MORE ON THE GENERALIZED MACAULAY THEOREM — II

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Let $k_1, k_2, \ldots, k_n$ be given positive integers and let $S$ denote the set of vectors $x = (x_1, x_2, \ldots, x_n)$ with integer components satisfying $0 \leq x_i \leq k_i$, $i = 1, 2, \ldots, n$. Let $X$ be a subset of $S$. $(I)X$ denotes the subset of $X$ consisting of vectors with component sum $I$; $F(m, X)$ denotes the lexicographically first $m$ vectors of $X$; $\Gamma X$ denotes the set of vectors in $S$ obtainable by subtracting 1 from a component of a vector in $X$; $|X|$ is the number of vectors in $X$. In this paper it is shown that $|\Gamma F(e, (I)S)|$ is an increasing function of $l$ for fixed $e$ and is a subadditive function of $e$ for fixed $l$.

1. Introduction

Let $n$ and integers $k_1, k_2, \ldots, k_n$ be given where $1 \leq k_1 \leq k_2 \leq \cdots \leq k_n$. $S = S(k_1, k_2, \ldots, k_n)$ denotes the set of vectors $x = (x_1, x_2, \ldots, x_n)$ satisfying $0 \leq x_i \leq k_i$, $i = 1, 2, \ldots, n$. We order $S$ lexicographically by defining for $x, y \in S$, $x < y$ if and only if $x_i < y_i$ for the smallest integer $i$ such that $x_i \neq y_i$. Let $S(k_1, \ldots, k_n)$ be arrayed in $K + 1$ columns numbered $0, 1, \ldots, K$, where $K = k_1 + k_2 + \cdots + k_n$ and $(k_1 + 1)(k_2 + 1) \cdots (k_n + 1)$ rows by writing its elements in increasing order from left to right, top to bottom with $k_n + 1$ elements in each row and element $x = (x_1, x_2, \ldots, x_n)$ in column $|x|$, where $|x| = x_1 + x_2 + \cdots + x_n$. We do not distinguish between the set $S$ and $S$ arrayed in this way. Considering these arrays will prove fruitful, as it often has in the past [2–8], since it gives one a geometrical grasp of complicated situations. $S(3), S(2, 3),$ and $S(2, 2, 3)$ are exhibited in Fig. 1.

The $k_n = 1$ case is of particular importance. $S$ is then the set of $2^n$ $n$-tuples of 0's and 1's and can be regarded as the set of subsets of an $n$ element set by identifying the vector $(1, 1, \ldots, 1)$ with the entire set, the vector $(0, 0, 0, \ldots, 0)$ with the empty set, etc.

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In the general case, the vectors in $S(k_1, \ldots, k_n)$ can be regarded as subsets of a finite set having $n$ different types of elements, $k_i$ of type $i, i = 1, 2, \ldots, n$. For example if we have a set of billiard balls, $k_i$ of color $i, i = 1, 2, \ldots, n$, then $x = (x_1, x_2, \ldots, x_n)$ may be identified with the set consisting of $x_1$ balls of color $i, i = 1, 2, \ldots, n$. The total number of balls in $x$ is $|x|$.

The arrays $S$ are also useful in coding problems. If one is dealing with a transmitter that either transmits or does not transmit during $n$ intervals of time, $S(1,1,\ldots,1)$ can be regarded as the set of all possible messages; however if the transmitter can transmit signals of $k_i - 1$ levels of intensity (counting not transmitting at all as one level) during the $i$th interval, then $S(k_1, k_2, \ldots, k_n)$ represents the set of all possible messages. (It was in the latter context that we first encountered the need for our Theorem 1 below.)

Now for $x = (x_1, x_2, \ldots, x_n) \in S(k_1, \ldots, k_n) = S$ let $\Gamma X$ be the family of sets in $S$ obtainable by removing one element from $x$; precisely,

$$\Gamma X = \{(x_1 - 1, x_2, \ldots, x_n), (x_1, x_2 - 1, \ldots, x_n), \ldots, (x_1, x_2, \ldots, x_{n-1}, 1, x_n), (x_1, x_2, \ldots, x_{n-1}, x_n - 1)\} \cap S.$$ 

For any subset $X$ of $S$ we define $\Gamma X$ to be $\bigcup_{x \in X} \Gamma x$ and use $|X|$ to denote the number of elements (of $S$) in $X$ (thus "|" is used in two ways). We use $(l)X$ to denote the vectors $x$ in $X$ which satisfy $|x| = l$, and if $e \leqslant |X|, F(e, X), C(e, X)$ and $L(e, X)$ denote respectively the lexicographically first $e$ vectors of $X$, some consecutive $e$ elements of $X$ and the last $e$ vectors of $X$. We will become involved with complicated but very explicit expressions. If one will read them completely, we believe he will find them a help rather than a hindrance. For example, "$(F(3,2), S(2,2,3))$" should be read "the first 3 vectors in column 2 of the $S(2,2,3)$ array". (From Fig. 1 it is seen that this set is $[002, 011, 020]$.)

