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Letter to the Editor

Invertibility of multipliers ☆

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ABSTRACT

In the present paper the invertibility of multipliers is investigated in detail. Multipliers are operators created by (frame-like) analysis, multiplication by a fixed symbol, and resynthesis. Sufficient and/or necessary conditions for invertibility are determined depending on the properties of the analysis and synthesis sequences, as well as the symbol. Examples are given, showing that the established bounds are sharp. If a multiplier is invertible, a formula for the inverse operator is determined and n-term error bounds are given. The case when one of the sequences is a Riesz basis is completely characterized.

1. Introduction

In modern life, applications of signal processing can be found in numerous technical items, for example in wireless communication or medical imaging. In these applications, "*time-invariant filters*", i.e. convolution operators, are used very often. Such operators are *Fourier multipliers* [1]. In the last decade *time-variant filters* have found more and more applications. A particular way to implement such filters are Gabor multipliers [2,3], also known as Gabor filters [4]. Such operators find application in psychoacoustics [5], computational auditory scene analysis [6], measurement of acoustical systems [7], and seismic data analysis [8]. In [9] the concept of Bessel multipliers, i.e. operators of the form

$$M_{(m_n),(\phi_n),(\psi_n)}h = \sum_n m_n \langle h, \psi_n\rangle \phi_n, \quad \forall h \in \mathcal{H},$$

with (ϕ_n) and (ψ_n) being Bessel sequences and m bounded, were introduced and investigated. Further, the similar concept for p-Bessel sequences is considered in [10].

From a theoretical point of view, it is very natural to investigate Bessel and frame multipliers. R. Schatten investigated such operators in [11] for orthonormal sequences. By the spectral theorem, every self-adjoint compact operator on a Hilbert space can be represented as a multiplier using an orthonormal system. Moreover, multipliers generalize the frame operators, as every frame operator S for a frame (ϕ_n) is the multiplier $M_{(1),(\phi_n),(\phi_n)}$. Therefore, the investigation of the invertibility of multipliers in the special case of the identity implies properties of dual systems.

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Multipliers have application as time-variant filters [12,5,4] in acoustical signal processing. Therefore, it is interesting to determine their inverses. If, for example, some operator can be well approximated by a multiplier, we could solve an operator equation numerically (like in computational acoustics, for example for the simulation of vibrations [13]) by inverting the multiplier.

Some properties of the invertibility of multipliers are known [9,14]. In the present paper, we investigate the invertibility of multipliers in much more details. In Section 2, we specify the notation and state the needed results for the paper. The later sections concern the question of the invertibility of multipliers $M_{(m_n),(\phi_n),(\psi_n)}$. Different cases for (ϕ_n) and (ψ_n) are considered. Sections 3, 4 and 5 deal with multipliers where at least one sequence is a Bessel sequence, a frame or a Riesz basis, respectively. Sufficient and/or necessary conditions for the invertibility of $M_{(m_n),(\phi_n),(\psi_n)}$ are given. If the multipliers are invertible, formulas for $M^{-1}_{(m_n),(\phi_n),(\psi_n)}$ are determined and n-term error bounds are given. Examples are given that show the sharpness of the established bounds as well as the independence of the results.

2. Notation and preliminaries

Throughout the paper $\mathcal{H}$ denotes a separable, infinite-dimensional Hilbert space and $(e_n)_{n=1}^{\infty}$ denotes an orthonormal basis of $\mathcal{H}$. The notion *operator* is used for linear mappings. The identity operator on $\mathcal{H}$ is denoted by $I_{\mathcal{H}}$. The operator $G : \mathcal{H} \to \mathcal{H}$ is called *invertible on* $\mathcal{H}$ if there exists a bounded operator $G^{-1} : \mathcal{H} \to \mathcal{H}$ such that $GG^{-1} = G^{-1}G = I_{\mathcal{H}}$ (equivalently, if G is bounded and bijective on $\mathcal{H}$). Throughout the paper, the set $\mathbb{N}$ of the natural numbers is used as an index set, also implicitly.

The notation Φ (resp. Ψ) is used to denote the sequence (ϕ_n) (resp. (ψ_n)) with elements from $\mathcal{H}$; $\Phi - \Psi$ denotes the sequence $(\phi_n - \psi_n)$; m denotes a complex scalar sequence (m_n), and $\overline{m}$ denotes the sequence of the complex conjugates of m_n; $m\Phi$ denotes the sequence $(m_n\phi_n)$. Recall that m is called *semi-normalized* if there exist constants a, b such that $0 < a \leqslant |m_n| \leqslant b < \infty$, $\forall n$. The sequence m is called *positive* (resp. *negative*) if $m_n > 0$, $\forall n$ (resp. $m_n < 0$, $\forall n$).

