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Existence of solutions for $p(x)$-Laplacian problems on a bounded domain

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Abstract

In this paper we study the following $p(x)$-Laplacian problem:

$$-\operatorname{div}\big(a(x)|\nabla u|^{p(x)-2}\nabla u\big)+b(x)|u|^{p(x)-2}u=f(x,u), \quad x\in\Omega,$$
$$u=0, \quad \text{on } \partial\Omega,$$

where $1<p_1\leqslant p(x)\leqslant p_2<n$, $\Omega\subset\mathbb{R}^n$ is a bounded domain and applying the mountain pass theorem we obtain the existence of solutions in $W_0^{1,p(x)}(\Omega)$ for the $p(x)$-Laplacian problems in the superlinear and sublinear cases.

Keywords: Existence; $p(x)$-Laplacian problem; Bounded domain

1. Introduction

After Kovacik and Rakosnik first discussed the $L^{p(x)}$ spaces and $W^{k,p(x)}$ spaces in [17], a lot of research has been done concerning these kinds of variable exponent spaces; see,

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for example, [9–11,13,20] for the properties of such spaces and [1–3,5,6,14,15] for the applications of variable exponent spaces on partial differential equations. In [19] Ruzicka presented the mathematical theory for the application of variable exponent spaces in electro-rheological fluids.

Inspired by their works, we want to study the $p(x)$-Laplacian problem:

$$\begin{aligned} &-\operatorname{div}\big(a(x)|\nabla u|^{p(x)-2}\nabla u\big)+b(x)|u|^{p(x)-2}u=f(x,u), \quad x\in\Omega,\\ &u=0, \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $0<a_0\leqslant a(x)\in L^\infty(\Omega)$, $0\leqslant b_0\leqslant b(x)\in L^\infty(\Omega)$, p is Lipschitz continuous on $\bar\Omega$ and satisfies

$$1<p_1\leqslant p(x)\leqslant p_2<n. \tag{1.2}$$

Our object is to obtain sufficient conditions on f for (1.1) to admit nontrivial and nonnegative solutions in the following prototype cases:

$$f(x,u)=\begin{cases} g(x)u^{\alpha(x)}, & p(x)-1<\alpha(x)<p^*(x)-1, & (1.3)\\ h(x)u^{\beta(x)}, & 0\leqslant\beta(x)<p(x)-1, & (1.4)\end{cases}$$

where $p^*(x)=\frac{np(x)}{n-p(x)}$.

When $p(x)$ is a constant function, there are a lot of studies for the case of bounded domains; see, for example, [4,7,8,12,16] and references therein. It is beyond our ability to write out all the works in this direction here. When $p(x)$ is a variable function, Fan and Zhang [14] studied the $p(x)$-Laplacian problems on bounded domains. Under some conditions, they established some results on the existence of solutions. Although we study the $p(x)$-Laplace problems on bounded domains and we apply mountain pass theorem as well, our method is a bit different from that in [14] and in some sense we discuss the $p(x)$-Laplacian problem in a more general setting than that in [14].

2. Preliminaries

In this section we first recall some facts on variable exponent spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$. For the details see [13,15,17].

Let $\mathbf{P}(\Omega)$ be the set of all Lebesgue measurable functions $p:\Omega\to[1,+\infty]$,

$$\rho_p(f)=\int\limits_{\Omega\setminus\Omega_\infty}\big|f(x)\big|^{p(x)}\,dx+\inf_{\Omega_\infty}\big|f(x)\big|, \tag{2.1}$$

$$\|f\|_p=\inf\left\{\lambda>0\colon\ \rho_p\left(\frac{f}{\lambda}\right)\leqslant 1\right\}, \tag{2.2}$$

where $\Omega_\infty=\{x\in\Omega\colon\ p(x)=\infty\}$. The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is the class of all functions f such that $\rho_p(\lambda f)<\infty$ for some $\lambda=\lambda(f)>0$. $L^{p(x)}(\Omega)$ is a Banach space endowed with the norm (2.2). $\rho_p(f)$ is called the modular of f in $L^{p(x)}(\Omega)$.

For a given $p(x) \in \mathbf{P}(\Omega)$ we define the conjugate function $p'(x)$ as

$$p'(x) = \begin{cases} \infty, & \text{if } x \in \Omega_1 = \{x \in \Omega\colon\ p(x) = 1\}, \\ 1, & \text{if } x \in \Omega_\infty, \\ \frac{p(x)}{p(x)-1}, & \text{for other } x \in \Omega. \end{cases}$$

Theorem 2.1. *Let $p \in \mathbf{P}(\Omega)$. Then the inequality*

$$\int_\Omega \left| f(x)g(x) \right| dx \leqslant r_p \|f\|_p \|g\|_{p'}$$

holds for every $f \in L^{p(x)}(\Omega)$ and $g \in L^{p'(x)}(\Omega)$ with the constant r_p depending on $p(x)$ and Ω only.

Theorem 2.2. *The topology of the Banach space $L^{p(x)}(\Omega)$ endowed by the norm* (2.2) *coincides with the topology of modular convergence if and only if $p \in L^\infty(\Omega)$.*

Theorem 2.3. *The dual space to $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$ if and only if $p \in L^\infty(\Omega)$. The space $L^{p(x)}(\Omega)$ is reflexive if and only if*

$$1 < \inf_\Omega p(x) \leqslant \sup_\Omega p(x) < \infty. \tag{2.3}$$

Next we assume that $\Omega \subset \mathbb{R}^n$ is a nonempty open set, $p \in \mathbf{P}(\Omega)$ and k is a given natural number.

Given a multiindex $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, we set $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$, where $D_i = \frac{\partial}{\partial x_i}$ is the generalized derivative operator.

The generalized Sobolev space $W^{k,p(x)}(\Omega)$ is the class of all functions f on Ω such that $D^\alpha f \in L^{p(x)}(\Omega)$ for every multiindex α with $|\alpha| \leqslant k$, endowed with the norm

$$\|f\|_{k,p} = \sum_{|\alpha| \leqslant k} \|D^\alpha f\|_p. \tag{2.4}$$

By $W_0^{k,p(x)}(\Omega)$ we denote the subspace of $W^{k,p(x)}(\Omega)$ which is the closure of $C_0^\infty(\Omega)$ with respect to the norm (2.4).