The generalized Macaulay theorem [1] due to the author and Lindström states that

$$|\Gamma(l)X| \geqslant |\Gamma F(|(l)X|, (l)S)|.$$  

Fig. 1

\begin{align*}
S(3): & 0 1 2 3 & A(3): & 0 1 1 1 \\
 00 & 01 & 02 & 03 & & & 0 & 1 & 1 & 1 \\
S(2,3) = S_{ae}(3): & 10 & 11 & 12 & 13 & & 0 & 1 & 1 & 2 \\
 20 & 21 & 22 & 23 & & & 0 & 1 & 1 & 2 \\
 000 & 001 & 002 & 003 & & 0 & 1 & 1 & 1 \\
 010 & 011 & 012 & 013 & & 0 & 1 & 1 & 2 \\
 020 & 021 & 022 & 023 & & 0 & 1 & 1 & 2 \\
 100 & 101 & 102 & 103 & & 0 & 1 & 1 & 1 \\
S(2,2,3) = S_{ae}(2,3): & 110 & 111 & 112 & 113 & & 0 & 1 & 1 & 2 \\
 120 & 121 & 122 & 123 & & 0 & 1 & 1 & 3 \\
 200 & 201 & 202 & 203 & & 0 & 1 & 1 & 1 \\
 210 & 211 & 212 & 213 & & 0 & 1 & 1 & 2 \\
 220 & 221 & 222 & 223 & & 0 & 1 & 1 & 3 \\
\end{align*}
The $k_0 = 1$ special case of (1) was discovered by Kruskal [12] and rediscovered by Katona [10]. These authors also gave an algorithm for calculating the right side of (1) in the $k_0 = 1$ case. The present author extended their algorithm to the general case [5] and has recently [8] discovered how to write explicit formulas for $|IF(e,(l)S)|$ as a function of $e$ in the $k_0 = 1$ special case.

For a listing of some applications of the Kruskal-Katona theorem the reader is referred to Katona's survey [11]. Applications involving (1) in its full generality are to be found in [2, 4, 6].

In this paper we establish two more basic inequalities concerning the function $|IF(e,(l)S(k_1,\ldots,k_n)|$.

**Theorem 1.** For fixed $l$, $|IF(e,(l)S)|$ is a subadditive function of $e$; that is, for given $1 \leq l \leq K-1$, and positive integers $e_1, e_2$, where $e_1 + e_2 = e \leq |(l)S|,$

$$|IF(e_1 + e_2,(l)S)| \leq |IF(e_1,(l)S)| + |IF(e_2,(l)S)|.$$

(2)

For fixed $e$, $|IF(e,(l)S)|$ is an increasing function of $l$; that is, for given $1 \leq l \leq K$, and given $1 \leq e \leq \min(|(l-1)S|,|(l)S|),$

$$|IF(e,(l-1)S)| \leq |IF(e,(l)S)|.$$

(3)

An application of (2) is given in [7]. In the special case $k_0 = 1$, the inequalities (2) and (3) are not new. For example we have given elementary but complicated proofs of them in our paper [2, p. 232], where, as we will see below, they appear in different notation as inequalities (8) and (9).

An uncomplicated proof of (2) in the $k_0 = 1$ case based on the Kruskal-Katona theorem is pointed out in [9, p. 247]. This proof generalizes without difficulty to the special case $k_1 = k_2 = \cdots = k_n = k$ where $k$ is an integer exceeding 1. Let $n, e_1, e_2,$ and $l \leq nk$ be given where $e_1 + e_2 = e \leq |(l)S|,$ where $S = S(k_1, k_2, \ldots, k_n)$ and each $k_i$ is $k$. Let $U$ be $F(e_1,(l)S)$ and $T$ be $F(e_2,(l)S)$. Let $U'$ be the result of preceding each element of $U$ by $n$ 0's and let $T'$ be the result of following each element of $T$ with $n$ 0's. Thus $U'$ and $T'$ are subsets of $(l)S'$ where $S' = S(k_1, k_2, \ldots, k_n)$ and each $k_i$ is $k$.

By construction $(U') \cap (T') = \emptyset$ (unless $l = 1$ in which case $IF(e_1 + e_2,(l)S) = 1 < 2 = |IF(e_1,(l)S)| + |IF(e_2,(l)S)|$). Also

$$|I'U'| = |IU'| = |IF(e_1,(l)S)|$$

and

$$|IT'| = |IT'| = |IF(e_2,(l)S)|.$$

Hence

$$|IF(e_1,(l)\setminus e_1) + |IF(e_2,(l)S)| = |IU'| + |IT'|$$

$$= |I(U' \cup T')| \geq |IF(e_1 + e_2,(l)S')| = |IF(e_1 + e_2,(l)S)|.$$
the first $e_1 + e_2$ $n$-tuples of $(1)S$ with each $n$-tuple preceded by $n$ 0's. Thus (2) is established if $k_1 = k_2 = \cdots = k_n$.

In the general case, $k_1 \leq k_2 \leq \cdots \leq k_n$, a parallel argument with $S' = S'_{k_1, k_2, \ldots, k_m}$ could be made if the generalized Macaulay theorem held such $S''$'s. But this is not necessarily so: in $S(1,2,1,2)$ one finds

$$2 = |\Gamma \{0202\}| \neq |\Gamma F(1,(4)S(1,2,1,2))| = |\Gamma \{0112\}| = 3.$$ Our elementary proof does however generalize.

The last several paragraphs may be regarded as an apology for the barbaric proof of our theorem.

