Existence of solutions for $p(x)$-Laplacian problems on a bounded domain

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Abstract
In this paper we study the following $p(x)$-Laplacian problem:

$$
- \text{div} (a(x)|\nabla u|^{p(x)-2} \nabla u) + b(x)|u|^{p(x)-2}u = f(x,u), \quad x \in \Omega,
$$

$$
u = 0, \quad \text{on } \partial \Omega,
$$

where $1 < p_1 \leq p(x) \leq p_2 < n$, $\Omega \subset \mathbb{R}^n$ is a bounded domain and applying the mountain pass theorem we obtain the existence of solutions in $W^{1,p(x)}_0(\Omega)$ for the $p(x)$-Laplacian problems in the superlinear and sublinear cases.

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1. Introduction

After Kovacik and Rakosnik first discussed the $L^{p(x)}$ spaces and $W^{k,p(x)}$ spaces in [17], a lot of research has been done concerning these kinds of variable exponent spaces; see,

Inspired by their works, we want to study the $p(x)$-Laplacian problem:

$$-\text{div}(a(x)|\nabla u|^{p(x)-2}\nabla u) + b(x)|u|^{p(x)-2}u = f(x,u), \quad x \in \Omega,$$

$$u = 0, \quad \text{on} \quad \partial \Omega,$$

where $\Omega$ is a bounded domain in $\mathbb{R}^n$, $0 < a_0 \leq a(x) \in L^\infty(\Omega)$, $0 \leq b_0 \leq b(x) \in L^\infty(\Omega)$, $p$ is Lipschitz continuous on $\overline{\Omega}$ and satisfies

$$1 < p_1 \leq p(x) \leq p_2 < n.$$ (1.2)

Our object is to obtain sufficient conditions on $f$ for (1.1) to admit nontrivial and nonnegative solutions in the following prototype cases:

$$f(x,u) = \begin{cases} g(x)u^{\alpha(x)}, & p(x) - 1 < \alpha(x) < p^*(x) - 1, \\ h(x)u^{\beta(x)}, & 0 \leq \beta(x) < p(x) - 1, \end{cases}$$ (1.3)

where $p^*(x) = \frac{np(x)}{n-p(x)}$.

When $p(x)$ is a constant function, there are a lot of studies for the case of bounded domains; see, for example, [4,7,8,12,16] and references therein. It is beyond our ability to write out all the works in this direction here. When $p(x)$ is a variable function, Fan and Zhang [14] studied the $p(x)$-Laplacian problems on bounded domains. Under some conditions, they established some results on the existence of solutions. Although we study the $p(x)$-Laplace problems on bounded domains and we apply mountain pass theorem as well, our method is a bit different from that in [14] and in some sense we discuss the $p(x)$-Laplacian problem in a more general setting than that in [14].

2. Preliminaries

In this section we first recall some facts on variable exponent spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$. For the details see [13,15,17].

Let $P(\Omega)$ be the set of all Lebesgue measurable functions $p : \Omega \to [1, +\infty]$.

$$\rho_p(f) = \int_{\Omega \setminus \Omega_\infty} |f(x)|^{p(x)} \, dx + \inf_{\Omega_\infty} |f(x)|,$$ (2.1)

$$\|f\|_p = \inf \left\{ \lambda > 0 : \rho_p \left( \frac{f}{\lambda} \right) \leq 1 \right\},$$ (2.2)

where $\Omega_\infty = \{ x \in \Omega : p(x) = \infty \}$. The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is the class of all functions $f$ such that $\rho_p(\lambda f) < \infty$ for some $\lambda = \lambda(f) > 0$. $L^{p(x)}(\Omega)$ is a Banach space endowed with the norm (2.2). $\rho_p(f)$ is called the modular of $f$ in $L^{p(x)}(\Omega)$. 
For a given \( p(x) \in P(\Omega) \) we define the conjugate function \( p'(x) \) as

\[
p'(x) = \begin{cases} 
\infty, & \text{if } x \in \Omega_1 = \{ x \in \Omega : p(x) = 1 \}, \\
1, & \text{if } x \in \Omega_\infty, \\
\frac{p(x)}{p(x) - 1}, & \text{for other } x \in \Omega.
\end{cases}
\]

**Theorem 2.1.** Let \( p \in P(\Omega) \). Then the inequality

\[
\int_{\Omega} |f(x)g(x)| \, dx \leq r_p \|f\|_p \|g\|_{p'},
\]

holds for every \( f \in L^{p(x)}(\Omega) \) and \( g \in L^{p'(x)}(\Omega) \) with the constant \( r_p \) depending on \( p(x) \) and \( \Omega \) only.

