ \alpha) = h \circ F\alpha$ and therefore ${}_b^* h_a' \circ \alpha = {}_b(h \circ F\alpha)_a$. Using that F is balanced we get that $E(a, b) \circ B(Fa, Fb)(h) = {}_b^*(h \circ F\alpha)_a = {}_b^* h_a' \circ \alpha = A(\alpha, b) \circ E(a', b)(h)$. \square

Given a k -linear functor $F : A \rightarrow B$ the composition $F_*F_\lambda : {}_A\text{Mod} \rightarrow {}_A\text{Mod}$ is connected to the identity $\mathbf{1}$ of ${}_A\text{Mod}$ by a canonical transformation $\varphi : F_*F_\lambda \rightarrow \mathbf{1}$ determined by $F_*F_\lambda A(a, -)(b) = \bigoplus_{Fb'=Fb} A(a, b') \rightarrow A(a, b)$, $(f_{b'}) \mapsto f_b$, see [1, p. 234]. Following [1], F is a *cleaving functor* if the canonical transformation φ admits a natural section $\varepsilon : \mathbf{1} \rightarrow F_*F_\lambda$ such that $\varphi(X)\varepsilon(X) = \mathbf{1}_X$ for each $X \in {}_A\text{Mod}$. The following statement, essentially from [1], yields Theorem 0.1 in the introduction.

Corollary 3.2. *Let $F : A \rightarrow B$ be a balanced covering, then F is a cleaving functor.*

Proof. Observe that F_*F_λ is exact, preserves direct sums and projectives (the last property holds since $F_*B(i, -) = \bigoplus_{Fa=i} A(a, -)$). Hence to define $\varepsilon : \mathbf{1} \rightarrow F_*F_\lambda$ it is enough to define $\varepsilon(A(a, -)) : A(a, -) \rightarrow F_*F_\lambda A(a, -)$ with the desired properties. For $b \in A_0$, consider $\varepsilon_b : A(a, b) \rightarrow \bigoplus_{Fb'=Fb} A(a, b') = F_*F_\lambda A(a, -)(b)$ the canonical inclusion. For $h \in A(b, c)$ we shall prove the commutativity of the following diagram:

$$\begin{array}{ccc} A(a, b) & \xrightarrow{\varepsilon_b} & \bigoplus_{Fb'=Fb} A(a, b') \\ \downarrow A(a, h) & & \downarrow (A(a, {}_c^* Fh_{b'})) \\ A(a, c) & \xrightarrow{\varepsilon_c} & \bigoplus_{Fc'=Fc} A(a, c') \end{array}$$

Let $f \in A(a, b)$, since F is balanced $A(a, {}_c^* Fh_{b'}) \circ \varepsilon_b(f) = {}_c^* Fh_b \circ f = {}_c^* Fh_b \circ f = \varepsilon_c \circ A(a, h)(f)$, since ${}_c^* Fh_b = h$ if $c' = c$ and it is 0 otherwise. This is what we wanted to show. \square

4. On the representation type of categories

4.1. Representation-finite case

Recall that a k -category A is said to be *locally representation-finite* if for each object a of A there are only finitely many indecomposable A -modules X , up to isomorphism, such that $X(a) \neq 0$. For a cleaving functor $F : A \rightarrow B$ it was observed in [1] that in case B is of locally representation-finite then so is A . In particular this holds when F is a Galois covering by [6]. We shall generalize this result for covering functors.

Part (a) of Theorem 0.2 in the introduction is the following:

Theorem 4.1. *Assume that k is algebraically closed and let $F : A \rightarrow B$ be a covering induced from a map of quivers with relations. Then B is locally representation-finite if and only if so is A . Moreover in this case the functor $F_\lambda : {}_A\text{mod} \rightarrow {}_B\text{mod}$ preserves indecomposable modules and Auslander–Reiten sequences.*

Proof. Let $F : A \rightarrow B$ be induced from $q : (Q', I') \rightarrow (Q, I)$ where $A = kQ'/I'$ and $B = kQ/I$. Let $\tilde{B} = k\tilde{Q}/\tilde{I}$ be the universal cover of B and $\tilde{F} : \tilde{B} \rightarrow B$ the universal covering functor. By Proposition 2.4 there is a covering functor $F' : \tilde{B} \rightarrow A$ such that $\tilde{F} = FF'$.

(1) Assume that B is a connected locally representation-finite category. Since F is induced by a map of quivers with relations, then Proposition 1.7 implies that F is balanced. Hence Corollary 3.2

implies that F is a cleaving functor. By [1, (3.1)], A is locally representation-finite; for the sake of completeness, recall the simple argument: each indecomposable A -module $X \in A\text{-mod}$ is a direct summand of $F_\bullet F_\lambda X = \bigoplus_{i=1}^n F_\bullet N_i^{n_i}$ for a finite family N_1, \dots, N_n of representatives of the isoclasses N of the indecomposable \tilde{B} -modules with $N(i) \neq 0$ for some $i = F(a)$ with $X(a) \neq 0$.

(2) Assume that A is a locally representation-finite category. First we show that B is representation-finite. Indeed, by case (1), since $F' : \tilde{B} \rightarrow A$ is a covering induced by a map of a quiver with relations, then \tilde{B} is locally representation-finite. By [10], B is representation-finite. In particular, [6] implies that \tilde{F}_λ preserves indecomposable modules, hence F_λ and F'_λ also preserve indecomposable modules.