Theorem 2.4. *The space $W^{k,p(x)}(\Omega)$ and $W_0^{k,p(x)}(\Omega)$ are Banach spaces, which are reflexive if p satisfies* (2.3).

We denote the dual space of $W_0^{k,p(x)}(\Omega)$ by $W^{-k,p'(x)}(\Omega)$, then we have

Theorem 2.5. *Let $p \in \mathbf{P}(\Omega) \cap L^\infty(\Omega)$. Then for every $G \in W^{-k,p'(x)}(\Omega)$ there exists a unique system of functions $\{g_\alpha \in L^{p'(x)}(\Omega)\colon\ |\alpha| \leqslant k\}$ such that*

$$G(f) = \sum_{|\alpha| \leqslant k} \int_\Omega D^\alpha f(x) g_\alpha(x)\, dx, \quad f \in W_0^{k,p(x)}(\Omega).$$

The norm of $W_0^{-k,p'(x)}(\Omega)$ is defined as

$$\|G\|_{-k,p'} = \sup\left\{\frac{|G(f)|}{\|f\|_{k,p}}\colon\ f \in W_0^{k,p(x)}(\Omega)\right\}.$$

Theorem 2.6. *If Ω is a bounded domain with the cone property, $p(x) \in C(\bar{\Omega})$ satisfies* (1.2) *and $q(x)$ is any Lebesgue measurable function defined on Ω with $p(x) \leqslant q(x)$ a.e. on $\bar{\Omega}$ and $\inf_{x\in\Omega}\{p^*(x) - q(x)\} > 0$, then there is a compact embedding $W^{1,p(x)}(\Omega) \to L^{q(x)}(\Omega)$.*

Theorem 2.7. *Let Ω be a domain with the cone property. If $p : \bar{\Omega} \to \mathbb{R}$ is Lipschitz continuous and satisfies* (1.2), *and $q(x) \in \mathbf{P}(\Omega)$ satisfies $p(x) \leqslant q(x) \leqslant p^*(x)$ a.e. on $\bar{\Omega}$, then there is a continuous embedding $W^{1,p(x)}(\Omega) \to L^{q(x)}(\Omega)$.*

Theorem 2.8. *Let Ω be a bounded domain. If $p \in L^\infty(\Omega)$ and $u \in W_0^{1,p(x)}(\Omega)$, then*

$$\int_\Omega |u|^{p(x)}\,dx \leqslant C \int_\Omega |\nabla u|^{p(x)}\,dx,$$

where C is a constant depending on Ω.

For the $p(x)$-Laplacian problems (1.1) we define two functionals $K(u)$ and $J(u)$ on Ω:

$$K(u) = \int_\Omega F(x,u)\,dx,$$

$$J(u) = \int_\Omega \frac{1}{p(x)}\big(a(x)|\nabla u|^{p(x)} + b(x)|u|^{p(x)}\big)\,dx - K(u),$$

where $F(x,t) = \int_0^t f(x,s)\,ds$.

Next we discuss the properties of $K(u)$ while f satisfies the following conditions:

(H1) $f \in C(\bar{\Omega} \times \mathbb{R})$, $f(x,t) > 0$ in $\Omega_0 \times (0,+\infty)$ for some nonempty open set $\Omega_0 \subseteq \Omega$ and $f(x,t) = 0$ for all $x \in \Omega$ and $t \leqslant 0$.

(H2) $|f(x,t)| \leqslant c_1 + c_2|t|^{\alpha(x)}$, $\alpha + 1 \in C(\bar{\Omega})$ with $\hat{a} = \inf_{x\in\Omega}\{\alpha(x) - p(x) + 1\} > 0$ and $a = \sup_{x\in\Omega}\{p^*(x) - \alpha(x) - 1\} > 0$. Here c_1, c_2 are positive constants.

(H3) $|f(x,t)| \leqslant \tilde{c}_1 + \tilde{c}_2|t|^{\beta(x)}$, $\beta + 1 \in \mathbf{P}(\Omega)$ with $0 \leqslant \beta(x)$ and $b = \sup_{x\in\Omega}\{p(x) - \beta(x) - 1\} > 0$. Here $\tilde{c}_1, \tilde{c}_2$ are positive constants.

Lemma 2.9. *Suppose that f satisfies* (H1) *and* (H2) *or* (H3). *Then $K(u)$ is weakly continuous on $W_0^{1,p(x)}(\Omega)$.*

Proof. Suppose that f satisfies (H1) and (H3). Let $u_j \to u$ weakly in $W_0^{1,p(x)}(\Omega)$. Then $\{u_j\}$ is bounded in $W_0^{1,p(x)}(\Omega)$. By Theorem 2.6 there is a compact embedding $W^{1,p(x)}(\Omega) \to L^{p(x)}(\Omega)$ while there is a continuous embedding $L^{p(x)}(\Omega) \to$

$L^{\beta(x)+1}(\Omega)$, so the embedding $W^{1,p(x)}(\Omega) \to L^{\beta(x)+1}(\Omega)$ is compact and $u_j \to u$ in $L^{\beta(x)+1}(\Omega)$. Then by Theorem 2.2 $u_j \to u$ in modular as well. From (H3) we get

$$\left|F(x,t)\right| \leqslant \tilde{c}_1|t| + \frac{\tilde{c}_2}{\beta(x)+1}|t|^{\beta(x)+1}.$$

Then by Vitali theorem (see [18]), we have

$$\int_\Omega F(x,u_j)\,dx \to \int_\Omega F(x,u)\,dx \quad \text{as } j \to \infty.$$

Similarly if f satisfies (H1) and (H2) the theorem is valid as well. □

Theorem 2.10. *Suppose that f satisfies* (H1) *and* (H2) *or* (H3). *Then $K(u)$ is differentiable on $W_0^{1,p(x)}(\Omega)$ with*

$$K'(u)\phi = \int_\Omega f(x,u)\phi\,dx \quad \forall \phi \in W_0^{1,p(x)}(\Omega).$$

Proof. For differentiability of K, we will show that for any $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon, u) > 0$ such that

$$\left|K(u+\phi) - K(u) - \int_\Omega f(x,u)\phi\,dx\right| = \left|\int_\Omega F(x,u+\phi) - F(x,u) - f(x,u)\phi\,dx\right| < \varepsilon\|\phi\|_{1,p}$$

for all $\phi \in W_0^{1,p(x)}(\Omega)$ with $\|\phi\|_{1,p} < \delta$.

