GEOMETRIC TOPOLOGICAL COMPLETIONS WITH UNIVERSEAL FINAL LIFTS

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Received 4 February 1983
Revised 7 November 1983

In this paper necessary and sufficient conditions are given on a concrete category over a category B so that it can be densely embedded (over B) into a geometric topological category E that admits certain universal final lifts. These conditions, as well as the class of universal final lifts, depend upon an a priori given full subcategory \( \Delta \) of B. For example, E may have, depending upon \( \Delta \) and B, universal coproducts or quotients or colimits. For appropriate \( \Delta \)'s, if B is cartesian closed then so is E.


1. Introduction

The category, \( \text{kTop} \), of compactly generated spaces [17] may be viewed as a model category for doing algebraic topology. The relevant features, for classical singular algebraic topology, are that \( \text{kTop} \) is cartesian closed, the forgetful functor \( U: \text{kTop} \rightarrow \text{Sets} \) is topological [7] and its left adjoint (discrete spaces) is exact (i.e. \( U \) is a geometric topological functor), and \( \text{Sets} \) is a topos with a natural number object. The details of formulating the machinery of singular algebraic topology in an appropriate geometric topological category \( U:E \rightarrow B \) are given in [13]. This paper is concerned with the problem of embedding concrete categories into such "convenient categories for algebraic topology". However, since many of the results of [13], in particular those concerning intervals, simplicial complexes, and geometric realization, do not directly stem from the cartesian closedness of E but depend instead on the facts that binary products of coequalizers are coequalizers and, especially if B is a Grothendieck topos, that countable coproducts are universal (i.e. are preserved by pullbacks), our primary interest is not in obtaining cartesian closed completions, but rather in finding geometric topological completions with...
enough universal final lifts. Cartesian closedness then follows as a special case. The main result of this paper, Theorem 5.2, gives necessary and sufficient conditions for such completions to exist. The conditions are expressed in terms of an a priori given full subcategory $\Delta$ of $B$ that turns out to be closely related to the discret objects of the completion. By making appropriate choices for $\Delta$ one can obtain geometric completions with certain predetermined types of universal final sinks.

Parts of this completion problem have been considered by others. Cartesian closed topological extensions of $\text{Top}$ and similar concrete categories over $\text{Sets}$ where considered by Spanier [16], Antoine [5], Day [6], and Wyler [19], among others, who, in general, neglected set theoretical difficulties. Herrlich, in [9], [10], gave a study of initial and final completions taking into account the legitimacy of the constructions. Cartesian closed initial completions, mainly over $\text{Sets}$, were also considered by Herrlich [8], Nel [14], and Herrlich and Nel [11]. In [2], [3] Adámek and Koubek gave necessary and sufficient conditions for certain concrete categories over $\text{Sets}$ to have cartesian closed final completions. Their completions, however, have several drawbacks; they are not generally amnestic, the morphisms are difficult to handle, and, although final lifts are easy to describe, initial lifts are not. An improved version of [3] is given in [4]. In these papers geometric completions are not considered (those of [3] are not geometric), little attention is generally paid to universal final lifts, and no explicit mention of $\Delta$ is made, although its existence, for $\text{Sets}$, is implied, for example, by the special assumptions on the empty and singleton sets (p. 442 [14], § 2.7 [19]) by the conditions on constant maps (2.1 [4], § 2.7 [19]) and by the results on final epi sinks ([8], p. 442 [14]).

It should be noted that the basic aim of this paper and of [4] is to construct topological completions and that while the completions obtained in both cases are categories of sieves, the sieves involved are technically different due to the divergent objectives of the papers, i.e. the construction of geometric completions with universal final lifts as opposed to the construction of cartesian closed completions.

General definitions and results are given in §§ 2, 3.4. The main results are stated and discussed in § 5 and proved in §§ 6, 7.

I thank the referee for many useful suggestions.

2. Topological $\Delta$-functors

A functor $U : E \to B$ is said to be concrete if it is faithful (i.e. $U$ is mono on hom sets) and amnestic (if $U(f) = 1$ and $f$ is an isomorphism then $f = 1$). Note that the fibers $U^{-1}(b)$, $b \in B$, of $U$ are naturally partially ordered (preordered since $U$ is faithful, antisymmetric since $U$ is amnestic) classes. The functor $U$ is said to be topological if it is concrete, has small (i.e. set) fibers and for which every $U$-source (i.e. family, possibly large, possibly empty, of pairs $(f, e)$, $f : b \to Ue$, $b$ a fixed object of $B$) has an initial lift or, equivalently, for which every $U$-sink (i.e. family $\{ (e, f) \}$, $f : U(e) \to b$) has a final lift (cf. [7]). Note that the fibers of a topological functor
are non empty complete posets. Each such topological functor $U$ has a left adjoint $D : B \to E$, where $D(b)$ is obtained as the final lift of the empty sink on $b$. The objects $D(b)$ are said to be discrete; they are characterized as the minimum elements of the fibers of $U$ or as those objects $d$ of $E$ for which every map $Ud \to Ue$ lifts to a map $d \to e$. The functor $U$ is said to be geometric (see 2.1 (1) below) topological if it is topological and if, for any finite (possibly empty) mono source $g : b \to b_j$ in $B$, $D(g) : D(b) \to D(b_j)$ is an initial lift. Note that if $U$ is geometric then $D$ preserves terminal objects since the empty source from the terminal is mono. Let $\Delta(U)$ be the full subcategory of $B$ generated by those objects $b$ for which $U^{-1}(b)$ has at most one element. If $U$ is topological then $\Delta(U)$ consists of those objects of $B$ that admit only a discrete structure. If $\Delta$ is a full subcategory of $B$ then $U : E \to B$ is said to be a $\Delta$-functor if all $\Delta$-maps lift. (3) If $U$ is a geometric $\Delta$-functor and $b \to d_j$ is a finite mono source in $B$ with $d_j \in \Delta$ then $b \in \Delta(U)$.

2.1. Proposition. Let $U : E \to B$ be a topological functor. (1) If $B$ has all finite limits and all monos in $B$ are equalizers (e.g. if $B$ is a topos) then $U$ is geometric iff $U$ is a geometric morphism (p. 26 [12]) in the sense that its left adjoint is left exact (i.e. preserves finite limits). Let $\Delta$ be a discrete structure on $B$. (2) $U$ is a $\Delta$-functor iff all $\Delta$-maps lift. (3) If $U$ is a geometric $\Delta$-functor and $b \to d_j$ is a finite mono source in $B$ with $d_j \in \Delta$ then $b \in \Delta(U)$.

Proof. (1) Under the assumptions on $B$, each finite mono source $g_j : b \to b_j$ is equivalent to an equalizer $(g_j) : b \rightrightarrows 1 b_j$. It readily follows that $D$ is left exact (i.e. preserves equalizers and finite products) iff $\{D(g_j)\}$ is an initial lift for all finite mono sources $\{g_j\}$. (2) If $U$ is a topological $\Delta$-functor and $f = f_2 f_1 : Ue_1 \to d \to Ue_2$, $d \in \Delta$, is a $\Delta$-map then, since $d \in \Delta(U)$, the final lift $\tilde{f}_1 : e_1 \to e$ of $f_1$ satisfies $e = D(d)$, and consequently $f = U(\tilde{f}_2 \tilde{f}_1)$, where $\tilde{f}_2 : D(d) \to e_2$ is the adjoint of $f_2$, i.e. $\Delta$-maps lift. Suppose $\Delta$-maps lift and $d \in \Delta$. If $Ue_1 = Ue_2 = d$ then the identity $Ue_1 = Ue_2$, being a $\Delta$-map, lifts to maps $e_1 \to e_2$, $e_2 \to e_1$. By concreteness, then, $e_1 = e_2$ and $d \in \Delta(U)$, i.e. $U$ is a $\Delta$-functor. (3) Suppose $Ue = b$ for the finite mono source $g_j : b \to d_j$, $d_j \in \Delta$. Since $U$ is a $\Delta$-functor, for each $j$ the final lift $\tilde{g}_j : e \to e_j$ of $g_j : Ue \to d_j$ satisfies $e_j = D(d_j)$. Since, by assumption, $D(g_j) : D(b) \to D(d_j)$ is initial, the identity $Ue = UD(b) = b$ lifts to a map $e \to D(b)$ and since it always lifts to a map $D(b) \to e$, $D(b) = e$ and $b \in \Delta(U)$.

