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Essential Kurepa trees versus essential Jech–Kunen trees

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Abstract

By an ω_1-tree we mean a tree of cardinality ω_1 and height ω_1. An ω_1-tree is called a Kurepa tree if all its levels are countable and it has more than ω_1 branches. An ω_1-tree is called a Jech–Kunen tree if it has κ branches for some κ strictly between ω_1 and 2^{ω_1}. A Kurepa tree is called an essential Kurepa tree if it contains no Jech–Kunen subtrees. A Jech–Kunen tree is called an essential Jech–Kunen tree if it contains no Kurepa subtrees. In this paper we prove that (1) it is consistent with CH and $2^{\omega_1} > \omega_2$ that there exist essential Kurepa trees and there are no essential Jech–Kunen trees, (2) it is consistent with CH and $2^{\omega_1} > \omega_2$ plus the existence of a Kurepa tree with 2^{ω_1} branches that there exist essential Jech–Kunen trees and there are no essential Kurepa trees. In the second result we require the existence of a Kurepa tree with 2^{ω_1} branches in order to avoid triviality.

Introduction

Our trees are always growing downward. We use T_α for the αth level of T and use $T\upharpoonright\alpha$ for $\bigcup_{\beta<\alpha} T_\beta$. For every $t \in T$ let $ht(t) = \alpha$ iff $t \in T_\alpha$. Let $ht(T)$, the height of T, be the least ordinal α such that $T_\alpha = \emptyset$. By a branch of T we mean a totally ordered subset of T which intersects every nonempty level of T. For any tree T let $m(T)$ be the set of all maximal nodes of T, i.e. $m(T) = \{t \in T\colon (\forall s \in T)(s \leqslant t \;\rightarrow\; s = t)\}$. All trees considered in this paper have cardinalities less than or equal to ω_1 so that, without loss of generality, we can assume all those trees are subtrees of $(\omega_1^{<\omega_1}, \supseteq)$, where $\omega^{<\omega_1}$ is the set of all functions from some countable ordinals to ω_1. Hence every tree here has a unique root $\emptyset$ and if $\{t_n\colon n \in \omega\} \subseteq T$ is a decreasing sequence of T, then $t = \bigcup_{n\in\omega} t_n$ is

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the only possible greatest lower bound of $\{t_n: n \in \omega\}$. We are also free to use either $\leqslant_T$ or $\supseteq$ for the order of a tree T, i.e. $s \leqslant_T t$ if and only if $s \supseteq t$.

By an ω_1-tree we mean tree of height ω_1 and cardinality ω_1. Notice that our definition of ω_1-tree is slightly different from the usual definition by not requiring every level to be countable. An ω_1-tree T is called a *Kurepa tree* if every level of T is countable and T has more than ω_1 branches. An ω_1-tree T is called a *Jech–Kunen tree* if T has κ branches for some κ strictly between ω_1 and 2^{ω_1}. We call a Kurepa tree *thick* if it has 2^{ω_1} branches. Obviously, a Kurepa non-Jech–Kunen tree must be thick, and a Jech–Kunen tree with every level countable is a Kurepa tree.

While Kurepa trees are better studied, Jech–Kunen trees are relatively less popular. It is Kunen [7, 8], who brought Jech–Kunen trees to people's attention by proving that: under CH and $2^{\omega_1} > \omega_2$, the existence of a compact Hausdorff space with weight ω_1 and cardinality strictly between ω_1 and 2^{ω_1} is equivalent to the existence of a Jech–Kunen tree. It is also easy to observe that: under CH and $2^{\omega_1} > \omega_2$, the existence of a (Dedekind) complete dense linear order with density ω_1 and cardinality strictly between ω_1 and 2^{ω_1} is also equivalent to the existence of a Jech–Kunen tree. Above results are interesting because those compact Hausdorff spaces and complete dense linear orders cannot exist if we replace ω_1 by ω, while the existence of a Jech–Kunen tree is undecidable. In this paper we would like to consider Jech–Kunen trees only under CH and $2^{\omega_1} > \omega_2$.

The consistency of a Jech–Kunen tree was given in [2], in which Jech constructed a generic Kurepa tree with less than 2^{ω_1} branches in a model of CH and $2^{\omega_1} > \omega_2$. By assuming the consistency of an inaccessible cardinal, Kunen proved the consistency of nonexistence of Jech–Kunen trees with CH and $2^{\omega_1} > \omega_2$ (see [7, Theorem 4.8]). In Kunen's model there are also no Kurepa trees. Kunen proved (see [7, Theorem 4.10]) also that the assumption of an inaccessible cardinal above is necessary. The differences between Kurepa trees and Jech–Kunen trees in terms of the existence have been studied in [4–6, 10, 11]. It was proved that the consistency of an inaccessible cardinal implies (1) it is consistent with CH and $2^{\omega_1} > \omega_2$ that there exist Kurepa trees but there are no Jech–Kunen trees [10], (2) it is consistent with CH and $2^{\omega_1} > \omega_2$ that there exist Jech–Kunen trees but there are no Kurepa trees [11].

What could we say without the presence of large cardinals? Instead of killing all Kurepa trees, which needs an inaccessible cardinal, while keeping some Jech–Kunen trees alive, or killing all Jech–Kunen trees, which needs again an inaccessible cardinal, while keeping some Kurepa trees alive, we can kill all Kurepa subtrees of a Jech–Kunen tree or kill all Jech–Kunen subtrees of a Kurepa tree without using large cardinals. Let's call a Kurepa tree T essential if T has no Jech–Kunen subtrees, and call a Jech–Kunen tree T essential if T has no Kurepa subtrees. In [4], the first author proved that it is consistent with CH and $2^{\omega_1} > \omega_2$, together with *Generalized Martin's Axiom* and the existence of a thick Kurepa tree, that no essential Kurepa trees and no essential Jech–Kunen trees. We required the presence of thick Kurepa trees in the model in order to avoid triviality. In [6], the first author proved that it is consistent with CH and $2^{\omega_1} > \omega_2$ that there exist both essential Kurepa trees and essential Jech–Kunen trees. A weak version of this result was proved in [4] with help of an inaccessible cardinal. This paper could be considered as a continuation of the research done in [4–6, 10, 11].

In Section 1, we prove that it is consistent with CH and $2^{\omega_1} > \omega_2$ that there exist essential Kurepa trees but there are no essential Jech–Kunen trees. In Section 2, we prove that it is consistent with CH and $2^{\omega_1} > \omega_2$ plus the existence of a thick Kurepa tree that there exist essential Jech–Kunen trees but there are no essential Kurepa trees. In Section 3, we simplify the proofs of two old results by using the forcing notion for producing a generic essential Jech–Kunen tree defined in Section 2.

We write $\dot{a}$ in the ground model for a name of an element a in the forcing extension. If a is in the ground model, we usually write a itself as a canonical name of a. The rest of the notation will be consistent with [9] or [3].

1. Yes essential Kurepa trees, no essential Jech–Kunen trees

In this section we are going to construct a model of CH and $2^{\omega_1} > \omega_2$ in which there exist essential Kurepa trees and there are no essential Jech–Kunen trees. Our strategy to do this can be described as follows: first, we take a model of CH and $2^{\omega_1} > \omega_2$ plus GMA (Generalized Martin's Axiom) as our ground model, so that in the ground model there are no essential Jech–Kunen trees, then, we add a generic Kurepa tree which has no Jech–Kunen subtrees. The hard part is to prove that the forcing adds no essential Jech–Kunen trees.

Let $\mathbb{P}$ be a poset. A subset S of $\mathbb{P}$ is called *linked* if any two elements in S are compatible in $\mathbb{P}$. A poset $\mathbb{P}$ is called *ω_1-linked* if $\mathbb{P}$ is the union of ω_1 linked subsets of $\mathbb{P}$. A subset S of $\mathbb{P}$ is called *centered* if every finite subset of S has a lower bound in $\mathbb{P}$. A poset $\mathbb{P}$ is called *countably compact* if every countable centered subset of $\mathbb{P}$ has a lower bound in $\mathbb{P}$. Now GMA is the following statement:

> Suppose $\mathbb{P}$ is an ω_1-linked and countably compact poset. For any $\kappa < 2^{\omega_1}$, if $\mathscr{D} = \{D_\alpha : \alpha < \kappa\}$ is a collection of κ dense subsets of $\mathbb{P}$, then there exists a filter G of $\mathbb{P}$ such that $G \cap D_\alpha \neq \emptyset$ for all $\alpha < \kappa$.

We choose the form of GMA from [1], where a model of CH and $2^{\omega_1} > \omega_2$ plus GMA can be found.

Let I be any index set. We write $\mathbb{K}_I$ for a poset such that p is a condition in $\mathbb{K}_I$ iff $p = (A_p, l_p)$ where A_p is a countable subtree of $(\omega_1^{<\omega}, \supseteq)$ of height $\alpha_p + 1$ and l_p is a function from a countable subset of I into $(A_p)_{\alpha_p}$, the top level of A_p. For any p, $q \in \mathbb{K}_I$, define $p \leqslant q$ iff

(1) $A_p \upharpoonright \alpha_p + 1 = A_q$,

(2) $dom(l_p) \supseteq dom(l_q)$,

(3) $(\forall \xi \in dom(l_q))(l_q(\xi) \subseteq l_p(\xi))$.

It is easy to see that $\mathbb{K}_I$ is countably closed (or ω_1-closed). If CH holds, then $\mathbb{K}_I$ is ω_1-linked. Let M be a model of CH and $\mathbb{K}_I \in M$. Suppose that G is a $\mathbb{K}_I$-generic filter over M and let $T_G = \bigcup_{p \in G} A_p$. Then in $M[G]$, the tree T_G is an ω_1-tree with every level countable and T_G has exactly $|I|$ branches. Furthermore, if for every $i \in I$ let

$$B(i) = \bigcup \{l_p(i)\colon p \in G \text{ and } i \in dom(l_p)\},$$

then $B(i) \neq B(i')$ for any $i, i' \in I$ and $i \neq i'$, and $\{B(i)\colon i \in I\}$ is the set of all branches of T_G in $M[G]$. Hence if $|I| > \omega_1$, then T_G will be Kurepa tree with $|I|$ branches in $M[G]$. $\mathbb{K}_I$ is the poset used in [2] for creating a generic Kurepa tree. All those facts above can also be found in [2] or [12].

For convenience we sometimes view $\mathbb{K}_I$ as an iterated forcing notion

$$\mathbb{K}_{I'} * Fn(I \setminus I', T_{\dot{G}_{I'}}, \omega_1),$$

for any $I' \subseteq I$, where $G_{I'}$ is a $\mathbb{K}_{I'}$-generic filter over the ground model and $Fn(I \setminus I', T_{G_{I'}}, \omega_1)$, *in* $M[G_{I'}]$, is the set of all functions from some countable subset of $I \setminus I'$ to $T_{G_{I'}}$ with the order defined by letting $p \leqslant q$ iff $dom(q) \subseteq dom(p)$ and for any $i \in dom(q)$, $p(i) \leqslant q(i)$. The poset $Fn(J, T_G, \omega_1)$ is in fact the countable support product of $|J|$-copies of T_G. We say two posets $\mathbb{P}$ and $\mathbb{Q}$ are forcing equivalent if there is a poset $\mathbb{R}$ such that $\mathbb{R}$ can be densely embedded into both $\mathbb{P}$ and $\mathbb{Q}$. The posets $\mathbb{K}_I$ and $\mathbb{K}_{I'} * Fn(I \setminus I', T_{\dot{G}_{I'}}, \omega_1)$ are forcing equivalent because the map

$$F\colon \mathbb{K}_I \mapsto \mathbb{K}_{I'} * Fn(I \setminus I', T_{\dot{G}_{I'}}, \omega_1)$$

such that for every $p \in \mathbb{K}_I$,

$$F(p) = ((A_p, l_p \upharpoonright I'), l_p \upharpoonright I \setminus I')$$

is a dense embedding.

