Stationary and transient resonant response of a spring pendulum

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Abstract

Plane motion of the spring pendulum is considered. The mathematical model of the system is transformed into dimensionless complex form and then the analytic approach using multiple time scale method is applied to solve the obtained initial value problem. The approximate analytical solution gives possibility to analyse steady-state and transient motion of the system for various parameters. Especially non stationary processes are discussed. The nonlinear transition of the dynamics and intersections of the tori are presented. For steady-state vibration the frequency response functions are derived. All analytical results are fully confirmed by numerical analysis.

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1. Introduction

The behavior of the nonlinear dynamical systems is of great interest of many researchers. Such systems are investigated mainly numerically due to mathematical complexity. In the numerical approach calculations have to be performed for certain set of parameters and initial conditions. For that reason the important phenomena occurring for only narrow range of some parameter can be omitted and un-noticed. On the other hand, analytical approach allows to obtain the solution as a function of some parameters which gives the opportunity to discuss results for full spectrum of parameters.

The steady-state motion is commonly observed in most of vibrating structures and machines and that regime of motion is mostly studied. However, in some situations the unsteady or transient behavior also should be taken into...
considerations. In many cases, in the non-steady vibrations intensive energy exchange between the system and either external source, or environment is observed. In multi-degree of freedom systems also energy exchange between parts of the structure or between modes may often occur. These phenomena are widely illustrated and discussed in references [3, 4, 6].

In the present paper, the dynamics of nonlinear spring pendulum is investigated. Such quite simple and intuitive system serves as a very good example of a study of non-linear phenomena exhibited by two DOEs mechanical systems. The mathematical model is fully dimensionless, which significantly improves its generality. The asymptotic method is used to solve the initial value problem. Such approach has been successfully applied earlier to solve non-linear ODEs describing behavior of lumped dynamical systems [1]. In general, the exact analytical solution cannot be obtained due to nonlinearity and couplings in the equations of motion. Main goal of the paper is to obtain approximate asymptotic solution of the non-steady state motion of the system. The multiple time scale method (MSM) is applied which has been used also in works [3, 6].

2. Formulation of the problem

The pendulum-type system investigated in this paper consists of the small body of mass \( m \) suspended at a fixed point on the nonlinear spring of the length \( L_0 \), whose elastic constants are denoted by \( k \) and \( k_1 \). Due to the introduced constraints, the body can move only in a fixed vertical plane, and is loaded by two harmonic forces \( F_1(t) = F_1 \cos(\Omega_1 t) \) and \( F_2(t) = F_2 \cos(\Omega_2 t) \) which act along and transversely to the spring, respectively. The viscous damping is assumed both for longitudinal and swing vibrations, where \( C_1 \) and \( C_2 \) are damping coefficients. The total spring elongation \( Z \) and the angle \( \phi \) describe unambiguously the position of the system.

The equations of motion have been obtained from Lagrange’s equations of the second type. Their dimensionless form is

\[
\ddot{z} + c_1 \dot{z} + z + \alpha z^3 + 3\alpha z_2^2 z + 3\alpha z_2 z^2 + w^2 (1 - \cos \phi) - (z + 1)\dot{\phi}^2 = f_1 \cos(p_1 \tau),
\]

\[
(z + 1) (z + 1) \dot{\phi} + w^2 \sin \phi + \dot{\phi}(c_2 + 2 \ddot{z}) = (z + 1) f_2 \cos(p_2 \tau).
\]

Where: \( L = L_0 + Z_\tau, Z_\tau \) denotes the static spring elongation, \( \omega_s = \sqrt{g / L}, \omega_\lambda = \sqrt{k / m}, z = Z / L \), dimensionless time \( \tau = t \omega_\lambda \), \( c_1 = C_1 / m \omega_\lambda \), \( c_2 = C_2 / L m \omega_\lambda \), \( w = \omega_\lambda / \omega_\lambda \), \( p_1 = \Omega_1 / \omega_\lambda \), \( p_2 = \Omega_2 / \omega_\lambda \), \( f_1 = F_1 / L \omega_\lambda \), \( f_2 = F_2 / L \omega_\lambda \), \( \alpha = k / k_1 \), \( z_\tau = Z / L \) and it satisfies the equation \( \alpha z_\tau^2 + z_\tau = w^2 \).
The following homogeneous initial conditions are taken
\[ z(0) = 0, \dot{z}(0) = 0, \phi(0) = 0, \dot{\phi}(0) = 0. \] (3)