Let $\Omega_1 = \{x \in \Omega_k\colon |u(x)| \geqslant h\}$, $\Omega_2 = \{x \in \Omega_k\colon |\phi(x)| \geqslant r\}$, $\Omega_3 = \{x \in \Omega_k\colon |u(x)| < h$ and $|\phi(x)| < r\}$, where h, r are constant which will be determined later.

Next we consider the case that f satisfies (H1) and (H3) only, the other case that f satisfies (H1) and (H2) can be treated similarly. First on Ω_1 we have

$$\begin{aligned}
&\left|\int_{\Omega_1} F(x,u+\phi) - F(x,u) - f(x,u)\phi\,dx\right| \\
&\quad \leqslant C\int_{\Omega_1} \left(\tilde{c}_1 + \tilde{c}_2\left(|u|+|\phi|\right)^{\beta(x)}\right)|\phi| + \left(\tilde{c}_1 + \tilde{c}_2|u|^{\beta(x)}\right)|\phi|\,dx \\
&\quad \leqslant C\int_{\Omega_1} \left(|\phi| + |u|^{\beta(x)}|\phi| + |\phi|^{\beta(x)+1}\right)dx \\
&\quad \leqslant C\|\chi_{\Omega_1}\|_{(p^*)'}\|\phi\|_{p^*} + I_1 + I_2,
\end{aligned}$$

since

$$\left(|u|+|\phi|\right)^{\beta(x)} \leqslant 2^{\beta(x)}\left(|u|^{\beta(x)} + |\phi|^{\beta(x)}\right) \leqslant 2^{\bar{\beta}}\left(|u|^{\beta(x)} + |\phi|^{\beta(x)}\right),$$

where $\bar{\beta} = \sup_\Omega \beta(x)$.

Because $u \in W^{1,p(x)}(\Omega)$, we can get

$$\infty > \int_{\Omega_1} |u|^{p(x)}\,dx \geqslant \int_{\Omega_1} h^{p(x)}\,dx \geqslant \min\{h^{p_1}, h^{p_2}\} \operatorname{meas} \Omega_1. \tag{2.5}$$

From (2.5), $\operatorname{meas} \Omega_1 \to 0$ as $h \to \infty$. Then we can get for sufficiently large h,

$$C\|\chi_{\Omega_1}\|_{(p^*)'}\|\phi\|_{p^*} \leqslant C(\operatorname{meas} \Omega_1)^{\frac{np_1}{np_1-n+p_1}}\|\phi\|_{1,p} < \frac{\varepsilon}{9}\|\phi\|_{1,p}.$$

For I_1 we have

$$I_1 \leqslant C\big\||u|^\beta\big\|_{(p^*)',\Omega_1}\|\phi\|_{p^*} \leqslant C\big\||u|^\beta\big\|_{(p^*)',\Omega_1}\|\phi\|_{1,p}.$$

As $\beta(x)(p^*(x))' = \frac{\beta(x)p^*(x)}{p^*(x)-1} < \beta(x)+1$, we have

$$\int_{\Omega_1} |u|^{\beta(x)(p^*(x))'}\,dx \leqslant \big\||u|^{\beta(p^*)'}\big\|_{\frac{\beta+1}{\beta(p^*)'},\Omega_1}\|\chi_{\Omega_1}\|_{(\frac{\beta+1}{\beta(p^*)'})'}.$$

In view of

$$\int_\Omega (\chi_{\Omega_1})^{(\frac{\beta(x)+1}{\beta(x)(p^*(x))'})'}\,dx = \operatorname{meas} \Omega_1 \leqslant \operatorname{meas} \Omega < \infty,$$

by Theorem 2.2,

$$\|\chi_{\Omega_1}\|_{(\frac{\beta+1}{\beta(p^*)'})'} < \infty.$$

In view of $\int_{\Omega_1} |u|^{\beta(x)(p^*(x))'\frac{\beta(x)+1}{\beta(x)(p^*(x))'}}\,dx = \int_{\Omega_1} |u|^{\beta(x)+1}\,dx$ and by Theorem 2.6 we conclude $\int_{\Omega_1} |u|^{\beta(x)+1}\,dx = \int_\Omega |u\chi_{\Omega_1}|^{\beta(x)+1}\,dx \to 0$ as $h \to \infty$ and $\|u^{\beta(p^*)'}\|_{\frac{\beta+1}{\beta(p^*)'},\Omega_1} \to 0$ as $h \to \infty$. Therefore $\int_{\Omega_1} |u|^{\beta(x)(p^*(x))'}\,dx \to 0$ as $h \to \infty$ and by Theorem 2.2 we can choose h so large that

$$I_1 < \frac{\varepsilon}{9}\|\phi\|_{1,p}.$$

Similarly for I_2 we can also show for sufficiently large h,

$$I_2 < \frac{\varepsilon}{9}\|\phi\|_{1,p}$$

and therefore

$$\left|\int_{\Omega_1} F(x, u+\phi) - F(x,u) - f(x,u)\phi\,dx\right| < \frac{\varepsilon}{3}\|\phi\|_{1,p}. \tag{2.6}$$

Second from $f \in C(\bar{\Omega} \times \mathbb{R})$ we have $F \in C^1(\bar{\Omega} \times \mathbb{R})$. For any $\varepsilon_1, h > 0$, there exists $r > 0$ such that

$$\big|F(x, \xi+\eta) - F(x,\xi) - f(x,\xi)\eta\big| < \varepsilon_1|\eta| \tag{2.7}$$

whenever $x \in \bar{\Omega}$, $|\xi| \leqslant h$ and $|\eta| < r$. From (2.7) we have

$$\int_{\Omega_3} \left|F(x, u+\phi) - F(x, u) - f(x, u)\phi\right| dx \leqslant \varepsilon_1 \|\phi\|_p \|\chi_\Omega\|_{p'}.$$

Choose ε_1 such that $\varepsilon_1 \|\chi_\Omega\|_{p'} < \frac{\varepsilon}{3}$, then

$$\int_{\Omega_3} \left|F(x, u+\phi) - F(x, u) - f(x, u)\phi\right| dx < \frac{\varepsilon}{3} \|\phi\|_{1,p}. \tag{2.8}$$

Here $\|\chi_\Omega\|_{p'} < \infty$ because $\int_\Omega (\chi_\Omega)^{p'(x)}\,dx = \operatorname{meas}\Omega < \infty$.

