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THE FUNDAMENTAL GROUP OF THE COMPLEMENT OF AN ARRANGEMENT OF COMPLEX HYPERPLANES

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1. INTRODUCTION

LET V be a finite dimensional vector space over the complex numbers. An *arrangement* $\mathscr{A}$ in V is a finite collection of affine hyperplanes. Let

$$N = N(\mathscr{A}) = \bigcup_{H \in \mathscr{A}} H$$

and consider the *complement*

$$M = M(\mathscr{A}) = V - N(\mathscr{A}).$$

This space is an open connected submanifold of V. We wish to give a presentation for its fundamental group $\pi_1(M)$. By standard methods it suffices to consider only affine arrangements $\mathscr{A}$ in $V = \mathbb{C}^2$.

The answer is known provided $\mathscr{A}$ is the complexification of a real arrangement [3, 4, 5]. In this paper we remove this restriction and thus the fundamental group of any complex arrangement can be determined.

A presentation for $\pi_1(M)$ is determined from a certain planar graph *with additional structure*. If $\mathscr{A}$ is a complexified real arrangement this planar graph needs no additional structure and is just the underlying *real arrangement* of $\mathscr{A}$. In this case our calculation of $\pi_1(M)$ agrees with [3, 4, 5].

We refer the reader to [2] for introductory material regarding arrangements.

2. PRELIMINARIES

We assume here and in the sequel that $\mathscr{A}$ is an affine arrangement in $V = \mathbb{C}^2$.

Choose coordinates z_1, z_2 for $\mathbb{C}^2$ and coordinates x_1, y_1, x_2, y_2 for $\mathbb{R}^4$. Identify $\mathbb{C}^2$ with $\mathbb{R}^4$ via

$$z_1 = x_1 + iy_1$$
$$z_2 = x_2 + iy_2.$$

We identify $\mathbb{R}^2$ with the real part of $\mathbb{C}^2$ and thus $\mathbb{R}^2$ has coordinates x_1, x_2.

A *multiple point* is the non-empty intersection of two or more distinct hyperplanes. If $\mathscr{A}$ is a complexified real arrangement then we refer to its underlying real arrangement as the *real graph* of $\mathscr{A}$.

If u and v are words in a free group we set $u^v = v^{-1}uv$. If a group Π is given by a set of generators G and a set of relators R then Π has presentation

$$\Pi = \langle G | R \rangle.$$

If $w_1, \ldots, w_k$ are words in a free group we set

$$[w_1, \ldots, w_k] = \{w_1 \ldots w_k = w_{\sigma(1)} \ldots w_{\sigma(k)} : \sigma\}$$

where σ ranges over all *cyclic* permutations of the tuple $(1, \ldots, k)$. The set on the right should be interpreted as a set of relators.

3. ANALOG OF THE REAL GRAPH - Γ^2

Intuition and Motivation

Suppose for the moment that $\mathscr{A}$ is a complexified real arrangement.

In [3, 4, 5] we find that a presentation for $\pi_1(M)$ is encoded in the *real graph* of $\mathscr{A}$. In particular there is a generator for each hyperplane and a subset of relators for each multiple point p of the form

$$R_p = [g_1^{w_1}, \ldots, g_k^{w_k}].$$

The symbols $g_1, \ldots, g_k$ are the generators associated to those hyperplanes which pass through p in the *order* indicated by the real graph. The conjugating words $w_1, \ldots, w_k$ are also determined by the real graph.

Our aim is to define for any complex arrangement an analog of the real graph of a complexified real arrangement.

Definition of the Analog

Assumption 1. No hyperplane in $\mathscr{A}$ is of the form $\{z_1 = c\}$ for any constant $c \in \mathbb{C}$.

Assumption 2. Each pair of distinct multiple points can be distinguished by their x_1-coordinates alone.

Remark 3.1. Both assumptions are valid after a suitable change of coordinates.

Let $p_1 < \ldots < p_r$ be the multiple points ordered by x_1-coordinate. Thus we have

$$x_1(p_1) < \ldots < x_1(p_r).$$

Set $P = \{p_i\}$.