The vibration of the system is investigated in the neighborhood of the static equilibrium position, hence the trigonometric functions of the angle \( \phi \) can be replaced by some first terms of their power series; namely, we take:
\[ \sin \phi \approx \phi - \frac{\phi^3}{6}, \quad \cos \phi \approx 1 - \frac{\phi^2}{2}. \]

3. Complex valued form of the problem

The important step in solving the problem relies on introduction of the following relations:
\[ z = \frac{\psi_z e^{i\tau} - \overline{\psi_z} e^{-i\tau}}{2i}, \quad \dot{z} = \frac{\psi_z e^{i\tau} + \overline{\psi_z} e^{-i\tau}}{2}, \quad \phi = \frac{\psi_\phi e^{i\omega t} - \overline{\psi_\phi} e^{-i\omega t}}{2wi}, \quad \dot{\phi} = \frac{\psi_\phi e^{i\omega t} + \overline{\psi_\phi} e^{-i\omega t}}{2w}, \] (4)

where \( \psi_z \) and \( \psi_\phi \) are some unknown complex functions, and \( \overline{\psi_z}, \overline{\psi_\phi} \) are their complex conjugate. As a result the equations of motion are transformed to the two differential equations of the first order
\[ \psi_z + 8i e^{-2i\tau} e^{2i\tau} \frac{(\psi_z - \overline{\psi_z})}{2} \frac{1}{2} c_1 (\psi_z + \overline{\psi_z} e^{2i\tau}) - \frac{3}{2} \alpha \frac{3}{2} \alpha z^2 (\psi_z - \overline{\psi_z} e^{2i\tau}) - \frac{3}{4} \alpha (\psi_z - \overline{\psi_z} e^{2i\tau})^2 \]
\[ - \frac{3}{8} w^2 e^{-i(2\tau + 2\omega)} (\overline{\psi_\phi}^2 + \psi_\phi^2 e^{4i\omega}) + 4 i w^2 e^{2i\tau} \psi_\phi \overline{\psi_\phi} (e^{2i\tau} \psi_z - \overline{\psi_z} + e^{i\tau}) \]
\[ + \frac{9}{8} w^2 e^{2i\tau} (e^{4i\omega} \psi_\phi^2 + \overline{\psi_\phi}^2) = f_1 e^{-i\tau} \cos (p_1 \tau). \] (5)
\[ w \psi_\phi + \frac{1}{2} e^{2i\tau} w \psi_\phi (\overline{\psi_z} - e^{2i\tau}) - \frac{1}{2} c_2 w (\psi_\phi - e^{2i\omega} \overline{\psi_\phi}) + \frac{1}{4} e^{i\tau} w (w + 2) \psi_\phi \psi_z \]
\[ - \frac{1}{4} e^{i(2\tau - 2\omega)} w (w - 2) \psi_\phi \overline{\psi_z} + \frac{1}{4} e^{i(2\tau + 2\omega)} w (w + 2) \psi_\phi \overline{\psi_\phi} - \frac{1}{4} i w^2 e^{4i\omega} (\psi_\phi e^{2i\omega} - \overline{\psi_\phi})^2 = f_2 e^{-i\omega} \cos (p_1 \tau). \] (6)

with the initial conditions
\[ \psi_z(0) = 0, \psi_\phi(0) = 0, \overline{\psi_z}(0) = 0, \overline{\psi_\phi}(0) = 0. \] (7)

4. Asymptotic solution near main resonance

The method of multiple scales (MSM) has been used to solve of the problem (5) – (7). Since a few of the parameters are assumed to be small, after introduction of the so-called small/perturbation parameter \( \varepsilon \), the following relations are used:
\[ c_1 = \tilde{c}_1 \varepsilon^2, c_2 = \tilde{c}_2 \varepsilon^2, z_\varepsilon = \tilde{z}_\varepsilon \varepsilon, f_1 = \tilde{f}_1 \varepsilon^3, f_2 = \tilde{f}_2 \varepsilon^3. \] (8)

Owing to the above assumptions, at least three time scales must be used to guarantee the preserving of the majority of the nonlinear terms as well as the most important couplings in the system (5) – (6). The time scales are defined in the following manner: \( \tau_0 = \tau \) is the “fast” time, whereas \( \tau_1 = \varepsilon \tau \) and \( \tau_2 = \varepsilon^2 \tau \) serve as the “slow” times. The
differential operator has the form 
\[ \frac{d}{d\tau} = \frac{\partial}{\partial \tau_0} + \varepsilon \frac{\partial}{\partial \tau_1} + \varepsilon^2 \frac{\partial}{\partial \tau_2} + \ldots + O(\varepsilon^4) \]. The solution is searched in the form of the power series regarding the small parameter \( \varepsilon \):

\[ \psi_{\varepsilon}(\tau; \varepsilon) = \sum_{k=1}^{k=3} \varepsilon^k \xi_k(\tau_0, \tau_1, \tau_2) + O(\varepsilon^4), \quad \psi_{\varepsilon}(\tau; \varepsilon) = \sum_{k=1}^{k=3} \varepsilon^k \xi_{\varepsilon k}(\tau_0, \tau_1, \tau_2) + O(\varepsilon^4). \]  