Recall that Φ is called a *Bessel sequence* (in short, *Bessel*) *for* $\mathcal{H}$ *with bound* B_Φ, if $B_\Phi \in (0, \infty)$ and $\sum |\langle h, \phi_n\rangle|^2 \leqslant B_\Phi \|h\|^2$ for every $h \in \mathcal{H}$. A Bessel sequence Φ with bound B_Φ is called a *frame for* $\mathcal{H}$ *with frame bounds* A_Φ, B_Φ, if $A_\Phi > 0$ and $A_\Phi \|h\|^2 \leqslant \sum |\langle h, \phi_n\rangle|^2$ for every $h \in \mathcal{H}$; A_Φ^{opt} and B_Φ^{opt} denote the optimal frame bounds for Φ. The sequence Φ is called a Riesz basis for $\mathcal{H}$ with bounds A_Φ, B_Φ, if Φ is complete in $\mathcal{H}$, $0 < A_\Phi \leqslant B_\Phi < \infty$ and $A_\Phi \sum |c_n|^2 \leqslant \|\sum c_n\phi_n\|^2 \leqslant B_\Phi \sum |c_n|^2$, $\forall (c_n) \in \ell^2$. Every Riesz basis for $\mathcal{H}$ with bounds A, B is a frame for $\mathcal{H}$ with bounds A, B. For a given sequence Φ, the sequence (ϕ_n^d) is called a *dual of* Φ if $h = \sum \langle h, \phi_n^d\rangle \phi_n = \sum \langle h, \phi_n\rangle \phi_n^d$ for all h in $\mathcal{H}$.

Let Φ be a frame for $\mathcal{H}$. The operator $S_\Phi : \mathcal{H} \to \mathcal{H}$ given by $S_\Phi h = \sum \langle h, \phi_n\rangle \phi_n$ is called the *frame operator for* Φ and fulfills $A_\Phi \|h\| \leqslant \|Sh\| \leqslant B_\Phi \|h\|$ for all h in $\mathcal{H}$. The sequence $\widetilde{\Phi} = (S_\Phi^{-1}\phi_n)$ is a dual frame of Φ, called *the canonical dual of* Φ, with frame operator S_Φ^{-1} and frame bounds $\frac{1}{B_\Phi}$ and $\frac{1}{A_\Phi}$. For standard references for frame theory and related topics see [15–17].

For any Φ, Ψ and any m (called *weight* or *symbol*), the operator $M_{m,\Phi,\Psi}$, given by

$$M_{m,\Phi,\Psi} h = \sum m_n \langle h, \psi_n\rangle \phi_n, \quad h \in \mathcal{H},$$

is called a *multiplier* [9]. Depending on m, Φ, and Ψ, the corresponding multiplier might not be well defined, i.e. might not converge for some $h \in \mathcal{H}$. If Φ and Ψ are Bessel sequences for $\mathcal{H}$ and $m \in \ell^\infty$, then $M_{m,\Phi,\Psi}$ is well defined from $\mathcal{H}$ into $\mathcal{H}$ and $\|M_{m,\Phi,\Psi}\| \leqslant \sqrt{B_\Phi B_\Psi}\,\|m\|_\infty$. Note that the well-definedness of a multiplier $M_{m,\Phi,\Psi}$ does not require Φ and Ψ to be Bessel sequences, examples can be found in [18]. We will need the following result:

Lemma 2.1. *(See [14, Lemma 4.4].) If Φ is a frame for $\mathcal{H}$ and m is positive* (*resp. negative*) *and semi-normalized, then $M_{m,\Phi,\Phi} = S_{(\sqrt{m_n}\,\phi_n)}$ (resp. $M_{m,\Phi,\Phi} = -S_{(\sqrt{|m_n|}\,\phi_n)}$) for the weighted frame $(\sqrt{m_n}\,\phi_n)$ and is therefore invertible on $\mathcal{H}$.*

Note that one should not expect invertibility of $M_{m,\Phi,\Phi}$ for all semi-normalized sequences. Consider for example the frame $\Phi = (e_1, e_1, e_2, e_2, e_3, e_3, \ldots)$ and $m = (1, -1, 1, -1, 1, -1, \ldots)$.

We will use the following criterion for the invertibility of operators:

Proposition 2.2. *Let $F : \mathcal{H} \to \mathcal{H}$ be invertible on $\mathcal{H}$. Suppose that $G : \mathcal{H} \to \mathcal{H}$ is a bounded operator and $\|Gh - Fh\| \leqslant \nu \|h\|$ for all h in $\mathcal{H}$, where $\nu \in [0, \frac{1}{\|F^{-1}\|})$. Then*

(i) *G is invertible on $\mathcal{H}$, $G^{-1} = \sum_{k=0}^{\infty} [F^{-1}(F - G)]^k F^{-1}$, and*

$$\left\| G^{-1} - \sum_{k=0}^{n} \left[F^{-1}(F - G)\right]^k F^{-1} \right\| \leqslant \left\|F^{-1}\right\| \sum_{k=n+1}^{\infty} \left\|F^{-1}(F - G)\right\|^k.$$

(ii) $\frac{1}{\nu + \|F\|} \|h\| \leqslant \|G^{-1}h\| \leqslant \frac{1}{(\frac{1}{\|F^{-1}\|} - \nu)} \|h\|$, $\forall h \in \mathcal{H}$.

Proof. (i) is proved in [19, Theorem 8.1 and Corollary 8.2]. For the upper inequality in (ii), observe that $\|Gh - Fh\| \leqslant \nu\|F^{-1}\|\|Fh\|$ and apply [20, Theorem 1] with $\lambda_1 = \nu\|F^{-1}\| < 1$ and $\lambda_2 = 0$. The lower inequality can be shown by a direct calculation. □

To shorten notation we will call the approximation error $\|G^{-1} - \sum_{k=0}^{n}[F^{-1}(F-G)]^k F^{-1}\|$ the *n-term error*.

