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# Timelike Bonnet surfaces in Lorentzian space forms

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**Abstract**

We study timelike surfaces in Lorentzian space forms which admit a one-parameter family of isometric deformations preserving the mean curvature.


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*Keywords:* Timelike surfaces; Bonnet surfaces; $B$-scrolls; Lorentz surfaces

## Introduction

Surfaces which admit a one-parameter family of isometric deformations preserving the mean curvature are called Bonnet surfaces after a result due to Bonnet [4]:

**Proposition.** *If a surface with constant mean curvature is not totally umbilic, then it admits a one-parameter family of isometric deformations preserving the mean curvature.*

In the following we assume that surfaces are sufficiently smooth and contain no umbilic points.

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Spacelike Bonnet surfaces in space forms have been studied by many differential geometers for long years (see [1,2,5,6,8,10,12,14] and references therein). On the contrary very little is known about timelike Bonnet surfaces in Lorentzian space forms, which we shall study in this paper.

An outline of this paper is as follows. As we shall see later, a timelike surface in $\mathfrak{M}^3_\nu(c)$, where $\mathfrak{M}^3_\nu(c)$ is the complete 3-dimensional space form of curvature $c$ with signature $(3-\nu, \nu)$, is given by a conformal immersion $F$ from a Lorentz surface $M$ to $\mathfrak{M}^3_\nu(c)$, and $F$ has two quadratic differentials $Q\,dx^2$ and $R\,dy^2$, called the Hopf differentials of $F$, where $(x, y)$ is a null coordinate on $M$ (see [15] for more about Lorentz surfaces). We prove that if $QR = 0$ then $F$ is a Bonnet surface and it is a $B$-scroll (cf. [7,11,13]). If $F$ is a Bonnet surface with $QR \neq 0$, then we can prove that $Q = \pm R$ for a suitable choice of a null coordinate (we call such a surface to be $\pm$isothermic), and $1/Q$ and $1/R$ are Lorentz-harmonic with respect to the above coordinate, which is a result analogous to that by Graustein [10] and Raffy [14] for Bonnet surfaces with definite induced metric. Furthermore, if the mean curvature $H$ is $\pm$Lorentz-holomorphic and $dH \neq 0$, then $\pm$isothermic parametrization implies that $F$ is flat and enables us to calculate the first and the second fundamental forms explicitly. In the case that $H$ is not $\pm$Lorentz-holomorphic and $H_x H_y \neq 0$, we can reduce the Gauss–Codazzi equations for $F$ to an ordinary differential equation of the third order, which is called a generalized Hazzidakis equation (see [1–3,5,6,12]). Following our previous paper [8], we shall also study timelike $\pm$isothermic Bonnet surfaces with constant curvature and find that they are parametrized by curves in 2-dimensional Riemannian or Lorentzian space forms with specific geodesic curvature.

## 1. Preliminaries

In order to study surfaces in the Lorentzian space form $\mathfrak{M}^3_\nu(c)$, we may assume that $c = 0, \pm 1$. In the following we consider only the case $\nu = 1$ for simplicity (we can carry out the similar computation for the case $\nu = 2$). We define a scalar product $\langle\, ,\rangle_c$ on $\mathbf{R}^4$ by

$$\langle a, b\rangle_0 = -a_1b_1 + a_2b_2 + a_3b_3,$$
$$\langle a, b\rangle_1 = -a_0b_0 + a_1b_1 + a_2b_2 + a_3b_3,$$
$$\langle a, b\rangle_{-1} = -a_0b_0 - a_1b_1 + a_2b_2 + a_3b_3,$$

where $a = (a_0, a_1, a_2, a_3)$, $b = (b_0, b_1, b_2, b_3) \in \mathbf{R}^4$. $\mathfrak{M}^3_1(c)$ is embedded in $\mathbf{R}^4$ by

$$\mathfrak{M}^3_1(0) = \mathbf{E}^3_1 = \left(\left\{p \in \mathbf{R}^4;\ p_0 = 0\right\}, \langle\, ,\rangle_0\right); \quad \text{the Minkowski 3-space},$$
$$\mathfrak{M}^3_1(1) = S^3_1 = \left\{p \in \mathbf{R}^4;\ \langle p, p\rangle_1 = 1\right\}; \quad \text{the de Sitter 3-space},$$
$$\mathfrak{M}^3_1(-1) = H^3_1 = \left\{p \in \mathbf{R}^4;\ \langle p, p\rangle_{-1} = -1\right\}; \quad \text{the anti de Sitter 3-space}.$$