Lemma 1 (Kunen). *Let M be a model of* CH. *Suppose that $\lambda > \omega_2$ is a cardinal in M and $\mathbb{K}_\lambda \in M$. Suppose G_λ is a $\mathbb{K}_\lambda$-generic filter over M and $T_{G_\lambda} = \bigcup_{p \in G_\lambda} A_p$. Then in $M[G_\lambda]$ the tree T_{G_λ} is a Kurepa tree with λ branches and T_{G_λ} has no subtrees with κ branches for any κ strictly between ω_1 and λ.*

Proof. Assume that T is a subtree of T_{G_λ} with more than ω_1 branches in $M[G_\lambda]$. We want to show that T has λ branches in $M[G_\lambda]$. Since $|T| = \omega_1$ and $\mathbb{K}_\lambda$ has ω_2-c.c., then there exists a subset $I \subseteq \lambda$ in M with cardinality $\leqslant \omega_1$ such that $T \in M[G_I]$, where

$$G_I = \{p \in G\colon dom(l_p) \subseteq I\}.$$

Notice that $T_{G_\lambda} = T_{G_I}$ (in fact $T_G = T_{G_\emptyset}$). Since in $M[G_I]$ the tree T_{G_I} has only $|I|$ branches, then the tree T can have at most ω_1 branches in $M[G_I]$. Let B be a branch of T in $M[G_\lambda]$ which is not in $M[G_I]$. Since $|B| = \omega_1$, there exists a subset J of $\lambda \setminus I$ with cardinality $\leqslant \omega_1$ such that $B \in M[G_I][H_J]$, where H_J is a $Fn(J, T_{G_I}, \omega_1)$-generic filter over $M[G_I]$. Now $\lambda \setminus I$ can be partitioned into λ-many subsets of cardinality ω_1 and for every subset $J' \subseteq \lambda \setminus (I \cup J)$ of cardinality ω_1 the poset $\mathbb{P}_J = Fn(J, T_{G_I}, \omega_1)$ is isomorphic to the poset $\mathbb{P}_{J'} = Fn(J', T_{G_I}, \omega_1)$ through an obvious isomorphism π induced by a bijection between J and J'. Let $\dot{B}$ be a $\mathbb{P}_J$-name for B. Then $\pi_*(\dot{B})$ is a $\mathbb{P}_{J'}$-name for a new branch of T. Forcing with $\mathbb{P}_J \times \mathbb{P}_{J'}$ will create two different branches $\dot{B}_{H_J}$ and $(\pi_*(\dot{B}))_{H_{J'}}$. *Hence forcing with* $Fn(\lambda \setminus I, T_{G_I}, \omega_1)$ will produce at least λ new branches of T. □

Next lemma is a simple fact which will be used later.

Lemma 2. *Suppose* $\mathbb{P}$ *is an* ω_1*-closed poset of cardinality* ω_1 *(hence* CH *must hold). Then the tree* $(\omega_1^{<\omega_1}, \supseteq)$ *can be densely embedded into* $\mathbb{P}$.

Proof Folklore. □

Lemma 3. *Let* M *be a model of* CH *and* $2^{\omega_1} > \omega_2$ *plus* GMA *and let* $\mathbb{P} = (\omega_1^{<\omega_1}, \supseteq) \in M$. *Suppose* G *is a* $\mathbb{P}$*-generic filter over* M. *Then in* $M[G]$ *every Jech–Kunen tree has a Kurepa subtree.*

Proof. Let T be a Jech–Kunen tree $M[G]$ with δ branches for $\omega_1 < \delta < \lambda = 2^{\omega_1}$. Without loss of generality we can assume that there is a regular cardinal κ such that $\omega_1 < \kappa \leqslant \delta$ and for every $t \in T$ there are at least κ branches of T passing through t in $M[G]$. Again in $M[G]$ let $f: \kappa \mapsto \mathscr{B}(T)$ be a one to one function such that for every $t \in T$ and for every $\alpha < \kappa$ there exists an $\beta \in \kappa \setminus \alpha$ such that $t \in f(\beta)$. Without loss of generality let us assume that

$$1_{\mathbb{P}} \Vdash (\dot{T} \text{ is a Jech–Kunen tree and } \dot{f}: \kappa \mapsto \mathscr{B}(\dot{T})$$

$$\text{is a one to one function such that} \quad (\forall t \in \dot{T})(\forall \alpha \in \kappa)(\exists \beta \in \kappa \setminus \alpha)(t \in \dot{f}(\beta))).$$

We want now to construct a poset $\mathbb{R}$ in M such that a filter H of $\mathbb{R}$ obtained by applying GMA in M will give us a $\mathbb{P}$-name for a Kurepa subtree of T in $M[G]$.

Let r be a condition of $\mathbb{R}$ iff $r = (I_r, \mathbb{P}_r, \mathscr{A}_r, \mathscr{S}_r)$ where I_r is a countable subtree of $(\omega_1^{<\omega_1}, \supseteq)$, $\mathbb{P}_r = \langle p_t^r: t \in I_r \rangle$, $\mathscr{A}_r = \langle A_t^r: t \in I_r \rangle$ and $\mathscr{S}_r = \langle S_t^r: t \in I_r \rangle$ such that

(1) $\mathbb{P}_r \subseteq \mathbb{P}$ and for every $t \in I_r$ the element A_t^r is a nonempty countable subtree of $(\omega_1^{<\omega_1}, \supseteq)$ of height $\alpha_t^r + 1$ (we will use some A_t^r's to generate a Kurepa subtree of T) and S_t^r is a nonempty countable subset of κ,

(2) $(\forall s, t \in I_r)(s \subseteq t \leftrightarrow p_t^r \leqslant p_s^r)$ (this implies that s and t are incompatible iff p_s^r and p_t^r are incompatible for all $s, t \in I_r$ because $\mathbb{P}$ is a tree),

(3) $(\forall s, t \in I_r)(s \subseteq t \rightarrow A_t^r \restriction ht(A_s^r) = A_s^r)$,

(4) $(\forall s, t \in I_r)(s \subseteq t \rightarrow S_s^r \subseteq S_t^r)$,

(5) $(\forall t \in I_r)(p_t^r \Vdash A_t^r \subseteq \dot{T})$,

(6) $(\forall t \in I_r)(\forall \alpha \in S_t^r)(\exists a \in (A_t^r)_{\alpha_t^r})(p_t^r \Vdash a \in \dot{f}(\alpha))$.

The order of $\mathbb{R}$: for any $r, r' \in \mathbb{R}$, let $r \leqslant r'$ iff $I_{r'} \subseteq I_r$ and for every $t \in I_{r'}$

$$p_t^{r'} = p_t^r, \; A_t^{r'} = A_t^r \quad \text{and } S_t^{r'} \subseteq S_t^r.$$

Claim 3.1. *The poset* $\mathbb{R}$ *is* ω_1*-linked.*

Proof of Claim 3.1. Let $r, r' \in \mathbb{R}$ such that $I_r = I_{r'}$, $\mathbb{P}_r = \mathbb{P}_{r'}$ and $\mathscr{A}_r = \mathscr{A}_{r'}$. Then the condition $r'' \in \mathbb{R}$ such that

$$I_{r''} = I_r, \qquad \mathbb{P}_{r''} = \mathbb{P}_r, \qquad \mathscr{A}_{r''} = \mathscr{A}_r \quad \text{and} \quad \mathscr{S}_{r''} = \langle S_t^r \cup S_t^{r'} : t \in I_{r''} \rangle$$

is a common lower bound of both r and r'. Since there are only ω_1 different $\langle I_r, \mathbb{P}_r, \mathscr{A}_r \rangle$'s and for each fixed $\langle I_{r_0}, \mathbb{P}_{r_0}, \mathscr{A}_{r_0} \rangle$ the set

$$\{ r \in \mathbb{R} : \langle I_r, \mathbb{P}_r, \mathscr{A}_r \rangle = \langle I_{r_0}, \mathbb{P}_{r_0}, \mathscr{A}_{r_0} \rangle \}$$

is linked, then $\mathbb{R}$ is the union of ω_1 linked subsets of $\mathbb{R}$. □

Claim 3.2. *The poset* $\mathbb{R}$ *is countably compact.*

Proof of Claim 3.2. Suppose that $\mathbb{R}'$ is a countable centered subset of $\mathbb{R}$. Notice that for any finite $\mathbb{R}'_0 \subseteq \mathbb{R}'$ and for any $t \in \bigcap \{ I_r : r \in \mathbb{R}'_0 \}$ all p_t^r's are same and all A_t^r are same for $r \in \mathbb{R}'_0$ because $\mathbb{R}'_0$ has a common lower bound in $\mathbb{R}$. We now want to construct a condition $\bar{r} \in \mathbb{R}$ such that $\bar{r}$ is a common lower bound of $\mathbb{R}'$. Let

(1) $I_{\bar{r}} = \bigcup_{r \in \mathbb{R}'} I_r$,
(2) $\mathbb{P}_{\bar{r}} = \langle p_t^{\bar{r}} : t \in I_{\bar{r}} \rangle$ where $p_t^{\bar{r}} = p_t^r$ for some $r \in \mathbb{R}'$ such that $t \in I_r$,
(3) $\mathscr{A}_{\bar{r}} = \langle A_t^{\bar{r}} : t \in I_{\bar{r}} \rangle$ where $A_t^{\bar{r}} = A_t^r$ for some $r \in \mathbb{R}'$ such that $t \in I_r$,
(4) $\mathscr{S}_{\bar{r}} = \langle S_t^{\bar{r}} : t \in I_{\bar{r}} \rangle$ where $S_t^{\bar{r}} = \bigcup_{s \subseteq t} S_s$ and $S_s = \bigcup \{ S_s^r : (r \in \mathbb{R}')(s \in I_r) \}$.

Notice that form the argument above all $p_t^{\bar{r}}$'s, $A_t^{\bar{r}}$'s and $S_t^{\bar{r}}$'s are well-defined. We need to show $\bar{r} \in \mathbb{R}$. It is obvious that $\bar{r}$ is a common lower bound of all elements in $\mathbb{R}'$ if $\bar{r} \in \mathbb{R}$.

It is easy to see that $\bar{r}$ satisfies (1), (2), (3), (4) and (5) in the definition of a condition in $\mathbb{R}$. Let's check (6).

Suppose $t \in I_{\bar{r}}$ and $\alpha \in S_t^{\bar{r}}$. We want to show that there exists an $a \in (A_t^{\bar{r}})_{\alpha_t^{\bar{r}}}$ *such that* $p_t^{\bar{r}} \Vdash a \in \dot{f}(\alpha)$. Let $r \in \mathbb{R}'$ be such that $t \in I_r$, let $r' \in \mathbb{R}'$ and $s \in I_{r'}$ be such that $s \subseteq t$ and $\alpha \in S_s^{r'}$. Since r and r' are compatible, then there exists an $r'' \in \mathbb{R}$ such that $r'' \leqslant r$ and $r'' \leqslant r'$. By the facts that

$$p_t^{\bar{r}} = p_t^r = p_t^{r''}, \qquad A_t^{\bar{r}} = A_t^r = A_t^{r''}, \qquad S_s^{r'} \subseteq S_s^{r''} \subseteq S_t^{r''}$$

and $r'' \in \mathbb{R}$ we have now that there exists an $a \in (A_t^{\bar{r}})_{\alpha_t^{\bar{r}}}$ *such that* $p_t^{\bar{r}} \Vdash a \in \dot{f}(\alpha)$. □

Next we are going to apply GMA in M to the poset $\mathbb{R}$ to construct a $\mathbb{P}$-name for a Kurepa subtree in $M[G]$.