(9)

We are focused on the case when two main resonances \( p_1 \approx 1 \) and \( p_2 \approx w \) appear simultaneously in our system. Introducing the detuning parameters \( \sigma_1 = \tilde{\sigma}_1 \varepsilon^2 \) and \( \sigma_2 = \tilde{\sigma}_2 \varepsilon^2 \), we can write

\[ p_1 = 1 + \sigma_1, \quad p_2 = w + \sigma_2. \]  

(10)

Substituting (8) – (10) into (5) – (6) we obtain the equations, having the small parameter \( \varepsilon \) in various powers. These equations must be satisfied for any value of \( \varepsilon \), hence after rearrangement each of them according to the powers of \( \varepsilon \) we obtain a system of successive recurrence equations.

Observe that the equations of order \( \varepsilon^1 \) have the form: \( \frac{\partial \xi_{\varepsilon 1}}{\partial \tau_0} = 0, \quad \frac{\partial \xi_{\varepsilon 1}}{\partial \tau_1} = 0 \), and hence the functions \( \xi_{\varepsilon 1} \) and \( \xi_{\varepsilon 1} \) do not depend on \( \tau_0 \). On the other hand, the equations of the second and third approximation (i.e. of order \( \varepsilon^2 \) and \( \varepsilon^3 \), respectively) contain a few secular terms, and the solution becomes unbounded with respect to time. All the secular terms should be omitted in order to get reliable solution of physical meaning. Assuming that the pendulum can vibrate far from the internal resonance \( 2w-1 = 0 \) and \( 2w+1 = 0 \), the following conditions for solvability of the equations of the second approximation \( \frac{\partial \xi_{\varepsilon 1}}{\partial \tau_0} = 0, \quad \frac{\partial \xi_{\varepsilon 1}}{\partial \tau_1} = 0 \) are obtained. Consequently, the unknown functions depend only on \( \tau_2 \): \( \xi_{\varepsilon 1} = \xi_{\varepsilon 1}(\tau_2) \) and \( \xi_{\varepsilon 1} = \xi_{\varepsilon 1}(\tau_2) \). According to the asymptotic procedure, the solutions of the equations of the second approximation are introduced into the equations of the third approximation. The equations obtained in this way contain also the secular terms leading to unbounded growth of the amplitude with time. In order to remove the secular terms, the following equations should be satisfied

\[ \frac{\partial \xi_{\varepsilon 1}}{\partial \tau_2} = \frac{1}{2} \xi_{\varepsilon 1} + \frac{1}{2} \xi_{\varepsilon 1} - \frac{3}{2} \frac{\tilde{c}_1}{\xi_{\varepsilon 1}} = \frac{3}{2} i \alpha \delta_1 r_{\varepsilon 1} + \quad \frac{3}{8} i \alpha |\xi_{\varepsilon 1}|^2 \xi_{\varepsilon 1} + \quad \frac{2 i w^3 (w^2 - 1)}{4 - 16 w^2} \xi_{\varepsilon 1} = \frac{1}{2} \xi_{\varepsilon 1} = \frac{1}{2} f_1 e^{\varepsilon \delta_1}, \]  

(11)

\[ \frac{\partial \xi_{\varepsilon 1}}{\partial \tau_2} = \frac{1}{2} w \xi_{\varepsilon 1} - \frac{1}{4} i w^3 |\xi_{\varepsilon 1}|^2 \xi_{\varepsilon 1} + \quad \frac{i w^3 (w^2 + 2)}{16 w^2 - 4} \xi_{\varepsilon 1} + \quad \frac{2 i w^3 (8 w^4 - 7 w^2 - 1)}{64 w^2 - 16} \xi_{\varepsilon 1} = \frac{1}{2} f_2 e^{\varepsilon \delta_2}. \]  

(12)

Let us introduce the polar representation of the functions \( \xi_{\varepsilon 1} \) and \( \xi_{\varepsilon 1} \) in the following form

\[ \xi_{\varepsilon 1}(\tau_2) = a_i(\tau_2)e^{i\delta_1(\tau_2)}, \quad \xi_{\varepsilon 1}(\tau_2) = a_i(\tau_2)e^{i\delta_1(\tau_2)}, \quad a_i = a_i e \quad \text{for} \quad i = 1, 2 \]  

(13)

where \( a_i \) and \( \delta_i \) are real-valued functions and represent the amplitude and the phase of oscillations, respectively.