3. $\Delta U$-sieves and sinks

In this (and the following) section we define certain types of sieves and sinks which are basic to the rest of the paper. Given $U : E \to B$ and $b \in B$ let $\Omega(b)$ be the class of $U$-sieves (i.e. saturated $U$-sinks) on $b$. Thus a family $S = \{(e, f : Ue \to b)\}$ is in $\Omega(b)$ iff for any $E$-map $g : e_1 \to e$, if $(e, f) \in S$ then $(e_1, f_\cup g) \in S$. $\Omega(b)$ can be
viewed as the collection of all subfunctors of the functor $B(U(-) b): E^{op} \to \text{Set}$ (compare p. 13 [12]) and, as such, is readily seen to have the structure of a complete, distributive, partially ordered class. More explicitly, for a family $\{S_j \in \Omega(b), j \in J\}$, $\bigcap_{j} S_j = \{(e, f): (e, f) \in S_j \text{ for all } j \in J\}$ and $\bigcup_{j} S_j = \{(e, f)(e, f) \in S_j \text{ for some } j \in J\}$. Further, for each $B$-map $g: b \to c$ the correspondence $S \mapsto \{(e, gf)|(e, f) \in S\}$ induces a functor $g_*: \Omega(b) \to \Omega(c)$. Moreover, the correspondence $S \mapsto \{(e, f)|(e, gf) \in S\}$ induces a functor $g^*: \Omega(c) \to \Omega(b)$ that is right adjoint to $g_*$. Note that $g^*$ is both continuous and cocontinuous while $g_*$ is only cocontinuous. Hence $\Omega$ can be viewed as a functor from $B$ to the category with complete, distributive, partially ordered classes as objects and with cocontinuous maps with cocontinuous right adjoints as morphisms.

We next introduce two important relations on $\Omega(b)$.

3.1. Definition. For $S_1, S_2 \in \Omega(b)$, define $S_1 \approx S_2$ if for all pairs $(g, e), g: b \to U(e)$, the map

$$U(e_1) \xrightarrow{f_1} b \xrightarrow{g} U(e)$$

lifts for all $(e_1, f_1) \in S_1$ iff the map

$$U(e_2) \xrightarrow{f_2} b \xrightarrow{g} U(e)$$

lifts for all $(e_2, f_2) \in S_2$. Define $S_1 \approx^* S_2$ iff for all maps $k: a \to b$ and all $U$-sieves $T$ on $a$, $T \cap k^* S_1 \approx T \cap k^* S_2$.

For example if $E = k\text{Top}$ then $S_1 \approx S_2$ means that $S_1$ and $S_2$ coinduce the same topology on $b$. In fact this observation holds for arbitrary topological functors and is the gist of lemma 7.5(1). Similarly, $S_1 \approx^* S_2$ means that the topologies coinduced on $b$ by any pair of subsieves $T \cap S_1, T \cap S_2$, of $S_1, S_2$, respectively, coincide, and this property is preserved under sieve pullback. See lemma 7.5(2) for a more general result.

In terms of the functor $g_*$ it is readily seen that $S_1 \approx S_2$ iff for all the pairs $(g, e), g_* S_1 \subseteq S(e)$ iff $g_* S_2 \subseteq S(e)$ where $S(e) = \{(e', f: U e' \to U e)|f \text{ lifts}\}$ is the $U$-sieve on $U(e)$ determined by the $B$-maps that lift.

3.2. Remarks. It should be noted, for $B = \text{Sets}$, that our relation $\approx$ corresponds to the relation $\sim$ of § 1.3 [3] in the sense that structured maps (in the terminology of § 1.3 [3]) $(e, f: U(e) \to b)$ and $(e', g: U(e') \to b)$ are $\sim$-equivalent iff the $U$-sieves $S_f$ and $S_g$ generated by these maps respectively are $\approx$-related. Furthermore, if $E$ has finite concrete products (in the terminology of § 1.1 [3]) then our relation $\approx^*$ implies the relation "productive equivalence" of § 1.4 [3] in the sense that if $S_f \approx^* S_g$ then $(e, f)$ and $(e', g)$ are productively equivalent. To see this note that if $\pi_1: b \times U(e) \to b$ denotes projection on the first factor and $T = \{(e, h: U(e) \to b \times U(e_1)) | \pi_2 h: U(e) \to b \times U(e) \to U(e_1) \text{ lifts}\}$ then $S_f \approx^* S_g$ implies, by definition, that $T \cap \pi_1^* S_f = T \cap \pi_1^* S_g$, from which it readily follows that $(e, f)$ and $(e', g)$ are productively equivalent.
In order to state the main result (5.2) a further definition is needed. To this end let $U_i : E_i \rightarrow B$ and let $i : (E, U) \rightarrow (E_1, U_1)$ be a functor over $B$ ($U = U_1 i$). The correspondence $S \rightarrow \{(e, g : U e \rightarrow b)\}$ there are $(e, g : U e \rightarrow b) \in S$ and a map $k : e \rightarrow i(e)$ such that $g = g U_1(k)$ clearly defines a cocontinuous functor $i^*_b : \Omega_1(b) \rightarrow \Omega_i(b) = \text{class of } U_i$-sieves on $b$. Moreover, the correspondence $S_1 \rightarrow \{(e, g) : (i(e), g) \in S_1\}$ defines a functor $i^*_b : \Omega_i(b) \rightarrow \Omega_1(b)$ that is right adjoint to $i_b$. If $i$ is full then $i^*_b i_b S = S$.

3.3. Definition. An $i\Delta U_1$-sieve is a $U_1$-sieve $S_1 \in \Omega_i(b)$ for which $i^*_b S_1 = i^*_b S_1 \cup \Delta(b)$, where $\Delta(b) = \{(e, f : U e \rightarrow b) | f \text{ factors through an object of } \Delta\}$ is the $U$-sieve of $\Delta$-maps. $S_1$ is said to be an $i\Delta^* U_1$-sieve if $i^*_b S_1 = *i^*_b S_1 \cup \Delta(b)$. An $i\Delta$-sink (resp. $i\Delta^*$-sink) is a sink in $E$ for which the $U_1$-sieve generated by it is an $i\Delta U_1$-sieve (resp. $i\Delta^* U_1$-sieve). When $i$ is the identity functor, i.e. $U_1 = U$, we refer to $i\Delta U_1$- and $i\Delta^* U_1$-sieves simply as $\Delta U$- and $\Delta^* U$-sieves respectively.

For future use we note the following result.

3.4. Lemma. If $\Delta$-maps lift for $U : E \rightarrow B$ then for any $U$-sieve $S$ on $b$, $S = S \cup \Delta(b)$.

Proof. For any $g : b \rightarrow U(e)$, clearly $g_*(S) \subseteq S(e)$ if $g_*(S \cup \Delta(b)) \subseteq S(e)$. On the other hand if $\Delta$-maps lift then $g_*(\Delta(b)) \subseteq S(e)$ and consequently $g_*(S \cup \Delta(b)) \subseteq S(e)$ if $g_*(S) \subseteq S(e)$.

4. Universal families of sinks

Let $B$ be a category with pullbacks and $U : E \rightarrow B$ a topological functor. By § 5 [7] $E$ also has pullbacks. Let $\{g_j : x \rightarrow x_j, j \in J\}$ be a source in $E$ and let $\sigma_j$ be a sink in $E$ on $x_j$ for each $j \in J$. If $J$ is finite and $f = (f_j : y_j \rightarrow x_j)_{j \in J}$ let $L(f)$ be the limit of the diagram

$$(x \leftarrow \cdots \leftarrow y_j, j \in J)$$

with $p(f) : L(f) \rightarrow x$ and $p_j(f) : L(f) \rightarrow y_j$ as the canonical projections. By the joint pullback of $\{\sigma_j\}$ along $\{g_j\}$ we mean the sink $\{p(f) : L(f) \rightarrow x\}$ on $x$, where $f$ ranges over $\Pi_j \sigma_j$. The joint pullback reduces to the pullback in case $J$ is a one point set. Recall that a sink $\sigma$ in $E$ is called final if it is the final lift of the $U$-sink $U \sigma = \{e, U g\} | (e, g) \in \sigma\}$. A finite family of final sinks is said to be universal if every joint pullback of it is a final sink. Universal finite families of $i\Delta$- and $i\Delta^*$-sinks are defined similarly by requiring the joint pullbacks to be $i\Delta$- and $i\Delta^*$-sinks respectively.