**Theorem 2.2.** The topology of the Banach space \( L^{p(x)}(\Omega) \) endowed by the norm (2.2) coincides with the topology of modular convergence if and only if \( p \in L^\infty(\Omega) \).

**Theorem 2.3.** The dual space to \( L^{p(x)}(\Omega) \) is \( L^{p'(x)}(\Omega) \) if and only if \( p \in L^\infty(\Omega) \). The space \( L^{p(x)}(\Omega) \) is reflexive if and only if

\[
1 < \inf_{\Omega} p(x) \leq \sup_{\Omega} p(x) < \infty.
\]  

(2.3)

Next we assume that \( \Omega \subset \mathbb{R}^n \) is a nonempty open set, \( p \in P(\Omega) \) and \( k \) is a given natural number.

Given a multiindex \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \), we set \( |\alpha| = \alpha_1 + \cdots + \alpha_n \) and \( D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n} \), where \( D_i = \frac{\partial}{\partial x_i} \) is the generalized derivative operator.

The generalized Sobolev space \( W^{k,p(x)}(\Omega) \) is the class of all functions \( f \) on \( \Omega \) such that \( D^\alpha f \in L^{p(x)}(\Omega) \) for every multiindex \( \alpha \) with \( |\alpha| \leq k \), endowed with the norm

\[
\|f\|_{k,p} = \sum_{|\alpha| \leq k} \|D^\alpha f\|_p.
\]  

(2.4)

By \( W_0^{k,p(x)}(\Omega) \) we denote the subspace of \( W^{k,p(x)}(\Omega) \) which is the closure of \( C_0^\infty(\Omega) \) with respect to the norm (2.4).

**Theorem 2.4.** The space \( W^{k,p(x)}(\Omega) \) and \( W_0^{k,p(x)}(\Omega) \) are Banach spaces, which are reflexive if \( p \) satisfies (2.3).

We denote the dual space of \( W_0^{k,p(x)}(\Omega) \) by \( W^{-k,p'(x)}(\Omega) \), then we have

**Theorem 2.5.** Let \( p \in P(\Omega) \cap L^\infty(\Omega) \). Then for every \( G \in W^{-k,p'(x)}(\Omega) \) there exists a unique system of functions \( \{g_\alpha \in L^{p'(x)}(\Omega) : |\alpha| \leq k \} \) such that

\[
G(f) = \sum_{|\alpha| \leq k} \int_{\Omega} D^\alpha f(x) g_\alpha(x) \, dx, \quad f \in W_0^{k,p(x)}(\Omega).
\]
The norm of $W^{-k,p'}(x)(\Omega)$ is defined as
\[ \| G \|_{-k,p'} = \sup \left\{ \frac{|G(f)|}{\|f\|_{k,p}} : f \in W_{0}^{k,p}(\Omega) \right\}. \]

**Theorem 2.6.** If $\Omega$ is a bounded domain with the cone property, $p(x) \in C(\overline{\Omega})$ satisfies (1.2) and $q(x)$ is any Lebesgue measurable function defined on $\Omega$ with $p(x) \leq q(x)$ a.e. on $\overline{\Omega}$ and $\inf_{x \in \Omega} \{ p^*(x) - q(x) \} > 0$, then there is a compact embedding $W^{1,p(x)}(\Omega) \to L^{q(x)}(\Omega)$.

**Theorem 2.7.** Let $\Omega$ be a domain with the cone property. If $p: \overline{\Omega} \to \mathbb{R}$ is Lipschitz continuous and satisfies (1.2), and $q(x) \in \mathcal{P}(\Omega)$ satisfies $p(x) \leq q(x) \leq p^*(x)$ a.e. on $\overline{\Omega}$, then there is a continuous embedding $W^{1,p(x)}(\Omega) \to L^{q(x)}(\Omega)$.

**Theorem 2.8.** Let $\Omega$ be a bounded domain. If $p \in L^\infty(\Omega)$ and $u \in W_{0}^{1,p(x)}(\Omega)$, then
\[ \int_{\Omega} |u|^{p(x)} \, dx \leq C \int_{\Omega} |\nabla u|^{p(x)} \, dx, \]
where $C$ is a constant depending on $\Omega$.