Note that a timelike surface in $\mathfrak{M}^3_1(c)$ is given by a conformal immersion $F$ from an oriented Lorentz surface $M$ to $\mathfrak{M}^3_1(c)$. Using a null coordinate $(x, y)$ on $M$, we can write the induced metric on $M$ as $\mathrm{e}^u dx\,dy$. Let $N$ be a unit normal to $F$. The Gauss–Codazzi equations for $F$ have the following form:

$$\begin{cases} u_{xy} + \frac{1}{2}(H^2 + c)\mathrm{e}^u - 2QR\mathrm{e}^{-u} = 0, \\ Q_y = \frac{1}{2}\mathrm{e}^u H_x, \\ R_x = \frac{1}{2}\mathrm{e}^u H_y, \end{cases} \tag{1.1}$$

where $\langle F_{xy}, N\rangle_c = \frac{1}{2}H\mathrm{e}^u$, $\langle F_{xx}, N\rangle_c = Q$ and $\langle F_{yy}, N\rangle_c = R$. The function $H$ and the quadratic differentials $Q\,dx^2$, $R\,dy^2$ are independent of the choice of $(x, y)$, which are called the mean curvature and the Hopf differentials of $F$, respectively. The (intrinsic Gaussian) curvature $K$ is defined by

$$K = H^2 + c - 4QR\mathrm{e}^{-2u}. \tag{1.2}$$

The extrinsic (Gaussian) curvature is $K - c$. Note that the first equation of (1.1) (the Gauss equation) implies that

$$K = -2u_{xy}\mathrm{e}^{-u}. \tag{1.3}$$

Since we assume that $F$ is umbilic-free, we have $Q$ or $R \neq 0$. In the following we divide our study into two cases: (i) $Q \neq 0$, $R \equiv 0$ or $Q \equiv 0$, $R \neq 0$, (ii) $Q, R \neq 0$.

## 2. $B$-scrolls

Before studying the case (i) in the previous section, we shall give several definitions.

**Definition 2.1.** A curve $\gamma$ in $\mathfrak{M}^3_1(c)$ is called a null Frenet curve if it admits a frame field $(A, B, C)$ such that

$$\begin{cases} \frac{dA}{dx} = \kappa C, \\ \frac{dB}{dx} = -c\gamma + \tau C, \\ \frac{dC}{dx} = -\tau A - \kappa B, \end{cases}$$

where $A = \frac{d\gamma}{dx}$, $\langle A, A\rangle_c = \langle B, B\rangle_c = 0$, $\langle A, B\rangle_c = 1$ and $C$ is a vector product of $A$ and $B$ on $T\mathfrak{M}^3_1(c)$. $\kappa$ and $\tau$ are called the curvature and the torsion of $\gamma$, respectively.

**Definition 2.2.** Let $\gamma$ be a null Frenet curve in $\mathfrak{M}^3_1(c)$. A surface $F$ in $\mathfrak{M}^3_1(c)$ defined by

$$F(x, y) = \gamma(x) + yB(x)$$

is called a $B$-scroll of $\gamma$.

Let $F$ be a timelike surface with $Q \neq 0$, $R \equiv 0$ or $Q \equiv 0$, $R \neq 0$. Then it follows that $F$ is a Bonnet surface.

**Theorem 2.1.** *$F$ is a $B$-scroll.*

**Proof.** We consider only the case in which $Q \neq 0$, $R \equiv 0$, for simplicity. Since the case $K = 0$ is due to Dajczer and Nomizu [7] and Graves [11], we have only to consider the case $K \neq 0$ (in the case of $K = 1$, $c = 0$, we shall give an alternative proof of McNertney's [13]).

Note that we can solve (1.1) explicitly:

$$u = \log \frac{-4\frac{df}{dx}\frac{dg}{dy}}{(H^2 + c)(f + g)^2}, \qquad Q = \frac{2\frac{dH}{dx}\frac{df}{dx}}{(H^2 + c)(f + g)} + v,$$

where $f$ and $g$ are functions of $x$ and $y$ only, respectively, such that

$$\frac{\frac{df}{dx}\frac{dg}{dy}}{H^2+c}<0$$

and $v$ is a function of $x$ only.