The following result gives some conditions for the existence of certain universal families of sinks.
4.1. Lemma. Let $U_1 : E_1 \rightarrow B$ be a topological functor over a category $B$ with pullbacks. (1) If all final sinks in $E_1$ are universal then the same is true of all finite families of final sinks. (2) If $U_1$ is a topological $\Delta$-functor then all finite families of $i\Delta^*$-sinks are universal.

Proof. (1) Let $g_j : x \rightarrow x_j$ be a source and $\sigma_j$ a final sink on $x_j$ for $j \in J = \{1, 2\}$. If $f = (f_1, f_2) \in \sigma_1 \times \sigma_2$ then the map $p(f) : L(f) \rightarrow x$ can be described as the composite

$$L(f) \xrightarrow{f_2^*} x_2 \xrightarrow{\tau} x,$$

where $f_2^*$ is the pullback of $f_2$ along $g_2$ and $f_2^*$ is the pullback of $f_2$ along $g_2 f_1^*$. By hypothesis, the sink of all $f_2^*$ is final and, for each $f_1$, the sink of all $f_2^*$ is final. From this it readily follows that the joint pullback $\{p(f) | f \in \sigma_1 \times \sigma_2\}$ is a final sink. Since the argument clearly extends to the finite $J$ case the result follows. The proof of (2) depends on a further analysis of $\approx^*$ and is given in §6 as corollary 6.2.

The following notion is useful in establishing cartesian closedness of certain topological categories.

4.2. Definition. A category $B$ is said to have enough $i\Delta^* U_1$-sieves for a functor $i : (E, U) \rightarrow (E_1, U_1)$ over $B$ if each $U_1$-sieve $S_1$ can be written $S_1 = g S$, where $g$ is a mono and $g S$ is an $i\Delta^* U_1$-sieve. Here $g, g^*$ is the adjoint pair of functors of §3.

4.3. Proposition. Let $B$ be cartesian closed and have enough $i\Delta^* U_1$-sieves where $U_1 : E_1 \rightarrow B$ is a topological functor. If final $i\Delta^*$-sinks in $E_1$ are universally final then $E_1$ is cartesian closed.

Proof. Since $B$ is cartesian closed and $U_1$ is topological, $E_1$ has finite products and $U_1$ preserves them (§5 [7]). For $Y, Z \in E_1$, let $b = U_1(Y)^{U_1(Z)}$. Write the $U_1$-sieve $S_1$ of all pairs $(e_1, f : U_1 e_1 \rightarrow b)$ for which the adjoint $U_1(e_1 \times Z) \rightarrow U_1(Y)$ of $f$ lifts, as $S_1 = g^*(g S)$, where $g : a \rightarrow b$ is a mono and $S = g^* S_1$ is an $i\Delta^* U$-sieve. Define $Y^Z$ via the final lift $\sigma = \{e_1 \rightarrow Y^Z\}$ of $S$. Clearly $\sigma$ is an $i\Delta^*$-sink and thus $\sigma \times Z = \{e_1 \times Z \rightarrow Y^Z \times Z\}$ is a final sink. From this it readily follows that the adjoint $U_1(Y^Z \times Z) = a \times U_1(Z) \rightarrow U_1(Y)$ of $g$ lifts to a map $Y^Z \times Z \rightarrow Y$ that defines the counit (evaluation) of the desired adjunction rendering $E_1$ cartesian closed.

5. The embedding theorems

In this section we state the main embedding theorem (5.2), derive some important corollaries (5.3 and 5.5), and discuss several special cases covered by the corollaries. A functor $i : (E, U) \rightarrow (E_1, U_1)$ over $B$ is said to be $\Delta^*$-dense if for any $e_1 \in E_1$, the
sink \( \{ f: ie \to e_1 | e \in E, f \in E_i \} \) is a final \( i\Delta^* \)-sink and to be an embedding if it is full, faithful, and injective on objects. Note that \( \Delta^* \)-dense implies dense and that any full \( \Delta^* \)-dense functor between concrete categories preserves initial lifts (cf. proposition 10 [11]). Similar statements hold with \( \Delta \) in place of \( \Delta^* \).

5.1. Definition. A functor \( U: E \to B \) is said to \( \Delta \)-small if the collection of \( \approx^* \) equivalence classes of \( \Delta^* U \)-sieves on \( b \) is a small set for all \( b \in B \). It is said to be \( \Delta^* \)-geometric if, for any finite mono source \( b \to d_j \) with \( d_j \in \Delta \), \( M(b) = \approx^* \Delta(b) \), where \( M(b) \) is the maximum \( U \)-sieve on \( b \). The notions of \( \Delta \)-small and \( \Delta^* \)-geometric functors are defined analogously with \( \approx^* \) replaced by \( \approx \).

The notions of \( \Delta \) and \( \Delta^* \)-small are technical assumptions that, as will be seen, are necessitated by the requirement that the fibers of a topological functor be small. It should be noted that, as a direct consequence of 5.1 and 3.1, \( \Phi^* \)-small (\( \Phi \) denotes the empty category) implies “strictly small fibered” in the sense of § 1.4 [3] and 1.12 [4] when \( E \) is concrete and has concrete finite products over \( B = \text{Sets} \). In particular, the nonstrictly small fibered topological functor of § 3 [3] is an example of a functor that is not \( \Phi^* \)-small. On the other hand “\( \Phi^* \)-small” clearly implies “\( \Delta \)-small” for any \( \Delta \). Moreover, if \( E \) is itself a small category then \( \Omega(b) \) is a set and consequently all functors \( U: E \to B \) are \( \Phi^* \)-small. In this case if \( U \) is topological then \( B \) is also small since it can be identified with the full subcategory of discrete objects of \( E \). Further the \( \Phi \)-small functors are strongly fibre-small in the sense of [1] and, by 3.4, a \( \Delta \)-small functor is \( \Phi \)-small if \( \Delta \)-maps lift. As will be seen, all topological functors are \( \Phi \)-small and thus, for example, the forgetful functor from the category of topological spaces is \( \Phi \)-small over \( \text{Sets} \) and its restriction to any small subcategory is \( \Phi^* \)-small, but not topological, over \( \text{Sets} \).

5.2. Theorem. Let \( U: E \to B \) be a concrete functor and \( \Delta \subseteq B \). (1) \( U \) can be \( \Delta \)-densely embedded into a topological \( \Delta \)-functor \( U_1 \) over \( B \) if \( U \) is \( \Phi^* \)-small and \( \Delta \)-maps lift. \( U_1 \) can be chosen to be geometric iff \( U \) is \( \Delta \)-geometric. Assume \( B \) has pullbacks. (2) There is a \( \Delta^* \)-dense embedding \( i: (E, U) \to (E_1, U_1) \) for which \( U_1 \) is a topological \( \Delta \)-functor and finite families of final \( i\Delta^* \)-sinks in \( E_1 \) are universal iff \( U \) is \( \Delta^* \)-small and \( \Delta \)-maps lift. Moreover \( U_1 \) is geometric iff \( U \) is \( \Delta^* \)-geometric.

We postpone the rather technical proof of 5.2 to § 7 and instead turn to a discussion of various aspects of the result. We begin with a closer look at the role of the category \( \Delta \).