Let $h : \mathbb{R} \to \mathbb{R}$ be an arbitrary continuous map. Now consider the 1-parameter family of 2-planes in $\mathbb{R}^4$ defined for $t \in \mathbb{R}$ by

$$K_t = \{q \in \mathbb{R}^4 : x_1(q) = t, y_1(q) = h(t)\}$$
$$= \{q \in \mathbb{C}^2 : z_1(q) = t + ih(t)\}.$$

Let $H \in \mathscr{A}$ be a hyperplane. By Assumption 1 the set $H \cap K_t$ is a single point in $\mathbb{R}^4$ whose coordinates are continuous in the parameter t. Now recall that $N = \bigcup_{H \in \mathscr{A}} H$ and let $n = |\mathscr{A}|$. Consider the set $N \cap K_t$. This set consists of n distinct points *unless* it should happen that some multiple point $p \in P$ is contained in it i.e. *both* $t = x_1(p)$ *and* $h(t) = y_1(p)$. In this case the set consists of $n - v(p) + 1$ points where $v(p)$ is the number of hyperplanes through p. By Assumption 2 at most one multiple point may lie in $N \cap K_t$.

Definition 3.2. The **graph** of $\mathscr{A}$ relative to the map h is the pair $(\mathbb{R}^4, \Gamma_h)$ where Γ_h is the subset of $\mathbb{R}^4$ defined by

$$\Gamma_h = \bigcup_{t \in \mathbb{R}} N \cap K_t.$$

Identify $\mathbb{R}^3$ as the span of the coordinates x_1, x_2, y_2 in $\mathbb{R}^4$ and recall that $\mathbb{R}^2$ is the span of x_1, x_2. Let ϕ^2 and ϕ^3 denote the natural projections

$$\phi^2 : \mathbb{R}^4 \to \mathbb{R}^2 \quad \phi^2(x_1, y_1, x_2, y_2) = (x_1, x_2)$$
$$\phi^3 : \mathbb{R}^4 \to \mathbb{R}^3 \quad \phi^3(x_1, y_1, x_2, y_2) = (x_1, x_2, y_2).$$

Now set

$$\Gamma_h^2 = \phi^2(\Gamma_h)$$
$$\Gamma_h^3 = \phi^3(\Gamma_h).$$

Call the pairs $(\mathbb{R}^2, \Gamma_h^2)$ and $(\mathbb{R}^3, \Gamma_h^3)$ the 2-**graph** and the 3-**graph** of $\mathscr{A}$ respectively. We refer to the function h as the **graphing map.**

Remark 3.3. We may occasionally suppress the graphing map and the ambient space and thus use Γ, Γ^3 or Γ^2 to denote the embedded graph.

Incidence Condition

In [3, 4, 5] we see that the multiple points play a principal role when we are considering a complexified real arrangement. Thus for complex arrangements we find it necessary to require that the graph Γ_h pass through each multiple point i.e.

$$p \in P \Rightarrow p \in \Gamma_h.$$

This is equivalent to the following *incidence condition* on the graphing map h.

$$p \in P \Rightarrow y_1(p) = h(x_1(p))$$

We henceforward require that all graphing maps satisfy this incidence condition.

Extension of the Real Graph

Example 3.4. Suppose that $\mathscr{A}$ is a complexified real arrangement and choose the graphing map h to be identically zero. In this case the 2-graph $(\mathbb{R}^2, \Gamma_h^2)$ is precisely the real graph of $\mathscr{A}$. The 3-graph $(\mathbb{R}^2, \Gamma_h^3)$ is the natural embedding of the real graph into $\mathbb{R}^3$. See Fig. 1(A).

Combinatorial Graphs

Observe that both of the sets $\Gamma_h \subseteq \mathbb{R}^4$ and $\Gamma_h^3 \subseteq \mathbb{R}^3$ are indeed *combinatorial graphs* in the sense that they consist of *vertices* and *edges*. The vertices are in one-to-one correspondence with the multiple points of $\mathscr{A}$. The edges are of two types: closed and half-open. Each closed edge joins two distinct vertices with a path. The half-open edges are paths starting at one vertex and running off to infinity. No two distinct edges intersect at an interior point.