Let X be an indecomposable A -module. We shall prove that X is isomorphic to $F'_\lambda N$ for some indecomposable \tilde{B} -module N . Since indecomposable projective A -modules are of the form $A(a, -) = F'_\lambda \tilde{B}(x, -)$ for some x in \tilde{B} , using the connectedness of Γ_A , we may assume that there is an irreducible morphism $Y \xrightarrow{f} X$ such that $Y = F'_\lambda N$ for some indecomposable \tilde{B} -module N . If N is injective, say $N = D\tilde{B}(-, j)$, there is a surjective irreducible map $(h_i) : N \rightarrow \bigoplus_i N_i$ such that all N_i are indecomposable modules and

$$0 \longrightarrow S_j \longrightarrow N \xrightarrow{(h_i)} \bigoplus_i N_i \longrightarrow 0$$

is an exact sequence. Then $Y = DA(-, F'j)$ and the exact sequence

$$0 \longrightarrow S_{F'j} \longrightarrow Y \xrightarrow{(F'_\lambda(h_i))} \bigoplus_i F'_\lambda(N_i) \longrightarrow 0$$

yields the irreducible maps starting at Y (ending at the indecomposable modules $F'_\lambda(N_i)$). Therefore $X = F'_\lambda(N_r)$ for some r , as desired. Next, assume that N is not injective and consider the Auslander–Reiten sequence $\xi : 0 \rightarrow N \xrightarrow{g} N' \xrightarrow{g'} N'' \rightarrow 0$ in $\tilde{B}\text{-mod}$. We shall prove that the push-down $F'_\lambda \xi : 0 \rightarrow F'_\lambda N \xrightarrow{F'_\lambda g} F'_\lambda N' \xrightarrow{F'_\lambda g'} F'_\lambda N'' \rightarrow 0$ is an Auslander–Reiten sequence in $A\text{-mod}$. This implies that there exists a direct summand \tilde{N} of N' such that $X \xrightarrow{\sim} F'_\lambda \tilde{N}$ which completes the proof of the claim.

To verify that $F'_\lambda \xi$ is an Auslander–Reiten sequence, let $h : F'_\lambda N \rightarrow Z$ be non-split mono in $A\text{-mod}$. Consider $\text{Hom}_A(F'_\lambda N, Z) \xrightarrow{\sim} \text{Hom}_{\tilde{B}}(N, F'_\bullet Z)$, $h \mapsto h'$ which is not a split mono (otherwise, then $\text{Hom}_{\tilde{B}}(F'_\bullet Z, N) \xrightarrow{\sim} \text{Hom}_A(Z, F'_\bullet N)$, $v \mapsto v'$ with $vh' = 1_{F'_\bullet Z}$. By Lemma 1.3, $F'_\lambda = F'_\rho$ and $v'h = 1_Z$). Then there is a lifting $\tilde{h} : N' \rightarrow F'_\bullet Z$ with $\tilde{h}g = h'$. Hence $\text{Hom}_{\tilde{B}}(N', F'_\bullet Z) \xrightarrow{\sim} \text{Hom}_A(F'_\lambda N', Z)$, $\tilde{h} \mapsto \tilde{h}'$ with $\tilde{h}'F'_\lambda g = h$.

We show that F_λ preserves Auslander–Reiten sequences. Let X be an indecomposable A -module of the form $X = F'_\lambda N$ for an indecomposable \tilde{B} -module N . Then $F_\lambda X = F_\lambda F'_\lambda N = \tilde{F}_\lambda N$. Since by [10], \tilde{F}_λ preserves indecomposable modules, then $F_\lambda X$ is indecomposable. Finally, as above, we conclude that F_λ preserves Auslander–Reiten sequences. \square

4.2. Tame representation case

Let k be an algebraically closed field. We recall that A is said to be of *tame representation type* if for each dimension $d \in \mathbb{N}$ and each object $a \in A_0$, there are finitely many $A - k[t]$ -bimodules M_1, \dots, M_s which satisfy:

- (a) M_i is finitely generated free as right $k[t]$ -module $i = 1, \dots, s$;
- (b) each indecomposable $X \in A\text{-mod}$ with $X(a) \neq 0$ and $\dim_k X = d$ is isomorphic to some module of the form $M_i \otimes_{k[t]} (k[t]/(t - \lambda))$ for some $i \in \{1, \dots, s\}$ and $\lambda \in k$.

In fact, it is shown in [11] that A is tame if (a) and (b) are substituted by the weaker conditions:

- (a') M_i is finitely generated as right $k[t]$ -module $i = 1, \dots, s$;
 (b') each indecomposable $X \in {}_A \text{mod}$ with $X(a) \neq 0$ and $\dim_k X = d$ is a direct summand of a module of the form $M_i \otimes_{k[t]} (k[t]/(t - \lambda))$ for some $i \in \{1, \dots, s\}$ and $\lambda \in k$.

The following statement covers claim (b) of Theorem 0.2 in the introduction.

Theorem 4.2. *Let $F : A \rightarrow B$ be a balanced covering functor. If B is tame, then A is tame.*

Proof. Let $a \in A_0$ and $d \in \mathbb{N}$. Let M_1, \dots, M_s be the $B - k[t]$ -bimodules satisfying (a) and (b): each indecomposable $M \in {}_B \text{mod}$ with $M(Fa) \neq 0$ and $\dim_k M \leq d$ is isomorphic to some $M_i \otimes_{k[t]} (k[t]/(t - \lambda))$ for some $i \in \{1, \dots, s\}$ and $\lambda \in k$.

By Corollary 3.2 each indecomposable $X \in {}_A \text{mod}$ with $X(a) \neq 0$ and $\dim_k X = d$ is a direct summand of some $F_*(M_i \otimes_{k[t]} (k[t]/(t - \lambda)))$, which is isomorphic to $F_* M_i \otimes_{k[t]} (k[t]/(t - \lambda))$, for some $i \in \{1, \dots, s\}$ and $\lambda \in k$. Hence A satisfies conditions (a') and (b'). \square

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