For each $t \in \omega_1^{<\omega_1}$ define

$$D_t = \{ r \in \mathbb{R} : t \in I_r \}.$$

For each $p \in \mathbb{P}$ define

$$E_p = \{ r \in \mathbb{R} : (\exists t \in I_r)(p_t^r \leqslant p) \}.$$

For each $\alpha < \omega_1$ define

$$F_\alpha = \{ r \in \mathbb{R} : (\forall s \in I_r)(\exists t \in I_r)(s \subseteq t \text{ and } ht(A_t^r) > \alpha) \}.$$

For each $\alpha < \kappa$ define

$$O_\alpha = \{r \in \mathbb{R}\colon (\forall s \in I_r)(\exists t \in I_r)(s \subseteq t \text{ and } [\alpha, \kappa) \cap S_t^r \neq \emptyset)\}.$$

Claim 3.3. *All those D_t, E_p, F_α and O_α's are dense in $\mathbb{R}$.*

Proof of Claim 3.3. Let r_0 be an arbitrary element in $\mathbb{R}$.

We show first that for every $t \in \omega_1^{<\omega_1}$ the set D_t is dense in $\mathbb{R}$, i.e. there is an $r \in D_t$ such that $r \leqslant r_0$. It's done if $t \in I_{r_0}$. Let's assume that $t \notin I_{r_0}$. Let

$$t_0 = \bigcup \{s \in I_{r_0}\colon s \subseteq t\}.$$

Case 1: $t_0 \in I_{r_0}$. Find a sequence $\{p_s\colon t_0 \subseteq s \subseteq t\}$ in $\mathbb{P}$ such that $p_{t_0} = p_{t_0}^{r_0}$ and

$$(\forall s, s')(t_0 \subseteq s \subseteq s' \subseteq t \leftrightarrow p_{s'} \leqslant p_s).$$

The sequence $\{p_s\colon t_0 \subseteq s \subseteq t\}$ exists because $\mathbb{P}$ is ω_1-closed. Let

$$I_r = I_{r_0} \cup \{s\colon t_0 \subseteq s \subseteq t\}.$$

For any $s \in I_r$, if $s \in I_{r_0}$, then let

$$p_s^r = p_s^{r_0}, \qquad A_s^r = A_s^{r_0} \quad \text{and} \quad S_s^r = S_s^{r_0}.$$

Otherwise let

$$p_s^r = p_s, \qquad A_s^r = A_{t_0}^{r_0} \quad \text{and} \quad S_s^r = S_{t_0}^{r_0}.$$

It is easy to see that $r \in D_t$ and $r \leqslant r_0$.

Case 2: $t_0 \notin I_{r_0}$, i.e. I_{r_0} has no least element which is above t.

Let

$$I_r = I_{r_0} \cup \{s\colon t_0 \subseteq s \subseteq t\}.$$

Again by ω_1-closedness we can find

$$\{p_s\colon t_0 \subseteq s \subseteq t\} \subseteq \mathbb{P}$$

such that p_{t_0} is a lower bound of

$$\{p_s^{r_0}\colon s \subseteq t_0 \text{ and } s \in I_{r_0}\}$$

and

$$(\forall s, s')(t_0 \subseteq s \subseteq s' \subseteq t \leftrightarrow p_{s'} \leqslant p_s).$$

Let

$$A'_{t_0} = \bigcup \{A_s^{r_0}\colon s \in I_{r_0} \text{ and } s \subseteq t_0\}$$

and let

$$S_{t_0} = \bigcup \{S_s^{r_0}\colon s \in I_{r_0} \text{ and } s \subseteq t_0\}.$$

If the height of A'_{t_0} is a successor ordinal, then let $A_{t_0} = A'_{t_0}$. If the height of A'_{t_0} is a limit ordinal, then we have to add one more level to A'_{t_0}. For any $\beta \in S_{t_0}$ let $s' \subseteq t_0$ and $s' \in I_{r_0}$ be such that $\beta \in S^{r_0}_{s'}$. Then for any $s \in I_{r_0}$ such that $s' \subseteq s \subseteq t_0$ there exists an $a_{s,\beta} \in (A^{r_0}_s)_{\alpha^{r_0}_s}$ such that $p^{r_0}_s \Vdash a_{s,\beta} \in \dot{f}(\beta)$. Now let

$$a_\beta = \bigcup \{a_{s,\beta}: s' \subseteq s \subseteq t_0 \text{ and } s \in I_{r_0}\}$$

and let

$$A_{t_0} = A'_{t_0} \cup \{a_\beta: \beta \in S_{t_0}\}.$$

It is easy to see that

(1) the height of A_{t_0} is a successor ordinal.

(2) for every $s \subseteq t_0$ the tree A_{t_0} is an end-extension of $A^{r_0}_s$, i.e.

$$A_{t_0} \restriction ht(A^{r_0}_s) = A^{r_0}_s,$$

(3) for every $\beta \in S_{t_0}$ there exists an a_β in the top level of A_{t_0} such that $p_{t_0} \Vdash a_\beta \in \dot{f}(\beta)$.

Now for every $s \in I_r$, if $r \in I_{r_0}$, then let

$$p^r_s = p^{r_0}_s, \qquad A^r_s = A^{r_0}_s \quad \text{and} \quad S^r_s = S^{r_0}_s.$$

Otherwise let

$$p^r_s = p_s, \qquad A^r_s = A_{t_0} \quad \text{and} \quad S^r_s = S_{t_0}.$$

It is easy to see that $r \in D_t$ and $r \leqslant r_0$.

We show now that for every $p \in \mathbb{P}$ the set E_p is dense in $\mathbb{R}$. We want to find an $r \in E_p$ such that $r \leqslant r_0$. If there exists an $t \in I_{r_0}$ such that $p^{r_0}_t \leqslant p$, then $r_0 \in E_p$. Let us assume that for every $t \in I_{r_0}$ $p^{r_0}_t \nleqslant p$. Let

$$t_0 = \bigcup \{t \in I_{r_0}: p \leqslant p^{r_0}_t\}.$$

Case 1: $t_0 \in I_{r_0}$. Let $t' = t_0{}^\frown\langle 0\rangle$, i.e. t' is a successor of t_0. It is clear that $t' \notin I_{r_0}$. Let $I_r = I_{r_0} \cup \{t'\}$. For every $t \in I_r$, if $t = t'$, then let

$$p^r_t = p, \qquad A^r_t = A^{r_0}_{t_0} \quad \text{and} \quad S^r_t = S^{r_0}_{t_0}.$$

Otherwise let

$$p^r_t = p^{r_0}_t, \qquad A^r_t = A^{r_0}_t \quad \text{and} \quad S^r_t = S^{r_0}_t.$$

Then we have $r \in E_p$ and $r \leqslant r_0$.

Case 2: $t_0 \notin I_{r_0}$. Let $I_r = I_{r_0} \cup \{t_0\}$. We construct S_{t_0}, A'_{t_0} and then A_{t_0} exactly same as we did in the proof of Case 2 about the denseness of the set D_t. For every $t \in I_r$, if $t = t_0$, then let

$$p^r_t = p, \qquad A^r_t = A_{t_0} \quad \text{and} \quad S^r_t = S_{t_0}.$$

Otherwise let

$$p^r_t = p^{r_0}_t, \qquad A^r_t = A^{r_0}_t \quad \text{and} \quad S^r_t = S^{r_0}_t.$$

Now $r \in E_p$ and $r \leqslant r_0$. Notice also that E_p is open, i.e.

$$(\forall p', p'' \in \mathbb{P})(p' \leqslant p'' \wedge p'' \in E_p \rightarrow p' \in E_p).$$

We show next that for every $\alpha \in \omega_1$ the set F_α is dense in $\mathbb{R}$. We need to find an $r \in F_\alpha$ such that $r \leqslant r_0$.

Let $I_r \supseteq I_{r_0}$ be such that I_r is a countable subtree of $\omega_1^{<\omega_1}$, $I_r \setminus I_{r_0}$ is an antichain and for every $s \in I_{r_0}$ there is a $t \in I_r \setminus I_{r_0}$ such that $s \subseteq t$. For every $t \in I_r \setminus I_{r_0}$ let $p_t \in \mathbb{P}$ be such that $p_t \leqslant p_s^{r_0}$ for every $s \in I_{r_0}$ and $s \subseteq t$, let

$$S_t^r = \bigcup \{S_s^{r_0}: s \in I_{r_0} \text{ and } s \subseteq t\}$$

and let

$$A_t' = \bigcup \{A_s^{r_0}: s \in I_{r_0} \text{ and } s \subseteq t\}.$$

If $ht(A_t')$ is a successor ordinal, then let $A_t = A_t'$. Otherwise let

$$A_t = A_t' \cup \{a_\beta: \beta \in S_t^r\}$$

where

$$a_\beta = \bigcup \{a \in A_t': p_t \Vdash a \in \dot{f}(\beta)\}.$$

Since S_t^r is countable and $\mathbb{P}$ is ω_1-closed, then there exists a $p_t^r \leqslant p_t$ such that for every $\beta \in S_t^r$ there exists an $a \in \omega_1^\alpha$ such that $p_t^r \Vdash a \in \dot{f}(\beta)$. Let

$$A_t^r = A_t \cup \{a \in \omega_1^{\leqslant \alpha}: (\exists \beta \in S_t^r)(p_t^r \Vdash a \in \dot{f}(\beta))\}.$$

Then $ht(A_t^r) \geqslant \alpha$ is a successor ordinal and for every $\beta \in S_t^r$ there exists an a in the top level of A_t^r such that $p_t^r \Vdash a \in \dot{f}(\beta)$. For every $t \in I_r \setminus I_{r_0}$ we have already defined p_t^r, A_t^r and S_t^r. If $t \in I_{r_0}$, then let

$$p_t^r = p_t^{r_0}, \qquad A_t^r = A_t^{r_0} \quad \text{and} \quad S_t^r = S_t^{r_0}.$$

Hence $r \in F_\alpha$ and $r \leqslant r_0$.

We show next the O_α for every $\alpha < \kappa$ is dense in $\mathbb{R}$, i.e, finding an $r \in O_\alpha$ such that $r \leqslant r_0$.

By imitating the proof of the denseness of F_α we can find an $r' \leqslant r_0$ such that $I_{r'} \setminus I_{r_0}$ is an antichain and for every $s \in I_{r'}$ there exists an $t \in I_{r'} \setminus I_{r_0}$ such that $s \subseteq t$. For every $t \in I_{r'} \setminus I_{r_0}$ fix a $\bar{t}$ which is an successor of t (for example $\bar{t} = t^\frown\langle 0 \rangle$). Let

$$I_r = I_{r'} \cup \{\bar{t}: t \in I_{r'} \setminus I_{r_0}\}.$$

For every $t \in I_{r'}$ let

$$p_t^r = p_t^{r'}, \qquad A_t^r = A_t^{r'} \quad \text{and} \quad S_t^r = S_t^{r'}.$$

For every $\bar{t}$ with $t \in I_{r'} \setminus I_{r_0}$ we want to construct $p_{\bar{t}}^r$, $A_{\bar{t}}^r$ and $S_{\bar{t}}^r$. If there is a $\beta \in S_t^{r'}$ which is greater than α, then let $p_{\bar{t}}^r$ be any proper extension of p_t^r, let $A_{\bar{t}}^r = A_t^{r'}$ and let $S_{\bar{t}}^r = S_t^{r'}$. Otherwise, first, pick an a in the top level of $A_t^{r'}$, then choose a $\beta \in \kappa \setminus \alpha$ and a $p \leqslant p_t^{r'}$ such that $p \Vdash a \in \dot{f}(\beta)$. This can be done because

$$1_{\mathbb{P}} \Vdash (\forall t \in \dot{T})(\forall \alpha \in \kappa)(\exists \beta \in \kappa \setminus \alpha)(t \in \dot{f}(\beta))$$

is true in M. Now let

$$p_t^r = p, \quad A_t^r = A_t^{r'} \quad \text{and} \quad S_t^r = S_t^{r'} \cup \{\beta\}.$$

It is easy to see that $r \in O_\alpha$ and $r \leqslant r_0$. $\square$

By applying GMA in M we can find an $\mathbb{R}$-filter H such that $H \cap D_t \neq \emptyset$ $H \cap F_\alpha \neq \emptyset$ and $H \cap E_p \cap O_{\alpha'} \neq \emptyset$ for each $t \in \omega_1^{<\omega_1}$, each $\alpha \in \omega_1$, each $p \in \mathbb{P}$ and each $\alpha' \in \kappa$.