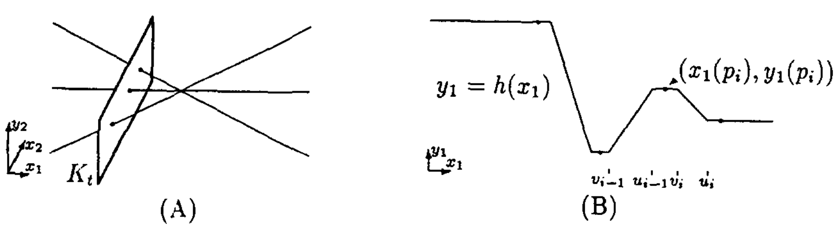

Fig. 1.

Assertion 3.5. The projection $\phi^3: \mathbb{R}^4 \to \mathbb{R}^3$ induces an isomorphism

$$\Gamma_h \simeq \Gamma_h^3.$$

Now consider $\Gamma_h^2 \subseteq \mathbb{R}^2$. This set is the projection of the graph Γ onto a 2-plane. The vertices project to distinct points. However self intersections may occur.

Definition 3.6. Call the 2-graph Γ^2 **regular** provided all self intersections are distinct, transverse and occur only on the interiors of edges.

Remark 3.7. After a change of coordinates and a change of graphing map if necessary we can assume without loss of generality that the 2-graph is regular.

Definition 3.8. If the 2-graph is regular we can refine its graph structure as follows.

We view each self intersection as a new vertex. Those vertices which arise from multiple points are referred to as **actual vertices**, and those which arise as self intersections are referred to as **virtual vertices.**

Each edge is subdivided into one or more new edges by those self intersections which lie on it. The new edges are referred to as **segments.**

Definition 3.9. Suppose that the 2-graph Γ^2 is regular. By comparison with the 3-graph Γ^3 each virtual vertex of Γ^2 may be marked to indicate whether it represents an under or overcrossing of two edges in $\mathbb{R}^3$. If Γ^2 is so marked we refer to it as the **marked 2-graph.**

The marking is made explicit by the following. Let $q \in \Gamma^2$ be a virtual vertex. Let $p, p' \in \Gamma$ be the two points with $\phi^2(p) = \phi^2(p') = q$. Let $H, H' \in \mathscr{A}$ be the two hyperplanes with $p \in H$ and $p' \in H'$. Now choose a real number $c < x_1(q)$ sufficiently close to $x_1(q)$. Recall that $H \cap K_c$ and $H' \cap K_c$ are points in Γ. Assume that p and p' were labelled so that $x_2(H \cap K_c) < x_2(H' \cap K_c)$. Set $t = x_1(q)$ and call the virtual vertex q **positive** if $y_2(H \cap K_t) < y_2(H' \cap K_t)$ and **negative** in the other case. See Fig. 2.

Remark 3.10. The 3-graph can be recovered up to ambient isotopy from the marked 2-graph.