For the $p(x)$-Laplacian problems (1.1) we define two functionals $K(u)$ and $J(u)$ on $\Omega$:
\[ K(u) = \int_{\Omega} F(x,u) \, dx, \]
\[ J(u) = \int_{\Omega} \frac{1}{p(x)} \left( a(x)|\nabla u|^{p(x)} + b(x)|u|^{p(x)} \right) \, dx - K(u), \]
where $F(x,t) = \int_{0}^{t} f(x,s) \, ds$.

Next we discuss the properties of $K(u)$ while $f$ satisfies the following conditions:

(H1) $f \in C(\overline{\Omega} \times \mathbb{R})$, $f(x,t) > 0$ in $\Omega_{0} \times (0, +\infty)$ for some nonempty open set $\Omega_{0} \subseteq \Omega$ and $f(x,t) = 0$ for all $x \in \Omega$ and $t \leq 0$.

(H2) $|f(x,t)| \leq c_{1} + c_{2}|t|^{\beta(x)}$, $\alpha + 1 \in C(\overline{\Omega})$ with $\alpha = \inf_{x \in \Omega} \{ \alpha(x) - p(x) + 1 \} > 0$ and $a = \sup_{x \in \Omega} \{ p^*(x) - \alpha(x) - 1 \} > 0$. Here $c_{1}, c_{2}$ are positive constants.

(H3) $|f(x,t)| \leq \tilde{c}_{1} + \tilde{c}_{2}|t|^{\beta(x)}$, $\beta + 1 \in \mathcal{P}(\Omega)$ with $0 \leq \beta(x)$ and $b = \sup_{x \in \Omega} \{ p(x) - \beta(x) - 1 \} > 0$. Here $\tilde{c}_{1}, \tilde{c}_{2}$ are positive constants.

**Lemma 2.9.** Suppose that $f$ satisfies (H1) and (H2) or (H3). Then $K(u)$ is weakly continuous on $W_{0}^{1,p(x)}(\Omega)$.

**Proof.** Suppose that $f$ satisfies (H1) and (H3). Let $u_{j} \to u$ weakly in $W_{0}^{1,p(x)}(\Omega)$. Then $\{ u_{j} \}$ is bounded in $W_{0}^{1,p(x)}(\Omega)$. By Theorem 2.6 there is a compact embedding $W^{1,p(x)}(\Omega) \to L^{p(x)}(\Omega)$ while there is a continuous embedding $L^{p(x)}(\Omega) \to$
$L^{\beta(x)+1}(\Omega)$, so the embedding $W^{1,p(x)}(\Omega) \to L^{\beta(x)+1}(\Omega)$ is compact and $u_j \to u$ in $L^{\beta(x)+1}(\Omega)$. Then by Theorem 2.2 $u_j \to u$ in modular as well. From (H3) we get

$$|F(x,t)| \leq \tilde{c}_1|t| + \frac{\tilde{c}_2}{\beta(x)}|t|^{\beta(x)+1}.$$  

Then by Vitali theorem (see [18]), we have

$$\int_{\Omega} F(x,u_j) \, dx \to \int_{\Omega} F(x,u) \, dx \text{ as } j \to \infty.$$  

Similarly if $f$ satisfies (H1) and (H2) the theorem is valid as well. □

**Theorem 2.10.** Suppose that $f$ satisfies (H1) and (H2) or (H3). Then $K(u)$ is differentiable on $W^{1,p(x)}_0(\Omega)$ with

$$K'(u)\phi = \int_{\Omega} f(x,u) \phi \, dx \quad \forall \phi \in W^{1,p(x)}_0(\Omega).$$  

**Proof.** For differentiability of $K$, we will show that for any $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon, u) > 0$ such that

$$\left| K(u + \phi) - K(u) - \int_{\Omega} f(x,u) \phi \, dx \right| < \varepsilon \, \|\phi\|_{1,p}$$

for all $\phi \in W^{1,p(x)}_0(\Omega)$ with $\|\phi\|_{1,p} < \delta$.

Let $\Omega_1 = \{x \in \Omega_k: |u(x)| \geq h\}$, $\Omega_2 = \{x \in \Omega_k: |\phi(x)| \geq r\}$, $\Omega_3 = \{x \in \Omega_k: |u(x)| < h \text{ and } |\phi(x)| < r\}$, where $h, r$ are constant which will be determined later.