Third similar to the above we have

$$\begin{aligned}
&\left|\int_{\Omega_2} F(x, u+\phi) - F(x, u) - f(x, u)\phi\,dx\right| \\
&\quad \leqslant C \int_{\Omega_2} |u|^{\beta(x)}|\phi| + |\phi|^{\beta(x)+1}\,dx \\
&\quad \leqslant C\left(\left\||u|^\beta\right\|_{\frac{\beta+1}{\beta},\Omega_2} + \left\||\phi|^\beta\right\|_{\frac{\beta+1}{\beta},\Omega_2}\right)\|\phi\|_{\beta+1,\Omega_2}.
\end{aligned}$$

For any $0 < \varepsilon_2 < 1$, we have

$$\int_{\Omega_2} \left(\frac{|\phi|}{\varepsilon_2 \|\phi\|_{p^*}}\right)^{\beta(x)+1} dx \leqslant \int_{\Omega_2} \left(\frac{|\phi|}{\|\phi\|_{p^*}}\right)^{p^*(x)} \left(\frac{\|\phi\|_{p^*}}{r}\right)^{p^*(x)-\beta(x)-1} \left(\frac{1}{\varepsilon_2}\right)^{\beta(x)+1} dx.$$

As $\bar{b} = \sup_{x\in\Omega}\{p^*(x) - \beta(x) - 1\} > 0$, we can choose $\|\phi\|_{1,p}$ sufficiently small such that

$$\int_{\Omega_2} \left(\frac{|\phi|}{\varepsilon_2 \|\phi\|_{p^*}}\right)^{\beta(x)+1} dx \leqslant \int_{\Omega_2} \left(\frac{|\phi|}{\|\phi\|_{p^*}}\right)^{p^*(x)} dx \leqslant 1$$

and further

$$\|\phi\|_{\beta+1,\Omega_2} \leqslant \varepsilon_2 \|\phi\|_{p^*} \leqslant C\varepsilon_2 \|\phi\|_{1,p}.$$

From $u \in W^{1,p(x)}(\Omega)$ and Theorem 2.6, $u \in L^{\beta(x)+1}(\Omega)$, so $\int_{\Omega_2} (|u|^{\beta(x)})^{\frac{\beta(x)+1}{\beta(x)}}\,dx < \infty$ and further $\||u|^\beta\|_{\frac{\beta+1}{\beta},\Omega_2} < \infty$. Similarly $\||\phi|^\beta\|_{\frac{\beta+1}{\beta},\Omega_2} < \infty$ if $\|\phi\|_{1,p} \leqslant 1$. Choose ε_2 such that

$$\int_{\Omega_2} \left|F(x, u+\phi) - F(x, u) - f(x, u)\phi\right| dx < \frac{\varepsilon}{3} \|\phi\|_{1,p}. \tag{2.9}$$

From (2.6), (2.8) and (2.9) we conclude that $K(u)$ is differentiable on $W_0^{1,p(x)}(\Omega)$ with

$$K'(u)\phi = \int_\Omega f(x, u)\phi\,dx \quad \forall \phi \in W_0^{1,p(x)}(\Omega). \qquad \square$$

Theorem 2.11. *Suppose that f satisfies* (H1) *and* (H2) *or* (H3)*. Then $K'(u)$ is a continuous and compact mapping from $W_0^{1,p(x)}(\Omega)$ to $W^{-1,p'(x)}(\Omega)$.*

Proof. From

$$\begin{aligned}\left|K'(u_j)\phi - K'(u)\phi\right| &\leqslant \int\limits_{\Omega_k} \left|f(x,u_j) - f(x,u)\right||\phi|\,dx \\ &\leqslant C\left\|f(x,u_j) - f(x,u)\right\|_{(p^*)'}\|\phi\|_{p^*} \\ &\leqslant C\left\|f(x,u_j) - f(x,u)\right\|_{(p^*)'}\|\phi\|_{1,p},\end{aligned}$$

we have

$$\left\|K'(u_j) - K'(u)\right\|_{-1,p'} \leqslant C\left\|f(x,u_j) - f(x,u)\right\|_{(p^*)'}.$$

Similarly to the differentiability of $K(u)$ we can get the result.

At last we show the compactness of $K'(u)$. We consider the case in which f satisfies (H1) and (H3) only and the other case can be treated similarly as well. Let $\{u_j\}$ be a bounded sequence in $W_0^{1,p(x)}(\Omega)$. As the embedding $W^{1,p(x)}(\Omega) \to L^{\beta(x)+1}(\Omega)$ is compact, then the boundedness of $\{u_j\}$ in $W^{1,p(x)}(\Omega)$ implies that $\{u_j\}$ has a Cauchy subsequence which is still denoted by $\{u_j\}$. Similar to the above we can choose j sufficiently large such that

$$\left\|f(x,u_j) - f(x,u_i)\right\|_{(p^*)'} < \varepsilon.$$

Then $\{K'(u_j)\}$ is a Cauchy sequence in $W^{-1,p'(x)}(\Omega)$ and the compactness of K' follows immediately. □

3. Existence of solutions

The critical points u of $J(u)$, i.e.,

$$J'(u)(\phi) = \int\limits_{\Omega} a(x)|\nabla u|^{p(x)-2}\nabla u\nabla\phi + b(x)|u|^{p(x)-2}u\phi - f(x,u)\phi\,dx = 0 \qquad (3.1)$$

for all $\phi \in W_0^{1,p(x)}(\Omega)$ are weak solutions of

$$-\operatorname{div}\left(a(x)|\nabla u|^{p(x)-2}\nabla u\right) + b(x)|u|^{p(x)-2}u = f(x,u).$$

So next we need only to consider the existence of nontrivial critical points of $J(u)$.

First we study the prototype case (1.3). To establish the existence of solutions, we suppose that f satisfies the following additional condition:

(H4) there exist constants $M > 0$ and $\mu > p(x)$ with $\sup_{x\in\Omega}\{\mu - p(x)\} > 0$ such that $\mu F(x,t) \leqslant tf(x,t)$ for $x \in \Omega$ and $|t| \geqslant M$, and $f(x,t) = o(|t|^{p(x)-1})$ as $t \to 0$.

Theorem 3.1. *Under conditions* (H1), (H2) *and* (H4) *the* $p(x)$*-Laplacian problem* (1.1) *has a nontrivial and nonnegative solution* $u \in W_0^{1,p(x)}(\Omega)$.