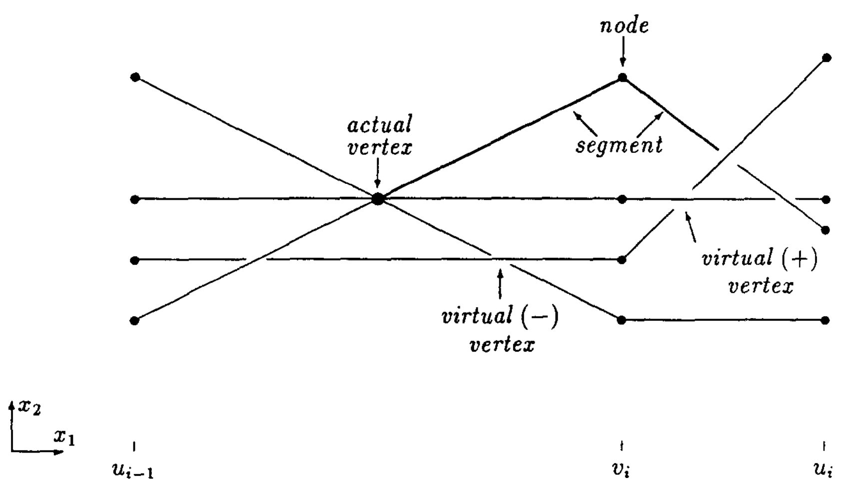

Fig. 2. A portion of an admissible 2-graph

Choosing the Graphing Map

We wish now to consider the marked 2-graph for an explicit choice of graphing map. Using a linear graphing map is in general precluded by the incidence condition. However a piecewise-linear (PL) map may be chosen. Since the multiple points play a principal role we would like the graphing map to be well-behaved near each of them. More precisely we will choose the graphing map to be constant on a neighborhood of each point $x_1(p)$ for $p \in P$.

Thus we choose real numbers $u_0, \ldots, u_r$ and $v_0, \ldots, v_r$ so that

$$\begin{aligned} -\infty < v_0 = u_0 < x_1(p_1) & \\ x_1(p_i) < v_i < u_i < x_1(p_{i+1}) & \quad i = 1, \ldots, r-1 \\ x_1(p_r) < v_r = u_r < \infty. & \end{aligned}$$

We define a PL graphing map h as follows.

$$\begin{aligned} -\infty < a \leq v_1 &\Rightarrow h(a) = y_1(p_1) \\ u_{i-1} \leq a \leq v_i &\Rightarrow h(a) = y_1(p_i) \\ u_{r-1} \leq a < \infty &\Rightarrow h(a) = y_1(p_r). \end{aligned}$$

On the complementary intervals we interpolate linearly. See Fig. 1(B).

Nodes

We now consider the 2-graph for the graphing map described above. First fix a real number $t \in \{u_i\} \cup \{v_i\}$. Recall the set $N \cap K_t \subseteq \mathbb{R}^4$ which was considered earlier. This set consists of n distinct points—one for each hyperplane. The x_1, x_2-projections of these points lie in the 2-graph. These projected points are referred to as *nodes*—not to be confused with vertices.

The Admissible 2-Graph

Definition 3.12. Call the 2-graph Γ^2 **admissible** if it is regular, marked, arises from a PL graphing map as described above and satisfies the following conditions on its vertices and nodes. Distinct vertices (both actual and virtual) are required to have distinct x_1-coordinates. Two nodes which share the same x_1-coordinate and arise from distinct hyperplanes are required to have distinct x_2-coordinates.

Remark 3.14. After a change of coordinates and a different choice of real numbers $\{u_i\} \cup \{v_i\}$ if necessary we can assume without loss of generality that the 2-graph is admissible.

Remark 3.15. The admissible 2-graph Γ^2 is our generalization of the real graph to any complex arrangement in $\mathbb{C}^2$.

4. DETERMINING $\pi_1(M)$ FROM Γ^2

PROPOSITION 4.1. *There is an algorithm for determining a presentation for $\pi_1(M)$ from an admissible 2-graph for $\mathscr{A}$.*

Remark 4.2. If $\mathscr{A}$ is a complexified real arrangement we may choose the graphing map to be identically zero. The 2-graph for this choice is the ordinary real graph i.e. there are no virtual vertices. In this case the algorithm which we give for determining $\pi_1(M)$ from the 2-graph agrees with Randell [3] and Salvetti [4].

Statement of Algorithm

Let Γ^2 be an admissible 2-graph for $\mathscr{A}$. Let $q_1 < \ldots < q_l$ be the vertices (actual and virtual) of Γ^2-ordered by x_1-coordinate. Choose a sequence of real numbers $t_0 < \ldots < t_l$ which separate the x_1-coordinates of the vertices. Thus we have

$$t_0 < x_1(q_1) < t_1 < x_1(q_2) < \ldots < x_1(q_l) < t_l.$$

Now recall that the segments of Γ^2 are just its edges. Let E_k denote the vertical line $\{x_1 = t_k\} \subseteq \mathbb{R}^2$. Let $F(k)$ denote the set of those segments which E_k crosses—ordered by x_2-coordinate.

Remark 4.3. The segments in $F(k)$ are in one-to-one correspondence with the hyperplanes in $\mathscr{A}$ and thus $F(k)$ has cardinality $n = |\mathscr{A}|$.