If \( \Delta_1 \subseteq \Delta_2 \) then a \( \Delta^* \)-small concrete functor \( U: E \to B \) is \( \Delta^*_1 \)-small, \( \Delta_1 \)-maps lift if \( \Delta_2 \)-maps do, and an \( i\Delta^*_1 \)-sink is an \( i\Delta^*_2 \)-sink. Thus the smaller \( \Delta \) is, the stronger is the \( \Delta \) and \( \Delta^* \)-smallness conditions on \( U \) but the more \( \Delta^* \)-sieves and \( i\Delta^* \)-sinks there are (if \( \Delta = \phi \) then every \( U \)-sieve is a \( \phi^* \)-sieve, \( B \) always has enough \( \phi^* \)-sieves (take \( g = \text{identity} \) in Definition 4.2), and \( \phi^* \)-sinks are just sinks). The larger \( \Delta \) is, the weaker is the \( \Delta \) and \( \Delta^* \)-smallness conditions on \( U \) (every \( U \) is...
A*-small if $\Delta = B$ but, if $\Delta$-maps lift, the closer $U$ is to being an embedding (if $\Delta = B$ then $U$ is full, and consequently, by concreteness, is an embedding). The geometric situation is more involved. The smaller $\Delta$ is, the smaller is $FM(\Delta) = \{ b \mid \text{there is a finite mono source } b \rightarrow d, d \in \Delta \}$ but the stronger is the condition on $U$ that $M(b) \approx \Delta(b)$. If $\Delta$-maps lift then $M(b) \approx \Delta(b)$ iff $\{ b \}$-maps lift. Hence, if $\Delta$-maps lift, $U$ is $\Delta$-geometric iff $FM(\Delta)$-maps lift. The $\Delta^*$-geometric case is generally more complicated. However, if the identity map on each $b \in FM(\Delta)$ is a $\Delta$-map (i.e. if $b$ is a retract of a $\Delta$-object, e.g. if $FM(\Delta) \subseteq b$) then, since $M(b) = \Delta(b)$, $U$ is $\Delta^*$-geometric. In fact, $FM$ is a closure operation ($\Delta \subseteq FM(\Delta)$, $FM(FM(\Delta)) = FM(\Delta)$, $FM(\Delta_1) \subseteq FM(\Delta_2)$ if $\Delta_1 \subseteq \Delta_2$) on the class of discrete structures on $B$ and any functor $U : E \rightarrow B$ is $\Delta^*$-geometric for any $FM$-closed discrete structure $\Delta$. Since $FM(\Delta) = \Delta$ for $\Delta = B$, all $U$ are $B^*$-geometric. For $\Delta = \phi$, $FM(\phi)$ consists of those objects of $B$ from which the empty source is mono, i.e. of the subterminal objects of $B$ (if $B$ has a terminal 1, then $b \in FM(\phi)$ iff $b \rightarrow 1$ is mono). It readily follows that $U$ is $\phi$-geometric iff constant (i.e. $FM(\phi)$) maps lift and that $U$ is $\phi^*$-geometric iff maps that factor through objects with support (i.e. objects that map to subterminals) lift. In particular, if $B$ has a terminal then $U$ is $\phi^*$-geometric iff $U$ is full (i.e. an embedding, if $U$ is concrete). In view of these observations, 5.2 and (the proof of) 4.3 give:

5.3. Corollary. Let $U : E \rightarrow B$ be a concrete functor. If $U$ is $\phi$-small (e.g. if $E$ is small) it can be densely embedded, over $B$, in a topological functor $U_1$ that is geometric if subconstant maps lift. Let $U$ be $\phi^*$-small and $B$ have pullbacks. $U$ can be densely embedded, over $B$, in a topological category $(E^*, U^*)$ in which final sinks are universal. If $B$ is cartesian closed then $E^*$ is cartesian closed and $U^*$ preserves exponentials. In this case, $U^*$ is geometric iff it is an isomorphism.

The next result is central to several further applications of 4.3 and 5.2. If $U : E \rightarrow B$, $\Delta \subseteq B$, and $\sigma$ is a sink on $b \in B$, let $\Delta(\sigma) = \{(e, h) \mid h = fk : U(e) \rightarrow a \rightarrow b, \text{ for some } \Delta\text{-map } k, \text{ and } f \in \sigma\}$. Note that $\Delta$ (identity on $b$) = $\Delta(b)$ and $\Delta(\sigma) \subseteq \Delta(b)$. Let $\mathcal{E}$ be a family of sinks in $B$. $B$ is said to have $\mathcal{E}$-factorizations if each sink in $B$ factors, via a mono, through a sink in $\mathcal{E}$. A category has $\mathcal{E}$-factorizations, for example, if it is an ($\mathcal{E}$-$M$)-category in the sense dual (i.e. for sinks instead of sources) to that in [7]. An $\mathcal{E}$-sink in $E$ is a sink $\sigma$ in $E$ with $\{ Uf | f \in \sigma \} \in \mathcal{E}$.

5.4. Lemma. Let $\mathcal{E}$ be a family of sinks in $B$ such that for all $b \in B$, and all $\sigma \in \mathcal{E}$ on $b$, $\Delta(\sigma) = \Delta(b)$, for $U : E \rightarrow B$ and $\Delta \subseteq B$. Let $i : (E, U) \rightarrow (E_1, U_1)$ be a functor over $B$ and suppose $\Delta$-maps lift through $U_1$. (1) If $B$ has $\mathcal{E}$-factorizations then $B$ has enough $i\Delta^*U_1$-sieves. (2) Each $\mathcal{E}$-sink $\sigma$ in $E_1$ is an $i\Delta^*$-sink.

Proof. Given a $U_1$-sieve $S$ on $b$, let $|S| = g\sigma$ be a factorization of the sink $|S| = \{ f | (e_1, f) \in S \text{ for some } e_1 \in E_1 \}$, where $g : a \rightarrow b$ is a mono and $\sigma \in \mathcal{E}$. To complete (1) it suffices, since $S = g^*g^*S$ and $\Delta(\sigma) = \Delta(b)$, to show that $\Delta(\sigma) \subseteq i^*g^*S$. But
if \((e, h) \in \Delta(\sigma)\) then \(h = f k : U e = U i(e) \to U i(e) \to a\), where \(k\) is a \(\Delta\)-map and \((e_1, g f) \in S\). Since \(\Delta\)-maps lift, \(k = U k_1\), for \(k_1 : i(e) \to e_1\). Thus \((i(e), gh)\) factors, via \(k_1\), through \((e_1, gf) \in S\) and \((i(e), h) \in g^* S\) i.e. \((e, h) \in i^* g^* S\). This shows (1). Part (2) is proved similarly by showing that \(\Delta(\{U i f | f \in \sigma\}) \subset i^* U i \sigma\) where \(\sigma\) is the sieve in \(F_i\) generated by \(\sigma\).

As a direct consequence of 4.3 and 5.2, in view of 5.4 and 2.1(2), we have:

5.5. Corollary. Let \(U\) be a concrete, \(\Delta^*\)-small functor over a cartesian closed category \(B\) with pullbacks and \(\varepsilon\)-factorizations. If \(\Delta\)-maps lift and \(\Delta(\sigma) = \Delta(b)\) for all \(b \in B\), \(\sigma \in \varepsilon\) on \(b\) then \(U\) has a cartesian closed topological completion, in the sense of 5.2, in which the joint pullback of final \(\varepsilon\)-sinks is final and which is geometric if \(FM(\Delta) = \Delta\).

We conclude this section by considering some special cases and situations covered by the previous results. Let \(Ep(\Delta)\) be the class of sinks in \(B\) that transforms to epi sinks under all the representable functors \(B(d, -) : B \to \text{Sets}, d \in \Delta\).

5.6. Corollary. Let \(B\) be a topos. If \(Ep(\Delta)\) contains the singleton epi sinks or the coproduct sinks then the completion of 5.2(2) has universal quotients (i.e. final singleton epi sinks) or universal coproducts respectively.

Proof. Since a topos has universal epis and coproducts the result is a direct consequence of 5.4(2) in view of the readily proved fact that \(\Delta(\sigma) = \Delta(b)\) for any sink \(\sigma \in Ep(\Delta)\) on \(b\) and for any \(U : E \to B\).

5.7. Corollary. Let \(\Delta\) consist of the subterminal objects of a Grothendieck topos \(B\) and let \(U : E \to B\) be a concrete functor with \(E\) a small category. If subconstant maps lift and \(Ep(\Delta)\) contains the epi sinks then \(U\) has a cartesian closed geometric topological completion with universal final epi sinks.

Proof. The result will follow from 5.5, with \(\varepsilon\) the class of epi sinks, once the hypotheses are verified. Since \(E\) is assumed to be a small category and \(B\) is a topos, \(U : E \to B\) is a concrete \(\Delta^*\)-small functor over a cartesian closed category with pullbacks. We next show that Grothendieck toposes have (epi sink)-factorizations. To this end let \(\sigma\) be a sink in \(B\). Since the class of subobjects of an object in a topos has a set of representatives there is a small subsink \(\sigma_1\) of \(\sigma\) so that each map in \(\sigma\) factors through the image of some map in \(\sigma_1\). Since a Grothendieck topos has set-indexed coproducts (p. 17 [12]) the map \(f = \sum f_i : \prod a_i \to b, f_i \in \sigma_1\) exists. If \(f = gh : \prod a_i \to b\) is the image factorization of \(f\) then \(\sigma_1\) clearly factors, via the mono \(g\), through the epi sink \(\{h_i : a_i \to a\}\) associated to \(h\). It readily follows that \(\sigma\) has the desired factorization. Moreover, as in the proof of 5.6, \(\Delta(\sigma) = \Delta(b)\) if \(\sigma \in Ep(\Delta)\).
Finally, since \(\Delta\) is closed under the formation of subproducts, \(FM(\Delta) = \Delta\) and the conditions of 5.5 are met.
The next result characterizes those spatial toposes that satisfy the various requirements of 5.6 and 5.7.