Substituting (13) into (11) – (12), going back to the original denotations, taking into account (8), and then separating real and imaginary parts of the equations, the following four modulation equations are derived:

\[ \frac{da_i}{d\tau} = -\frac{1}{2} c_i a_1 + \frac{1}{2} f_i \cos \theta_1, \]  

(14)
\[ a_1 \frac{d\theta_1}{d\tau} = -\frac{3}{2} c_2 a_1 + \frac{3w^7}{4-16w^2} a_1^2 - \frac{1}{2} f_2 \sin \theta_1, \]  
(15)  
\[ \frac{da_1}{d\tau} = -\frac{1}{2} c_2 a_2 + \frac{1}{2w} f_2 \cos \theta_2, \]  
(16)  
\[ a_2 \frac{d\theta_2}{d\tau} = \sigma_2 a_2 + \frac{3w^7}{4-16w^2} a_2^2 + \frac{w(8w^5 - 7w^4 - 1)}{64w^2 - 16} a_2^3 - \frac{1}{2w} f_2 \sin \theta_2, \]  
(17)  

where $\theta_1, \theta_2$ are the modified phases defined as follows: $\delta_1(\tau_2) = \tau_2 \sigma_1 - \theta_1(\tau_2), \quad \delta_2(\tau_2) = \tau_2 \sigma_2 - \theta_2(\tau_2)$.

The system governed by (14) - (17) becomes now an autonomous one. The unknown functions $a_1, a_2$ and $\theta_1, \theta_2$ can be understood as variables of a four-dimensional space $\Lambda$, where all possible dynamic states of the autonomous system are presented.

5. Results – non stationary motion

Near the resonance, the system is very sensitive to any change of the parameter values. Investigating the non-stationary vibration, we focus firstly on the sensitivity of the system on the parameter $\alpha$ which describes the spring nonlinearity. The transition through a critical value $\alpha_c$ causes the sudden qualitative change of the system behaviour. For $\alpha < \alpha_c$, the vibration can be considered as quasi-linear, while for $\alpha > \alpha_c$ it becomes strongly-nonlinear. For the further increase in the value of $\alpha$ other qualitative changes appear consisting in the alteration of the number of the fixed points of the system (14) – (17). The smooth modifications of the shape of amplitude modulation are then also observed. However, the spectacular qualitative change of the system dynamics occurs only for $\alpha = \alpha_c$. The value of $\alpha_c$ depends on all parameters of the system, and unfortunately it cannot be derived analytically due to the nonlinearities occurring in the modulation equations.

The results of calculations for the values of parameters collected in SET1={$\sigma_1 = 0.01, \sigma_2 = 0.012, f_1 = 0.0007, f_2 = 0.0005, c_1 = 0.002, c_2 = 0.0025, w = 0.175$} are presented in Figs. 2 - 6. For these parameters $\alpha_c \approx 1.13$. In Figs 2 – 3 the system behaviour is illustrated for $\alpha < \alpha_c$. Projections of the trajectory in space $\Lambda$ onto the plane ($a_1, \theta_1$) and ($a_2, \theta_2$) are shown. Three fixed points which can be theoretically predicted for the values from SET1 are also depicted. One of the points to which both trajectories strive is stable. In Fig. 3, both continuous bold lines present the shape of the amplitude modulations. Both areas plotted in grey colour are, indeed, the graphs of the oscillations which are obtained by the numerical integration of the problem (1) – (3).

![Fig. 2. Phase plane trajectories of the longitudinal (a) and swing (b) vibration for $\alpha = 1.12 < \alpha_c$ (just before the transition); red points denote fixed points.](image)
Fig. 3. Amplitude modulation (blue and green curves) and time histories (gray) of the longitudinal (a) and swing (b) vibration obtained for $\alpha = 1.12 < \alpha_c$ (just before the transition).

The transitional vibration, where the intensive energy exchange between modes and external sources is observed, is followed by the steady state-motion. One can notice that the above mentioned fixed point is stable.

Fig. 4. Phase plane trajectories of the longitudinal (a) and swing (b) vibration obtained for $\alpha = 1.14 > \alpha_c$ (just after the transition); red points indicate fixed points.