Let $\gamma$ be a null Frenet curve such that $\kappa \circ w = [(H^2+c)^2 v]/[4(df/dx)^2]$, $\tau \circ w = H$. Then a direct computation shows that $F$ is given by

$$F = \gamma \circ w + \frac{1}{f+g} B \circ w,$$

where $w = \int \frac{2\frac{df}{dx}}{H^2+c}\,dx$.

## 3. Timelike ±isothermic Bonnet surfaces

If $F$ is a timelike Bonnet surface with $Q, R \neq 0$ then there exists a real-valued function $\lambda$ of $x$ and $y$ such that

$$\begin{cases} (\lambda Q)_y = \frac{1}{2}\mathrm{e}^u H_x, \\ (\lambda^{-1} R)_x = \frac{1}{2}\mathrm{e}^u H_y. \end{cases}$$

Combining the second and the third equations of (1.1) (the Codazzi equations), we have

$$\begin{cases} (\lambda - 1)Q = f(x), \\ (\lambda^{-1} - 1)R = g(y), \end{cases}$$

where $f$ and $g$ are real-valued functions of $x$ and $y$, respectively. Then we obtain $gQ + fR = -fg$. Changing the null coordinate, we may assume that $\rho Q + \sigma R = 1$, where $\rho, \sigma = \pm 1$. Since we have $Q = \frac{f(\sigma g+1)}{\rho f - \sigma g}$ and $R = \frac{-g(\rho f+1)}{\rho f - \sigma g}$, changing the null coordinate again, we may assume that $Q = R = \frac{-fg}{f+g}$ or $Q = -R = \frac{fg}{f-g}$.

**Definition 3.1.** A timelike surface $F : M \to \mathfrak{M}^3_1(c)$ is said to be isothermic (respectively anti-isothermic) if $Q = R$ (respectively $Q = -R$) for a suitable choice of a null coordinate, which is called an isothermic (respectively an anti-isothermic) coordinate.

Computations as above leads to the following:

**Theorem 3.1.** *$F : M \to \mathfrak{M}^3_1(c)$ is a timelike Bonnet surface with $Q, R \neq 0$ if and only if it is ±isothermic, i.e., isothermic or anti-isothermic and $1/Q$, $1/R$ are Lorentz-harmonic with respect to the ±isothermic coordinate $(x, y)$, i.e., $(1/Q)_{xy} = (1/R)_{xy} = 0$.*

## 4. Timelike ±isothermic Bonnet surfaces with ±Lorentz-holomorphic mean curvature

Let $F : M \to \mathfrak{M}^3_1(c)$ be a timelike ±isothermic Bonnet surface and $(x, y)$ a ±isothermic coordinate. In this section we consider the case that $Q$ and $R$ are ±Lorentz-holomorphic, i.e., they are functions of $x$ or $y$ only, which implies automatically that $1/Q$ and $1/R$ are Lorentz-harmonic. From the Codazzi equations, we have

**Proposition 4.1.** *$Q$ and $R$ are Lorentz-holomorphic (respectively anti Lorentz-holomorphic) if and only if $H$ is anti Lorentz-holomorphic (respectively Lorentz-holomorphic).*

From Proposition 4.1 and (1.3), we have

**Proposition 4.2.** *If $dH \neq 0$, then $F$ is flat.*

In the following we assume that $dH \neq 0$. For simplicity we consider only the case that $Q = \varepsilon R = \frac{1}{2} f(x)$ and $H = g(y)$, where $\varepsilon = \pm 1$, and $f$ and $g$ are functions of $x$ and $y$, respectively. Then (1.1) is equivalent to

$$\begin{cases} (g^2 + c)(\frac{df}{dx})^2 = \varepsilon (f \frac{dg}{dy})^2, \\ \frac{df}{dx} = \varepsilon \mathrm{e}^u \frac{dg}{dy}. \end{cases} \tag{4.1}$$

Note that $f$, $g^2 + c$, $\frac{df}{dx}$, $\frac{dg}{dy} \neq 0$ by the assumption. It is obvious to see that $\varepsilon = 1$ if $c = 0, 1$. Solving (4.1) directly, we obtain