Proof. By (H4), for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $F(x,t) \leqslant \frac{\varepsilon}{2}|t|^{p(x)}$ whenever $x \in \Omega$ and $|t| < \delta$. By (H2) there exists a constant $A = A(\delta) > 0$ such that $F(x,t) \leqslant A|t|^{\alpha(x)+1}$ whenever $x \in \Omega$ and $|t| \geqslant \delta$. Combining the two inequalities, we get

$$F(x,t) \leqslant \frac{\varepsilon}{2}|t|^{p(x)} + A|t|^{\alpha(x)+1}.$$

Then

$$J(u) \geqslant \int_\Omega \frac{a_0}{p_2}|\nabla u|^{p(x)}\,dx - \frac{\varepsilon}{2}\int_\Omega |u|^{p(x)}\,dx - \int_\Omega \left(\frac{c_2}{p_1} + A\right)|u|^{\alpha(x)+1}\,dx.$$

By Theorem 2.8 we can take ε sufficiently small such that

$$\frac{\varepsilon}{2}\int_\Omega |u|^{p(x)}\,dx \leqslant \frac{a_0}{2p_2}\int_\Omega |\nabla u|^{p(x)}\,dx.$$

Then

$$J(u) \geqslant \frac{a_0}{2p_2}\int_\Omega |\nabla u|^{p(x)}\,dx - C\int_\Omega |u|^{\alpha(x)+1}\,dx. \tag{3.2}$$

By Theorems 2.6 and 2.8 we have $\|u\|_{\alpha+1} \leqslant C\|\nabla u\|_p$. If $\|\nabla u\|_p < 1$ is sufficiently small such that $C\|\nabla u\|_p < 1$, then $\|u\|_{\alpha+1} < 1$.

For any $x \in \bar{\Omega}$, as $p, \alpha \in C(\bar{\Omega})$ we can get $Q_R(x) = \{y = (y^1, \ldots, y^n)\colon |y^i - x^i| < R, i = 1, \ldots, n\}$ such that $|p(y) - p(x)| < \varepsilon$ and $|\alpha(y) - \alpha(x)| < \varepsilon$ whenever $y \in Q_R(x) \cap \bar{\Omega}$. Take $\varepsilon = \frac{1}{4}(\alpha(x) - p(x) + 1)$, then

$$p_{x2} = \sup_{x \in Q_R(x)\cap\bar{\Omega}} p(x) \leqslant \alpha_{x1} + 1 = \inf_{x \in Q_R(x)\cap\bar{\Omega}} \alpha(x) + 1.$$

$\{Q_R(x)\}_{x\in\bar{\Omega}}$ is an open covering of $\bar{\Omega}$. As $\bar{\Omega}$ is compact, we can pick a finite subcovering $\{Q_{R_i}(x_i)\}_{i=1}^m$ for $\bar{\Omega}$ from the covering $\{Q_R(x)\}_{x\in\bar{\Omega}}$. If $Q_{R_i}(x_i) \not\subset \Omega$ we define $u = 0$ on $Q_{R_i}(x_i) \setminus \Omega$ and then Theorem 2.6 is still valid for $u \in W_0^{1,p(x)}(\Omega)$ on $Q_{R_i}(x_i)$. We can use all the hyperplanes, for each of which there exists at least one hypersurface of some $Q_{R_i}(x_i)$ lying on it, to divide $\bigcup_{i=1}^m Q_{R_i}(x_i)$ into finite open hypercubes $\{Q_j\}_{j=1}^J$ which mutually have no common points. It is obvious that $\bar{\Omega} \subseteq \bigcup_{j=1}^J \bar{Q}_j$ and for each Q_j there exists at least one $Q_{R_i}(x_i)$ such that $Q_j \subseteq Q_{R_i}(x_i)$. By Theorems 2.6, 2.8 and [11],

$$\int_{Q_j\cap\Omega} |u|^{\alpha(x)+1}\,dx \leqslant \left(C\|u\|_{1,p,Q_j\cap\Omega}\right)^{\alpha_{j1}+1}, \tag{3.3}$$

where $\alpha_{j1} = \inf_{x\in Q_j\cap\Omega}\alpha(x)$. As $\|\nabla u\|_{p,Q_j\cap\Omega} < 1$, we have

$$\int_{Q_j\cap\Omega} |\nabla u|^{p(x)}\,dx \geqslant \frac{1}{1+C}\|u\|_{1,p,Q_j\cap\Omega}^{p_{j2}}, \tag{3.4}$$

where $p_{j2} = \sup_{x \in Q_j \cap \Omega} p(x)$. From (3.2)–(3.4) we have

$$\frac{a_0}{2p_2} \int\limits_{Q_j \cap \Omega} |\nabla u|^{p(x)}\,dx - C \int\limits_{Q_j \cap \Omega} |u|^{\alpha(x)+1}\,dx$$
$$\geqslant C_1\big(\|u\|_{1,p,Q_j\cap\Omega}\big)^{p_{j2}} - C_2\big(\|u\|_{1,p,Q_j\cap\Omega}\big)^{\alpha_{j1}+1}.$$

As $\alpha_{j1} + 1 > p_{j2}$, there exists r_j such that

$$C_1\big(\|u\|_{1,p,Q_j\cap\Omega}\big)^{p_{j2}} - C_2\big(\|u\|_{1,p,Q_j\cap\Omega}\big)^{\alpha_{j1}+1} \geqslant C_r > 0$$

if $\|u\|_{1,p,Q_j\cap\Omega} = r$ and $0 < r \leqslant r_j < \frac{1}{1+C}$, where C is the embedding constant in $\|u\|_{\alpha+1} \leqslant C\|\nabla u\|_p$. Take $r_0 = \min_{1\leqslant j\leqslant J}\{r_j\}$. As

$$\|u\|_{1,p} = \left\| u \sum_{j=1}^{J} \chi_{Q_j\cap\Omega} \right\|_{1,p} \leqslant \sum_{j=1}^{J} \|u\chi_{Q_j\cap\Omega}\|_{1,p} = \sum_{j=1}^{J} \|u\|_{1,p,Q_j\cap\Omega}.$$

So there exists at least one $\|u\|_{1,p,Q_j\cap\Omega}$ satisfying

$$\frac{r_0}{J} \leqslant \|u\|_{1,p,Q_j\cap\Omega} \leqslant r_0.$$

Then we have

$$J(u) \geqslant \sum_{j=1}^{J}\left(\frac{a_0}{2p_2} \int\limits_{Q_j\cap\Omega} |\nabla u|^{p(x)}\,dx - C \int\limits_{Q_j\cap\Omega} |u|^{\alpha(x)+1}\,dx \right)$$
$$\geqslant C_1\left(\frac{r_0}{J}\right)^{p_2}\left(1 - \frac{C_2}{C_1} r_0^{\frac{\hat{a}}{2}}\right) > 0$$

if $\|u\|_{1,p} = r_0 > 0$.