Fig. 5. Amplitude modulation (blue and green lines) and time histories (gray) of the longitudinal (a) and swing (b) vibration obtained for $\alpha = 1.14 > \alpha_c$ (just after the transition).

Figures 4-5 correspond to the situation just after the spectacular changing of the vibration kind. As a result of the qualitative transition the vibration becomes strongly-nonlinear. Moreover, the motion stabilizes about the fixed point, but it possesses much larger amplitude in comparison to the previous case. Almost perfect fit of the modulation curves and time histories, obtained from the problem (1) – (3), presented in Figs. 3 and 5 confirms correctness of the asymptotic analysis. An interesting phenomenon which shows the evolution of the Poincaré maps (projected on the plane $(\alpha_1, \alpha_2)$) with increasing the values of the detuning parameter $\sigma_z$ is illustrated in Fig. 6.
The numerically confirmed occurrence of the tori indicates the quasi-linear character of the resonant vibration. This phenomenon cannot be obtained using standard procedure for steady-state motion (problems related to quasi-periodic solutions tori and their bifurcations are addressed in reference [5]).

6. Results – stationary motion

The modulation equations (14) – (17) allow also to analyse the steady state vibration. The fixed points for these equations correspond to the situation when all time derivatives become equal to zero. The modified phases can be eliminated from the modulation equations, which leads to the following frequency response of the system:

\[
a_1^2 c_1^2 + \frac{1}{16} a_1^2 \left( 3a_1^2 a_2^2 + \frac{6a_2^2 w^2 (w^2 - 1)}{4 w^2 - 1} + 12 \alpha a_1^2 z_1^2 - 8 \sigma_1 \right)^2 = f_1^2 ,
\]

\[
w^2 \left( 64 a_2^2 c_2^2 + a_2^2 \left( -12 a_1^2 w (w^2 - 1) + a_1^2 w (8 w^4 - 7 w^2 - 1) + 16 \sigma_2 \left( 4 w^2 - 1 \right)^2 \right) \right) = 64 f_2^2 .
\]

Up to seven roots of the algebraic system (18) – (19) can occur for fixed values both of \( \sigma_1 \) and \( \sigma_2 \). There are indeed two frequency responses: \( a_1(\sigma_1, \sigma_2) \) and \( a_2(\sigma_1, \sigma_2) \) due to simultaneous occurrence of two main resonances. Their intersection projected on the planes \( (a_1, \sigma_1) \) and \( (a_2, \sigma_1) \), while \( \sigma_2 \) is constant, is presented in Fig. 7. In Fig. 8 projections for \( \sigma_1 \) constant are shown.

Fig. 7. Intersection of the surfaces \( a_1(\sigma_1, \sigma_2) \) and \( a_2(\sigma_1, \sigma_2) \), for \( \alpha=0.2 \) and \( \sigma_2=0.01 \) (red color denotes stable branches).
Fig. 8. Intersection of the surfaces $a_1(\sigma_1,\sigma_2)$ and $a_2(\sigma_1,\sigma_2)$, for $\alpha=0.2$ and $\sigma_i=0.01$ (red color denotes stable branches).

The results presented in Figs 6 and 8 are calculated for the parameters taken from SET1. Standard stability analysis using the Hurwitz criterion has been utilized. Comparing the results shown in Fig. 6 and Fig. 8 one may notice that the amplitudes do not stabilize for some values of $\sigma_2$ (approximately for $-0.006 < \sigma_2 < 0.001$).

7. Conclusions

The study of the resonant behavior of the spring pendulum has been carried out. The initial value problem has been transformed into the complex form, and then the multiple scale method has been applied in order to examine the resonant response of the pendulum for two main resonances occurring simultaneously. According to the assumptions, the nonlinear effects in the system are regarded as weak. At least three time scales have to be used in order to consider an impact of the system nonlinearities and the couplings in the mathematical model properly.

The modulation equations obtained analytically provide the possibility of discussing the system behavior for various parameters. It has been demonstrated, in particular, for the parameter $\alpha$ describing the spring nonlinearity.

There is a critical value $\alpha_c$ of the parameter for which the qualitative transition in the modulation process observed in slow time occurs. For $\alpha<\alpha_c$ the vibration is qualitatively similar to the linear case, whereas for $\alpha>\alpha_c$ the shape of modulation dramatically changes, and vibrations take place with much larger amplitude.

The frequency response equations have been derived for the steady-state motion in the same case of the resonance, and the obtained results have been presented as the two-dimensional projections of the amplitudes surfaces in the frequency domain.

All the analytical results have been confirmed by numerical analysis.

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