2. Geometric preliminaries

The proof of Theorem 1 is made by exploiting certain geometrical facts concerning the array $S(k_1, \ldots, k_n)$ and the array $A(k_1, \ldots, k_n)$ which we introduce now. $A(k_1, \ldots, k_n)$ is the array formed by replacing each vector $x$ in the array $S(k_1, k_2, \ldots, k_n)$ by the integer

$$[\Gamma X] - [\Gamma X']$$

where $X$ is the set of all vectors in column $|x|$ of $S(k_1, \ldots, k_n)$ having lexicographical order not exceeding that of $x$ and $X'$ is $X - \{x\}$. $A(3)$, $A(2,3)$, and $A(2,2,3)$ are exhibited in Fig. 1. Notations involving $\Lambda(k_1, \ldots, k_n)$ inherit their meanings from the corresponding notations involving $S(k_1, \ldots, k_n)$. For example, “$F(3,(2)A(2,2,3))$” should be read “the first 3 integers in column 2 of the $A(2,2,3)$ array”. If $H$ is a subset of $A(k_1, \ldots, k_n)$, we do not distinguish between arrays and the set of elements, which form them), $\Sigma H$ means $\Sigma_{h \in H} h$. (Thus from Fig. 1 we have $\Sigma F(3,(2)A(2,2,3)) = 1 + 1 + 0$.)

We now develop geometrical facts concerning these arrays. Contemplation of Fig. 1 shows how to construct $S(k_1, \ldots, k_n)$ if $S(k_2, \ldots, k_n)$ has been constructed and $k_1 \leq k_2 \leq k_3$ is given: one writes a first copy of the $S(k_2, \ldots, k_n)$ array in which each entry is preceded by a 0, follows it with a copy of $S(k_2, \ldots, k_n)$ translated one column to the right in which each entry is preceded by a 1, follows it with a copy of $S(k_2, \ldots, k_n)$ translated one more column to the right in which each entry is preceded by a 2, etc., until a $(k_1 + 1)$st copy of $S(k_2, \ldots, k_n)$ is written in which each entry is preceded by a $k_1$. We denote these altered copies of $S(k_2, \ldots, k_n)$ by $S_{\Lambda(k_2, \ldots, k_n)}$, $i = 0, 1, \ldots, k_1$, and will use the notation

$$S(k_1, \ldots, k_n) = S_{\Lambda(k_2, \ldots, k_n)}$$

(4) to recall this construction. We have used the “\" to suggest the diagonal nature of the deployment of the arrays $S(k_2, \ldots, k_n)$, $i = 0, 1, \ldots, k_1$.

Similar notation and remarks are applicable to the construction of $A(k_1, \ldots, k_n)$ given $A(k_2, \ldots, k_n)$ and $k_1 \leq k_1 \leq k_2$ (see Fig. 1): one writes a copy of
A \( (k_2, \ldots, k_n) \), follows it with a copy of \( A \( k_2, \ldots, k_n \) \) translated one column to the right in which the single right-most entry has been increased by 1, follows it with a copy of \( A \( k_2, \ldots, k_n \) \) translated one more column to the right in which the single right-most entry has been increased by 1, etc., until a \( (k_1 + 1) \)st copy of \( A \( k_2, \ldots, k_n \) \) in which the single rightmost entry has been increased by 1 is written. We denote the first copy of \( A \( k_2, \ldots, k_n \) \) involved here by \( A_{m(k)} \( k_2, \ldots, k_n \) \) and the remaining altered copies of \( A \( k_2, \ldots, k_n \) \) by \( A_i \( k_2, \ldots, k_n \) \), \( i = 1, 2, \ldots, k_1 \). We use the notation

\[
A \( k_1, \ldots, k_n \) = A_{m_1} \( k_2, \ldots, k_n \)
\]

to recall this construction. These constructions are the basis of the induction proof of our theorem.

We will use \( A_{m(k+1)} \( k_2, \ldots, k_n \) \) to denote the result of deleting \( A_k \( k_2, \ldots, k_n \) \) from \( A_{m(k)} \( k_2, \ldots, k_n \) \) and will write

\[
A_{m(k)} \( k_2, \ldots, k_n \) - A_k \( k_2, \ldots, k_n \) = A_{m(k+1)} \( k_2, \ldots, k_n \),
\]

e tc. In particular

\[
A_{m0} \( k_2, \ldots, k_n \) = A_0 \( k_2, \ldots, k_n \) = A \( k_2, \ldots, k_n \).
\]

Certain features of the arrays \( A \( k_1, \ldots, k_n \) \) are by now quite apparent:

I. The result of deleting \( A_k \( k_2, \ldots, k_n \) \) from \( A \( k_1, \ldots, k_n \) \) is \( A \( k_1 - 1, k_2, \ldots, k_n \) \): \( A \( k_1, \ldots, k_n \) - A_k \( k_2, \ldots, k_n \) = A \( k_1 - 1, k_2, \ldots, k_n \) \).

II. The result of deleting \( A_0 \( k_1, \ldots, k_n \) \) from \( A \( k_1, \ldots, k_n \) \) is \( A \( k_1, \ldots, k_n \) \) and decreasing the right-most entry in \( A_i \( k_2, \ldots, k_n \) \) by 1 is \( A \( k_1 - 1, k_2, \ldots, k_n \) \).