Note that having zero elements at "appropriate places" of Φ, Ψ and m, one can get any desired multiplier, for example, the invertible identity operator and the zero operator. Observe that if $M_{m,\Phi,\Psi}$ is invertible on $\mathcal{H}$, then Φ must be complete in $\mathcal{H}$. Therefore, without loss of generality, from now on we consider only sequences m, Φ, and Ψ, which do not contain zero elements, and $\mathbb{N}$ is the index set.

3. Necessary conditions for invertibility of multipliers for Bessel sequences

If the multiplier $M_{(1),\Phi,\Psi}$ is invertible and one of the sequences Ψ and Φ is Bessel, then the other one does not need to be Bessel. For example, consider the sequence $\Phi = (\frac{1}{2}e_1, e_2, \frac{1}{2^2}e_1, e_3, \frac{1}{2^3}e_1, e_4, \ldots)$, which is Bessel for $\mathcal{H}$, and $\Psi = (e_1, e_2, e_1, e_3, e_1, e_4, \ldots)$, which is non-Bessel for $\mathcal{H}$; they satisfy $M_{(1),\Phi,\Psi} = M_{(1),\Psi,\Phi} = I_{\mathcal{H}}$. Below we observe that if one of the sequences is Bessel, invertibility of $M_{(1),\Phi,\Psi}$ implies that the other one must satisfy the lower frame condition.

Proposition 3.1. *Let $M_{m,\Phi,\Psi}$ be invertible on $\mathcal{H}$.*

- (i) *If Ψ (resp. Φ) is a Bessel sequence for $\mathcal{H}$ with bound B, then $m\Phi$ (resp. $m\Psi$) satisfies the lower frame condition for $\mathcal{H}$ with bound $\frac{1}{B\|M_{m,\Phi,\Psi}^{-1}\|^2}$.*
- (ii) *If Ψ (resp. Φ) and $m\Phi$ (resp. $m\Psi$) are Bessel sequences for $\mathcal{H}$, then they are frames for $\mathcal{H}$.*
- (iii) *If Ψ (resp. Φ) is a Bessel sequence for $\mathcal{H}$ and $m \in \ell^\infty$, then Φ (resp. Ψ) satisfies the lower frame condition for $\mathcal{H}$.*
- (iv) *If Ψ and Φ are Bessel sequences for $\mathcal{H}$ and $m \in \ell^\infty$, then Ψ, Φ, $m\Phi$, and $m\Psi$ are frames for $\mathcal{H}$.*

Proof. (i) For brevity, the multiplier $M_{m,\Phi,\Psi}$ will be denoted by M.

First step: $m = (1)$. Assume that Ψ is a Bessel sequence for $\mathcal{H}$ with bound B_Ψ. For those $g \in \mathcal{H}$, for which $\sum|\langle g, \phi_n\rangle|^2 = \infty$ or $g = 0$, clearly the lower frame condition holds. Now let $g \in \mathcal{H}$ be such that $\sum|\langle g, \phi_n\rangle|^2 < \infty$ and $g \neq 0$. For every $f \in \mathcal{H}$,

$$\left|\langle Mf, g\rangle\right| \leqslant \sqrt{B_\Psi}\,\|f\| \Big(\sum\left|\langle \phi_n, g\rangle\right|^2\Big)^{1/2}.$$

For $f = M^{-1}g$, it follows that $\|g\| \leqslant \sqrt{B_\Psi}\,\|M^{-1}\|(\sum|\langle\phi_n, g\rangle|^2)^{1/2}$. Therefore, Φ satisfies the lower frame condition with bound $\frac{1}{B_\Psi\,\|M^{-1}\|^2}$.

The case, when Φ is a Bessel sequence, can be shown in a similar way.

Second step: general m. Apply the first step to the multiplier $M_{(1),m\Phi,\Psi}$ (resp. $M_{(1),\Phi,\overline{m}\Psi}$).

(ii)–(iv) follow now easily. □

Note that Proposition 3.1(i) generalizes one direction of [21, Prop. 3.4], which states that every dual of a Bessel sequence fulfills the lower frame condition. Furthermore, [16, Lemma 5.6.2] states that if a dual of a Bessel sequence is also Bessel, then both sequences are frames. This is a special case of Proposition 3.1(iv).

Note that the boundedness of m is essential for Proposition 3.1(iv) – if $m \notin \ell^\infty$, then a Bessel multiplier $M_{m,\Phi,\Psi}$ can be invertible also in cases when the Bessel sequences Φ and Ψ are not frames. Examples are given in [18].

4. Sufficient conditions for invertibility of multipliers for frames

We begin with a more general statement than Lemma 2.1, allowing different sequences Φ and Ψ in the multiplier.

Proposition 4.1. *Let Φ be a frame for $\mathcal{H}$. Assume that*

$\mathcal{P}_1$: $\exists\mu \in [0, \frac{A_\Phi^2}{B_\Phi})$ *such that* $\sum|\langle h, \psi_n - \phi_n\rangle|^2 \leqslant \mu\|h\|^2, \forall h \in \mathcal{H}$.