Next we consider the case that $f$ satisfies (H1) and (H3) only, the other case that $f$ satisfies (H1) and (H2) can be treated similarly. First on $\Omega_1$ we have

$$\int_{\Omega_1} F(x,u + \phi) - F(x,u) - f(x,u) \phi \, dx$$

$$\leq C \int_{\Omega_1} (\tilde{c}_1 + \tilde{c}_2 (|u| + |\phi|)^{\beta(x)}) |\phi| + (\tilde{c}_1 + \tilde{c}_2 |u|^{\beta(x)}) |\phi| \, dx$$

$$\leq C \int_{\Omega_1} (|\phi| + |u|^{\beta(x)} |\phi| + |\phi|^{\beta(x)+1}) \, dx$$

$$\leq C \|\chi_{\Omega_1}\|_{(p^*)'} \|\phi\|_{p^*} + I_1 + I_2,$$

since

$$|u|^{\beta(x)} \leq 2^{\beta(x)} (|u|^{\beta(x)} + |\phi|^{\beta(x)}) \leq 2^{\tilde{\beta}} (|u|^{\beta(x)} + |\phi|^{\beta(x)}),$$

where $\tilde{\beta} = \sup_{\Omega} \beta(x)$.  

Because \( u \in W^{1,p(x)}(\Omega) \), we can get

\[
\int_{\Omega_1} |u|^{p(x)} \, dx \geq \int_{\Omega} h^{p(x)} \, dx \geq \min \{ h^{p_1}, h^{p_2} \} \, \text{meas} \, \Omega_1.
\]

(2.5)

From (2.5), \( \text{meas} \, \Omega_1 \to 0 \) as \( h \to \infty \). Then we can get for sufficiently large \( h \),

\[
C \| \chi_{\Omega_1} \|_{(p^*)'} \| \phi \|_{p^*} \leq C (\text{meas} \, \Omega_1)^{\frac{np_1}{np_1-n+p_1}} \| \phi \|_{1,p} < \frac{\varepsilon}{9} \| \phi \|_{1,p}.
\]

(2.6)

Second from \( f \in C(\Omega \times \mathbb{R}) \) we have \( F \in C^1(\Omega \times \mathbb{R}) \). For any \( \varepsilon_1, h > 0 \), there exists \( r > 0 \) such that

\[
|F(x, \xi + \eta) - F(x, \xi) - f(x, \xi) \eta| < \varepsilon_1 |\eta|
\]

(2.7)
whenever \( x \in \bar{\Omega} \), \(|\xi| \leq h\) and \(|\eta| < r\). From (2.7) we have
\[
\int_{\Omega} |F(x, u + \phi) - F(x, u) - f(x, u)\phi| \, dx \leq \varepsilon_1 \|\phi\|_p \|\chi_{\Omega}\|_{p'}.
\]
Choose \( \varepsilon_1 \) such that \( \varepsilon_1 \|\chi_{\Omega}\|_{p'} < \frac{\varepsilon}{3} \), then
\[
\int_{\Omega} |F(x, u + \phi) - F(x, u) - f(x, u)\phi| \, dx < \frac{\varepsilon}{3} \|\phi\|_{1,p}.
\] (2.8)

Here \( \|\chi_{\Omega}\|_{p'} < \infty \) because \( \int_{\Omega} (\chi_{\Omega})^{p'} \, dx = \text{meas} \Omega < \infty \).

Third similar to the above we have
\[
\left| \int_{\Omega} F(x, u + \phi) - F(x, u) - f(x, u)\phi \, dx \right| 
\leq C \int_{\Omega} |u|^{\beta(x)}|\phi| + |\phi|^{\beta(x) + 1} \, dx 
\leq C \left( \|u\|^{\beta} \|\chi_{\Omega}\|_{p',\Omega} + \|\phi\|^{\beta + 1} \|\chi_{\Omega}\|_{p',\Omega} \right) \|\phi\|_{\beta + 1,\Omega}.
\]

For any \( 0 < \varepsilon_2 < 1 \), we have
\[
\int_{\Omega} \left( \frac{|\phi|}{\varepsilon_2 \|\phi\|_{p'}} \right)^{\beta(x) + 1} \, dx \leq \int_{\Omega} \left( \frac{|\phi|}{\|\phi\|_{p'}} \right)^{p^*(x)} \left( \frac{\|\phi\|_{p'}}{r} \right)^{p^*(x) - \beta(x) - 1} \left( \frac{1}{\varepsilon_2} \right)^{\beta(x) + 1} \, dx.
\]