III. Let \( n \geq 2, 1 \leq l \leq k \) be given and let

\[
m = | \( (l)A \( k_1, \ldots, k_n \) \) - | \( (l)A_0 \( k_2, \ldots, k_n \) \) |
\]
or, in alternative notation, let

\[
m = | \( (l)A_{1:k} \( k_2, \ldots, k_n \) \) |.
\]

For \( j \) such that \( 1 \leq j \leq m \),

\[
\sum F(j, (l - 1)A \( k_1, \ldots, k_n \)) \leq \sum F(j, (l)A_{1:k} \( k_2, \ldots, k_n \))
\]

with equality if and only if \( l - 1 \neq k_2 + k_3 + \cdots + k_n \).

IV. With \( S(\kappa_1, \ldots, \kappa_n) = S \) , \( A(\kappa_1, \ldots, \kappa_n) = A \) and \( k_1 + k_2 + \cdots + k_n = K \) the sequence \( | \( (l)S \) |, l = 0, 1, 2, \ldots, K \), or, what is the same thing, the sequence \( | \( (l)A \) |, l = 0, 1, 2, \ldots, K \) is unimodal; that is [4; p. 1287].

\[
| \( (l)S \) | = | \( (K - l)S \) |, \quad l = 0, 1, 2, \ldots, K
\]

and

\[
| \( (l - 1)S \) | \leq | \( (l)S \) | \quad 1 \leq l \leq [K/2].
\]
3. The proof

In order to take advantage of the geometrically inspired facts which we have developed above, we now reformulate our theorem.

Upon writing $\Gamma F(e, (l)S)$ as the disjoint union

$$(\Gamma F(1, (l)S) - \Gamma F(0, (l)S)) \cup (\Gamma F(2, (l)S) - \Gamma (1, (l)S)) \cup \cdots \cup$$

$$(\Gamma F(e, (l)S) - \Gamma F(e - 1, (l)S))$$

one sees that

$$|\Gamma F(e, (l)S)| = \sum_{i=1}^{e} |\Gamma F(i, (l)S) - \Gamma F(i - 1, (l)S)| = \sum F(e, (l)A).$$

The inequality (3) thus may be rewritten $\sum F(e, (l-1)A) \leq \sum F(e, (l)A)$.

Similarly, if $C(e_2, (l)A)$ denotes the set of $(e_1 + 1)$th, $(e_1 + 2)$th, $\ldots$, $(e_1 + e_2)$th integers in $(l)A$, then

$$|\Gamma F(e_2 + e_1, (l)S)| - |\Gamma F(e_1, (l)S)| = \sum C(e_2, (l)A)$$

and (2) may be rewritten $\sum C(e_2, (l)A) \leq |\Gamma F(e_2, (l)S)| = \sum F(e_2, (l)A)$. Thus Theorem 1 may be reformulated as:

**Theorem 2.** Let $n \geq 1$ and integers $k_1, k_2, \ldots, k_n$ which satisfy $1 \leq k_1 \leq k_2 \leq \cdots \leq k_n$ be given and let $l, e$ be integers satisfying $1 \leq l \leq k_1 + k_2 + \cdots + k_n$, $1 \leq e \leq |(l)A|$. Then

$$\sum C(e, (l)A) \leq \sum F(e, (l)A),$$

where $C(e, (l)A)$ denotes any $e$ consecutive elements of $(l)A$, and

$$\sum F(e, (l - 1)A) \leq \sum F(e, (l)A)$$

assuming $e \leq \min(|(l - 1)A|, |(l)A|)$. Also duals (using last elements in place of first elements) of these inequalities hold:

$$\sum L(e, (l)A) \leq \sum C(e, (l)A),$$

$$\sum L(e, (l - 1)A) \leq \sum L(e, (l)A).$$

The reader will perhaps find it helpful to verify instances of these inequalities in $A(2, 2, 3)$ (see Fig. 1.) The elementary but somewhat complicated arguments which follow were all inspired by the study of such arrays. Also, upon comparing Theorem 2 and Theorem 1 of [2; p. 232] he will see that we are in the process of generalizing
the latter theorem from the case.  

Since the proofs of (8') and (9') may be obtained from the proofs of (8) and (9) by generally replacing "first" by "last", "increases" by "decreases" "\( \leq \)" by "\( \geq \)" etc., we write detailed proofs only for (8) and (9). We present their proofs as separate lemmas. We will refer to \( n \)-tuples \( k_1, k_2, \ldots, k_n \) of integers \( \leq \) admissible provided they satisfy \( 1 \leq k_1 \leq k_2 \leq \cdots \leq k_n \).

**Lemma 1.** Suppose that the theorem holds for all admissible \((n - 1)\)-tuples, that \((k_1, \ldots, k_n)\) is an admissible \( n \)-tuple with \( k_1 \geq 2 \) and that theorem \((8')\) holds for \( n \)-tuples \((k_{k_1}, k_{k_2}, \ldots, k_{k_n})\), \( k = 1, 2, \ldots, k_1 - 1 \). Then (8) holds for the \( n \)-tuple \((k_1, \ldots, k_n)\). (The \( k_1 = 1 \) situation is discussed following the proof of the lemma.)