By (H1) and (H4) we have

$$F(x,t) \geqslant a_1 t^{\mu} - a_2, \quad |t| \geqslant M,$$

where $(x,t) \in \Omega_0 \times \mathbb{R}$ and $a_1, a_2 > 0$ are constant. Pick $x_0 \in \Omega_0$ and $B_{2R}(x_0) = \{x\colon |x - x_0| < 2R\} \subset \Omega_0$ with $2R < 1$. Let $\phi \in C_0^{\infty}(B_{2R}(x_0))$ such that $\phi \equiv 1$, $x \in B_R(x_0)$; $0 \leqslant \phi(x) \leqslant 1$ and $|\nabla\phi| \leqslant \frac{1}{R}$. Denote $l = \inf_{x\in\Omega}\{\mu - p(x)\}$. Then for $s > 1$,

$$J(s\phi) \leqslant \int\limits_{B_{2R}(x_0)} \frac{s^{p(x)}}{p(x)}\big(a(x)|\nabla\phi|^{p(x)} + b(x)|\phi|^{p(x)}\big)\,dx - \int\limits_{B_{2R}(x_0)} s^{\mu} a_1 |\phi|^{\mu}\,dx$$
$$+ \int\limits_{B_{2R}(x_0)\cap\{x\in\Omega\colon s|\phi|\leqslant M\}} s^{\mu} a_1 |\phi|^{\mu}\,dx - \int\limits_{B_{2R}(x_0)\cap\{x\in\Omega\colon s|\phi|\leqslant M\}} f(x,s\phi)\,dx$$
$$+ a_2 \operatorname{meas} B_{2R}(x_0)$$
$$\leqslant C\left(\frac{1}{R^{p_2}} + 1\right) \int\limits_{B_{2R}(x_0)} s^{p(x)}\,dx - s^{\mu} a_1 \int\limits_{B_{2R}(x_0)} |\phi|^{\mu}\,dx + \bar{C} \operatorname{meas} B_{2R}(x_0)$$

$$= \int\limits_{B_{2R}(x_0)} s^{p(x)}\left(\frac{C}{R^{p_2}} + C - \bar{\bar{C}} s^{\mu - p(x)}\right) dx + \bar{C} \operatorname{meas} B_{2R}(x_0)$$
$$\leqslant \int\limits_{B_{2R}(x_0)} s^{p(x)}\left(\frac{C}{R^{p_2}} + C - \bar{\bar{C}} s^{l}\right) dx + \bar{C} \operatorname{meas} B_{2R}(x_0) < 0$$

if s is sufficiently large. Here

$$\bar{\bar{C}} = \frac{\int_{B_{2R}(x_0)} a_1 |\phi|^{\mu}\, dx}{\operatorname{meas} B_{2R}(x_0)}.$$

Next we show that the (PS) condition (i.e., any sequence $\{u_i\} \subset W_0^{1,p(x)}(\Omega)$ for which $J(u_i) \leqslant C$ and $J'(u_i) \to 0$ as $i \to \infty$ in $W^{-1,p'(x)}(\Omega)$ possesses a convergent subsequence) holds. Suppose that $\{u_i\} \subset W_0^{1,p(x)}(\Omega)$ is a sequence such that $J(u_i) \leqslant C$ and $J'(u_i) \to 0$ in $W^{-1,p'(x)}(\Omega)$. By (H4) we have

$$J(u_i) \geqslant \int\limits_{\Omega} \frac{a(x)}{p(x)} |\nabla u_i|^{p(x)} + \frac{b(x)}{p(x)} |u_i|^{p(x)}\, dx - \int\limits_{\Omega} \frac{1}{\mu} f(x, u_i) u_i\, dx$$
$$- \int\limits_{\Omega \cap \{x \in \Omega:\ |u_i| \leqslant M\}} F(x, u_i)\, dx + \int\limits_{\Omega \cap \{x \in \Omega:\ |u_i| \leqslant M\}} \frac{1}{\mu} f(x, u_i) u_i\, dx$$
$$\geqslant \int\limits_{\Omega} \left(\frac{1}{p(x)} - \frac{1}{\mu}\right)\left(a(x) |\nabla u_i|^{p(x)} + b(x) |u_i|^{p(x)}\right) dx$$
$$+ \frac{1}{\mu} \int\limits_{\Omega} \left(a(x) |\nabla u_i|^{p(x)} + b(x) |u_i|^{p(x)} - f(x, u_i) u_i\right) dx - C$$
$$\geqslant \frac{l}{\mu p_2} \int\limits_{\Omega} a_0 |\nabla u_i|^{p(x)} + b_0 |u_i|^{p(x)}\, dx - \frac{1}{\mu} \left\| J'(u_i) \right\|_{-1,p'} \|u_i\|_{1,p} - C.$$

In the following we consider two cases to show that $\{u_i\}$ is bounded in $W^{1,p(x)}(\Omega)$:

(1) If $\|\nabla u_i\|_p \leqslant 1$, it is immediate that $\|u_i\|_{1,p} \leqslant C$ in view of Theorem 2.8.

(2) If $\|\nabla u_i\|_p > 1$, then $\|\nabla u_i\|_p \leqslant \int_{\Omega} |\nabla u_i|^{p(x)}\, dx$. For i sufficiently large we have

$$\frac{1}{\mu} \left\| J'(u_i) \right\|_{-1,p'} < \frac{a_0 l}{2\mu p_2}$$

and then $\int_{\Omega} |\nabla u_i|^{p(x)}\, dx \leqslant C$ and $\int_{\Omega} |u_i|^{p(x)}\, dx \leqslant C$ as well by Theorem 2.8. Therefore by Theorem 2.2 $\|u_i\|_{1,p} \leqslant C$.