**Theorem 4.1.**

$$f = C\mathrm{e}^{\pm\alpha x}, \qquad g = \begin{cases} C_1 \mathrm{e}^{\alpha y} + C_2 \mathrm{e}^{-\alpha y} & \text{if } F \text{ is isothermic}, \\ C_1 \cos \alpha y + C_2 \sin \alpha y & \text{if } c = -1,\ F \text{ is anti-isothermic}, \end{cases}$$

*where $C \in \mathbf{R} \setminus \{0\}$, $\alpha > 0$ and $(C_1, C_2) \in \mathbf{R}^2 \setminus \{0\}$ such that*

$$\begin{cases} 4C_1C_2 + c = 0, \quad \frac{df}{dx}\frac{dg}{dy} > 0 & \text{if } F \text{ is isothermic}, \\ C_1^2 + C_2^2 = 1, \quad \frac{df}{dx}\frac{dg}{dy} < 0 & \text{if } c = -1,\ F \text{ is anti-isothermic}. \end{cases}$$

**Remark 4.1.** When $c = 0$, we can solve the Gauss–Weingarten formulas for $F$ explicitly. See Section 6 Remark 6.3.

## 5. Timelike ±isothermic Bonnet surfaces with non ±Lorentz-holomorphic mean curvature

Let $F : M \to \mathfrak{M}_1^3(c)$ be a timelike ±isothermic Bonnet surface and $(x, y)$ a ±isothermic coordinate as in the previous section. We assume that $H$ is not ±Lorentz-holomorphic and $H_x, H_y \neq 0$. Then by Theorem 3.1 we have $Q = \varepsilon R = \frac{1}{f(x)+g(y)}$, where $\varepsilon = \pm 1$, $f$ and $g$ are functions of $x$ and $y$ respectively such that $f$, $g$, $\frac{df}{dx}$, $\frac{dg}{dy} \neq 0$. From the Codazzi equations, we have $\frac{df}{dx} H_x = \varepsilon \frac{dg}{dy} H_y$. Hence if we put

$$s = \int \frac{dx}{\frac{df}{dx}} + \varepsilon \int \frac{dy}{\frac{dg}{dy}}$$

then $H$ depends on $s$ only. The Codazzi equations become

$$\mathrm{e}^u = -\frac{2\frac{df}{dx}\frac{dg}{dy}}{(f+g)^2 H_s}. \tag{5.1}$$

Then the Gauss equation becomes a generalized Hazzidakis equation:

$$\left\{\left(\frac{H_{ss}}{H_s}\right)_s - H_s\right\}S^2 = \varepsilon\left(2 - \frac{H^2+c}{H_s}\right), \tag{5.2}$$

where

$$S = \frac{f+g}{\frac{df}{dx}\frac{dg}{dy}}.$$

In the following we shall consider two cases: (i) $2 - \frac{H^2+c}{H_s} = 0$, (ii) $2 - \frac{H^2+c}{H_s} \neq 0$.

Note that (5.2) is always satisfied in the case of (i). Combining this with the condition (5.1), we have

**Theorem 5.1.** *If* $2 - \frac{H^2+c}{H_s} = 0$ *then one of the following holds*:

(i) $c = 0$, $\frac{df}{dx}\frac{dg}{dy} < 0$, $H = -\frac{2}{s+C}$, *where* $C \in \mathbf{R}$,
(ii) $c = 1$, $\frac{df}{dx}\frac{dg}{dy} < 0$, $H = \arctan(\frac{1}{2}s + C)$, *where* $C \in \mathbf{R}$,
(iii) $c = -1$, $\frac{df}{dx}\frac{dg}{dy} < 0$, $H = \frac{1+Ce^s}{1-Ce^s}$, *where* $C > 0$,
(iv) $c = -1$, $\frac{df}{dx}\frac{dg}{dy} > 0$, $H = \frac{1+Ce^s}{1-Ce^s}$, *where* $C < 0$.