5.8. Proposition. Let $\Delta$ consist of the subterminal objects in the spatial topos $B$ of sheaves on a topological space $X$. (1) $Ep(\Delta)$ contains the singleton epi sinks if and only if supports split in $B$ (e.g. if $X$ is separable and zero dimensional, problem 4 p. 163 [12]). (2) $Ep(\Delta)$ contains the coproduct sinks if and only if $X$ is hereditarily connected (i.e. all open subsets are connected). (3) $Ep(\Delta)$ contains the epi sinks if and only if the open subsets of $X$ are compact and are linearly ordered by inclusion.

Proof. (1) Since, in any topos, the pullback of any epi $f: a \to b$ along any map $g: d \to b$ is an epi, it has, if supports split, a section for all $d \in \Delta$. Hence each $g \in B(d, b)$ factors through $f$ and consequently $f \in Ep(\Delta)$. Conversely, if the epi $f: a \to d$ represents the support of a then $f \in Ep(\Delta)$ and consequently the identity on $d$ factors through $f$, i.e. $f$ splits. (2) In any topos the pullback of any coproduct $f: b \to b$ along any map $g: d \to b$ is again a coproduct. Hence if all $d \in \Delta$ admit only the trivial coproduct decompositions, as is the case, then each $g \in B(d, b)$ factors through at least one of the maps $f_i$, i.e. $\{f_i\} \in Ep(\Delta)$. Conversely, if $d_1 \sqcup d_2 \to d$ is a separation of $d \in \Delta$ then $\{d_i \to d\}$ is in $Ep(\Delta)$ and consequently the identity on $d$ factors through $d_i$, say, i.e. $d_2 = 0$, a contradiction. Hence all open subsets of $X$ are connected. (3) Since the pullback $\sigma^*$ of an epi sink $\sigma$ on $b$ along a map $g: d \to b$ is an epi sink and since, for $d \in \Delta$, $d$ is a compact space with the set of open subsets linearly ordered, there is an epi map $f^* \subset \sigma^*$. Further, since $f^*$ is a local homeomorphism it must have a section and consequently $g$ factors through the map $f \in \sigma$, i.e. $\sigma \in Ep(\Delta)$. Conversely suppose $Ep(\Delta)$ contains all epi sinks. If $\{d_i\}$ is an open cover of $d \in \Delta$ then the identity map on $d$ factors through some inclusion $d_i \to d$ of the epi sink $\{d_i \to d\}$, i.e. $d$ is compact. Similarly, given $d_1$ and $d_2 \in \Delta$, either $d_1 \cup d_2 \subset d$ or $d_1 \cap d_2 \subset d_2$, i.e. either $d_1 \subset d_2$ or $d_2 \subset d_1$. This completes the proof.

Among the situations covered by 5.8(3) is the classical one in which $X$ is a point, i.e. $B = \text{Sets}$, and $\Delta = \{\emptyset, 1\}$. In this case 5.7 yields not only an embedding theorem but also a method for generating Set-valued topological functors with very nice properties.

5.9. Corollary. Let $U: E \to \text{Sets}$ be a concrete functor from a small category $E$. If constant maps lift then $U$ admits a cartesian closed geometric topological completion with universal final epi sinks.

6. Properties of $\approx$ and $\approx^*$

In this section are set forth certain properties of the relations $\approx$ and $\approx^*$ of 3.1 which are necessary for the proof of 5.2 in § 7.
6.1. Lemma. Let $S, S', S_n, S_j' \in \Omega(b)$, $j \in J$. (1) If $S \simeq S'$ then $S \subseteq S'$. (2) If $S(e) \subseteq S$ then $S \subseteq S(e)$. (3) If $S_j \simeq S_j'(S_j' \simeq S_j)$, $j \in J$, $J$ arbitrary, then $\bigcup_j S_j \simeq \bigcup_j S_j'$. (4) If $S_j \simeq S_j'$, $j \in J$, $J$ a finite set, then $\bigcap_j S_j \simeq \bigcap_j S_j'$. (5) If $S \simeq S'$ and $f : b \to c$ then $f \ast S \simeq f \ast S'$. (6) If $S \simeq S'$ and $f : a \to b$ then $f \ast S \simeq f \ast S'$.

Proof. (1) is trivial. (2) This follows from the definition (3.1) with $(g, e) =$ (identity on $U(e)$, $e$). (3) This is a direct consequence of the equations $g \ast (\bigcup_j S_j) = \bigcup_j g \ast S_j$ and $T \cap k \ast \bigcup_j S_j = (T \cap \bigcup_j k \ast S_j) = \bigcup_j (T \cap k \ast S_j)$. (4) If $S_j \simeq S_j'$ then $S \cap S_j \simeq S \cap S_j'$ since $T \cap k \ast (S \cap S') = (T \cap k \ast S) \cap k \ast S'$ for $S' = S_j$ and $S_j'$. Hence, if $S_j \simeq S_j'$ then $S_j \cap S_j \simeq S_j' \cap S_j'$; and the result follows by induction. (5) and (6) follow from the equations $g \ast (f \ast S) = (g \ast f) \ast S$ and $T \cap k \ast (f \ast S) = T \cap (f \ast k) \ast S$ respectively.

6.2. Corollary. If $U_i : E_i \to B$ is a topological $\Delta$-functor over a category $B$ with pullbacks and $i : (E, U) \to (E_i, U_i)$ is a functor over $B$ then any finite family of $i \Delta^*$-sinks is universal.

Proof. Let $\sigma_i$ be an $i \Delta^*$-sink on $x_j$ and $g_j : x \to x_j$ a map in $E_i$ for $j \in J$, a finite set. Let $A_i$ be $i \Delta_i^*$ of the $U_i$-sieve generated by the joint pullback of $\{\sigma_i\}$ along $\{g_i\}$ and let $A_i$ be $i \Delta_i^*$ of the $U_i$-sieve generated by $\sigma_i$, where $b = U_i(x)$ and $b_j = U_i(x_j)$. We shall show that $A_j = *A_j \cup \Delta(b)$, $j \in J$, implies $A = *A \cup \Delta(b)$ and consequently, by 3.3, the result. By 6.1(6) and the continuity of $(U_i, g_i)^\ast$ we have that $B_j = (U_i g_j)^\ast A_j$. Since $(U_i g_j)^\ast (\Delta(b)) \subseteq \Delta(b)$, adjointness gives $\Delta(b) \subseteq \bigcup_j B_j = \bigcup_j B_j \cap \bigcup_j \Delta(b)$ and thus, by 6.1(3), $B_j = \bigcup_j B_j \cap \Delta(b) = (U_i g_j)^\ast (\Delta(b))$. By 6.1(3) and (4) we have $\bigcup_j B_j = \bigcup_j (B_j \cap \Delta(b)) = \bigcup_j (B_j \cap \bigcup_j \Delta(b)) = * \bigcup_j B_j$.

6.3. Lemma. For $U : E \to B$, let $g : b \to b_j$ be a map in $B$ and let $\sigma$ be a family of pairs $(S, f)$, with $f$ a map in $B$ to $b$, and $S \in \Omega$ (domain $f$). If $B$ has pullbacks and $T \in \Omega(b)$ then $g^\ast (\bigcup S) \cap T = \bigcup(f^\ast (g^\ast S \cap f^\ast T))$, where the union is taken over all $j$.

The following result has several useful consequences.
Proof. Clearly \((e, h)\) is in the left side \(U\)-sieve iff \((e, h) \in T\) and \((e, gh) \in \bigcup f_{*} S\) i.e. iff \((1) \ (e, h) \in T\) and there is \((S, f) \in \sigma\) and a map \(k: U(e) \to \text{domain } (f)\) such that \(gh = fk\) and \((e, k) \in S\) while \((2) \ (e, h) \in T\) there is \((S, f) \in \sigma\) and a map \(g'\) to \(b\) with \(h = f g'\), \((e, g g') \in S\) and \((e, h) \in T\). If \(1)\) holds then there is a map \(g'\) to \(b\) such that \(h = f g'\) and \(g g' = k\). Since \((e, k) \in S\) condition \(2)\) holds. On the other hand if \(2)\) holds then \(gh = g f g' = f (f g')\) and \((e, g g') \in S\) i.e. condition \(1)\) holds with \(k = g g'\). Thus \(1)\) and \(2)\) are equivalent and the result follows.