**Proof.** Let \( e \) and \( l \leq k_1 + k_2 + \cdots + k_n = K \) be given where \( 1 \leq e \leq |(l)A(k_1, \ldots, k_n)| \). If \( l = K \), \( e \) is necessarily 1 and (8) holds with equality. We henceforth assume \( l < K \). Let some set \( C(e, (l)A(k_1, \ldots, k_n)) = C \) be given. The constructions \( A(k_1, k_2, \ldots, k_n) = A_{0,k_1}(k_2, \ldots, k_n) \) and \( A_{0,k_1-1}(k_2, \ldots, k_n) = A(k_1 - 1, k_2, \ldots, k_n) \) suggest dividing the proof into 3 parts:

(a) If \( C \subseteq (l)A_{0,(k_1-1)}(k_2, \ldots, k_n) = (l)A(k_1 - 1, k_2, \ldots, k_n), \) (8) holds by hypothesis.

(b) If \( C \subseteq (l)A_{0,k_1}(k_2, \ldots, k_n) \), we replace \( C \) by \( F(e, (l)A_{0,k_1}(k_2, \ldots, k_n)) \). Since, except for its (single) right-most entry, the array \( A_{0,k_1}(k_2, \ldots, k_n) \) coincides with \( A(k_1, k_2, \ldots, k_n) \) (although columns are numbered differently) and (8) holds by hypothesis in \( A(k_1, k_2, \ldots, k_n) \), it also holds in \( A_{0,k_1}(k_2, \ldots, k_n) \). Thus

\[
\sum F(e, (l)A_{0,k_1}(k_2, \ldots, k_n)) \geq \sum C
\]

and replacing the given set \( C \) with \( F(e, (l)A_{0,k_1}(k_2, \ldots, k_n)) \) leaves us with a set having a larger sum-of-elements. Here and in similar situations below we will say that the replacement does not decrease \( \Sigma \), etc.

Since \( C \subseteq (l)A_{0,k_1}(k_2, \ldots, k_n) \), we have

\[
e = |C| \leq |(l)A_{0,k_1}(k_2, \ldots, k_n)| = |(l - k_1)A(k_2, \ldots, k_n)|.
\]

Now if

(i) also \( e \leq |(l)A(k_1, \ldots, k_n)| \), it follows by unimodality (IV) that

\[
e \leq |(j)A(k_2, \ldots, k_n)|, \quad j = l - k_1, \quad l - k_1 + 1, \ldots, l.
\]

Replacing \( F(e, (l)A_{0,k_1}(k_2, \ldots, k_n)) \) by \( F(e, (l - k_1)A(k_2, \ldots, k_n)) \) does not change \( \Sigma \) since \( A(k_1, \ldots, k_n) = A_{0,k_1}(k_2, \ldots, k_n) \) and \( l \neq K \). Then replacing \( F(e, (l - k_1)A(k_2, \ldots, k_n)) \) successively by \( F(e, (j)A(k_2, \ldots, k_n)) \), \( j = l - k_1, \ldots, l \), is possible in view of (10) and does not decrease \( \Sigma \) since (9) holds in \( A(k_2, \ldots, k_n) \) by hypothesis. On the other hand, if

(ii) \( e > |(l)A(k_2, \ldots, k_n)| \), we replace the first \(|(l)A(k_2, \ldots, k_n)| \) elements of \( F(e, (l)A_{0,k_1}(k_2, \ldots, k_n)) \) by \( F(|(l)A(k_2, \ldots, k_n)|, (l - k_1)A(k_2, \ldots, k_n)) \). Since \( A(k_1, \ldots, k_n) = A_{0,k_1}(k_2, \ldots, k_n) \) and \( l \neq K \), this replacement does not alter \( \Sigma \).
Since \(|(l)A(k_2, \ldots, k_n)| < e < |(l)A_{k_1}(k_2, \ldots, k_n)| = |(l - k_1)A(k_2, \ldots, k_n)|\), we have by unimodality

\[ |(l)A(k_2, \ldots, k_n)| \leq |(i)A(k_2, \ldots, k_n)|, \quad i = l - k_1, l - k_1 + 1, \ldots, l, \]

so it is possible to replace \(F(|(l)A(k_2, \ldots, k_n)|, (l - k_1)A(k_2, \ldots, k_n))\) successively by

\[ F(|(l)A(k_2, \ldots, k_n)|, (i)A(k_2, \ldots, k_n)), \quad i = l - k_1 + 1, l - k_1 + 2, \ldots, l \]

and these replacements do not decrease \(\Sigma\) since (9) holds in \(A(k_2, \ldots, k_n)\). Finally we replace the remaining \(t = e - |(l)A(k_2, \ldots, k_n)|\) elements of \(F(e, (l)A_{k_1}(k_2, \ldots, k_n))\) by \(F(t, (l)A_{k_2}(k_2, \ldots, k_n))\), where, recall, \((l)A_{k_1}(k_2, \ldots, k_n)\) is an abbreviation for \((l)A(k_1, \ldots, k_n) - (l)A(k_2, \ldots, k_n)\). Since (8) holds by hypothesis in \(A(k_1 - 1, k_2, \ldots, k_n)\), it holds a fortiori in the present situation in view of II; hence this replacement does not decrease \(\Sigma\). It also completes the replacement of \(C\) by \(F(e, (l)A(k_1, \ldots, k_n))\).