In the case of (ii), $S$ depends on $s$ only, i.e., $\frac{df}{dx}S_x = \varepsilon\frac{dg}{dy}S_y$. Hence we have

$$\frac{d^2f}{dx^2} - \varepsilon\frac{d^2g}{dy^2} = \frac{1}{f+g}\left\{\left(\frac{df}{dx}\right)^2 - \varepsilon\left(\frac{dg}{dy}\right)^2\right\}. \tag{5.3}$$

**Proposition 5.1.**

$$\begin{cases} (\frac{df}{dx})^2 = Af^2 + 2Bf + C, \\ \varepsilon(\frac{dg}{dy})^2 = Ag^2 - 2Bg + C, \end{cases} \tag{5.4}$$

*where* $A, B, C \in \mathbf{R}$ *which satisfy one of the following*:

(i) $\varepsilon = 1$, $A > 0$, (ii) $\varepsilon = 1$, $A = 0$, $B \neq 0$, (iii) $\varepsilon = 1$, $A = B = 0$, $C > 0$, (iv) $\varepsilon = 1$, $A < 0$, $AC - B^2 < 0$, (v) $\varepsilon = -1$, $AC - B^2 < 0$.

**Proof.** Note that (5.3) is equivalent to

$$\begin{cases} \frac{d^2f}{dx^2} - \varepsilon\frac{d^2g}{dy^2} = \varphi, \\ (\frac{df}{dx})^2 - \varepsilon(\frac{dg}{dy})^2 = (f+g)\varphi, \end{cases}$$

where $\varphi$ is an unknown function which is to be determined. A direct computation shows that $\varphi = C_1(f-g) + C_2$, where $C_1, C_2 \in \mathbf{R}$. This completes the proof. □

Solving (5.4) directly, we obtain

**Theorem 5.2.**

$$S^2 = \begin{cases} \frac{1}{AC-B^2}\sin^2(\sqrt{AC-B^2}\,s+D) & \text{if } \varepsilon = 1,\ AC > B^2, \\ (s+D)^2 & \text{if } \varepsilon = 1,\ AC = B^2, \\ \frac{1}{B^2-AC}\sinh^2(\sqrt{B^2-AC}\,s+D) & \text{if } \varepsilon = 1,\ AC < B^2, \\ \frac{1}{B^2-AC}\cosh^2(\sqrt{B^2-AC}\,s+D) & \text{if } \varepsilon = -1,\ AC < B^2, \end{cases}$$

*where* $D \in \mathbf{R}$.

**Remark 5.1.** Theorem 5.2 implies that (5.2) is equivalent to the generalized Hazzidakis equation obtained in [1–3,5,6,12].

## 6. Timelike ±isothermic Bonnet surfaces with constant curvature

The following proposition can be obtained in a way to similar [6, Proposition 2.1].

**Proposition 6.1.** *Let* $F : M \to \mathfrak{M}^3_1(c)$ *be a timelike ±isothermic Bonnet surface with constant curvature* $K$*. Then* $K = c$ *or* 0*, i.e.,* $F$ *is extrinsically or intrinsically flat.*

First we consider the case that $F : M \to \mathfrak{M}^3_1(c)$ is a timelike ±isothermic Bonnet surface with flat extrinsic curvature. If we use ±isothermic parametrization, then (1.2) implies that $F$ is isothermic and $4Q^2 = e^{2u}H^2$. Let $(x, y)$ be an isothermic coordinate. Computations similar to those as in [8] leads to the following:

**Theorem 6.1.**

$$(u, Q, H) = \left(u(\eta), \frac{e^{\frac{1}{2}u(\eta)}}{2\psi(\xi)}, \frac{\mu e^{-\frac{1}{2}u(\eta)}}{\psi(\xi)}\right),$$

*where* $\mu = \pm 1$, $\eta = x - \mu y$, $\xi = x + \mu y$,

$$(u, \psi) = \begin{cases} \begin{cases} \left(\log \frac{4\alpha^2}{\sinh^2(\alpha\eta+\beta)}, C_1 e^{\alpha\xi} + C_2 e^{-\alpha\xi}\right), \\ \left(\log \frac{4\alpha^2}{\cos^2(\alpha\eta+\beta)}, C_1 \cos\alpha\xi + C_2 \sin\alpha\xi\right), & \text{if } c = \mu, \\ \left(\log \frac{4}{(\eta+\beta)^2}, C_1\xi + C_2\right), \end{cases} \\ \left(\log \frac{4\alpha^2}{\cosh^2(\alpha\eta+\beta)}, C_1 e^{\alpha\xi} + C_2 e^{-\alpha\xi}\right), & \text{if } c = -\mu, \\ (\beta, C_1\eta + C_2),\ \left(\alpha\eta+\beta, C_1 e^{\frac{\alpha}{2}\xi} + C_2 e^{-\frac{\alpha}{2}\xi}\right), & \text{if } c = 0 \end{cases}$$

*for* $\alpha > 0$, $\beta \in \mathbf{R}$ *and* $(C_1, C_2) \in \mathbf{R}^2 \setminus \{0\}$.