6.4. Corollary. Let \(B\) have pullbacks. If \(S = *S'\) in \(\Omega(b)\) and \(f: b \to c\) then \(f_{*} S = *f_{*} S'\).

Proof. Given \(k: a \to c, T \in \Omega(a)\) we must show \(T \cap k^{*} f_{*} S = T \cap k^{*} f_{*} S'\) since \(S \approx S'\) it follows from 6.1 \((6, 4, 1, S)\) that \(f_{*} (k^{*} S \cap f^{*} T) \approx f_{*} (k^{*} S' \cap f^{*} T)\) where

\[
\begin{array}{ccc}
\kappa & \Downarrow \varepsilon & \\
\alpha & \Downarrow \theta & \\
\eta & \Downarrow \delta & \\
\end{array}
\]

is a pullback. The result now follows since from 6.3 with \(g = k\) and \(\sigma = (S, f)\), \(T \cap k^{*}(f_{*} S) = f_{*} (k^{*} S \cap f^{*} T)\) and similarly for \(\sigma = (S', f)\).

7. Proof of 5.2.

The “if” part of 5.2, which follows from propositions 7.1, 7.3 and 7.4, is proved by explicitly constructing enveloping categories and functors \(E_{\Delta} \to B\) and \(E_{\Delta}^{*} \to B\). The “only if” part of 5.2 follows from proposition 7.6.

Let \(U: E \to B\) and \(\Delta \subset B\). A \(U\)-sieve \(S \in \Omega(b)\) is said to be \(\approx\)-closed if \(S \approx S'\) implies \(S' \subset S\). Denote the class of \(\approx\)-closed sieves on \(b\) that contain \(\Delta(b)\) by \(\Omega_{\Delta}(b)\).

Let \(E_{\Delta}\) be the category in which the objects are all pairs \((b, S)\) with \(S \in \Omega_{\Delta}(b)\) and the set of morphisms \((b, S) \to (b', S')\) is the set of maps \(g: b \to b'\) with \(g_{*}(S) = S'\).

Let \(U_{\Delta}: E_{\Delta} \to B\) be the functor induced by \((b, S) \mapsto b\). The notion of \(\approx\)-closed sieve and the corresponding functor \(U_{\Delta}^{*}: E_{\Delta}^{*} \to b\) are defined analogously.

7.1. Proposition. Let \(U: E \to B\) be a concrete functor and \(\Delta \subset B\). (1) If \(U\) is \(\Delta\)-small then \(U_{\Delta}\) is a topological \(\Delta\)-functor. In addition, if \(U\) is \(\Delta\)-geometric then \(U_{\Delta}\) is geometric.

Assume \(B\) has pullbacks. (2) If \(U\) is \(\Delta^{*}\)-small then \(U_{\Delta}^{*}\) is a topological \(\Delta\)-functor. In addition if \(U\) is \(\Delta^{*}\)-geometric then \(U_{\Delta}^{*}\) is geometric.
Proof. We begin with a consequence of 6.1(3). For $S \in \Omega(b)$ let $\bar{S}$ be the union of all $S'$ with $S' \approx S$.

7.2. Lemma. $\bar{S} \approx S$, $\bar{S}$ is $\approx$-closed, and the correspondence $S \mapsto \bar{S}$ defines a closure operation on $\Omega(b)$. An analogous result holds for $\approx^*$-closure.

Proof. By 6.1(3) $\bar{S} = \bigcup S' = \bigcup S = S$. If $\bar{S} \approx S'$ then $S \approx S'$ and consequently $S' \subset \bar{S}$. Thus $\bar{S}$ is $\approx$-closed by definition. Clearly $S \subset \bar{S}$ and if $S_1 = S_2$ then $\bar{S}_1 = \bar{S}_2$ and consequently $\bar{S} = \bar{S}$ follows from $\bar{S} \approx S$. It remains to show $\bar{S}_1 \subset \bar{S}_2$ if $S_1 \subset S_2$. Clearly $\bar{S}_1 \subset S_1 \subset \bar{S}_2 \subset \bar{S}_2$ and thus, by 6.1(3), $\bar{S}_1 \cup \bar{S}_2 \approx S_1 \cup \bar{S}_2 = \bar{S}_2$. Since $\bar{S}_2$ is $\approx$-closed, $\bar{S}_1 \subset \bar{S}_1 \cup \bar{S}_2 = \bar{S}_2$ and the first part of 7.2 is proved. The corresponding result for $\approx^*$ follows similarly since 6.1(3) holds for $\approx^*$.

By definition and 7.2, $\Omega_{\Delta}(b) = \{S | \bar{S} \subset S \text{ and } S \subset \Delta(b)\} \subset \Omega(b)$ and $\bigcup S_j$ is the least upper bound $\bigvee S_j$ in $\Omega_{\Delta}(b)$ of any family $\{S_j\}$ of elements of $\Omega_{\Delta}(b)$. Thus $\Omega_{\Delta}(b)$ is a complete, partially ordered class that is a set if $U$ is $\Delta$-small (see 5.1 and note that since $\bar{S}_1 \subset \bar{S}_2$ if $S_1 \approx S_2$, the collection of $\approx$-equivalence classes $[S]$ of $\Delta U$-sieves $S$ on $b$ can be identified, by $[S] \mapsto \bar{S}$, with $\Omega_{\Delta}(b)$). Further, for each map $f: b \to c$ it is not difficult to show, in view of 7.2, 6.1(3) and 6.1(5), that the correspondence $S \mapsto f \circ S \cup \Delta(c)$ defines a map $\Omega_{\Delta}(f): \Omega_{\Delta}(b) \to \Omega_{\Delta}(c)$ that respects $\subset$, preserves $\bigvee$, sends $\Delta(b)$ to $\Delta(c)$, and preserves composition in the sense that $\Omega_{\Delta}(g f) = \Omega_{\Delta}(g) \Omega_{\Delta}(f)$. Thus if $U$ is $\Delta$-small then $\Omega_{\Delta}: B \to \text{CmplP}$ is a functor where CmplP is the category with complete posets as objects and with cocontinuous maps as morphisms. Similarly, if $U$ is $\Delta^*$-small then, by replacing $\approx$ by $\approx^*$ and 6.1(5) by 6.4 in the foregoing discussion, we obtain a functor $\Omega_{\Delta}^*: B \to \text{CmplIH}$, where $\Omega_{\Delta}^*(b)$ is the set of $\approx^*$-closed sieves on $b$ that contain $\Delta(b)$. Moreover, in this case, since the greatest lower bound $\bigwedge S_j$ in $\Omega_{\Delta}^*(b)$ of a finite set $\{S_j\}$ of elements of $\Omega_{\Delta}^*(b)$ is $\bigwedge S_j$ and since $\Omega_{\Delta}(b)$ is distributive, $\Omega_{\Delta}^*(b)$ is, in view of 7.2 (for $\approx^*$-closure), 6.1(3) and 6.1(4), a complete distributive poset, i.e. a complete Heyting algebra (5.12 [12]). Further, for $f: b \to c$, the correspondence $S \mapsto f^* S$ is, by 6.1(6) and the fact that $\Delta(b) \subset f^* \Delta(c)$, a morphism $\Omega_{\Delta}^*(c) \to \Omega_{\Delta}^*(b)$ that is right adjoint to $\Omega_{\Delta}^*(f)$ and preserves $\bigvee$, but does not, in general, send $\Delta(c)$ to $\Delta(b)$. We have, therefore, a functor $\Omega_{\Delta}^*: B \to \text{CmplIH}$, where CmplIH is the category with complete Heyting algebras as objects and with cocontinuous maps with right adjoints that preserve $\bigvee$ as morphisms.

Recall that each functor $F: B \to \text{CmplP}$ determines a topological functor $\bar{U}: \bar{F} \to B$ as follows: the objects of $\bar{F}$ are the pairs $(b, x)$ with $x \in F(b)$, a morphism $(b, x) \to (b', x')$ is a map $g: b \to b'$, with $F(g)(x) \approx x'$, and $\bar{U}(b, x) = b$. Thus the $\bar{U}$-fiber of $b$ is $F(b)$, the final lift of $g: (b, x) \to b$ is given by $(b, \bigvee F(g)(x))$, and the initial lift of $g: b \to \bar{U}(b, x)$ is given by $(b, \bigwedge F(g)^*(x))$, where $F(g)^*$ is right adjoint to $F(g)$. Note that each functor $F: B \to \text{CmplP}$ can be equivalently described in an "adjoint-opposite" form by $(g: a \to b) \mapsto (F(g)^{op}: F(b)^{op} \to F(a)^{op}): B^{op} \to \text{CmplP}$. As such, it has the structure of an indexed category (1.4 [15]) of complete posets. This is essentially the point of view taken in [18].