From (1)–(2) we conclude that $\{u_i\}$ is bounded in $W^{1,p(x)}(\Omega)$ and by Theorem 2.11 there exists a subsequence of $\{u_i\}$ (we still denote it by $\{u_i\}$) such that $K'(u_i)$ is a Cauchy sequence in $W^{-1,p'(x)}(\Omega)$.

Divide Ω into two parts:

$$\Omega_1 = \{x \in \Omega:\ p(x) < 2\}, \qquad \Omega_2 = \{x \in \Omega:\ p(x) \geqslant 2\}.$$

From (3.1) it is easy to get

$$\begin{aligned}
&\int_\Omega a(x)\big(|\nabla u_i|^{p(x)-2}\nabla u_i - |\nabla u_j|^{p(x)-2}\nabla u_j\big)(\nabla u_i - \nabla u_j) \\
&\qquad + b(x)\big(|u_i|^{p(x)-2}u_i - |u_j|^{p(x)-2}u_j\big)(u_i - u_j)\,dx \\
&\quad \leqslant \big|J'(u_i)(u_i - u_j)\big| + \big|J'(u_j)(u_i - u_j)\big| \\
&\qquad + \left|\int_\Omega \big(f(x,u_i) - f(x,u_j)\big)(u_i - u_j)\,dx\right| \\
&\quad \leqslant C\big(\|J'(u_i)\|_{-1,p'} + \|J'(u_j)\|_{-1,p'} + \|K'(u_i) - K'(u_j)\|_{-1,p'}\big) \\
&\quad \to 0.
\end{aligned} \tag{3.5}$$

On Ω_1 we have

$$\begin{aligned}
&\int_{\Omega_1} |\nabla u_i - \nabla u_j|^{p(x)}\,dx \\
&\quad \leqslant \int_{\Omega_1} \Big(\big(|\nabla u_i|^{p(x)-2}\nabla u_i - |\nabla u_j|^{p(x)-2}\nabla u_j\big)(\nabla u_i - \nabla u_j)\Big)^{\frac{p(x)}{2}} \\
&\qquad \times \big(|\nabla u_i|^{p(x)} + |\nabla u_j|^{p(x)}\big)^{\frac{2-p(x)}{2}}\,dx \\
&\quad \leqslant \Big\|\Big(\big(|\nabla u_i|^{p(x)-2}\nabla u_i - |\nabla u_j|^{p(x)-2}\nabla u_j\big)(\nabla u_i - \nabla u_j)\Big)^{\frac{p(x)}{2}}\Big\|_{\frac{2}{p},\Omega_1} \\
&\qquad \times \Big\|\big(|\nabla u_i|^{p(x)} + |\nabla u_j|^{p(x)}\big)^{\frac{2-p(x)}{2}}\Big\|_{\frac{2}{2-p},\Omega_1}.
\end{aligned}$$

From (3.5) and Theorem 2.2 we get

$$\Big\|\Big(\big(|\nabla u_i|^{p(x)-2}\nabla u_i - |\nabla u_j|^{p(x)-2}\nabla u_j\big)(\nabla u_i - \nabla u_j)\Big)^{\frac{p(x)}{2}}\Big\|_{\frac{2}{p},\Omega_1} \to 0. \tag{3.6}$$

As $\int_{\Omega_1} (|\nabla u_i|^{p(x)} + |\nabla u_j|^{p(x)})^{\frac{2-p(x)}{2}\cdot\frac{2}{2-p(x)}}\,dx$ is bounded, by Theorem 2.2 and (3.6),

$$\int_{\Omega_1} |\nabla u_i - \nabla u_j|^{p(x)}\,dx \to 0. \tag{3.7}$$

On Ω_2, by (3.5) we have

$$\begin{aligned}
&\int_{\Omega_2} |\nabla u_i - \nabla u_j|^{p(x)}\,dx \\
&\quad \leqslant C\int_{\Omega_2} \big(|\nabla u_i|^{p(x)-2}\nabla u_i - |\nabla u_j|^{p(x)-2}\nabla u_j\big)(\nabla u_i - \nabla u_j)\,dx \\
&\quad \to 0.
\end{aligned} \tag{3.8}$$

Combining (3.7) with (3.8) and by Theorem 2.2 we conclude $\|\nabla u_i - \nabla u_j\|_p \to 0$ and moreover $\|u_i - u_j\|_{1,p} \to 0$ by Theorem 2.8. Thus the (PS) condition holds.

The mountain pass theorem (see [21]) guarantees that J has a nontrivial critical point u. Let $\phi = \max\{-u(x), 0\}$ in (3.1) we arrive at the conclusion that $u \geqslant 0$ in Ω. □

Second we study the prototype (1.4). To establish the existence of solutions, we suppose that f satisfies:

(H5) $f(x,t) \geqslant \tilde{c}_3|t|^{\beta_0}$ as $t \to 0^+$, $0 \leqslant \beta_0 \leqslant \beta(x)$, where $\tilde{c}_3 > 0$.

Theorem 3.2. *Under conditions* (H1), (H3) *and* (H5), *the $p(x)$-Laplacian problem* (1.1) *has a nontrivial and nonnegative solution $u \in W_0^{1,p(x)}(\Omega)$.*

Proof. By condition (H3) the functional J is weakly continuous and differentiable.

Next we show that J is bounded below. In view of Theorem 2.8 it is easy to calculate that

$$
\begin{aligned}
J(u) &\geqslant \int_\Omega \frac{1}{p(x)}\left(a_0|\nabla u|^{p(x)} + b_0|u|^{p(x)}\right)dx - \int_\Omega \tilde{c}_1|u| + \frac{\tilde{c}_2}{\beta(x)+1}|u|^{\beta(x)+1}\,dx \\
&\geqslant \frac{1}{p_2}\int_\Omega a_0|\nabla u|^{p(x)} + b_0|u|^{p(x)} - p_2\tilde{c}_1|u| - p_2\tilde{c}_2|u|^{\beta(x)+1}\,dx \\
&\geqslant \frac{1}{p_2}\int_\Omega |u|\left(\frac{C|u|^{p(x)-1}}{2} - p_2\tilde{c}_1\right) + |u|^{\beta(x)+1}\left(\frac{C|u|^{p(x)-\beta(x)-1}}{2} - p_2\tilde{c}_2\right)dx.
\end{aligned}
$$