Since D_t is dense for every $t \in \omega_1^{<\omega_1}$, then

$$I_H = \bigcup \{I_r: r \in H\} = \omega_1^{<\omega_1}.$$

Let

$$\mathbb{P}_H = \bigcup \{\mathbb{P}_r: r \in H\}$$

and let

$$\mathscr{A}_H = \bigcup \{\mathscr{A}_r: r \in H\}.$$

Notice that for any $r, r' \in H$ and for any $t \in I_r \cap I_{r'}$ we have $p_t^r = p_t^{r'}$ and $A_t^r = A_t^{r'}$ because r and r' are compatible. So now for every $t \in I_H$ we can define $p_t = p_t^r$ for some $r \in H$ and define $A_t = A_t^r$ for some $r \in H$. It is clear that the map $t \mapsto p_t$ is an isomorphism between I_H and $\mathbb{P}_H$, i.e. for any $s, t \in I_H$ we have $s \subseteq t$ iff $p_t \leqslant p_s$. It is also clear that the map $t \mapsto A_t$ is a homomorphism from I_H to $\mathscr{A}_H$, i.e. for any $s, t \in I_H$ we have $s \subseteq t$ implies $A_t \upharpoonright ht(A_s) = A_s$.

Claim 3.4. *For each $t \in I_H$ the set $\{p_{t^\frown\langle\gamma\rangle}: \gamma \in \omega_1\}$ is a maximal antichain below p_t in $\mathbb{P}$.*

Proof of Claim 3.4. Let γ and γ' be two ordinals in ω_1. Since $I_H = \omega_1^{<\omega_1}$ and H is a filter, there exists an $r \in H$ such that $t^\frown\langle\gamma\rangle, t^\frown\langle\gamma'\rangle \in I_r$. Hence $p^r_{t^\frown\langle\gamma\rangle}$ and $p^r_{t^\frown\langle\gamma'\rangle}$ are incompatible. So $\{p_{t^\frown\langle\gamma\rangle}: \gamma \in \omega_1\}$ is an antichain.

Suppose that $p \in \mathbb{P}$ and $p \leqslant p_t$ such that p is incompatible with any of $p_{t^\frown\langle\gamma\rangle}$'s. Let $r \in H \cap E_p$. Then there is an $s \in I_r$ such that $p_s = p_s^r \leqslant p$. Since $p_s \in \mathbb{P}_H$, then $p_s < p_t$ implies $t \subsetneq s$. Hence there exists an $\gamma \in \omega_1$ such that $t^\frown\langle\gamma\rangle \subseteq s$. This means that $p_s < p_{t^\frown\langle\gamma\rangle}$, i.e. p and $p_{t^\frown\langle\gamma\rangle}$ are compatible, a contradiction. $\square$

We now work in $M[G]$. Since G is a $\mathbb{P}$-generic filter over M, then $\mathbb{P}_H \cap G$ is a linearly ordered subset of $\mathbb{P}_H$. Let $T_G = \bigcup \{A_t: p_t \in G\}$.

Claim 3.5. *T_G is a Kurepa subtree of T in $M[G]$.*

Proof of Claim 3.5. Since for every $p_t \in G$ we have $p_t \Vdash A_t \subseteq \dot{T}$, it is clear that $T_G \subseteq T$ in $M[G]$. For any $p_s, p_t \in G$ we have $p_t \leqslant p_s$ implies $s \subseteq t$ which implies $A_t \upharpoonright ht(A_s) = A_s$. Hence T_G is an end-extension of A_t for every $p_t \in G$. This implies that every level of T_G is a level of some A_t, hence is countable.

We want to show now that T_G has at least κ branches. Suppose $|\mathscr{B}(T_G)| < \kappa$. Then there exists an $\alpha \in \kappa$ such that for every $\beta \in \kappa \setminus \alpha$ the function value $f(\beta)$ is not a branch of T_G. So there is a $p \in \mathbb{P}_H$ and there is an $\alpha \in \kappa$ such that

$$p \Vdash (\forall \beta \in \kappa \backslash \alpha)(\dot{f}(\beta) \text{ is not a branch of } T_{\dot{G}}).$$

On the other hand, since $H \cap E_p \cap O_\alpha \neq \emptyset$, then there exists an $r \in H \cap O_\alpha \cap E_p$. In M let $s \in I_r$ be such that $p_s \leqslant p$ and there is a $\beta \in S^r_s$ such that $\beta > \alpha$. Then for every $t \in I_H$, $s \subseteq t$, there is an $t' \in I_H$, $t \subseteq t'$, such that

$$p_{t'} \Vdash a \in \dot{f}(\beta)$$

for some $a \in (A_{t'})_{ht(A_{t'})}$. This shows that

$$p_s \Vdash \dot{f}(\beta) \text{ is a branch of } T_{\dot{G}},$$

which contradicts $p_s \leqslant p$ and

$$p \Vdash (\forall \beta \in \kappa \backslash \alpha)(\dot{f}(\beta) \text{ is not a branch of } T_{\dot{G}}).$$

Hence T_G has at least κ branches in $M[G]$. □

Now we conclude that $M[G] \models T$ has a Kurepa subtree T_G, which proves Lemma 3. □

Theorem 4. *It is consistent with* CH *and* $2^{\omega_1} > \omega_2$ *that there exist essential Kurepa trees and there are no essential Jech–Kunen trees.*

Proof. Let M be a model of CH and $2^{\omega_1} = \lambda > \omega_2$ plus GMA. Let $\mathbb{K}_\lambda \in M$. Suppose G_λ is a $\mathbb{K}_\lambda$-generic filter over M. We are going to show that $M[G_\lambda]$ is a model of CH and $2^{\omega_1} > \omega_2$ in which there exist essential Kurepa trees and there are no essential Jech–Kunen trees.

It is easy to see that $M[G_\lambda]$ satisfies CH and $2^{\omega_1} > \omega_2$. Lemma 1 implies that there exist essential Kurepa trees. We need only to show that in $M[G_\lambda]$ there are no essential Jech–Kunen trees.

Assume T is a Jech–Kunen tree in $M[G_\lambda]$. We need to show that T has a Kurepa subtree $M[G_\lambda]$. Since $|T| = \omega_1$, then there is an $I \subseteq \lambda$ of cardinality ω_1 in M such that $T \in M[G_I]$, where

$$G_I = \{p \in G_\lambda \colon dom(l_p) \subseteq I\}.$$

We claim that

$$\mathscr{B}(T) \cap M[G_\lambda] \subseteq M[G_I].$$

If the claim is true, then T is a Jech–Kunen tree in $M[G_I]$. Suppose that $B \in \mathscr{B}(T) \cap (M[G_\lambda] \backslash M[G_I])$. Then there is a $J \subseteq \lambda \backslash I$ such that $B \in M[G_I][H_J]$ where H_J is a $Fn(J, T_{G_I}, \omega_1)$-generic filter over $M[G_I]$. Let $\dot{B}$ be a $Fn(J, T_{G_I}, \omega_1)$-name for B. For any $J' \subseteq \lambda \backslash (I \cup J)$ such that $|J'| = |J|$ there is an isomorphism π from $Fn(J, T_{G_I}, \omega_1)$ to $Fn(J', T_{G_I}, \omega_1)$ induced by a bijection between J and J'. Since in

$M[G_\lambda]$, the branches $(\dot{B})_{H_{I'}}$ *and* $(\pi_*(\dot{B}))_{H_{I'}}$ are different, then T has at least λ branches. This contradicts that T is a Jech–Kunen tree. Let T have δ branches in $M[G_I]$. Since $\mathbb{K}_I$ has cardinality ω_1 and is ω_1-closed, then it contains a dense subset which is isomorphic to $\mathbb{P} = (\omega_1^{<\omega_1}, \supseteq)$ in M. Hence there is a $\mathbb{P}$-generic filter G over M such that $M[G] = M[G_I]$. By Lemma 3, the tree T has a Kurepa subtree in $M[G]$. Obviously, the Kurepa subtree is still a Kurepa subtree in $M[G_\lambda]$, so T is not an essential Jech–Kunen tree in $M[G_\lambda]$. □

2. Yes essential Jech–Kunen trees, no essential Kurepa trees

In this section we will construct a model of CH and $2^{\omega_1} > \omega_2$ plus the existence of a thick Kurepa tree, in which there are essential Jech–Kunen trees and there are no essential Kurepa trees. The arguments in this section are a sort of "symmetric" to the arguments in the last section.

We first take a model M of CH and $2^{\omega_1} = \lambda > \omega_2$ plus a thick Kurepa tree, where $\lambda^{<\lambda} = \lambda$ in M, as our ground model. We then extend M to a model $M[G]$ of CH and $2^{\omega_1} = \lambda > \omega_2$ plus GMA by a λ-stage iterated forcing (see [1] for the model and forcing). It has been proved in [4] that in $M[G]$ there are neither essential Jech–Kunen trees nor essential Kurepa trees. Instead of taking a model of GMA as our ground model as we did in Section 1, we consider this λ-stage iterated forcing as a part of our construction because it will be needed later (see also [4], Theorem5]). Next we force with an ω_1-closed poset $\mathbb{J}_{S,\kappa}$ in $M[G]$ to create a generic essential Jech–Kunen tree, where S is a stationary–costationary subset of ω_1. Again, the hard part is to prove that forcing with $\mathbb{J}_{S,\kappa}$ over $M[G]$ will not create any essential Kurepa trees.

Recall that for T, a tree, $m(T)$ denotes the set

$$\{t \in T\colon (\forall s \in T)(s \leqslant_T t \ \rightarrow\ s = t)\}.$$

Let I be any index set and let S be a subset of ω_1. We define a poset $\mathbb{J}_{S,I}$ such that p is a condition in $\mathbb{J}_{S,I}$ iff $p = (A_p, l_p)$ where

(1) A_q is a countable subtree of $\omega_1^{<\omega_1}$,

(2) l_p is a function from some countable subset of I to $m(A_p)$.

For any $p, q \in \mathbb{J}_{S,I}$ define $p \leqslant q$ iff

(1) $A_q \subseteq A_p$,

(2) for every $t \in A_p \setminus A_q$ either there is an $s \in m(A_q)$ such that $s \subseteq t$ or that $\alpha < ht(A_q)$ and $\alpha \in S$ is a limit ordinal imply

$$\alpha \neq \bigcup\{ht(s)\colon s \in A_q \text{ and } s \subseteq t\}.$$

(3) $dom(l_q) \subseteq dom(l_p)$ and $(\forall \alpha \in dom(l_q))(l_q(\alpha) \subseteq l_p(\alpha))$.