Let M denote any one of $M_{m,\Phi,\Psi}$ and $M_{m,\Psi,\Phi}$. For every positive (or negative) semi-normalized sequence m, satisfying

$$0 < a \leqslant |m_n| \leqslant b,\ \forall n, \quad \text{and} \quad \frac{b}{a}\sqrt{\mu} < \frac{A_\Phi}{\sqrt{B_\Phi}}, \tag{1}$$

it follows that Ψ is a frame for $\mathcal{H}$, M is invertible on $\mathcal{H}$ and

$$\frac{1}{bB_\Phi + b\sqrt{\mu B_\Phi}}\|h\| \leqslant \left\|M^{-1}h\right\| \leqslant \frac{1}{aA_\Phi - b\sqrt{\mu B_\Phi}}\|h\|,$$

$$M^{-1} = \begin{cases} \sum_{k=0}^{\infty} [S^{-1}_{(\sqrt{m_n}\phi_n)}(S_{(\sqrt{m_n}\phi_n)} - M)]^k S^{-1}_{(\sqrt{m_n}\phi_n)}, & \text{if } m_n > 0,\ \forall n, \\ -\sum_{k=0}^{\infty} [S^{-1}_{(\sqrt{|m_n|}\phi_n)}(S_{(\sqrt{|m_n|}\phi_n)} + M)]^k S^{-1}_{(\sqrt{|m_n|}\phi_n)}, & \text{if } m_n < 0,\ \forall n, \end{cases} \tag{2}$$

where the n-term error is bounded by $(\frac{1}{aA_\Phi})^{n+1} \cdot \frac{(b\sqrt{\mu B_\Phi})^{n+1}}{aA_\Phi - b\sqrt{\mu B_\Phi}}$.

Proof. If $\mu = 0$, the statement is given in Lemma 2.1. Let $\mu > 0$. First note that we need $\mu < \frac{A_\Phi^2}{B_\Phi}$ in order to be able to fulfill (1). Since Φ is a frame for $\mathcal{H}$ and $B^{opt}_{\Psi-\Phi} < \frac{A_\Phi^2}{B_\Phi} \leqslant A_\Phi$, the perturbation result [16, Corollary 15.1.5] implies that Ψ is also a frame for $\mathcal{H}$. Assume that m is positive and (1) holds. Thus, $M_{m,\Phi,\Psi}$ is well defined on $\mathcal{H}$. The sequence $(\sqrt{m_n}\phi_n)$ is a frame for $\mathcal{H}$ with lower bound aA_Φ and upper bound bB_Φ (see [14, Lemma 4.3]), and thus, $\|S^{-1}_{(\sqrt{m_n}\phi_n)}\| \leqslant \frac{1}{aA_\Phi}$. For every $h \in \mathcal{H}$,

$$\|M_{m,\Phi,\Psi}h - S_{(\sqrt{m_n}\phi_n)}h\| = \|M_{m,\Phi,\Psi-\Phi}\, h\| \leqslant b \cdot \sqrt{\mu B_\Phi}\, \|h\|.$$

Since $b \cdot \sqrt{\mu B_\Phi} < aA_\Phi \leqslant \frac{1}{\|S^{-1}_{(\sqrt{m_n}\phi_n)}\|}$, one can apply Proposition 2.2 to complete the proof.

If m is negative, apply what is already proved to the multiplier $M_{-m,\Phi,\Psi}$.

An analogous proof can be used for the invertibility of $M_{m,\Psi,\Phi}$ and the conclusions for $M^{-1}_{m,\Psi,\Phi}$. $\square$

Remark. The bound for μ in $\mathcal{P}_1$ is sharp. For every $\mu \geqslant \frac{A_\Phi^2}{B_\Phi}$ there exist multipliers $M_{m,\Phi,\Psi}$ which are non-invertible on $\mathcal{H}$, see Example 6.1. Note that invertible multipliers exist for any value of μ, see Example 6.2.

The bound for b/a in (1) is also sharp. If $\mathcal{P}_1$ holds with $\mu > 0$ and $\frac{\sup_n |m_n|}{\inf_n |m_n|} = \frac{A_\Phi}{\sqrt{\mu B_\Phi}}$, the multiplier $M_{m,\Phi,\Psi}$ can be non-invertible on $\mathcal{H}$. For example, consider the sequences Φ and Ψ from Example 6.3 with $k \in (0, \frac{1}{2})$ and $m = (1/k, -1/k, 2, 2, 2, 2, \ldots)$; clearly, $M_{m,\Phi,\Psi}$ is not surjective.

Note that $\mathcal{P}_1$ is equivalent to the following two conditions (see [16]):

- $\exists \nu \in [0, \frac{A_\Phi}{\sqrt{B_\Phi}})$ such that $\|\sum c_n(\psi_n - \phi_n)\| \leqslant \nu \|(c_n)\|_2$ for all finite scalar sequences (c_n) (and thus, for all $(c_n) \in \ell^2$);
- $\exists \mu \in [0, \frac{A_\Phi^2}{B_\Phi})$ such that $\sum |\langle h, \psi_n - \phi_n\rangle|^2 \leqslant \mu \|f\|^2$, for all h in a dense subset of $\mathcal{H}$.

The above result assumed the symbol to be positive or negative, and therefore real. Now we give sufficient conditions for invertibility of multipliers allowing m to be complex.