(c) The remaining case is that in which

\[ |C \cap (l)A_{k_1}(k_2, \ldots, k_n)| = e_1 > 0 \]

and

\[ |C \cap (i)A_{k_1}(k_2, \ldots, k_n)| = e_2 > 0 \]

where

\[ e_1 + e_2 = e. \]

Let \(j = |(l)A_{k_1}(k_2, \ldots, k_n) - C)|\). (If \(l > k_2 - \cdots + k_m\), \(j\) is 0.) Now

(i) If \(e_2 \geq j\), we replace \(F(j, C \cap (l)A_{k_1}(k_2, \ldots, k_n))\) successively by \(F(j, (l - k_1)A_{k_2}(k_2, \ldots, k_n))\) and then \(F(i, i)A_{k_2}(k_2, \ldots, k_n)), \quad i = l - k_1 + 1, \ldots, l.\)

That these replacements are possible and do not decrease \(\Sigma\) follows as in case b. We next replace the remaining \(t = e - j\) elements of \(C \cap (l)A_{k_1}(k_2, \ldots, k_n)\) by \(F(t, (l)A_{k_1}(k_2, \ldots, k_n))\). Since \(l \not\in K_1, (l)A_{k_1}(k_2, \ldots, k_n)\) is the same as \((l - k_1)A_{k_2}(k_2, \ldots, k_n)\), or, what is the same thing, \((l - k_1)A_{k_2}(k_2, \ldots, k_n)\) (But (8) holds in \(A(k_2, \ldots, k_n)\) by hypothesis so this replacement does not decrease \(\Sigma\). Thus far we have replaced \(C\) with \((l)A_{k_1}(k_2, \ldots, k_n)\) and a subset \(C'\) of \(e_1 + e_2 - j = e - j\) consecutive elements of \((l)A_{k_1}(k_2, \ldots, k_n)\). (If \(j = 0, C'\) remains \(C\).) We now replace \(C'\) by \(F(e - j, (l)A_{k_2}(k_2, \ldots, k_n))\). Since (8) holds in \(A(k_1 - 1, k_2, \ldots, k_n)\) by hypothesis, it holds a fortiori in \(A_{k_2}(k_2, \ldots, k_n)\) in view of II. This replacement therefore does not decrease \(\Sigma\). It also completes the replacement of \(C\) by \(F(e, (l)A(k_1, \ldots, k_n))\).

(ii) If \(e_2 < j\), we replace all \(e_2\) elements of \(C \cap (l)A_{k_1}(k_2, \ldots, k_n)\) by \(F(e_2, (l)A_{k_1}(k_2, \ldots, k_n))\). It follows as in case i above that this replacement does not decrease \(\Sigma\). We next replace the \(e_1\) elements of \(C \cap (l)A_{k_1}(k_2, \ldots, k_n)\) or, what is the same thing in view of I, \(C \cap (l)A(k_1 - 1, k_2, \ldots, k_n)\) by \(F(e_1, (l)A(k_1 - 1, k_2, \ldots, k_n)) - F(e_2, (l)A(k_1 - 1, k_2, \ldots, k_n))\). In view of (8'), which holds in \(A(k_1 - 1, k_2, \ldots, k_n)\) by
hypothesis, this replacement does not decrease $\Sigma$. It also completes the replacement of $C$ by $F(e, (l)A(k_1, \ldots, k_n))$. Thus Lemma 1 is proved.

We remark that under the hypothesis that the theorem holds for all admissible $(n-1)$-tuples $k_2, \ldots, k_n$ the proof of Lemma 1, upon replacing all occurrences of $k_1$ by 1 and all occurrences of $A(k_1 - 1, k_2, \ldots, k_n)$ by $A(k_2, \ldots, k_n)$ becomes a proof that (8) holds for all admissible $n$-tuples having $k_1 = 1$. A parallel remark also applies to the following lemma.

**Lemma 2.** Suppose that (8) holds in $A(k_1, k_2, \ldots, k_n)$ for all admissible $k_1, k_2, \ldots, k_n$ and that (9) holds for all admissible $(n - 1)$-tuples $k_2, k_3, \ldots, k_n$. Let admissible $k_1, k_2, \ldots, k_n$ be given where $k_1 \geq 2$ and suppose that (9) also holds for $k, k_2, \ldots, k_n$, $k = 1, 2, \ldots, k_1 - 1$. Then (9) holds for $k, k_2, \ldots, k_n$.

**Proof.** Let $l, 1 \leq l \leq K$, and $e \leq \min(\lfloor (l - 1)A\rfloor, \lfloor (l)A\rfloor)$ be given where $A = A(k_1, k_2, \ldots, k_n)$. The fact that the arrays $A_i(k_2, \ldots, k_n)$, $i = 1, 2, \ldots, k_1$, are not quite translated copies of the array $A_0(k_2, \ldots, k_n)$ seems to necessitate a proof by cases depending on what happens when $l$ is near the number of the exceptional column — that is, when $l$ is near $k_2 + k_3 + \cdots + k_n$.