Lemma 5 (CH). $\mathbb{J}_{S,I}$ is *ω_1-closed and ω_1-linked.*

Proof. We show first that $\mathbb{J}_{S,I}$ is ω_1-linked. For any $p, q \in \mathbb{J}_{S,I}$, if $A_p = A_q$, then the condition $(A_p, l_p \cup l_q)$ is a common extension of p and q. Because there are only ω_1 different countable subtrees of $\omega_1^{<\omega_1}$, it is clear that $\mathbb{J}_{S,I}$ is the union of ω_1 linked sets.

We now show that $\mathbb{J}_{S,I}$ is ω_1-closed. Let $\{p_n: n \in \omega\}$ be a decreasing sequence in $\mathbb{J}_{S,I}$. Let $A = \bigcup_{n\in\omega} A_{p_n}$ and let $D = \bigcup_{n\in\omega} dom(l_{p_n})$. For each $i \in D$ let

$$l(i) = \bigcup\{l_{p_n}(i): n \in \omega \text{ and } i \in dom(l_{p_n})\}.$$

Define a condition $p \in \mathbb{J}_{S,I}$ such that

$$A_p = A \cup \{l(i): i \in D\} \quad \text{and} \quad l_p = l.$$

We claim that p is a lower bound of the sequence $\{p_n: n \in \omega\}$. It suffices to show that for any n and for any $t \in A_p \setminus A_{p_n}$ either there exists an $s \in m(A_{p_n})$ such that $s \subseteq t$ or that $\alpha < ht(A_{p_n})$ and $\alpha \in S$ is a limit ordinal imply

$$\alpha \neq \bigcup\{ht(s): s \in A_{p_n} \text{ and } s \subseteq t\}.$$

If $t \in A$, then there is an $k > n$ such that $t \in A_{p_k}$. Hence either there is an $s \in m(A_{p_n})$ such that $s \subseteq t$ or that $\alpha < ht(A_{p_n})$ and $\alpha \in S$ is a limit ordinal imply

$$\alpha \neq \bigcup\{ht(s): s \in A_{p_n} \text{ and } s \subseteq t\}$$

because $p_k \leqslant p_n$. If $t = l(i)$ for some $i \in D$, then, by assuming $\langle l_{p_n}: n \in \omega\rangle$ is not eventually constant, there is a $k > n$ and there is a $t' \in A_{p_k} \setminus A_{p_n}$ such that $t' \subseteq t$. Hence either there is an $s \in m(A_{p_n})$ such that $s \subseteq t' \subseteq t$ or that $\alpha < ht(A_{p_n})$ *and* $\alpha \in S$ is a limit ordinal imply

$$\alpha \neq \bigcup\{ht(s): s \in A_{p_n} \text{ and } s \subseteq t'\}$$

because $p_k \leqslant p_n$. □

Remark. Again, we may consider the poset $\mathbb{J}_{S,I}$ as a two-step iterated forcing $\mathbb{J}_{S,I'} * Fn(I \setminus I', T_{\dot{G}_{I'}}, \omega_1)$, where I' is a subset of I, $T_{G_{I'}} = \bigcup\{A_p: p \in G_{I'}\}$ for a generic filter $G_{I'}$ of $\mathbb{J}_{S,I'}$ and $Fn(I \setminus I', T_{\dot{G}_{I'}}, \omega_1)$ is a countable support product of $|I \setminus I'|$-copies of $T_{\dot{G}_{I'}}$. The map

$$p = (A_p, l_p) \mapsto ((A_p, l_p \restriction I'), l_p \restriction I \setminus I')$$

is a dense embedding from $\mathbb{J}_{S,I}$ to $\mathbb{J}_{S,I'} * Fn(I \setminus I', T_{\dot{G}_{I'}}, \omega_1)$.

We now define S-completeness of a tree T. Let α be a limit ordinal and let T be a tree with $ht(T) = \alpha$. Let S be a subset of α. Then T is called S-complete if for every limit ordinal $\beta \in S$ and every $B \in \mathscr{B}(T \restriction \beta)$ the union $\bigcup B \in T_\beta$, i.e. every strictly decreasing sequence of T has a greatest lower bound b in T if $ht(b) \in S$.

Lemma 6. *Let M be a model of* CH *and let $\mathbb{J}_{S,I} \in M$ where $S \subseteq \omega_1$ and I is an index set in M. Suppose G is a $\mathbb{J}_{S,I}$-generic filter over M. Then the tree $T_G = \bigcup_{p\in G} A_p$ is $(\omega_1 \setminus S)$-complete in $M[G]$.*

Proof. Let $\alpha \in \omega_1 \setminus S$ be a limit ordinal and let B be a branch of $T_G \restriction \alpha$. We need to show that $t = \bigcup B \in T_G$. The set B is in M because $\mathbb{J}_{S,I}$ is ω_1-closed and B is countable. Let $p_0 \in G$ be such that $B \subseteq A_{p_0}$. It is clear that

$$p_0 \Vdash B \subseteq T_{\dot{G}}.$$

Let

$$D_B = \{p \in \mathbb{J}_{S,I}\colon p \leqslant p_0 \text{ and } t = \bigcup B \in A_p\}.$$

Then D_B is dense below p_0 because for any $p \leqslant p_0$ the element $p' = (A_p \cup \{\bigcup B\}, l_p)$ is a condition in $\mathbb{J}_{S,I}$ and $p' \leqslant p$ (here we use the fact that $\alpha \in \omega_1 \setminus S$). Since $p_0 \in G$, then there is a $p \in G \cap D_B$. Hence $t = \bigcup B \in \mathrm{T}_G$. □

Lemma 7. *Let M be a model of* CH. *In M let U be a stationary subset of ω_1, let T be an ω_1-tree which is U-complete and let I be any index set. Let $K \in M$ be any ω_1-tree such that every level of K is countable. Suppose $\mathbb{P} = Fn(I, T, \omega_1) \in M$ and G is a $\mathbb{P}$-generic filter over M. Then*

$$\mathscr{B}(K) \cap M[G] \subseteq M,$$

i.e. the forcing adds no new branches of K.

Proof. Suppose that B is a branch of K in $M[G] \setminus M$. Without loss of generality, let us assume that

$$1_{\mathbb{P}} \Vdash \dot{B} \in (\mathscr{B}(K) \setminus M).$$

By a standard argument (see [9, p. 259]) the statements

$$(\forall p \in \mathbb{P})(\forall \alpha \in \omega_1)(\exists t \in \omega_1^{\alpha})(\exists p' \leqslant p)(p' \Vdash t \in \dot{B})$$

and

$$(\forall p \in \mathbb{P})(\forall \alpha \in \omega_1)(\forall t \in \omega_1^{\alpha})(p \Vdash t \in \dot{B}) \to (\forall \beta \in \omega_1 \setminus \alpha)(\exists \gamma \in \omega_1 \setminus \beta)$$
$$(\exists t_j \in \omega_1^{\gamma})(t_0 \neq t_1)(\exists p_j \leqslant p)(p_j \Vdash t_j \in \dot{B}))$$

for $j = 0, 1$, are true in M.

Let's work in M. Let θ be a large enough cardinal and let N be a countable elementary submodel of $(H(\theta), \in)$ such that $K, \mathbb{P}, \dot{B} \in N$. Let $\delta = N \cap \omega_1 \in U$ (such N exists because U is stationary). In M we choose an increasing sequence of ordinals $\{\delta_n\colon n \in \omega\}$ such that $\bigcup_{n \in \omega} \delta_n = \delta$. Again in M we construct a set

$$\{p_s\colon s \in 2^{<\omega}\} \subseteq \mathbb{P} \cap N$$

and a set

$$\{t_s\colon s \in 2^{<\omega}\} \subseteq K \cap N$$

such that

(1) $(\forall s, s' \in 2^{<\omega})(s \subseteq s' \leftrightarrow p_{s'} \leqslant p_s \leftrightarrow t_{s'} \leqslant t_s)$,

(2) $(\forall s \in 2^{<\omega})(p_s \Vdash t_s \in \dot{B})$,

(3) $ht(t_s) \geqslant \delta_{|s|}$,

(4) $(\forall i \in dom(p_s))(ht(p_s(i)) \geqslant \delta_{|s|})$,

where $|s|$ means the length of the finite sequence s.

Let $p_\emptyset = 1_{\mathbb{P}}$ and let $t_\emptyset = \emptyset$, the root of K. Assume that we have found $\{p_s : s \in 2^{\leqslant n}\}$ and $\{t_s : s \in 2^{\leqslant n}\}$ which satisfy (1), (2), (3) and (4) relative to $2^{\leqslant n}$. Pick any $s \in 2^n$. Since the sentence

$$(\forall p \in \mathbb{P})(\forall \alpha \in \omega_1)(\forall t \in \omega_1^\alpha)(p \Vdash t \in \dot{B} \rightarrow (\forall \beta \in \omega_1 \setminus \alpha)(\exists \gamma \in \omega_1 \setminus \beta)(\exists t_j \in \omega_1^\gamma)$$

$$(t_0 \neq t_1)(\exists p_j \leqslant p)(p_j \Vdash t_j \in \dot{B}))$$

for $j = 0, 1$, is true in M, then it is true in N. Since $p_s, t_s \in N$, then in N there exist $p^0, p^1 \leqslant p_s$ and there exist $t^0, t^1 \in \omega_1^\gamma$, $t^0 \neq t^1$, for some $\gamma \in \delta \setminus \delta_{|s|+1}$ such that

$$p^j \Vdash t^j \in \dot{B}$$

for $j = 0, 1$. Again in N we can extend p^0 and p^1 to $p_{s^\frown\langle 0\rangle}$ and $p_{s^\frown\langle 1\rangle}$ respectively so that

$$(\forall i \in dom(p_{s^\frown\langle j\rangle}))(ht(p_{s^\frown\langle j\rangle}(i)) \geqslant \delta_{|s|+1})$$

for $j = 0, 1$. Since T is U-complete and for every $f \in 2^\omega$, for every $i \in \bigcup_{n \in \omega} dom(p_{f \restriction n})$ we have

$$\bigcup \{ht(p_{f \restriction n}(i)) : n \in \omega \text{ and } i \in dom(p_{f \restriction n})\} = \delta \in U,$$

then the condition p_f such that $dom(p_f) = \bigcup_{n \in \omega} dom(p_{f \restriction n})$ and

$$p_f(i) = \bigcup \{p_{f \restriction n}(i) : n \in \omega \text{ and } i \in dom(p_f)\}$$

for every $i \in dom(p_f)$ is a lower bound of $\{p_{f \restriction n} : n \in \omega\}$ *in* $\mathbb{P}$. Here we use the fact that T is U-complete so that $p_f(i) \in T$ for every $i \in dom(p_f)$. Let $t_f = \bigcup_{n \in \omega} t_{f \restriction n}$. Then $ht(t_f) = \delta$. Since

$$p_f \Vdash t_{f \restriction n} \in \dot{B}$$

for every $n \in \omega$, then

$$p_f \Vdash t_f \in \dot{B} \cap K_\delta.$$

It is easy to see that if $f, f' \in 2^\omega$ are different, then t_f and $t_{f'}$ are different. Hence K_δ is uncountable, a contradiction. $\square$

Lemma 8. *Let M be a model of* CH *and $2^{\omega_1} = \lambda > \omega_2$ and let $\mathbb{J}_{S,\kappa} \in M$ where κ is a cardinal in M such that $\omega_1 < \kappa < \lambda$ and S is a stationary subset of ω_1. Suppose that G is a $\mathbb{J}_{S,\kappa}$-generic filter over M. Then in $M[G]$ the tree $T_G = \bigcup_{p \in G} A_p$ is an essential Jech–Kunen tree with κ branches.*

Proof. It is easy to see that T_G is an ω_1 tree. We will divide the lemma into two claims.