As \( \bar{b} = \sup_{x \in \Omega} \{ p^*(x) - \beta(x) - 1 \} > 0 \), we can choose \( \|\phi\|_{1,p} \) sufficiently small such that
\[
\int_{\Omega} \left( \frac{|\phi|}{\varepsilon_2 \|\phi\|_{p'}} \right)^{\beta(x) + 1} \, dx \leq \int_{\Omega} \left( \frac{|\phi|}{\|\phi\|_{p'}} \right)^{p^*(x)} \, dx \leq 1
\]
and further
\[
\|\phi\|_{\beta + 1,\Omega} \leq \varepsilon_2 \|\phi\|_{p'} \leq C \varepsilon_2 \|\phi\|_{1,p}.
\]

From \( u \in W^{1,p(x)}(\Omega) \) and Theorem 2.6, \( u \in L^{\beta(x) + 1}(\Omega) \), so \( \int_{\Omega} |u|^{\beta(x) + 1} \, dx < \infty \) and further \( \|u\|^{\beta} \|\chi_{\Omega}\|_{p',\Omega} < \infty \). Similarly \( \|\phi\|^{\beta + 1} \|\chi_{\Omega}\|_{p',\Omega} < \infty \) if \( \|\phi\|_{1,p} < 1 \). Choose \( \varepsilon_2 \) such that
\[
\int_{\Omega} |F(x, u + \phi) - F(x, u) - f(x, u)\phi| \, dx < \frac{\varepsilon}{3} \|\phi\|_{1,p}.
\] (2.9)

From (2.6), (2.8) and (2.9) we conclude that \( K(u) \) is differentiable on \( W^{1,p(x)}_0(\Omega) \) with
\[
K'(u)\phi = \int_\Omega f(x, u)\phi \, dx \quad \forall \phi \in W^{1,p(x)}_0(\Omega).
\]
Theorem 2.11. Suppose that $f$ satisfies (H1) and (H2) or (H3). Then $K'(u)$ is a continuous and compact mapping from $W^{1,p(x)}_0(\Omega)$ to $W^{-1,p'(x)}(\Omega)$.

Proof. From
\[
|K'(u_j)\phi - K'(u)\phi| \leq \int_\Omega |f(x,u_j) - f(x,u)| |\phi| \, dx \\
\leq C \left\| f(x,u_j) - f(x,u) \right\|_{p^*} \|\phi\|_{p^*} \\
\leq C \left\| f(x,u_j) - f(x,u) \right\|_{p^*} \|\phi\|_{1,p},
\]
we have
\[
\left\| K'(u_j) - K'(u) \right\|_{-1,p'} \leq C \left\| f(x,u_j) - f(x,u) \right\|_{p^*}.
\]
Similarly to the differentiability of $K(u)$ we can get the result.

At last we show the compactness of $K'(u)$. We consider the case in which $f$ satisfies (H1) and (H3) only and the other case can be treated similarly as well. Let $\{u_j\}$ be a bounded sequence in $W^{1,p(x)}_0(\Omega)$. As the embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{p(x)+1}(\Omega)$ is compact, then the boundedness of $\{u_j\}$ in $W^{1,p(x)}(\Omega)$ implies that $\{u_j\}$ has a Cauchy subsequence which is still denoted by $\{u_j\}$. Similar to the above we can choose $j$ sufficiently large such that
\[
\left\| f(x,u_j) - f(x,u_i) \right\|_{p^*} < \varepsilon.
\]
Then $\{K'(u_j)\}$ is a Cauchy sequence in $W^{-1,p'(x)}(\Omega)$ and the compactness of $K'$ follows immediately. \qed

3. Existence of solutions

The critical points $u$ of $J(u)$, i.e.,
\[
J'(u)(\phi) = \int_\Omega a(x)|\nabla u|^{p(x)-2}\nabla u\nabla \phi + b(x)|u|^{p(x)-2}u\phi - f(x,u)\phi \, dx = 0 \quad (3.1)
\]
for all $\phi \in W^{1,p(x)}_0(\Omega)$ are weak solutions of
\[
-\text{div}(a(x)|\nabla u|^{p(x)-2}\nabla u) + b(x)|u|^{p(x)-2}u = f(x,u).
\]
So next we need only to consider the existence of nontrivial critical points of $J(u)$.