Proposition 4.2. *Let Φ be a frame for $\mathcal{H}$ and $\mathcal{P}_1$ hold. Let (m_n) satisfy*

$$|m_n - 1| \leqslant \lambda < \frac{A_\Phi - \sqrt{\mu B_\Phi}}{B_\Phi + \sqrt{\mu B_\Phi}}, \quad \forall n \in \mathbb{N}, \tag{3}$$

for some λ, and let M denote any one of $M_{m,\Phi,\Psi}$ and $M_{m,\Psi,\Phi}$. Then Ψ is a frame for $\mathcal{H}$, $M_{m,\Phi,\Phi}$ and M are invertible on $\mathcal{H}$, and

$$\frac{1}{(\lambda+1)B_\Phi}\|h\| \leqslant \left\|M^{-1}_{m,\Phi,\Phi}h\right\| \leqslant \frac{1}{A_\Phi - \lambda B_\Phi}\|h\|,$$
$$\frac{1}{(\lambda+1)(B_\Phi + \sqrt{\mu B_\Phi})}\|h\| \leqslant \left\|M^{-1}h\right\| \leqslant \frac{1}{A_\Phi - \lambda B_\Phi - (\lambda+1)\sqrt{\mu B_\Phi}}\|h\|,$$
$$M^{-1}_{m,\Phi,\Phi} = \sum_{k=0}^{\infty}\left[S_\Phi^{-1}(S_\Phi - M_{m,\Phi,\Phi})\right]^k S_\Phi^{-1},$$

where the n-term error is bounded by $(\frac{\lambda B_\Phi}{A_\Phi})^{n+1} \cdot \frac{1}{A_\Phi - \lambda B_\Phi}$, *and*

$$M^{-1} = \sum_{k=0}^{\infty}\left[M^{-1}_{m,\Phi,\Phi}(M_{m,\Phi,\Phi} - M)\right]^k M^{-1}_{m,\Phi,\Phi},$$

where the n-term error is bounded by $(\frac{(\lambda+1)\sqrt{\mu B_\Phi}}{A_\Phi - \lambda B_\Phi})^{n+1} \cdot \frac{1}{A_\Phi - \lambda B_\Phi - (\lambda+1)\sqrt{\mu B_\Phi}}$.

Proof. *First step:* $\Psi = \Phi$. By the assumptions (3), one can apply Proposition 4.1 to the multiplier $M_{(1),m\Phi,\Phi}$. This implies validity of the conclusions for $M^{-1}_{m,\Phi,\Phi}$.

Second step: general Ψ. If $\mathcal{P}_1$ holds with $\mu = 0$, then $\Psi = \Phi$ and the statement is already proved. Assume that $\mathcal{P}_1$ holds with $\mu > 0$. As in Proposition 4.1, it follows that Ψ is a frame for $\mathcal{H}$. For every $h \in \mathcal{H}$,

$$\|M_{m,\Phi,\Psi}h - M_{m,\Phi,\Phi}h\| = \|M_{m,\Phi,\Psi-\Phi}h\| \leqslant (\lambda+1)\sqrt{\mu B_\Phi}\|h\|.$$

Since

$$(\lambda+1)\sqrt{\mu B_\Phi} < A_\Phi - \lambda B_\Phi \leqslant \frac{1}{\|M^{-1}_{m,\Phi,\Phi}\|},$$

Proposition 2.2 completes the proof. Similar arguments hold for $M_{m,\Psi,\Phi}$. $\square$

Remark. As in Proposition 4.1, the bound for μ in Proposition 4.2 is sharp. If $\mathcal{P}_1$ holds with $\mu = 0$, then the bound for λ in Proposition 4.2 is sharp – if the assumptions hold with $\lambda = \frac{A_\Phi}{B_\Phi}$, then the multiplier $M_{m,\Phi,\Phi}$ might be non-invertible on $\mathcal{H}$. Consider for example $M_{(\frac{1}{n}),(e_n),(e_n)}$ which is injective but not surjective.

Proposition 4.3. *Let Φ be a frame for $\mathcal{H}$, $G : \mathcal{H} \to \mathcal{H}$ be a bounded bijective operator and $\psi_n = G\phi_n$, $\forall n$, i.e. Φ and Ψ are equivalent frames.*[1] *Let m be semi-normalized and satisfy one of the following three conditions: m is positive; m is negative; or there exists λ with $|m_n - 1| \leqslant \lambda < A_\Phi/B_\Phi$, $\forall n \in \mathbb{N}$. Then Ψ is a frame for $\mathcal{H}$, the multipliers $M_{m,\Phi,\Psi}$ and $M_{m,\Psi,\Phi}$ are invertible on $\mathcal{H}$, $M^{-1}_{m,\Phi,\Psi} = (G^{-1})^* M^{-1}_{m,\Phi,\Phi}$ and $M^{-1}_{m,\Psi,\Phi} = M^{-1}_{m,\Phi,\Phi} G^{-1}$.*

Proof. By [16, Corollary 5.3.2], Ψ is a frame for $\mathcal{H}$. Observe that $M_{m,\Phi,\Psi} = M_{m,\Phi,\Phi}G^*$ and $M_{m,\Psi,\Phi} = GM_{m,\Phi,\Phi}$. Now the conclusions follow from Lemma 2.1 and Proposition 4.2. $\square$

As stated in the Introduction and in the comments after Proposition 3.1, the invertibility of multipliers is related to the topic of dual frames. Note that the above proposition covers the case when Ψ is the canonical dual of Φ and does not cover any other dual frame of Φ. Indeed, if $\Psi = (G\phi_n)$ is a dual frame of Φ for some bounded operator G, then G must coincide with S_Φ^{-1}, and thus, Ψ must be the canonical dual of Φ, see [23, pp. 19–20]. For other duals, the following statement can be used.