Claim 8.1. *For every $\xi \in \kappa$ let*

$$B(\xi) = \bigcup \{l_p(\xi) \colon p \in G \text{ and } \xi \in dom(l_p)\}.$$

Then

$$B(T_G) = \{B(\xi) \colon \xi \in \kappa\}$$

and for any two different ξ and ξ' in κ the branches $B(\xi)$ and $B(\xi')$ are different.

Proof of Claim 8.1. Since in M, for every $\xi \in \kappa$ and for every $\alpha \in \omega_1$ the set

$$D_{\xi,\alpha} = \{p \in \mathbb{J}_{S,\kappa} \colon \xi \in dom(l_p) \text{ and } ht(l_p(\xi)) > \alpha\}$$

is dense in $\mathbb{J}_{S,\kappa}$, then $B(\xi)$ is a branch of T_G. For any two different $\zeta, \zeta' \in \kappa$ the set

$$D_{\xi,\xi'} = \{p \in \mathbb{J}_{S,\kappa} \colon \xi, \xi' \in dom(l_p) \text{ and } l_p(\xi) \neq l_p(\xi')\}$$

is also dense in $\mathbb{J}_{S,\kappa}$. So the branches $B(\xi)$ and $B(\xi')$ are different.

We now want to show that all branches of T_G in $M[G]$ are exactly those $B(\xi)$'s. Suppose that in $M[G]$ the tree T_G has a branch B which is not in the set

$$\{B(\xi) \colon \xi \in \kappa\}.$$

Without loss of generality, let us assume that

$$1_{\mathbb{J}_{S,\kappa}} \Vdash \dot{B} \in (\mathscr{B}(T_{\dot{G}}) \setminus \{\dot{B}(\xi) \colon \xi \in \kappa\}).$$

Work in M. Let θ be a large enough cardinal and let N be an elementary submodel of $(H(\theta), \in)$ such that $\kappa, S, \dot{B}, \mathscr{B} = \{\dot{B}(\xi) \colon \xi \in \kappa\}$, $\mathbb{J}_{S,\kappa} \in N$ and if $p \in N \cap \mathbb{J}_{s,\kappa}$, then $dom(l_p) \subseteq N$. Let $\delta = N \cap \omega_1 \in S$. In M we choose an increasing sequence of countable ordinals $\{\delta_n \colon n \in \omega\}$ such that $\delta = \bigcup_{n \in \omega} \delta$. We now want to find a decreasing sequence $\{p_n \colon n \in \omega\} \subseteq \mathbb{J}_{s,\kappa} \cap N$ such that $p_0 = 1_{\mathbb{J}_{S,\kappa}}$ and for each $n \in \omega$

(1) $(\forall \xi \in dom(l_{p_n}))(\exists t \in A_{p_{n+1}})(p_{n+1} \Vdash t \in \dot{B}(\xi) \setminus \dot{B})$,

(2) $(\exists t \in A_{p_{n+1}} \setminus A_{p_n})(ht(t) \geqslant ht(A_{p_n})$ and $p_{n+1} \Vdash t \in \dot{B}$,

(3) $ht(A_{p_n}) \geqslant \delta_n$.

Assume we have found $\{p_0, p_1, \ldots, p_n\}$. We now work in N. Let

$$dom(l_p) = \{\xi_k \colon k \in \omega\}$$

which is an enumeration in N. Choose $q_0 = p_n \geqslant q_1 \geqslant \cdots$ such that for every $k \in \omega_1$ there is a $t \in A_{q_k}$ such that

$$q_k \Vdash t \in \dot{B}(\xi_k) \setminus \dot{B}.$$

Assume, in N, that we have found $\{q_0, q_1, \ldots, q_k\}$. Since the sentence

$$q_k \Vdash (\exists t \in T_{\dot{G}})(t \in \dot{B}(\xi_k) \setminus \dot{B})$$

is true in N (because it is true in $H(\theta)$ and $\xi_k \in N$), then there is a $t \in \omega_1^{<\omega_1} \cap N = \delta^{<\delta}$ and there is a $q' \leqslant q_k$ such that

$$q' \Vdash (t \in T_{\dot{G}} \text{ and } t \in \dot{B}(\xi_k) \setminus \dot{B}).$$

Since

$$q' \Vdash A_{q'} \subseteq T_{\dot{G}},$$

then there is a $q_{k+1} \leqslant q'$ such that $t \in A_{q_{k+1}}$. Since $N \models$ "$\mathbb{J}_{S,\kappa}$ is ω_1-closed" and $\{q_k: k \in \omega\}$ is constructed in N, then there is a $q \in \mathbb{J}_{S,\kappa}$ in N such that q is a lower bound of $\{q_k: k \in \omega_1\}$. Let $\alpha = \max\{ht(A_{p_n}), \delta_{n+1}\}$. Notice that $\alpha \in \delta$ because $p_n \in N$. Since in N

$$q \Vdash \dot{B} \text{ is a branch of } T_{\dot{G}},$$

then

$$q \Vdash (\exists t \in (T_{\dot{G}})_{\alpha+1})(t \in \dot{B}).$$

Hence there is a $\bar{q} \leqslant q$ and there is a $t \in \omega_1^{\alpha+1} \cap N$ such that

$$\bar{q} \Vdash t \in \dot{B}.$$

We can also assume that $t \in A_{\bar{q}}$.

We now go back to M and let $p_{n+1} = \bar{q}$. This finishes the construction of $\{p_n: n \in \omega\}$.

Let $p \in \mathbb{J}_{S,\kappa}$ be such that

$$dom(l_p) = \bigcup_{n \in \omega} dom(l_{p_n}),$$

for every $\xi \in dom(l_p)$

$$l_p(\xi) = a_\xi = \bigcup \{l_{p_n}(\xi): n \in \omega \text{ and } \xi \in dom(l_{p_n})\}$$

and

$$A_p = (\bigcup_{n \in \omega} A_{p_n}) \cup \{a_\xi: \xi \in dom(l_p)\}.$$

By the construction of p_n's we have

$$\bigcup \{ht(t): t \in A_p \text{ and } p \Vdash t \in \dot{B}\} = \delta \in S.$$

Pick any $t \in A_p$. If $t \neq a_\xi$ for any $\xi \in dom(l_p)$, then we can find a $\gamma \in \omega_1$ such that $t^\frown\langle\gamma\rangle \notin A_p$. Extend $t^\frown\langle\gamma\rangle$ to $\bar{t} \in \omega_1^\delta$. Define $\bar{p}$ such that

$$A_{\bar{p}} = A_p \cup \{u: t \subseteq u \subseteq \bar{t}\}$$

and $l_{\bar{p}} = l_p$. If $t = a_\xi$ for some $\xi \in dom(l_p)$, then simply extend t to $b_\xi \in \omega_1^\delta$ (if $ht(a_\xi) = \delta$, then $b_\xi = a_\xi$). Define $\bar{p}$ such that

$$A_{\bar{p}} = A_p \cup \{u: t \subseteq u \subseteq b_\xi\}$$

and

$$l_{\bar{p}} = (l_p \restriction dom(l_p) \setminus \{\xi\}) \cup \{(\xi, b_\xi)\}.$$

It is easy to see that $\bar{p} \leqslant p$ and $ht(A_{\bar{p}}) = \delta + 1$. Let

$$a = \bigcup \{t \in A_{\bar{p}}: \bar{p} \Vdash t \in \dot{B}\}.$$

It is also easy to see that for any $q \leqslant \bar{p}$ the element a is not in A_q. Here we use the fact $\delta \in S$, δ is a limit ordinal and $ht(A_{\bar{p}}) > \delta$. Hence

$$\bar{p} \Vdash \dot{B} \cap T_{\dot{G}} \subseteq \dot{B} \cap A_{\bar{p}}.$$

This contradicts that

$$\bar{p} \Vdash \dot{B} \text{ is a branch of } T_{\dot{G}}. \quad \square$$

Claim 8.2. *T_G has no Kurepa subtree in $M[G]$.*

Proof of Claim 8.2. Suppose that T_G has a Kurepa subtree K in $M[G]$. Since $|K| = \omega_1$, then there is an $I \subseteq \kappa$ such that $|I| \leqslant \omega_1$ and $K \in M[G_I]$, where

$$G_I = \{p \in G: dom(l_p) \subseteq I\}.$$

Notice that G_I is a $\mathbb{J}_{S,I}$-generic filter over M. Since $\mathbb{J}_{S,\kappa}$ is forcing equivalent to $\mathbb{J}_{S,I} * Fn(\kappa \setminus I, T_{G_I}, \omega_1)$ and T_{G_I} is $(\omega_1 \setminus S)$-complete in $M[G_I]$ (notice that S is still stationary–costationary), then by Lemma 7, the set of all branches of K in $M[G_I]$ is same as the set of all branches of K in $M[G]$. Hence K is a Kurepa tree in $M[G_I]$. But by Claim 8.1, the tree $T_G = T_{G_I}$ has only $|I|$ branches in $M[G_I]$ and K is a subtree of T_G. Hence K has at most ω_1 branches in $M[G_I]$. This contradicts that K is a Kurepa tree in $M[G_I]$. $\square$

Lemma 9. *Let M be a model of* CH *and $2^{\omega_1} = \lambda > \omega_2$ with $\lambda^{<\lambda} = \lambda$. In M let $((\mathbb{P}_\alpha: \alpha \leqslant \lambda), (\dot{\mathbb{Q}}_\alpha: \alpha < \lambda))$ be a λ-stage iterated forcing notion used in* [1] *for a model of* GMA. *Suppose that G_λ is a $\mathbb{P}_\lambda$-generic filter over M. In $M[G_\lambda]$ let $\mathbb{P} = (\omega_1^{<\omega_1}, \supseteq)$ and let H be a $\mathbb{P}$-generic filter over $M[G_\lambda]$. Then in $M[G_\lambda][H]$ there are no essential Kurepa trees.*

Proof. For any $\alpha < \lambda$ the poset $\mathbb{P}_\lambda$ can be factored to $\mathbb{P}_\alpha * \mathbb{P}^\alpha$ and G_λ can also be written as $G_\alpha * G^\alpha$ such that G_α is a $\mathbb{P}_\alpha$-generic filter over M and G^α is a $\mathbb{P}^\alpha$-generic filter over $M[G_\alpha]$. Suppose T is a Kurepa tree in $M[G_\lambda][H]$ with λ branches. Without loss of generality, let's assume that for every $t \in T$ there are exactly λ branches of T passing through t in $M[G_\lambda][H]$. In $M[G_\lambda][H]$ let $f: \omega_2 \mapsto \mathscr{B}(T)$ be a one to one function such that for every $t \in T$ and for every $\alpha < \omega_2$ there exists a $\beta \in \omega_2 \setminus \alpha$ such that $t \in f(\beta)$. Notice that ω_2 here can be replaced by any regular cardinal κ satisfying $\omega_2 \leqslant \kappa < \lambda$. Without loss of generality, let us assume that

$1_{\mathbb{P}} \Vdash$ ($\dot{T}$ is a Kurepa tree and $\dot{f}: \omega_2 \mapsto \mathscr{B}(\dot{T})$

is a one to one function such that $(\forall t \in \dot{T})(\forall \alpha \in \omega_2)(\exists \beta \in \omega_2 \setminus \alpha)(t \in \dot{f}(\beta))$).

We want now to construct a poset $\mathbb{R}'$ in $M[G_\lambda]$ such that a filter $\bar{G}$ of $\mathbb{R}'$ obtained by applying a forcing argument similar to GMA in $M[G_\lambda]$ will give us a $\mathbb{P}$-name for a Jech–Kunen subtree of T in $M[G_\lambda][H]$.