**Remark 6.1.** Using Theorem 6.1, we can solve the Gauss–Weingarten formulas for $F$ explicitly. Then we find that $F$ is parametrized by a curve or a timelike curve in a 2-dimensional Riemannian or Lorentzian space form respectively whose geodesic curvature is defined by $\psi$. We will not go into further details (cf. [8]).

If $F: M \to \mathfrak{M}_1^3(c)$ is a flat timelike ±isothermic Bonnet surface, then using a ±isothermic coordinate $(x, y)$, (1.1) is equivalent to

$$\begin{cases} 4\varepsilon Q^2 = \mathrm{e}^{2u}(H^2 + c), \\ Q_y = \frac{1}{2}\mathrm{e}^u H_x, \\ \varepsilon Q_x = \frac{1}{2}\mathrm{e}^u H_y, \end{cases} \tag{6.1}$$

where $\varepsilon = 1$ or $-1$ if $F$ is isothermic or anti isothermic respectively. Note that $\varepsilon = 1$ if $c = 0, 1$. (6.1) implies that $F$ is a flat ±isothermic timelike HIMC surface (see [8, §3]). Since $F$ is a HIMC surface, we can write $H = \frac{f(x)g(y)-c}{f(x)+g(y)}$, where $f$ and $g$ are functions of $x$ and $y$ only respectively. Then (6.1) becomes

$$\begin{cases} 4\varepsilon Q^2 = \mathrm{e}^{2u}\frac{(f^2+c)(g^2+c)}{(f+g)^2}, \\ Q_y = \frac{1}{2}\mathrm{e}^u \frac{g^2+c}{(f+g)^2}\frac{df}{dx}, \\ \varepsilon Q_x = \frac{1}{2}\mathrm{e}^u \frac{f^2+c}{(f+g)^2}\frac{dg}{dy}. \end{cases} \tag{6.2}$$

If we denote by $\mathcal{C}_{H,c}$ the set of flat timelike ±isothermic Bonnet surfaces in $\mathfrak{M}_1^3(c)$ with $H = \frac{fg-c}{f+g}$, then (6.2) leads to the following:

**Theorem 6.2.** *If $M$ is simply connected, then $\mathcal{C}_{H,0} \cong \mathcal{C}_{H,1} \cong \mathcal{C}_{H,-1}$, i.e., flat timelike ±isothermic Bonnet surfaces in one Lorentzian space form correspond locally to those in the others.*

**Remark 6.2.** Theorem 6.2 is a special case of Lawson's correspondence obtained in [9].

When $c = 0$, using the same notation as in Theorem 6.1, we can obratin

**Theorem 6.3.**

$$(u, \psi) = (\beta, C_1\eta + C_2) \quad \text{or} \quad \bigl(\alpha\eta + \beta, C_1\mathrm{e}^{\frac{\alpha}{2}\xi} + C_2\mathrm{e}^{-\frac{\alpha}{2}\xi}\bigr).$$

**Remark 6.3.** (1) If $(u, \psi) = (\beta, C_1\xi + C_2)$, then $F$ is parametrized by a curve or a timelike curve in the Euclidean or the Minkowski 2-space, respectively. If $(u, \psi) = (\alpha\eta + \beta, C_1\mathrm{e}^{\frac{\alpha}{2}\xi} + C_2\mathrm{e}^{-\frac{\alpha}{2}\xi})$, the $F$ is parametrized by a curve or a timelike curve in the hyperbolic or the de Sitter 2-space, respectively. In each case the geodesic curvature of the (timelike) curve is defined by $\psi$.

(2) A simple computation shows that $H$ is ±Lorentz-holomorphic and $dH \neq 0$ if and only if $(u, \psi) = (\alpha\eta + \beta, C\mathrm{e}^{\pm\frac{\alpha}{2}\xi})$ for $C \in \mathbf{R}\backslash\{0\}$.

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