Proposition 4.4. *Let Φ be a frame for $\mathcal{H}$ and $\Phi^d = (\phi_n^d)$ be a dual frame of Φ. Let M denote any one of M_{m,Φ,Φ^d} and $M_{m,\Phi^d,\Phi}$, and let (m_n) be such that $|m_n - 1| \leqslant \lambda < \frac{1}{\sqrt{B_\Phi B_{\Phi^d}}}$ for all n in $\mathbb{N}$, for some λ. Then M is invertible on $\mathcal{H}$,*

$$\frac{1}{1+\lambda\sqrt{B_\Phi B_{\Phi^d}}}\|h\| \leqslant \left\|M^{-1}h\right\| \leqslant \frac{1}{1-\lambda\sqrt{B_\Phi B_{\Phi^d}}}\|h\|, \quad \forall h \in \mathcal{H},$$

$$M^{-1}_{m,\Phi,\Phi^d} = \sum_{k=0}^{\infty}(M_{(1-m_n),\Phi,\Phi^d})^k \quad and \quad M^{-1}_{m,\Phi^d,\Phi} = \sum_{k=0}^{\infty}(M_{(1-m_n),\Phi^d,\Phi})^k, \tag{4}$$

and the n-term error is bounded by $\frac{(\lambda\sqrt{B_\Phi B_{\Phi^d}})^{n+1}}{1-\lambda\sqrt{B_\Phi B_{\Phi^d}}}$.

Proof. The case $\lambda = 0$ is trivial. Consider the case $\lambda > 0$. For every $h \in \mathcal{H}$,

$$\|M_{m,\Phi,\Phi^d}h - h\| = \|M_{(m_n-1),\Phi,\Phi^d}h\| \leqslant \lambda\sqrt{B_\Phi B_{\Phi^d}}\|h\|.$$

Now apply Proposition 2.2. Since Φ is a dual of Φ^d, the conclusions for $M_{m,\Phi^d,\Phi}$ follow directly from what is already proved. $\square$

Remark. The bound for λ in Proposition 4.4 is sharp – if the assumptions hold with $\lambda = \sqrt{B_\Phi B_{\Phi^d}}$, then the multiplier might be non-invertible on $\mathcal{H}$. Consider for example the multiplier $M_{(\frac{1}{n}),(e_n),(e_n)}$ which is injective but not surjective.

For multipliers in the form $M_{m,\Phi,\widetilde{\Phi}}$ we could use either Proposition 4.3 or 4.4. The second one gives, in general, the more convenient inversion formula.

By Proposition 4.1, when Ψ is a perturbation of Φ, the inverse operator of $M_{(1),\Psi,\Phi}$ is given by $M^{-1}_{(1),\Psi,\Phi} = \sum_{k=0}^{\infty}[S_\Phi^{-1}(S_\Phi - M_{(1),\Psi,\Phi})]^k S_\Phi^{-1}$. A simpler representation for $M^{-1}_{(1),\Psi,\Phi}$ can be obtained if Ψ is a perturbation of a dual frame of Φ:

[1] For a treatise of equivalence of frames see [22] for the continuous and [15] for the discrete setting.

Proposition 4.5. *Let Φ be a frame for $\mathcal{H}$. Assume that*

$$\mathcal{P}_2: \quad \exists\mu \in [0, \tfrac{1}{B_\Phi}) \text{ such that } \sum |\langle h, m_n\psi_n - \phi_n^d\rangle|^2 \leqslant \mu\|h\|^2,\ \forall h \in \mathcal{H},$$

for some dual frame $\Phi^d = (\phi_n^d)$ of Φ. Let M denote any one of $M_{\overline{m},\Phi,\Psi}$ and $M_{m,\Psi,\Phi}$. Then $m\Psi$ is a frame for $\mathcal{H}$, M is invertible on $\mathcal{H}$ and

$$\frac{1}{1+\sqrt{\mu B_\Phi}}\|h\| \leqslant \|M^{-1}h\| \leqslant \frac{1}{1-\sqrt{\mu B_\Phi}}\|h\|, \quad \forall h \in \mathcal{H}.$$

Furthermore, $M^{-1} = \sum_{k=0}^{\infty}(I_{\mathcal{H}} - M)^k$, and the n-term error is bounded by $\frac{(\sqrt{\mu B_\Phi})^{n+1}}{1-\sqrt{\mu B_\Phi}}$.

Proof. The case $\mu = 0$ is trivial. Assume that $\mu > 0$. Since Φ^d is a dual frame of Φ, the number $\frac{1}{B_\Phi}$ is a lower bound for Φ^d (see the proof of [21, Prop. 3.4]). Since $\mu < A^{opt}_{\Phi^d}$, it follows from [16, Corollary 15.1.5] that $m\Psi$ is a frame for $\mathcal{H}$. Therefore, $M_{\overline{m},\Phi,\Psi}$ is well defined on $\mathcal{H}$. For every $h \in \mathcal{H}$,

$$\|M_{\overline{m},\Phi,\Psi}h - h\| = \|M_{(1),(\phi_n),(m_n\psi_n-\phi_n^d)}h\| \leqslant \sqrt{\mu B_\Phi}\|h\|$$

and similarly, $\|M_{m,\Psi,\Phi}h - h\| \leqslant \sqrt{\mu B_\Phi}\|h\|$. Now apply Proposition 2.2. □

Similar equivalences as for $\mathcal{P}_1$ hold for $\mathcal{P}_2$.