Let r be a condition in $\mathbb{R}'$ iff $r = (I_r, \mathbb{P}_r, \mathscr{A}_r, \mathscr{S}_r)$ where I_r is a countable subtree of $(\omega_1^{<\omega_1}, \supseteq)$, $\mathbb{P}_r = \langle p_t^r\colon t \in I_r\rangle$, $\mathscr{A}_r = \langle A_t^r\colon t \in I_r\rangle$ and $\mathscr{S}_r = \langle S_t^r\colon t \in I_r\rangle$ such that

(1) $\mathbb{P}_r \subseteq \mathbb{P}$, and for every $t \in I_r$ the element A_t^r is a nonempty countable subtree of $(\omega_1^{<\omega_1}, \supseteq)$ of height $\alpha_t^r + 1$ (we will use some A_t^r's to generate a Jech–Kunen subtree of T) and S_t^r is a nonempty countable subset of ω_2 (the requirement "$S_t^r \subseteq \omega_2$" makes $\mathbb{R}'$ different from $\mathbb{R}$ defined in Lemma 3),

(2) $(\forall s, t \in I_r)(s \subseteq t \leftrightarrow p_t^r \leqslant p_s^r)$,

(3) $(\forall s, t \in I_r)(s \subseteq t \rightarrow A_t^r \upharpoonright ht(A_s^r) = A_s^r)$,

(4) $(\forall s, t \in I_r)(s \subseteq t \rightarrow S_s^r \subseteq S_t^r)$,

(5) $(\forall t \in I_r)(p_t^r \Vdash A_t^r \subseteq \dot{T})$,

(6) $(\forall t \in I_r)(\forall \alpha \in S_t^r)(\exists a \in (A_t^r)_{\alpha_t^r})(p_t^r \Vdash a \in \dot{f}(\alpha))$.

For any $r, r' \in \mathbb{R}'$, let $r \leqslant r'$ iff $I_{r'} \subseteq I_r$, and for every $t \in I_{r'}$

$$p_t^{r'} = p_t^r, \qquad A_t^{r'} = A_t^r \quad \text{and} \quad S_t^{r'} \subseteq S_t^r.$$

Claim 9.1. *The poset $\mathbb{R}'$ is ω_1-linked.*

Proof of Claim 9.1. Same as the proof of Claim 3.1. □

Claim 9.2. *The poset $\mathbb{R}'$ is countably compact.*

Proof of Claim 9.2. Same as the proof of Claim 3.2. □

For each $t \in \omega_1^{<\omega_1}$ define

$$D_t = \{r \in \mathbb{R}'\colon t \in I_r\}.$$

For each $p \in \mathbb{P}$ define

$$E_p = \{r \in \mathbb{R}'\colon (\exists t \in I_r)(p_t^r \leqslant p)\}.$$

For each $\alpha < \omega_1$ define

$$F_\alpha = \{r \in \mathbb{R}'\colon (\forall s \in I_r)(\exists t \in I_r)(ht(A_t^r) > \alpha)\}.$$

For each $\alpha < \omega_2$ define

$$O_\alpha = \{r \in \mathbb{R}'\colon (\forall s \in I_r)(\exists t \in I_r)(s \subseteq t \text{ and } [\alpha, \omega_2) \cap S_t^r \neq \emptyset)\}.$$

Claim 9.3. *All those D_t, E_p, F_α and O_α's are dense in $\mathbb{R}'$.*

Proof of Claim 9.3. Same as the proof of Claim 3.3. □

Note that $|\mathbb{R}'| = \omega_2$. Note also that $M[G_\lambda][H] = M[H][G_\lambda]$. By the construction of $\mathbb{P}_\lambda$ there exists an $\beta < \lambda$ such that those dense sets D_t, E_p, F_α and O_α are in $M[G_\beta]$, the tree T is in $M[G_\beta][H]$ or $\dot{T}$ is in $M[G_\beta]$ and

$$1_{\mathbb{P}_\beta} \Vdash \dot{\mathbb{Q}}_\beta = \mathbb{R}',$$

i.e. $\mathbb{R}'$ is the poset used in βth step forcing in the λ-stage iteration.

Let U_β be a $\mathbb{Q}_\beta$-generic filter over $M[G_\beta]$ such that $G_\beta * U_\beta = G_{\beta+1}$.

Since D_t is dense for every $t \in \omega_1^{<\omega_1}$, then

$$I_{U_\beta} = \bigcup\{I_r : r \in U_\beta\} = \omega_1^{<\omega_1}.$$

Let

$$\mathbb{P}_{U_\beta} = \bigcup\{\mathbb{P}_r : r \in U_\beta\}$$

and let

$$\mathscr{A}_{U_\beta} = \bigcup\{\mathscr{A}_r : r \in U_\beta\}.$$

Notice that for any $r, r' \in U_\beta$ and for any $t \in I_r \cap I_{r'}$ we have $p_t^r = p_t^{r'}$ and $A_t^r = A_t^{r'}$ because r and r' are compatible. So now for every $t \in I_{U_\beta}$ we can define $p_t = p_t^r$ for some $r \in U_\beta$ and define $A_t = A_t^r$ for some $r \in U_\beta$. It is clear that the map $t \mapsto p_t$ is an isomorphism between I_{U_β} and $\mathbb{P}_{U_\beta}$, i.e. for any $s, t \in I_{U_\beta}$ we have $s \subseteq t$ iff $p_t \leqslant p_s$. It is also clear that the map $t \mapsto A_t$ is a homomorphism from I_{U_β} to $\mathscr{A}_{U_\beta}$, i.e. for any $s, t \in I_{U_\beta}$ we have $s \subseteq t$ implies $A_t \restriction ht(A_s) = A_s$.

Claim 9.4. *For each $t \in I_{U_\beta}$ the set $\{p_{t^\frown\langle\gamma\rangle} : \gamma \in \omega_1\}$ is a maximal antichain below p_t in $\mathbb{P}$.*

Proof of Claim 9.4. Same as the proof of Claim 3.4. □

The next claim is something different from Lemma 3. Let $T_H = \bigcup\{A_t : p_t \in H\}$ where H is the $\mathbb{P}$-generic filter over $M[G_\lambda]$.

Claim 9.5. *T_H is a Jech–Kunen subtree of T in $M[G_\lambda][H]$.*

Proof of Claim 9.5. By the proof of Claim 3.5, it is easy to see that T_H is a subtree of T with more than ω_1 branches. It suffices to show that T_H has exactly ω_2 branches.

Suppose that T_H has more than ω_2 branches. Then there is a branch B in $M[G_\lambda][H]$ which is not in the range of the function f. Without loss of generality, let us assume that

$$1_{\mathbb{P}} \Vdash (\forall \alpha \in \omega_2)(\dot{B} \neq \dot{f}(\alpha))$$

where $\dot{B}$ is a $\mathbb{P}$-name for B and let

$$D_{\dot{B}} = \{r \in \mathbb{R}' : (\forall s \in I_r)(\exists t \in I_r)(s \subseteq t \text{ and } ht(\dot{B} \cap A_t^r) < ht(A_t^r))\}.$$

Since $M[G_\lambda][H] = M[G_\beta][H][G^\beta]$ and $\mathbb{P}^\beta$ is ω_1-closed in $M[G_\beta][H]$, then B is in $M[G_\beta][H]$ because any ω_1-closed forcing will not add any new branches to the Kurepa tree T. We assume also that the $\mathbb{P}$-name $\dot{B}$ is in $M[G_\beta]$. Hence the set $D_{\dot{B}}$ is in $M[G_\beta]$. Let

$$E_{\dot{B}} = \{p_t^r \in \mathbb{P}_{U_\beta}: r \in D_{\dot{B}} \cap U_\beta \text{ and } p_t^r \Vdash ht(\dot{B} \cap A_t^r) < ht(A_t^r)\}.$$

Subclaim 9.5.1. *$D_{\dot{B}}$ is dense in $\mathbb{R}'$.*

Proof of Claim 9.5.1. Let r_0 be any element in $\mathbb{R}'$. It suffices to show that there is an element r in $D_{\dot{B}}$ such that $r \leqslant r_0$. Let's first extend r_0 to r' such that for every $s \in I_{r_0}$ there is a $t \in m(I_{r'})$ such that $s \subseteq t$. Let $t \in m(I_{r'})$. For every $\alpha \in S_t^{r'}$ let $a_\alpha \in (A_t^{r'})_{\alpha_t^{r'}}$ such that $p_t^{r'} \Vdash a_\alpha \in \dot{f}(\alpha)$. Since we have

$$p_t^{r'} \Vdash (\exists u \in \dot{T})(u \in \dot{f}(\alpha) \setminus \dot{B})$$

and $\mathbb{P}$ is ω_1-closed, then there is a $u_\alpha \supseteq a_\alpha$ in $\omega_1^{<\omega_1}$ for every $\alpha \in S_t^{r'}$ and a $p_t \leqslant p_t^{r'}$ such that for every $\alpha \in S_t^{r'}$

$$p_t \Vdash u_\alpha \in \dot{f}(\alpha) \setminus \dot{B}.$$

Without loss of generality, we can assume that there is a $\gamma \in \omega_1$ such that $ht(u_\alpha) = \gamma$ and

$$p_t \Vdash \dot{B} \text{ differs from all } \dot{f}(\alpha) \text{ below } \gamma$$

for every $\alpha \in S_t^{r'}$. Let

$$I_r = I_{r'} \cup \{\bar{t}: \bar{t} \text{ is a successor of } t \text{ for } t \in m(I_{r'})\}.$$

For every $t \in I_{r'}$ let

$$p_t^r = p_t^{r'}, \quad A_t^r = A_t^{r'} \quad \text{and} \quad S_t^r = S_t^{r'}.$$

For every $\bar{t} \in I_r \setminus I_{r'}$ let

$$p_{\bar{t}}^r = p_t, \quad A_{\bar{t}}^r = A_t^{r'} \cup \{s: s \subseteq u_\alpha \text{ for some } \alpha \in S_t^{r'}\} \quad \text{and} \quad S_{\bar{t}}^r = S_t^{r'}.$$

Now it is easy to see that $r \leqslant r_0$ and $r \in D_{\dot{B}}$. □

Subclaim 9.5.2. *$E_{\dot{B}}$ is dense in $\mathbb{P}_{U_\beta}$.*

Proof of Subclaim 9.5.2. Let $p_0 \in \mathbb{P}_{U_\beta}$. We need to show that there is a $p \in \mathbb{P}_{U_\beta}$ such that $p \leqslant p_0$ and $p \in E_{\dot{B}}$.