If $l - 1 > k_2 + \cdots + k_n + 1$, neither $(l - 1)A$ nor $(l)A$ involves elements of $A_0(k_2, \ldots, k_n)$ or of $A_1(k_2, \ldots, k_n)$. Hence if we delete $A_0(k_2, \ldots, k_n)$ from $A(k_1, \ldots, k_n) = A_{nk_1}(k_2, \ldots, k_n)$ and decrease the right-most entry of $A_1(k_2, \ldots, k_n)$ by 1, the situation is not changed. But then in view of II, we are in $A(k_1 - 1, k_2, \ldots, k_n)$ where (9) holds by hypothesis.

If $l - 1 = k_2 + \cdots + k_n + 1$, neither $(l - 1)A(k_1, \ldots, k_n)$ nor $(l)A(k_1, \ldots, k_n)$ involves elements of $A_0(k_2, \ldots, k_n)$, so the situation is not changed if we delete $A_0(k_2, \ldots, k_n)$ from $A(k_1, \ldots, k_n) = A_{nk_1}(k_2, \ldots, k_n)$ except that we are now dealing with columns $k_2 + \cdots + k_n$ and $k_2 + \cdots + k_n + 1$ of the resulting array instead of columns $l - 1 = k_2 + \cdots + k_n + 1$ and $l = k_2 + \cdots + k_n + 2$ in $A(k_1, \ldots, k_n)$. In view of II, the resulting array is exactly $A(k_1 - 1, k_2, \ldots, k_n)$ except that the first entry in its column $k_2 + \cdots + k_n$ is too large by 1. We now apply III to $A(k_1 - 1, k_2, \ldots, k_n)$ with $k_2 + \cdots + k_n$ in the role of $l - 1$. Since $(k_2 + \cdots + k_n + 1)A_0(k_2, \ldots, k_n)$ is empty (7) may be written

$$\sum F(j, (l - 1)A(k_1 - 1, k_2, \ldots, k_n)) < \sum F(j, (l)A(k_1 - 1, k_2, \ldots, k_n)) \quad (11)$$

for $j$ such that $1 \leq j \leq m$ where $m = \lfloor (k_2 + \cdots + k_n + 1)A_{nk_1 - 1}(k_2, \ldots, k_n)\rfloor$. Since $(k_2 + \cdots + k_n + 1)A_0(k_2, \ldots, k_n)$ is empty, we can also write

$$m = \lfloor (k_2 + \cdots + k_n + 1)A_{nk_1 - 1}(k_2, \ldots, k_n)\rfloor$$

$$= \lfloor (k_2 + \cdots + k_n + 1)A(k_1 - 1, k_2, \ldots, k_n)\rfloor$$

$$= \lfloor (k_2 + \cdots + k_n + 2)A(k_1, \ldots, k_n)\rfloor = \lfloor (l)A(k_1, k_2, \ldots, k_n)\rfloor.$$

With $(l - 1) = k_2 + \cdots + k_n + 1$, unimodality (IV) shows that
\[ |(I)A(k_1, \ldots, k_n)| = \min(|(I-1)A(k_1, \ldots, k_n)|, |(I)A(k_1, \ldots, k_n)|). \]

The first entry of column \( k_2 + \cdots + k_n \) of the array with which we are dealing exceeds by \( I \) the corresponding entry of the array to which (11) applies; otherwise the two arrays are the same. Taking this into account yields (9) in the special case \( l-1 = \sum k_i + 1 \). We henceforth assume \( l-1 \leq k_2 + \cdots + k_n \).

If \( l-1 = k_2 + \cdots + k_n \), then \((I)A_0(k_2, \ldots, k_n)\) is empty and (7) reads

\[ \sum F(j, (I-1)A(k_1, \ldots, k_n)) < \sum F(j, (I)A(k_2, \ldots, k_n)), \]

\[ 1 \leq j \leq m = |(I)A_{1k_2}(k_2, \ldots, k_n)| = |(I)A_{0k_2}(k_2, \ldots, k_n)| = |(I)A(k_1, \ldots, k_n)| \]

\[ = \min(|(I-1)A(k_1, \ldots, k_n)|, |(I)A(k_1, \ldots, k_n)|). \]

The last equality here follows from unimodality (IV). We henceforth assume \( l-1 < k_2 + \cdots + k_n \). Under this assumption, both \((I-1)A(k_1, \ldots, k_n)\) and \((I)A(k_1, \ldots, k_n)\) involve elements of \( A_0(k_2, \ldots, k_n) \). We also assume

\[ e > \min(|(I-1)A(k_1-1, k_2, \ldots, k_n)|, |(I)A_0(k_2, \ldots, k_n)|), \] (12)

since in the contrary case the situation would not be altered by deleting \( A_n(k_2, \ldots, k_n) \) from \( A(k_1, \ldots, k_n) = A_{0k_1}(k_2, \ldots, k_n) \) leaving us in \( A_{0k_1}(k_2, \ldots, k_n) = A(k_1-1, k_2, \ldots, k_n) \), an array in which (9) holds by hypothesis. From (12) we have a fortiori

\[ e > \min(|(I-1)A_0(k_2, \ldots, k_n)|, |(I)A_0(k_2, \ldots, k_n)|) \]

\[ = \min(|(I-1)A(k_2, \ldots, k_n)|, |(I)A(k_2, \ldots, k_n)|). \]