Remark. The bound for μ in $\mathcal{P}_2$ is sharp. For every $\mu \geqslant 1/B_\Phi$ there exist multipliers $M_{\overline{m},\Phi,\Psi}$ and $M_{m,\Psi,\Phi}$ which are non-invertible on $\mathcal{H}$, see Example 6.1. Note that invertible multipliers exist for any value of μ, see Example 6.2.

Remark. When Φ is a tight frame with $A = B = 1$ (so-called Parseval frame), then Φ is self-dual and both Propositions 4.1 and 4.5 can be applied to multipliers in the form $M_{(1),\Phi,\Psi}$. Note that Propositions 4.1 and 4.5 do not cover the same classes of sequences. Example 6.3 (resp. 6.4) shows a multiplier $M_{(1),\Phi,\Psi}$ for which Proposition 4.1 applies (resp. does not apply), but Proposition 4.5 does not apply (resp. applies).

5. Sufficient and necessary conditions for invertibility of multipliers for Riesz bases

For two Riesz bases and a semi-normalized symbol, the multipliers are always invertible [9, Prop. 7.7]. If Φ is a Riesz basis for $\mathcal{H}$, m is real and semi-normalized, and Ψ is a frame for $\mathcal{H}$, then the multiplier $M_{m,\Phi,\Psi}$ (resp. $M_{m,\Psi,\Phi}$) is invertible on $\mathcal{H}$ if and only if Ψ is a Riesz basis for $\mathcal{H}$ [24, Prop. 4.2]. What can be said about the cases, when one of the sequences has the Riesz property, m is complex and not necessarily semi-normalized, and Ψ is not necessarily a frame? The answer is given in the following assertion, whose proof is quite technical and lengthy, so we only give the sketch for a proof.

Theorem 5.1. *Let Φ be a Riesz basis for $\mathcal{H}$. Then the following hold.*

(i) *If Ψ is a Riesz basis for $\mathcal{H}$, then $M_{m,\Phi,\Psi}$ (resp. $M_{m,\Psi,\Phi}$) is invertible on $\mathcal{H}$ if and only if m is semi-normalized.*
(ii) *If m is semi-normalized, then $M_{m,\Phi,\Psi}$ (resp. $M_{m,\Psi,\Phi}$) is invertible on $\mathcal{H}$ if and only if Ψ is a Riesz basis for $\mathcal{H}$.*
(iii) *If m is not semi-normalized, then $M_{m,\Phi,\Psi}$ (resp. $M_{m,\Psi,\Phi}$) can be invertible on $\mathcal{H}$ only in the following cases:*
($\mathcal{R}_1$): *Ψ is Bessel for $\mathcal{H}$, which is not a frame for $\mathcal{H}$ and not norm-bounded from below, m is norm-bounded from below and $m \notin \ell^\infty$;*
($\mathcal{R}_2$): *Ψ is non-Bessel for $\mathcal{H}$ which is norm-bounded from below and not norm-bounded, $m \in \ell^\infty$, and m is not norm-bounded from below;*
($\mathcal{R}_3$): *Ψ is non-Bessel for $\mathcal{H}$ which is neither norm-bounded from above nor norm-bounded from below, m is not norm-bounded from below, and $m \notin \ell^\infty$.*

In the cases of invertibility, $M^{-1}_{m,\Phi,\Psi} = M_{(1),\widetilde{\overline{m}\Psi},\widetilde{\Phi}}$ and $M^{-1}_{m,\Psi,\Phi} = M_{(1),\widetilde{\Phi},\widetilde{m\Psi}}$. For the cases (i) *and* (ii) *this is equivalent to $M^{-1}_{m,\Phi,\Psi} = M_{(\frac{1}{m_n}),\widetilde{\Psi},\widetilde{\Phi}}$ and $M^{-1}_{m,\Psi,\Phi} = M_{(\frac{1}{m_n}),\widetilde{\Phi},\widetilde{\Psi}}$.*

Sketch of the proof. (ii) Let m be semi-normalized. If Ψ is not Bessel for $\mathcal{H}$, then $M_{m,\Phi,\Psi}$ (resp. $M_{m,\Psi,\Phi}$) is not well defined. If Ψ is Bessel for $\mathcal{H}$, which is not a frame for $\mathcal{H}$, then $M_{m,\Phi,\Psi}$ (resp. $M_{m,\Psi,\Phi}$) is not invertible on $\mathcal{H}$, see Proposition 3.1. If Ψ is an overcomplete frame for $\mathcal{H}$ and m is real, the non-invertibility of the multipliers is proved in [24, Prop. 4.2]. Similar arguments can be used in the cases when m is complex. If Ψ is a Riesz basis for $\mathcal{H}$, see [9, Prop. 7.7].