Since $p_0 \in \mathbb{P}_{U_\beta}$, then there is an $r \in U_\beta$ such that $p_0 = p_s^r$. Since $D_{\dot{B}}$ is dense and $r \in U_\beta$, then there is an $r' \leqslant r$ such that $r' \in U_\beta \cap D_{\dot{B}}$. Since $p_s^r = p_s^{r'}$ and $r' \in D_{\dot{B}}$, then there is a $t \in I_{r'}$ such that $s \subseteq t$ and

$$p_t^{r'} \Vdash ht(\dot{B} \cap A_t^{r'}) < ht(A_t^{r'}).$$

Hence we have $p_t^{r'} \leqslant p_s^r = p_0$ and $p_t^{r'} \in E_{\dot{B}}$. □

We prove Claim 9.5 now. Assume B be a branch of T and B is not in the range of f. We want to show that B is not a branch of T_H. Suppose B is a branch of T_H. Then there is a $p \in H$ such that

$$p \Vdash \dot{B} \in \mathscr{B}(T_{\dot{H}}).$$

Since $E_{\dot{B}}$ is dense in $\mathbb{P}$, then we can find a $p_t^r \in E_{\dot{B}}$ such that $p_t^r \leqslant p$. Hence we derived a contradiction because we have

$$p_t^r \Vdash \dot{B} \in \mathscr{B}(T_{\dot{H}}),$$

$$p_t^r \Vdash (T_{\dot{H}}) \upharpoonright \alpha_t^r + 1 = A_t^r$$

and

$$p_t^r \Vdash ht(\dot{B} \cap A_t^r) < ht(A_t^r).$$

Hence in $M[G_\lambda][H]$ the tree T_H has ω_2 branches because $\mathscr{B}(T_H) \subseteq f''\omega_2$ (the range of f). □

By Claim 9.5 there are no essential Kurepa trees in $M[G_\lambda][H]$. This proves Lemma 9. □

Theorem 10. *It is consistent with* CH *and* $2^{\omega_1} > \omega_2$ *plus the existence of a thick Kurepa tree that there exist essential Jech–Kunen trees and there are no essential Kurepa trees.*

Proof. Let M be a model of CH and $2^{\omega_1} = \lambda > \omega_2$ such that in M, $\lambda^{<\lambda} = \lambda$ and there is a thick Kurepa tree. Such model exists by Lemma 1. In M let

$$((\mathbb{P}_\alpha\colon \alpha \leqslant \lambda), (\dot{\mathbb{Q}}_\alpha\colon \alpha < \lambda))$$

be the λ-stage iterated forcing notion used in [1] for a model of GMA. Suppose G_λ is a $\mathbb{P}_\lambda$-generic filter over M. Then

$$M[G_\lambda] \Vdash \mathrm{CH} + 2^{\omega_1} = \lambda > \omega_2 + \mathrm{GMA}.$$

In $M[G_\lambda]$ let κ be a cardinal such that $\omega_2 \leqslant \kappa < \lambda$ and let S be a stationary–costationary subset of ω_1. Suppose that H is a $\mathbb{J}_{S,\kappa}$-generic filter over $M[G_\lambda]$. Then by Lemma 8, the tree $T_H = \bigcup\{A_p\colon p \in H\}$ is an essential Jech–Kunen tree in $M[G_\lambda][H]$. It is obvious that the thick Kurepa trees in M are still thick Kurepa trees in $M[G_\lambda][H]$. We need only to show that there are no essential Kurepa trees in $M[G_\lambda][H]$.

Suppose that K is an essential Kurepa tree in $M[G_\lambda][H]$. Since $|K| = \omega_1$, then there exists an $I \subseteq \kappa$ such that $|I| = \omega_1$ and $K \in M[G_\lambda][H_I]$, where

$$H_I = H \cap \mathbb{J}_{S,I} = \{p \in H\colon dom(l_p) \subseteq I\}.$$

Since $\mathbb{J}_{S,\kappa}$ is forcing equivalent to

$$\mathbb{J}_{S,I} * Fn(\kappa \backslash I, T_{\dot{H}_I}, \omega_1))$$

and by Lemma 6, the tree T_{H_I} is $(\omega_1 \setminus S)$-complete, then by Lemma 7, there are no new branches of K in $M[G_\lambda][H]$ which are not in $M[G_\lambda][H_I]$. So K is still a Kurepa tree in $M[G_\lambda][H_I]$. But the poset $\mathbb{J}_{S,I}$ is ω_1-closed and has cardinality ω_1. So by Lemma 2, the poset $\mathbb{J}_{S,I}$ is forcing equivalent to $(\omega_1^{<\omega_1}, \supseteq)$. Hence by Lemma 9, the Kurepa tree K has a Jech–Kunen subtree K' in $M[G_\lambda][H_I]$. Since every branch of K' is a branch of K and the set of branches of K stays the same in $M[G_\lambda][H_I]$and in $M[G_\lambda][H]$, then K' is still a Jech–Kunen subtree of K in $M[G_\lambda][H]$. This contradicts that K is an essential Kurepa tree in $M[G_\lambda][H]$. $\square$

Remark. It is quite easy to build a model of CH and $2^{\omega_1} > \omega_2$ in which there exist essential Jech–Kunen trees and there are no essential Kurepa tree without requiring the existence of a thick Kurepa tree. Let M be a model of GCH. First, increase 2^{ω_1} to ω_3 by an ω_1-closed Cohen forcing. Then, force with the poset $\mathbb{J}_{S,\omega_2}$. In the resulting model CH and $2^{\omega_1} = \omega_3$ hold and there is an essential Jech–Kunen tree. It can be shown easily that there are no thick Kurepa trees in the resulting model. Hence it is trivially true that there are no essential Kurepa trees in that model.

3. New proofs of two old results

In [10], we proved that, assuming the consistency of an inaccessible cardinal, it is consistent with CH and $2^{\omega_1} > \omega_2$ that there exist Jech–Kunen trees and there are no Kurepa trees. The model for that is constructed by taking Kunen's model for non-existence of Jech–Kunen trees as our ground model and then forcing with a countable support product of ω_2 copies of a "carefully pruned" tree T. The way that the tree T is pruned guarantees that (1) the forcing is ω-distributive, (2) forcing does not add any Kurepa trees, (3) T becomes a Jech–Kunen tree in the resulting model. In [6], this pruning technique was also used to construct a model of CH and $2^{\omega_1} > \omega_2$ in which there exist essential Kurepa trees and there exist essential Jech–Kunen trees. Here we realize that the Jech–Kunen tree obtained by forcing with that carefully pruned tree in [10] and [6] can be replaced by a generic Jech–Kunen tree obtained by forcing with $\mathbb{J}_{S,\kappa}$, the poset defined in Section 2. So now we can reprove those two results in [10] and [6] without going through a long and tedious construction of a "carefully pruned" tree.

Let $Lv(\kappa, \omega_1)$, the countable support Lévy collapsing order, denote a poset defined by letting $p \in Lv(\kappa, \omega_1)$ iff p is a function from some countable subset of $\kappa \times \omega_1$ to κ such that $p(\xi, \eta) \in \xi$ for every $(\xi, \eta) \in dom(p)$ and ordered by reverse inclusion.

Let $Fn(\lambda, 2, \omega_1)$, the countable support Cohen forcing, denote a poset defined by letting $p \in Fn(\lambda, 2, \omega_1)$ iff p is a function from some countable subset of λ to 2 and ordered by reverse inclusion.

Theorem 11. *Let κ and λ be two cardinals in a model M such that κ is strongly inaccessible and $\lambda > \kappa$ is regular in M. Let $S \in M$ be a stationary–costationary subset of*

ω_1 and let $\mathbb{J}_{S,\kappa} \in M$ be the poset defined in Section 2. Let $Lv(\kappa, \omega_1)$ and $Fn(\lambda, 2, \omega_1)$ be in M. Suppose that $G \times H \times F$ is a $(Lv(\kappa, \omega_1) \times Fn(\lambda, 2, \omega_1) \times \mathbb{J}_{S,\kappa})$-generic filter over M. Then $M[G][H][F] \models (\mathrm{CH} + 2^{\omega_1} > \omega_2 +$ there exist Jech–Kunen trees + there are no Kurepa trees).

Proof. It is easy to see that

$$M[G][H][F] \models (\mathrm{CH} + 2^{\omega_1} = \lambda > \kappa = \omega_2).$$

It is also easy to see that ω_1 and all cardinals greater than or equal to κ in M are preserved. By Lemma 8, the tree $T_F = \bigcup_{p \in F} A_p$ is a Jech–Kunen tree. We now need only to show that there are no Kurepa trees in $M[G][H][F]$. Suppose that K is a Kurepa tree in $M[G][H][F]$. Since $|K| = \omega_1$, then there exists an $I \subseteq \kappa$ with $|I| = \omega_1$ such that $K \in M[G][H][F_I]$ where $F_I = F \cap \mathbb{J}_{S,I}$ (recall that the poset $\mathbb{J}_{S,\kappa}$ is forcing equivalent to $\mathbb{J}_{S,I} * Fn(\kappa \backslash I, T_{\dot{F}_I}, \omega_1)$). By Lemma 7, the tree K is still a Kurepa tree in $M[G][H][F_I]$. Since the poset $\mathbb{J}_{S,I}$ is ω_1-closed and has cardinality ω_1, then by Lemma 2, $\mathbb{J}_{S,I}$ is forcing equivalent to $Fn(\omega_1, 2, \omega_1)$. By a standard argument we know that $Fn(\lambda, 2, \omega_1) \times Fn(\omega_1, 2, \omega_1)$ is isomorphic to $Fn(\lambda, 2, \omega_1)$. Hence there is a $Fn(\lambda, 2, \omega_1)$-generic filter H' over $M[G]$ such that $M[G][H][F_I] = M[G][H']$. But it is easy to see that in $M[G][H']$ there are neither Kurepa trees nor Jech–Kunen trees. So we have a contradiction that K is a Kurepa tree in $M[G][H']$. □

Theorem 12. *Let M be a model of* GCH. *Let κ and λ be two regular cardinals in M such that $\lambda > \kappa > \omega_1$ and let S be a stationary subset of ω_1 in M. In M let $\mathbb{K}_\lambda$ and $\mathbb{J}_{S,\kappa}$ be two posets defined in Sections* 1 *and* 2, *respectively. Suppose that $G \times H$ is a $\mathbb{K}_\lambda \times \mathbb{J}_{S,\kappa}$-generic filter over M. Then*

$$M[G \times H] \models (\mathrm{CH} + 2^{\omega_1} = \lambda > \kappa > \omega_1$$

$$+ \text{ there exist essential Kurepa trees}$$

$$+ \text{ there exist essential Jech–Kunen trees}).$$

Proof. It is easy to see that $M[G \times H]$ is a model of CH and $2^{\omega_1} = \lambda > \kappa > \omega_1$. Since $\mathbb{K}_\lambda$ and $\mathbb{J}_{S,\kappa}$ are ω_1-closed, then $\mathbb{K}_\lambda$ is absolute with respect to M and $M[H]$, and $\mathbb{J}_{S,\kappa}$ is absolute with respect to M and $M[G]$. By Lemma 8, the tree $T_H = \bigcup_{p \in G} A_p$ is an essential Jech–Kunen tree in $M[G][H]$. By Lemma 1, the tree $T_G = \bigcup_{p \in G} A_p$ is an essential Kurepa tree because $M[G][H] = M[H][G]$. □

References

[1] J. Baumgartner, Iterated forcing, in: A.R.D. Mathias ed., Surveys in Set Theory (Cambridge University Press, Cambridge, 1983) 1–59.

[2] T. Jech, Trees, J. Symbolic Logic 36 (1971) 1–14.
[3] T. Jech, Set Theory (Academic Press, New York, 1978).
[4] R. Jin, Some independence results related to the Kurepa tree, Notre Dame J. Formal Logic 32 (3) (1991) 448–457.
[5] R. Jin, A model in which every Kurepa tree is thick, Notre Dame J. Formal Logic 33 (1) (1992) 120–125.
[6] R. Jin, The differences between Kurepa trees and and Jech–Kunen trees, Arch. Math. Logic 32 (1993) 369–379.
[7] I. Juhász, Cardinal functions II, in: K. Kunen and J.E. Vaughan eds., Handbook of Set Theoretic Topology (North-Holland, Amsterdam, 1984) 63–110.
[8] K. Kunen, On the cardinality of compact spaces, Notices Amer. Math. Soc. 22 (1975) 212.
[9] K. Kunen, Set Theory, An Introduction To Independence Proofs (North-Holland, Amsterdam, 1980).
[10] S. Shelah and R. Jin, A model in which there are Jech–Kunen trees but there are no Kurepa trees, Israel J. Math. 84 (1993) 1–16.
[11] S. Shelah and R. Jin, Planting Kurepa trees and killing Jech–Kunen trees in a model by using one inaccessible cardinal, Fund. Math. 141 (1992) 287–296.
[12] S. Todorčević, Trees and linearly ordered sets, in: K. Kunen and J.E. Vaughan eds, Handbook of Set Theoretic Topology (North-Holland, Amsterdam, 1984) 235–293