We now relate $F(m+1)$ to $F(m)$. For $f \in F(m)$ let $f' \in F(m+1)$ be that segment which is associated to the same hyperplane as f. Let $q = q_{m+1}$ be the vertex between E_m and E_{m+1}. Now write $F(m) = (f_1, \ldots, f_n)$ in increasing order. Let j be the first index for which the segment f_j has the vertex q as endpoint and let k be the last such index. Note that $\{f_i : j \le i \le k\}$ are precisely those segments in $F(m)$ which have the vertex q as endpoint. We also note that if $i < j$ or $i > k$ then $f'_i = f_i$. We now observe that

$$F(m+1) = (f_1, \ldots, f_{j-1}, \underbrace{f'_k, f'_{k-1}, \ldots, f'_{j+1}, f'_j}, f_{k+1}, \ldots, f_n)$$

where the indicated subsequence is obtained from the appropriate subsequence of $F(m)$ by priming and reversing the order. See Fig. 3(A) wherein the depicted vertices may be either actual or virtual.

Remark 4.4. If q is a virtual vertex then $k = j + 1$ since only two hyperplanes contribute to the segments with endpoint q.

Let $G = \{g_1, \ldots, g_n\}$ be the generators for the free group of rank n.

We associate a word in the symbols G to each segment as follows.

First write $F(0) = (f_1, \ldots, f_n)$ in increasing order. Now associate $f_i \leftrightarrow g_i$.

Proceeding by induction on m assume we have associated the sequence of words $W(m) = (w_1, \ldots, w_n)$ to the segments in $F(m) = (f_1, \ldots, f_n)$ coordinatewise. Now write $F(m+1)$ as above.

$$F(m+1) = (f_1, \ldots, f_{j-1}, \underbrace{f'_k, f'_{k-1}, \ldots, f'_{j+1}, f'_j}, f_{k+1}, \ldots, f_n).$$

Fig. 3.

Set $q = q_{m+1}$. Note that q is the vertex between the vertical lines E_m and E_{m+1}.

There are three cases to consider:

(1) q is an actual vertex.

(2) q is a positive virtual vertex.

(3) q is a negative virtual vertex.

Case 1. q is an actual vertex. In this case we associate the sequence of words

$$W(m+1) = (w_1, \ldots, w_{j-1}, \underbrace{w'_k, w'_{k-1}, \ldots, w'_{j+1}, w'_j}, w_{k+1}, \ldots, w_n)$$

to the segments in $F(m+1)$ coordinatewise where

$$\begin{aligned} w'_j &= w_j \\ w'_{j+1} &= w_{j+1}^{w_j} \\ w'_{j+2} &= w_{j+2}^{w_{j+1}w_j} \\ &\vdots \\ w'_i &= w_i^{w_{i-1}\cdots w_j} \\ &\vdots \end{aligned}$$

See Fig. 3(B) where for simplicity we use a, b, c, d in place of $w_j, \ldots, w_k$.

Case 2. q is a positive virtual vertex. In this case we know that $k = j+1$ and thus

$$F(m+1) = (f_1, \ldots, f_{j-1}, \underbrace{f'_{j+1}, f'_j}, f_{j+2}, \ldots, f_n).$$

Now associate the sequence of words

$$W(m+1) = (w_1, \ldots, w_{j-1}, \underbrace{w'_{j+1}, w'_j}, w_{j+2}, \ldots, w_n)$$

to the segments in $F(m+1)$ coordinatewise where

$$\begin{aligned} w'_j &= w_j \\ w'_{j+1} &= w_{j+1}^{w_j}. \end{aligned}$$

See Fig. 4(A) where for simplicity we use a and b in place of w_j and w_{j+1}.