If this minimum is \( t = |(I)A(k_2, \ldots, k_n)| \), we replace the first \( t \) elements of \( F(e, (I-1)A(k_1, \ldots, k_n)) \) by \( F(t, (I)A(k_1, \ldots, k_n)) \) or, what is the same thing, we replace the first \( t \) elements of \((I-1)A(k_1, \ldots, k_n)\) by \((I)A(k_2, \ldots, k_n)\). Since (9) holds by hypothesis in \( A(k_2, \ldots, k_n) \), this replacement does not decrease \( \Sigma \). We next replace the remaining \( s = e-t \) elements of \( F(e, (I-1)A(k_1, \ldots, k_n)) \) by \( F(s, (I-1)A(k_1, \ldots, k_n)) \). Since (8) holds in \( A(k_1, \ldots, k_n) \), this replacement does not decrease \( \Sigma \). Finally, we replace \( F(s, (I-1)A(k_1, \ldots, k_n)) \) by \( F(s, (I)A(k_2, \ldots, k_n)) \). Since

\[ e \leq \min(|(I-1)A(k_1, \ldots, k_n)|, |(I)A_0(k_2, \ldots, k_n)|) \leq |(I)A(k_1, \ldots, k_n)|, \]

\[ s = e - |(I)A_0(k_2, \ldots, k_n)| \]

\[ \leq |(I)A(k_1, \ldots, k_n)| - |(I)A_0(k_2, \ldots, k_n)| = |(I)A_{1k_2}(k_2, \ldots, k_n)| \]

so this replacement is possible and, in accordance with III. does not decrease \( \Sigma \).

If

\[ |(I-1)A(k_2, \ldots, k_n)| < |(I)A(k_2, \ldots, k_n)|, \] (13)

we consider two cases. First, in addition to the assumptions \( l-1 \leq k_2 + \cdots + k_n \) and \( e > |(I-1)A(k_2, \ldots, k_n)| \), suppose that \( l > k_n \). From the inequalities \( 2l > k_n + k_n \Rightarrow \)
Let \( j \) be the smallest of these integers. Both \( j = 1 \) and \( j = 2 \) are impossible since \( j = 2 \) along with unimodality implies \(|(l-1)A(k_2, \ldots, k_n)| \geq |(l)A(k_2, \ldots, k_n)|\) contrary to (13). Since \( j > 2 \) and \( e > |(l-1)A(k_2, \ldots, k_n)|\), a fortiori \( e > |(l-1)A(k_n, \ldots, k_n)|\). Unimodality and (14) implies

\[|(l-1)A(k_n, \ldots, k_n)| \geq |(l)A(k_n, \ldots, k_n)|.\]

We replace the first \(|(l)A(k_n, \ldots, k_n)|\) elements of \( F(e, (l-1)A(k_1, \ldots, k_n))\) with \((l)A(k_n, \ldots, k_n)\). Since (9) holds in \( A(k_n, \ldots, k_n)\), this replacement does not decrease \( \Sigma \). We then replace the remaining \( s = e - |(l)A(k_n, \ldots, k_n)|\) elements of \( F(e, (l-1)A(k_n, \ldots, k_n))\) by \( F(s, (l-1)A(k_n, \ldots, k_n))\). Since (8) holds in \( A(k_1, \ldots, k_n)\), this replacement does not decrease \( \Sigma \).

If \( s \leq u = |(l)A(k_{j-1}, \ldots, k_n) - (l)A(k_n, \ldots, k_n)|\), we replace \( F(u, (l-1)A(k_{j-1}, \ldots, k_n))\) with \((l)A(k_{j-1}, \ldots, k_n) - (l)A(k_n, \ldots, k_n)\). As before, III shows that this replacement does not decrease \( \Sigma \). We now replace the remaining \( s' = s - u \) elements of \( F(s, (l-1)A(k_1, \ldots, k_n))\) by \( F(s', (l-1)A(k_1, \ldots, k_n))\), etc., until the original set \( F(e, (l-1)A(k_1, \ldots, k_n))\) has been replaced by \( F(e, (l)A(k_1, \ldots, k_n))\) without decreasing \( \Sigma \). Since \( e \leq \min(|(l-1)A(k_1, \ldots, k_n)|, |(l)A(k_1, \ldots, k_n)|) \leq |(l)A(k_1, \ldots, k_n)|\), this replacement is possible and (9) follows.

In the second case, \( l \leq k_n \), the argument is similar to the one just completed except that in place of \( A(k_n, \ldots, k_n)\) one begins with \( A(k_n)\). \((A(k_n))\) is a single row consisting of 0 followed by \( k_n \)'s; unimodality degenerates to \(|(l-1)A(k_n)| = |(l)A(k_n)|, l = 1, 2, \ldots, k_n\). Thus Lemma 2 is proved.

Theorem 2 now follows easily by induction. For \( n = 1 \), it is obvious. If the theorem holds for \((n-1)\)-tuples of \( k\)'s it follows from Lemma 1 and the remark following Lemma 1 that (8) holds for admissible \( n \)-tuples of \( k\)'s. Then Lemma 2 and a remark analogous to the remark following Lemma 1 show that (9) holds for admissible \( n \)-tuples. This completes the proof of the theorem.

References