First we study the prototype case (1.3). To establish the existence of solutions, we suppose that $f$ satisfies the following additional condition:

(H4) there exist constants $M > 0$ and $\mu > p(x)$ with $\sup_{x \in \Omega} \{\mu - p(x)\} > 0$ such that $\mu F(x,t) \leq tf(x,t)$ for $x \in \Omega$ and $|t| \geq M$, and $f(x,t) = o(t^{p(x)-1})$ as $t \to 0$. 

Theorem 3.1. Under conditions (H1), (H2) and (H4) the $p(x)$-Laplacian problem (1.1) has a nontrivial and nonnegative solution $u \in W_0^{1,p(x)}(\Omega)$.

Proof. By (H4), for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $F(x, t) \leq \frac{\varepsilon}{2} |t|^{p(x)}$ whenever $x \in \Omega$ and $|t| < \delta$. By (H2) there exists a constant $A = A(\delta) > 0$ such that $F(x, t) \leq A|t|^{p(x)+1}$ whenever $x \in \Omega$ and $|t| \geq \delta$. Combining the two inequalities, we get

$$F(x, t) \leq \frac{\varepsilon}{2} |t|^{p(x)} + A|t|^{p(x)+1}.$$ 

Then

$$J(u) \geq \int_\Omega \frac{a_0}{p_2} |\nabla u|^{p(x)} \, dx - \frac{\varepsilon}{2} \int_\Omega |u|^{p(x)} \, dx - \int_\Omega \left( \frac{c_2}{p_1} + A \right) |u|^{p(x)+1} \, dx.$$ 

By Theorem 2.8 we can take $\varepsilon$ sufficiently small such that

$$\frac{\varepsilon}{2} \int_\Omega |u|^{p(x)} \, dx \leq \frac{a_0}{2p_2} \int_\Omega |\nabla u|^{p(x)} \, dx.$$ 

Then

$$J(u) \geq \frac{a_0}{2p_2} \int_\Omega |\nabla u|^{p(x)} \, dx - C \int_\Omega |u|^{p(x)+1} \, dx. \quad (3.2)$$ 

By Theorems 2.6 and 2.8 we have $\|u\|_{p+1} \leq C\|\nabla u\|_p$. If $\|\nabla u\|_p < 1$ is sufficiently small such that $C\|\nabla u\|_p < 1$, then $\|u\|_{p+1} < 1$.

For any $x \in \Omega$, as $p, \alpha \in C(\tilde{\Omega})$ we can get $Q_R(x) = \{y = (y^1, \ldots, y^n) : |y^i - x^i| < R, \ i = 1, \ldots, n\}$ such that $|p(y) - p(x)| < \varepsilon$ and $|\alpha(y) - \alpha(x)| < \varepsilon$ whenever $y \in Q_R(x) \cap \tilde{\Omega}$. Take $\varepsilon = \frac{\delta}{2}(\alpha(x) - p(x) + 1)$, then

$$p_{x,2} = \sup_{x \in Q_R(x) \cap \tilde{\Omega}} p(x) \leq \alpha_{x,1} + 1 = \inf_{x \in Q_R(x) \cap \tilde{\Omega}} \alpha(x) + 1.$$ 

\{Q_R(x)\}_{x \in \tilde{\Omega}} is an open covering of $\tilde{\Omega}$. As $\tilde{\Omega}$ is compact, we can pick a finite subcovering $\{Q_{R_i}(x_i)\}_{i=1}^m$ for $\tilde{\Omega}$ from the covering $\{Q_R(x)\}_{x \in \tilde{\Omega}}$. If $Q_{R_i}(x_i) \subseteq \Omega$ we define $u = 0$ on $Q_{R_i}(x_i) \setminus \Omega$ and then Theorem 2.6 is still valid for $u \in W_0^{1,p(x)}(\Omega)$ on $Q_{R_i}(x_i)$. We can use all the hyperplanes, for each of which there exists at least one hypersurface of some $Q_{R_i}(x_i)$ lying on it, to divide $\bigcup_{i=1}^m Q_{R_i}(x_i)$ into finite open hypercubes $\{Q_j\}_{j=1}^J$ which mutually have no common points. It is obvious that $\tilde{\Omega} \subseteq \bigcup_{j=1}^J \tilde{Q}_j$ and for each $Q_j$ there exists at least one $Q_{R_i}(x_i)$ such that $Q_j \subseteq Q_{R_i}(x_i)$. By Theorems 2.6, 2.8 and [11],