Case 3. q is a negative virtual vertex. This case is identical to case 2 *except* that we have instead

$$\begin{aligned} w'_j &= w_j^{w_{j+1}^{-1}} \\ w'_{j+1} &= w_{j+1}. \end{aligned}$$

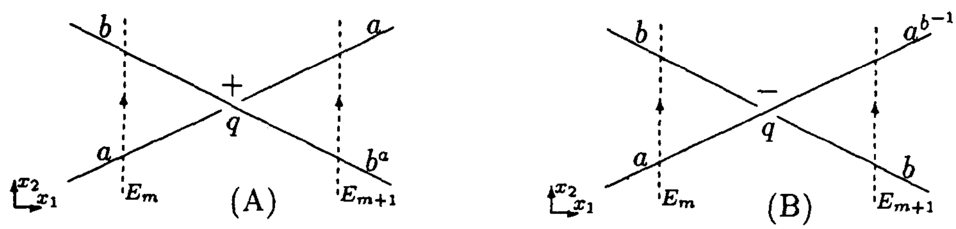

Fig. 4.

See Fig. 4(B) where again for simplicity we use a and b in place of w_j and w_{j+1}.

And thus iteratively we associate a word with each segment.

Now to each *actual* vertex $q = q_{m+1}$ we associate a set of relators R_q as follows. As before let $f_j, \ldots, f_k \in F(m)$ be those segments which have q as endpoint where $j < k$. Let $w_j, \ldots, w_k$ be the words associated to these segments. Set

$$R_q = [w_k, \ldots, w_j]$$

Note that the w_i are written in *reverse* order.

In Fig. 3(B) we have $R_q = [d, c, b, a]$.

We may now give our main result.

THEOREM 4.7. *Let $\mathscr{A}$ be an arrangement of complex hyperplanes in $\mathbb{C}^2$. Let M be the complement of $\mathscr{A}$. Let Γ^2 be an admissible 2-graph for $\mathscr{A}$. Then*

$$\pi_1(M) = \langle G \mid \bigcup_q R_q \rangle$$

where q ranges over the actual vertices of Γ^2.

Proof. We abuse notation slightly and set $K_i = K_{t_i}$. Let J be a negative integer of sufficiently large magnitude. Let B_0 be the contractible subspace of M given by

$$B_0 = \{m \in \mathbb{R}^4 : x_1(m) = t_0, y_1(m) = h(t_0), y_2(m) = J\}.$$

Note that $B_0 \subseteq K_0$. Identify the generating symbols $g_1, \ldots, g_n$ with the loops depicted in Fig. 5. Each of these loops runs around a point of the form $K_0 \cap H$ for $H \in \mathscr{A}$.

Now move these loops along the graph Γ in the direction of increasing x_1-coordinate. The local behavior at a multiple point (actual vertex) is the same as the complexified real case [3, 4, 5]. At the virtual vertices any *braiding* that occurs between multiple points is rectified as in [1].

No other generators or relators are needed by the work of Zariski and van Kampen [6, 7] on the complements of algebraic curves in $\mathbb{C}^2$. □

Remark 4.9. Consider again the pencil of four lines in Fig. 3(B)–viewed as a portion of an admissible 2-graph.

We observe that the generators (or words) a, b, c, d suffer the depicted conjugations as they pass through the actual vertex q. In particular the first line through q is unconjugated, the second conjugated by the first, the third by the second then the first, and so forth. Since $R_q = [d, c, b, a]$ and thus $d^{cba} = d$ we may simplify (remove) the conjugation of the last line through q. However no other simplifications are apparent.

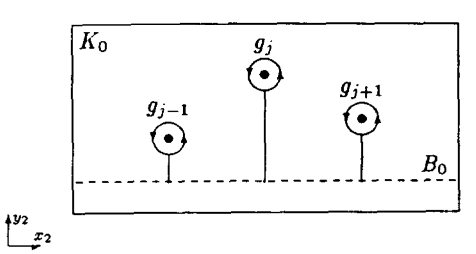

Fig. 5.

If we generalize this observation to pencils of arbitrarily many lines the association of words to segments in the algorithm may be somewhat simplified i.e. the *last* line through an actual vertex need not be conjugated.

Remark 4.10. Let $q = q_{m+1}$ be the first actual vertex which occurs in the list of vertices. Since no relators arise from the portion of the 2-graph to the left of the vertical line E_m one may use the symbols associated to the segments in $F(m)$ as a generating set. In other words we may discard those virtual vertices which come before q and start the algorithm at $F(m)$ in place of $F(0)$ and proceed to the right from there.

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