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Obviously $\tilde{\Omega}_\Delta$ and $E_\Delta$ have the same objects and $g : b \to b_1$ is a map $(b, S) \to (b_1, S_1)$ in $E_\Delta$ if $g_*(S) \subseteq S_1$ while it is a map in $\Omega_\Delta$ if $\Omega_\Delta(g)(S) \subseteq S_1$, i.e. $g_*(S) \cup \Delta(b_1) \subseteq S_1$. Clearly these conditions on $g$ are equivalent and consequently $E_\Delta = \tilde{\Omega}_\Delta$. Thus if $U$ is $\Delta$-small then $U_\Delta$ is a topological functor. Analogously if $U$ is $\Delta^*$-small then $F_\Delta^* = \tilde{\Omega}_\Delta^*$ and $U^*_\Delta$ is topological. Further, if $d \in \Delta$ then $\Delta(d)$ is the maximum $U$-sieve on $d$. Hence $\Omega_\Delta(d)$ is a singleton and $U_\Delta$ is a $\Delta$-functor. Moreover, if $U$ is $\Delta$-geometric then the initial lift of any finite mono source $g_j : b \to d_j = U_\Delta D(d_j)$, $d_j \in \Delta$, coincides with $D(g_j) : D(b) \to D(d_j)$ since, by definition 5.1, $b$ admits only the discrete $E_\Delta$-structure. Thus $U_\Delta$ is geometric. Since a similar argument holds for $U^*_\Delta$, proposition 7.1 is proved.

**7.3. Proposition.** If $\Delta$-maps lift for a concrete functor $U : E \to B$ then $E$ can be $\Delta$-densely ($\Delta^*$-densely) embedded in $E_\Delta$ ($E^*_\Delta$, resp.) over $B$.

**Proof.** For $e \in E$ let $i(e) = (U(e), S(e)) \in E_\Delta$. If $f : e \to e_1$ then $U(f)_*(S(e)) \subseteq S(e_1)$ and consequently $U(f)$ lifts to a unique map $i(f) : i(e) \to i(e_1)$. Thus $i$ is readily seen to be a faithful functor over $B$. Further if $g : U(e) \to U(e_1)$ lifts to a map $i(e) \to i(e_1)$ then, by definition (and 6.1(1)) $g_*(S(e)) \subseteq S(e_1)$ and thus, by 6.1(2), $g_*(S(e)) \subseteq S(e_1)$. Since $g \in g_*(S(e))$, $g$ lifts to a map $e \to e_1$. Thus $i$ is full. Moreover, if $i(e) \to i(e_1)$ then the identity map $U(e) = U(e_1)$ lifts to maps $e \to e_1$, $e_1 \to e$, which, since $U$ is faithful, are inverses, and thus, since $U$ is amnestic $e = e_1$. Thus $i$ is an embedding. Finally, for $(b, S) \in E_\Delta$, the sink $\sigma = \{f | f : i(e) \to (b, S), e \in E\}$ is a final $i\Delta$-sink. To see this note that the final lift of the $U_\Delta$-sink $U_\Delta \sigma = \{(i(e), f) | f \in \sigma\}$ is given by $(b, \bigvee (\Omega_\Delta(f)(S(e)) | f \in \sigma))$ and that $\bigvee \Omega_\Delta(f)(S(e)) = S$ since, for $(e, f) \in S$, $f$ is in $\sigma$ and $\bigcup (f_*(S(e)) | (e, f) \in S) = S$. Thus $\sigma$ is final. Since $\Delta(b) \subseteq S \subseteq i^*_\Delta U_\Delta(\sigma)$, $\sigma$ is also an $i\Delta$-sink and the result follows for the $\Delta$-case. The proof of the $\Delta^*$-case is analogous.

**7.4. Proposition.** Let $U : E \to B$ be a $\Delta^*$-small concrete functor. If $B$ has pullbacks and $\Delta$-maps lift then finite families of final $i\Delta^*$-sinks in $E^*_\Delta$ are universal, where $i$ is the embedding of 7.3.

**Proof.** By 4.1, in view of 7.1, it suffices to show that the pullback of final $i\Delta^*$-sinks in $E^*_\Delta$ are final. To this end suppose $\sigma = \{f_j : (b_j, S_j) \to (b_1, S_1)\}$ is a final $i\Delta^*$-sink and $g : (b, S) \to (b_1, S_1)$ a map in $E^*_\Delta$. The pullback $\sigma'$ of $\sigma$ is given by $\sigma' = \{f'_j : (b'_j, S'_j) \to (b, S)\}$ where

$$
\begin{array}{ccc}
\begin{array}{c}
\begin{array}{c}
b'_j \xrightarrow{g_j} b_j \\
\downarrow r'_j \\
b \\
\downarrow g \\
b_1
\end{array}
\end{array}
\end{array}
$$

is a pullback and $S'_j = \Omega^*_\Delta(f'_j)^*(S) \cap \Omega^*_\Delta(g_j)^*(S_1)$. $(\Omega^*_\Delta(f)^*)$ denotes the right adjoint
to $\Omega^*_A(f)$. We must show that $\sigma'$ is final i.e. that

$$S = \bigvee_i \Omega^*_A(f_i)(S_i).$$  \hfill (1)

Since $\bigvee_i \Omega^*_A(f_i)(S_i) \subseteq \bigvee_i \Omega^*_A(f_i)(S_i) \subseteq \bigvee_i S = S$ and since $\Omega^*_A(g)(S) \subseteq S_1$, i.e. $S \subseteq \Omega^*_A(g)^*(S_1)$ it follows that $S = \Omega^*_A(g)^*(S_1) \cap S$ and (1) is equivalent to

$$\Omega^*_A(g)^*(S_1) \cap S \subseteq \bigvee_i \Omega^*_A(f_i)(S_i).$$  \hfill (2)

To show (2) it suffices to show that

$$S_1 \subseteq \bigcup_i (f_i)_*(S_i)$$  \hfill (3)

for then

$$\Omega^*_A(g)^*(S_1) \cap S \subseteq \bigvee_i \Omega^*_A(g)^*(\bigcup_i (f_i)_*(S_i)) = g^*(\bigcup_i (f_i)_*(S_i) \cap S)$$

$$= \bigvee_i (f_i)_*(g^*(S_i) \cap (f_i)^*(S)) \subseteq \bigvee_i \Omega^*_A(f_i)(g^*(S_i) \cap (f_i)^*(S))$$

$$= \bigvee_i \Omega^*_A(f_i)(\Omega^*_A(g)^*(S_i) \cap \Omega^*_A(f_i)^*(S_i)) = \bigvee_i \Omega^*_A(f_i)(S_i),$$

where the first equality is due to the definition of $\Omega^*_A(g)$ and 6.1 (6 and 4), the second equality is due to 6.3, the remaining equalities are due to the definitions of $\bigvee, \land, \Omega^*_A$ etc., and the last $\subseteq$ is due to the general fact that $h^*(S) \subseteq \Omega^*_A(h)(S)$. Let $A$ be $i_\sigma^*$ of the $U_\sigma^*$-sieve generated by $\sigma$. To show (3) it suffices to show

$$A \subseteq \bigcup_i (f_i)_*(S_i)$$  \hfill (4)

for then we have $A = \bigvee A \cup \Delta(b_1)$ ($\sigma$ is an $i\Delta^*$-sink) and thus, by 6.1 (3), $\bigcup_i (f_i)_*(S_i) = \bigcup_i (f_i)_*(S_i) \cup A \supseteq A \cup \Delta(b_1) = \bigcup_i ((f_i)_*(S_i) \cup \Delta(b_1))$ from which (3) follows since

$$\bigcup_i (f_i)_*(S_i) = \bigvee_i \Omega^*_A(f_i)(S_i) = S_1,$$

the last equality because $\sigma'$ is final. To show (4) note that $(e, h) \in A$ iff there is some $f_0: (b_0, S_0) \rightarrow (b_1, S_1)$ in $\sigma$ and some $k: U(e) \rightarrow b_0$ with $\Omega^*_A(k)(S(e)) \subseteq S_0$ and $h = f_0 k$. Clearly $(e, k) \in \Omega^*_A(k)(S(e))$ and thus $(e, h) = (f_0)_*(e, k) \in (f_0)_*(S_0) \subseteq \bigcup_i (f_i)_*(S_i)$. This shows (4) and consequently 7.4.