$$\int_{Q_j \cap \Omega} |u|^{p(x)+1} \, dx \leq (C\|u\|_{1,p, Q_j \cap \Omega})^{d_{j,1}+1}, \quad (3.3)$$

where $d_{j,1} = \inf_{x \in Q_j \cap \Omega} \alpha(x)$. As $\|\nabla u\|_{p, Q_j \cap \Omega} < 1$, we have

$$\int_{Q_j \cap \Omega} |\nabla u|^{p(x)} \, dx \geq \frac{1}{1 + C\|u\|_{1,p, Q_j \cap \Omega}}. \quad (3.4)$$
where \( p_{j2} = \sup_{x \in Q_j \cap \Omega} p(x) \). From (3.2)–(3.4) we have
\[
\frac{\alpha_0}{2p_2} \int_{Q_j \cap \Omega} |\nabla u|^{p(x)} \, dx - C \int_{Q_j \cap \Omega} |u|^{\alpha(x)+1} \, dx \\
\geq C_1 (\|u\|_{1,p,Q_j \cap \Omega})^{p_{j2}} - C_2 (\|u\|_{1,p,Q_j \cap \Omega})^{\alpha_{j1}+1}.
\]
As \( \alpha_{j1} + 1 > p_{j2} \), there exists \( r_j \) such that
\[
C_1 (\|u\|_{1,p,Q_j \cap \Omega})^{p_{j2}} - C_2 (\|u\|_{1,p,Q_j \cap \Omega})^{\alpha_{j1}+1} \geq C_r > 0
\]
if \( \|u\|_{1,p,Q_j \cap \Omega} = r \) and \( 0 < r < \frac{1}{1+C_r} \), where \( C \) is the embedding constant in \( \|u\|_{\alpha+1} \leq C \|\nabla u\|_p \). Take \( r_0 = \min_{1 \leq j \leq J} \{ r_j \} \). As
\[
\|u\|_{1,p} = \left\| u \sum_{j=1}^J \chi_{Q_j \cap \Omega} \right\|_{1,p} \leq \sum_{j=1}^J \|u\chi_{Q_j \cap \Omega}\|_{1,p} = \sum_{j=1}^J \|u\|_{1,p,Q_j \cap \Omega}.
\]
So there exists at least one \( \|u\|_{1,p,Q_j \cap \Omega} \) satisfying
\[
r_0 \leq \|u\|_{1,p,Q_j \cap \Omega} \leq r_0.
\]
Then we have
\[
J(u) \geq \sum_{j=1}^J \left( \frac{\alpha_0}{2p_2} \int_{Q_j \cap \Omega} |\nabla u|^{p(x)} \, dx - C \int_{Q_j \cap \Omega} |u|^{\alpha(x)+1} \, dx \right) \\
\geq C_1 \left( \frac{r_0}{J} \right)^{p_{j2}} \left( 1 - \frac{C_2}{C_1} \right) > 0
\]
if \( \|u\|_{1,p} = r_0 > 0 \).

By (H1) and (H4) we have
\[
F(x,t) \geq a_1 t^\mu - a_2, \quad |t| \geq M,
\]
where \((x,t) \in \Omega_0 \times \mathbb{R} \) and \( a_1, a_2 > 0 \) are constant. Pick \( x_0 \in \Omega_0 \) and \( B_{2R}(x_0) = \{ x : |x-x_0| < 2R \} \subset \Omega_0 \) with \( 2R < 1 \). Let \( \phi \in C_0^\infty(B_{2R}(x_0)) \) such that \( \phi \equiv 1 \), \( x \in B_R(x_0) \), \( 0 \leq \phi(x) \leq 1 \) and \( |\nabla \phi| \leq \frac{1}{R} \). Denote \( l = \inf_{x \in \Omega} \{ \mu - p(x) \} \). Then for \( s > 1 \),
\[
J(s\phi) \leq \int_{B_{2R}(x_0)} \frac{s^{p(x)}}{p(x)} (a(x)|\nabla \phi|^{p(x)} + b(x)|\phi|^{p(x)}) \, dx - \int_{B_{2R}(x_0)} s^\mu a_1 |\phi|^\mu \, dx \\
+ \int_{B_{2R}(x_0) \cap \{ x \in \Omega : s|\phi| \leq M \}} s^\mu a_1 |\phi|^\mu \, dx \\
- \int_{B_{2R}(x_0) \cap \{ x \in \Omega : s|\phi| \leq M \}} f(x,s\phi) \, dx \\
+ a_2 \text{ meas } B_{2R}(x_0) \\
\leq C \left( \frac{1}{R^{p_2}} + 1 \right) \int_{B_{2R}(x_0)} s^{p(x)} \, dx - s^\mu a_1 \int_{B_{2R}(x_0)} |\phi|^\mu \, dx + C \text{ meas } B_{2R}(x_0)
\[
\int_{B_2} s^{p(x)} \left( \frac{C}{R_{p/2}} + C - \tilde{C} s^{-p(x)} \right) dx + \tilde{C} \text{ meas } B_{2R}(x_0) \\
\leq \int_{B_2} s^{p(x)} \left( \frac{C}{R_{p/2}} + C - \tilde{C} \right) dx + \tilde{C} \text{ meas } B_{2R}(x_0) < 0
\]