Denote

$$
L = \max\left\{1, \left(\frac{2p_2\tilde{c}_1}{C}\right)^{\frac{1}{p_1-1}}, \left(\frac{2p_2\tilde{c}_2}{C}\right)^{\frac{1}{b}}\right\}
$$

and $\Omega_1 = \{x \in \Omega\colon |u(x)| \geqslant L\}$, $\Omega_2 = \{x \in \Omega\colon |u(x)| < L\}$. Then from

$$
\int_{\Omega_1} |u|^{p(x)} - p_2\tilde{c}_1|u| - p_2\tilde{c}_2|u|^{\beta(x)+1}\,dx \geqslant 0
$$

and

$$
\begin{aligned}
&\left|\int_{\Omega_2} |u|^{p(x)} - p_2\tilde{c}_1|u| - p_2\tilde{c}_2|u|^{\beta(x)+1}\,dx\right| \\
&\quad\leqslant \int_{\Omega_2} L^{p(x)} + p_2\tilde{c}_1 L + p_2\tilde{c}_2 L^{\beta(x)+1}\,dx \\
&\quad\leqslant 2\left(L^{p_2} + p_2\tilde{c}_1 L + p_2\tilde{c}_2 L^{p_2}\right)\operatorname{meas}\Omega,
\end{aligned}
$$

we conclude that J is bounded below. Thus J has a critical point u: $J(u) = \inf\{J(v)\colon v \in W_0^{1,p(x)}(\Omega)\}$ and u is a weak solution of (1.1).

At last we show u is nontrivial and nonnegative. Pick $x_0 \in \Omega$ and $B_{2R}(x_0) \subset \Omega$ with $2R < 1$. Let $\phi \in C_0^\infty(B_{2R}(x_0))$ such that $\phi \equiv 1$, $x \in B_R(x_0)$; $0 \leqslant \phi(x) \leqslant 1$ and $|\nabla \phi| \leqslant \frac{1}{R}$. Then for $s < 1$,

$$\begin{aligned} J(s\phi) &\leqslant \int\limits_{B_{2R}(x_0)} \frac{s^{p(x)}}{p(x)} \big(a(x)|\nabla\phi|^{p(x)} + b(x)|\phi|^{p(x)}\big)\,dx - \int\limits_{B_{2R}(x_0)} \frac{s^{\beta_0+1}\tilde{c}_3}{\beta_0+1}|\phi|^{\beta_0+1}\,dx \\ &\leqslant C\left(\frac{1}{R^{p_2}}+1\right) \int\limits_{B_{2R}(x_0)} s^{p(x)}\,dx - \frac{s^{\beta_0+1}\tilde{c}_3}{\beta_0+1} \int\limits_{B_{2R}(x_0)} |\phi|^{\beta_0+1}\,dx \\ &\leqslant \int\limits_{B_{2R}(x_0)} s^{\beta_0+1}\left(\left(\frac{C}{R^{p_2}}+C\right)s^b - C'\right)dx < 0 \end{aligned}$$

if s is sufficiently small. Therefore u is nontrivial and similar to Theorem 3.1 we can conclude that u is nonnegative. □

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References

[1] E. Acerbi, G. Mingione, Regularity results for a class of functionals with non-standard growth, Arch. Rational Mech. Anal. 156 (2001) 121–140.
[2] E. Acerbi, G. Mingione, Regularity results for stationary electro-rheological fluids, Arch. Rational Mech. Anal. 164 (2002) 213–259.
[3] Y. Alkhutov, The Harnack inequality and the Hölder property of solutions of nonlinear elliptic equations with a nonstandard growth conditions, Differential Equations 33 (1997) 1653–1662.
[4] T. Bartsch, Z. Liu, On a superlinear elliptic p-Laplacian equation, J. Differential Equations 198 (2004) 149–175.
[5] V. Chiad Piat, A. Coscia, Hölder continuity of minimizers of functionals with variable growth exponent, Manuscripta Math. 93 (1997) 283–299.
[6] A. Coscia, G. Mingione, Hölder continuity of the gradient of $p(x)$-harmonic mappings, C. R. Acad. Sci. Paris Ser. 328 (1999) 363–368.
[7] P. De Napoli, M. Mariani, Mountain pass solutions to equations of p-Laplacian type, Nonlinear Anal. 54 (2003) 1205–1219.
[8] G. Dinca, P. Jebelean, J. Mawhin, Variational and topological methods for Dirichlet problems with p-Laplacian, Portugal. Math. 58 (2001) 339–378.
[9] D. Edmunds, J. Lang, A. Nekvinda, On $L^{p(x)}$ norms, Roy. Soc. London Proc. Sect. A 1981 (1999) 219–225.
[10] D. Edmunds, J. Rakosnik, Sobolev embedding with variable exponent, Studia Math. 143 (2000) 267–293.
[11] D. Edmunds, J. Rakosnik, Sobolev embedding with variable exponent II, Math. Nachr. 246–247 (2002) 53–67.
[12] H. Egnell, Existence and nonexistence results for m-Laplace equations involving critical Sobolev exponents, Arch. Rational Mech. Anal. 104 (1988) 57–77.
[13] X. Fan, J. Shen, D. Zhao, Sobolev embedding theorems for spaces $W^{k,p(x)}(\Omega)$, J. Math. Anal. Appl. 262 (2001) 749–760.

[14] X. Fan, Q. Zhang, Existence of solutions for $p(x)$-Laplacian Dirichlet problem, Nonlinear Anal. 52 (2003) 1843–1852.
[15] Y. Fu, The existence of solutions for elliptic systems with nonuniform growth, Studia Math. 151 (2002) 227–246.
[16] Y. Huang, Existence of positive solutions for a class of the p-Laplace equations, J. Austral. Math. Soc. Sect. B 36 (1994) 249–264.
[17] O. Kovacik, J. Rakosnik, On spaces $L^{p(x)}$ and $W^{1,p(x)}$, Czechoslovak Math. J. 41 (1991) 592–618.
[18] W. Rudin, Real and Complex Analysis, McGraw–Hill Science, New York, 1986.
[19] M. Ruzicka, Electro-Rheological Fluids: Modeling and Mathematical Theory, Springer-Verlag, 2000.
[20] S. Samko, Convolution and potential type operators in $L^{p(x)}(\mathbb{R}^n)$, Integral Transform. Spec. Funct. 7 (1998) 261–284.
[21] M. Willem, Minimax Theorems, Birkhäuser, Boston, 1996.