Clearly 7.1, 7.3, 7.4, and the fact that "$\Delta$-small" is equivalent to "$\phi$-small" if $\Delta$-maps lift (see 3.4) combine to give the "if" part of 5.2. In order to prove the "only if" part a further result about the relations $\approx$ and $\approx^*$ is needed.

**7.5. Lemma.** Let $U_i: E_i \rightarrow B$ be a topological functor and $i: (E_i, U_i) \rightarrow (E_i, U_1)$ a full functor over $B$. If, for $U$-sieves $S_i$ on $b \in B$, the final lifts $\sigma_i$ of $i_\sigma$ define, for $i = 1, 2$, the same $E_i$-structure $x$ on $b$ (i.e., both $\sigma_1$ and $\sigma_2$ are sinks to $x$), then (1) $S_1 = S_2$. If, in addition, $B$ has pullbacks, finite families of final $i\Delta^*$-sinks are universal in $E_i$, $\Delta$-maps lift, $i$ is dense and $\Delta(b) \subseteq S_1 \cap S_2$, then (2) $S_1 = \approx S_2$. 

Proof. For (1) we must show that for any $h : b = U_i \xrightarrow{e_i} U_i$ with $h_n S_n \subseteq S(e)$. If $h_n S_n \subseteq S(e)$, then, since any $g \in \sigma_1$ has the form $g = f_1 k : \gamma_i \xrightarrow{e_i} \gamma_i$ with $(\gamma_i, U_i(\gamma_i)) \in S_1$, and since $h U_i(f_i) = U_i(k)$ for some $k_i : \gamma_i \xrightarrow{e_i} \gamma_i$, $h U_i(g) = U_i((i(k_i))) k_i$ lifts for all $g \in \sigma_1$. Thus, by finality of $\sigma_1$, $h$ lifts to a map $\tilde{h} : \gamma \xrightarrow{e_i} \gamma_i$. If $(e_2, f_2) \in S_2$, then $f_2$ lifts to a map $\tilde{f}_2 : i(e_2) \xrightarrow{e_2} \gamma_i$ in $\sigma_2$ and thus since $i$ is full, $\tilde{h} \tilde{f}_2 = i(k_2)$, for some $k_2 : \gamma_2 \xrightarrow{e_2} \gamma$. Hence, $S_1 \approx S_2$ and (1) is proved.

To show (2) we must show, by definition, that $T \cap k^* S_1 \approx T \cap k^* S_2$ for any $k : a \to b$, $T \in \Omega(a)$. However, since $\Delta(a) \subseteq k^* S_n$, $i = 1, 2$ (because $\Delta(b) \subseteq S_1 \cap S_2$) and since $\Delta$-maps lift, it suffices, by Lemma 3.4, to consider $T$ with $\Delta(a) \subseteq T$. Hence, by part (1) it suffices to show that, for $k = a \to b$, $\Delta(a) \subseteq T \in \Omega(a)$, the final lifts of $i_n(T \cap k^* S_1)$, $i = 1, 2$, define the same $E_1$-structure on $a$. To this end, let $g_i : a \to b_i$ be a finite source in $B$ and let $\Delta(b_i) \subseteq S_i \subseteq \Omega(b_i)$. If $\sigma_j$ is the final lift of $i_n S_n$, then, since $i^*_n i_n b_i \cap S_j \supset \Delta(b_j)$, $\sigma_j$ is a final $i A^*$-sink, and, as such, the joint pullback $\{p(f) : L(f) \to x| f \in \square[\sigma_j]\}$ of $\{\sigma_j\}$ along $\{g_i\}$ is a final lift, where $\tilde{g}_i : x \xrightarrow{e_i} \gamma_i$ is the initial lift of $g_i : a \to b_i = U_i(\gamma_i)$ and $\gamma_i$ is the structure on $b_i$ defined by the final lift $\sigma_j$. We claim that $x$ is also the structure defined on $a$ by the final lift $\sigma_i$ of $i_n(S = \bigcap g_i^* S_i)$. To see this, note that if $(h : e_i \to y) \in \sigma$ then $(e_i, U_i(h)) \in i_n(S)$ and $(e_i, g_i U_i(h)) \in i_n S_i$, i.e. $g_i U_i(h)$ lifts to a map $e_i \xrightarrow{e_i} \gamma_i$ in $\sigma_i$, all $i$. Thus, since $\{g_i\}$ is initial, $U_i(h)$ lifts to a map $\tilde{h} : e_i \xrightarrow{e_i}$ for all $h \in \sigma$ and consequently $y \subseteq x$. To show $x \subseteq y$ it clearly suffices to show that $U_i$ of each map in the final sink $\{p(f) k : i_k \to x| f \in \square[\sigma_i]\}$ lifts to a map in $\sigma$. (Note that both $\{p(f) f \in \square[\sigma_i]\}$ and $\{k : i_k \to L(f)\}$ are final sinks.) However, since each $f = \{f_i\} \in \square[\sigma_i]$ has the form $f_i = f'_j k_j : y_i \xrightarrow{e_i} \gamma_i$, with $(e_i, U_i(f'_j)) \in S_j$ given any $k : i_k \to L(f)$, the map $k p(f) k_i : i(e_i) \xrightarrow{e_i}$ lifts, since $i$ is full, to a map $\tilde{k}_i : e_i \to e_j$. Thus, $(e_i, g_i U_i(p(f) k))$ factors, via $\tilde{k}_i$, through $(e_i, U_i(f'_j)) \in S_j$, all $i$, and consequently $(i(e), U_i(p(f) k)) \in i_n S_i$, i.e. $U_i(p(f) k)$ lifts to a map in $\sigma$. Hence $x = y$. In particular, if $i_n S_i$ define the same structure on $b$, $i = 1, 2$, for $\Delta(b) \subseteq S_1 \cap S_2$ and $\Delta(a) \subseteq T \in \Omega(a)$ then, by taking $g_1 = \text{identity on } a$, $g_2 = k : a \xrightarrow{e_i} b_i$, $i_n(T \cap k^* S_1)$ and $i_n(T \cap k^* S_2)$ are seen to define the same structure on $a$ and (2) follows from (1).

7.6. Proposition. Let $U$ be a concrete functor to $B \supseteq \Delta$. (1) If $U$ can be embedded, over $B$, in a topological $\Delta$-functor $U_1$ then $U$ is $\delta$-small and $\Delta$-maps lift. If $U_1$ is geometric then $U$ is $\Delta$-geometric. (2) If, over $B$, there is a dense embedding $i$ of $U$ into a topological $\Delta$-functor $U_i : E_i \to B$ where $B$ has pullbacks and finite families of final $i \Delta^*$-sinks in $E_i$ are universal then $U$ is $\Delta^*$ small and $\Delta$-maps lift. Moreover, if $U_1$ is geometric then $U$ is $\Delta^*$-geometric.

Proof. In view of 2.1(2) and the requirement that embeddings be full, $\Delta$-maps lift. Further, the smallness conditions follow from 7.5 since the correspondence $\Omega(b) \to U_1^{-1}(b)$ that sends $S$ to the structure on $b$ defined by the final lift of $i_n S$ induces an injection from the appropriate collection of $\equiv$ or $\equiv^*$-equivalence classes of $U$-sieves into the small set $U_1^{-1}(b)$. Finally, if $U_1$ is a geometric $\Delta$-functor and $g_j : b \to d_j$,
$d_j \in \Delta$, is a finite mono source then, by 2.1(3), $b \in \Delta(U_1)$, and consequently the final lifts of $i_b(M(b))$ and $i_b(\Delta(b))$ must define the same structure (the discrete) on $b$. Hence, by 7.5(1) or (2) $U$ is $\Delta$-geometric or $\Delta^\omega$-geometric respectively.

We conclude with the observation that the embeddings $i$ of 7.3 are maximal in the sense that if $j: (E, U) \to (E_1, U_1)$ is a full functor over $B$ and $\Delta$-maps lift through $U_1$ then the correspondence

$$e_1 \mapsto j_\Delta^* S(e_1)$$

defines functors $j_\Delta: E_1 \to E_\Delta$, $j_\Delta^*: E_1 \to E_\Delta^*$ over $B$ such that $i = j_\Delta j$ and $i = j_\Delta^* j$.

References