(iii) Assume that m is not semi-normalized. By (ii), $M_{m,\Phi,\Psi}$ (resp. $M_{m,\Psi,\Phi}$) is invertible on $\mathcal{H}$ if and only if $\overline{m}\Psi$ (resp. $m\Psi$) is a Riesz basis for $\mathcal{H}$ if and only if $m\Psi$ is a Riesz basis for $\mathcal{H}$. When m is real, it is proved in [24, Prop. 3.2] that

$m\Psi$ can be a Riesz basis for $\mathcal{H}$ only in the case $(\mathcal{R}_1)$ or in the case $(\widetilde{\mathcal{R}})$: *$\Psi$ is non-Bessel for $\mathcal{H}$ which is not norm-bounded from above and m is not norm-bounded from below with $m_n \neq 0$, $\forall n$.* Similar arguments can be used for the cases when m is complex. Furthermore, one can prove that among the combinations in $(\widetilde{\mathcal{R}})$, $m\Psi$ can be a Riesz basis only in the cases $(\mathcal{R}_2)$ and $(\mathcal{R}_3)$.

(i) follows from (ii) and (iii).

The representations for $M^{-1}_{m,\Phi,\Psi}$ and $M^{-1}_{m,\Psi,\Phi}$ follow from [9, Prop. 7.7]. $\square$

6. Examples

In this section we list some examples, which we refer to throughout the paper.

Example 6.1. Let $\Phi = (e_n)$ and $\Psi = (ke_1, \frac{1}{2}e_2, \frac{1}{3}e_3, \frac{1}{4}e_4, \ldots)$ for some real number k. The sequence $\Psi - \Phi (= \Psi - \Phi^d)$ is Bessel for $\mathcal{H}$ with optimal bound

$$B^{opt}_{\Psi-\Phi} = \begin{cases} |k-1|^2 > 1 = 1/B_\Phi = A^2_\Phi/B_\Phi, & \text{when } |k-1| > 1, \\ 1 = 1/B_\Phi = A^2_\Phi/B_\Phi, & \text{when } |k-1| \leqslant 1, \end{cases}$$

which shows that the example fulfills $\mathcal{P}_1$ (resp. $\mathcal{P}_2$) with any $\mu \geqslant A^2_\Phi/B_\Phi$ (resp. $\mu \geqslant 1/B_\Phi$). The multipliers $M_{(1),\Phi,\Psi}$ and $M_{(1),\Psi,\Phi}$ are non-invertible on $\mathcal{H}$.

Example 6.2. Let $\Phi = (e_n)$ and $\Psi = (ke_1, e_2, e_3, e_4, \ldots)$, where $k \neq 0$. The sequence $\Psi - \Phi (= \Psi - \Phi^d)$ is Bessel for $\mathcal{H}$ with optimal bound $\mu = |k-1|^2$. The multipliers $M_{(1),\Phi,\Psi}$ and $M_{(1),\Psi,\Phi}$ are invertible on $\mathcal{H}$.

Example 6.3. Consider the frames $\Phi = (e_1, e_1, e_2, e_3, e_4, e_5, \ldots)$ and $\Psi = ((k+1)\phi_n)$, where $k \in (0, \frac{1}{2})$. The sequence $\Psi - \Phi$ is Bessel for $\mathcal{H}$ with the optimal bound $B^{opt}_{\Psi-\Phi} = 2k^2 < \frac{(A^{opt}_\Phi)^2}{B^{opt}_\Phi}$. Furthermore, $1 < \frac{1}{2k} = \frac{A^{opt}_\Phi}{\sqrt{B^{opt}_{\Psi-\Phi} B^{opt}_\Phi}}$. Thus, Proposition 4.1 implies the invertibility of $M_{(1),\Phi,\Psi}$ and $M_{(1),\Psi,\Phi}$.

Now observe that all the dual frames of Φ are among the sequences $(h, e_1 - h, e_2, e_3, e_4, \ldots)$, $h \in \mathcal{H}$. Let $\Phi^d = (\phi^d_1, e_1 - \phi^d_1, e_2, e_3, e_4, \ldots)$ be an arbitrary chosen dual frame of Φ and denote $\langle e_1, \phi^d_1 \rangle = x + iy$. Assume that $\Psi - \Phi^d$ is Bessel for $\mathcal{H}$ with bound $B_{\Psi-\Phi^d} \leqslant \frac{1}{2} = \frac{1}{B^{opt}_\Phi}$. Applying the Bessel inequality to the element e_1, we obtain $2k^2 + 2x^2 + 2k - 2x + 2y^2 + 1 \leqslant \frac{1}{2}$, which never holds. Thus, the invertibility of $M_{(1),\Phi,\Psi}$ and $M_{(1),\Psi,\Phi}$ cannot be concluded from Proposition 4.5.

Example 6.4. Consider the frame $\Phi = (e_1, e_1, e_1, e_2, e_2, e_2, e_3, e_3, e_3, \ldots)$ and its dual frame $\Psi = (e_1, e_2, -e_2, e_2, e_3, -e_3, e_3, e_4, -e_4, \ldots)$. Clearly, $M_{(1),\Phi,\Psi} = M_{(1),\Psi,\Phi} = I_\mathcal{H}$ and the assumptions of Proposition 4.5 are fulfilled with $\mu = 0$. Since $\frac{(A^{opt}_\Phi)^2}{B^{opt}_\Phi} = 3$ and $B^{opt}_{\Psi-\Phi} \geqslant 4$, Proposition 4.1 does not apply.

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