if \( s \) is sufficiently large. Here \( \tilde{C} = \int_{B_2} a_1 |\phi|^\mu dx / \text{ meas } B_{2R}(x_0) \).

Next we show that the (PS) condition (i.e., any sequence \( \{u_i\} \subset W^{1, p(x)}(\Omega) \) for which \( J(u_i) \leq C \) and \( J'(u_i) \to 0 \) as \( i \to \infty \) in \( W^{-1, p'(x)}(\Omega) \) possesses a convergent subsequence) holds. Suppose that \( \{u_i\} \subset W^{1, p(x)}(\Omega) \) is a sequence such that \( J(u_i) \leq C \) and \( J'(u_i) \to 0 \) in \( W^{-1, p'(x)}(\Omega) \). By (H4) we have

\[
J(u_i) \geq \int_\Omega \frac{a(x)}{p(x)}(|\nabla u_i|^{p(x)} + \frac{b(x)}{p(x)} |u_i|^{p(x)}) dx - \int_\Omega \frac{1}{\mu} f(x, u_i) u_i dx \\
- \int_{\Omega \cap \{x \in \Omega: |u_i| \leq M\}} F(x, u_i) dx + \int_{\Omega \cap \{x \in \Omega: |u_i| \leq M\}} \frac{1}{\mu} f(x, u_i) u_i dx \\
\geq \int_\Omega \left( \frac{1}{p(x)} - \frac{1}{\mu} \right) \left( a(x)|\nabla u_i|^{p(x)} + b(x)|u_i|^{p(x)} \right) dx \\
+ \frac{1}{\mu} \int_\Omega \left( a(x)|\nabla u_i|^{p(x)} + b(x)|u_i|^{p(x)} - f(x, u_i) u_i \right) dx - C
\]

In the following we consider two cases to show that \( \{u_i\} \) is bounded in \( W^{1, p(x)}(\Omega) \):

1. If \( \|\nabla u_i\|_p \leq 1 \), it is immediate that \( \|u_i\|_{1,p} \leq C \) in view of Theorem 2.8.

2. If \( \|\nabla u_i\|_p > 1 \), then \( \|\nabla u_i\|_p \leq \int_\Omega |\nabla u_i|^{p(x)} dx \). For \( i \) sufficiently large we have

\[
\frac{1}{\mu} \|J'(u_i)\|_{-1,p'} \leq \frac{a_0}{2\mu p_2}
\]

and then \( \int_\Omega |\nabla u_i|^{p(x)} dx \leq C \) and \( \int_\Omega |u_i|^{p(x)} dx \leq C \) as well by Theorem 2.8. Therefore by Theorem 2.2 \( \|u_i\|_{1,p} \leq C \).

From (1)–(2) we conclude that \( \{u_i\} \) is bounded in \( W^{1, p(x)}(\Omega) \) and by Theorem 2.11 there exists a subsequence of \( \{u_i\} \) (we still denote it by \( \{u_i\} \)) such that \( K'(u_i) \) is a Cauchy sequence in \( W^{-1, p'(x)}(\Omega) \).

Divide \( \Omega \) into two parts:

\( \Omega_1 = \{ x \in \Omega: p(x) < 2 \} \), \( \Omega_2 = \{ x \in \Omega: p(x) \geq 2 \} \).
From (3.1) it is easy to get
\[
\begin{aligned}
\int_{\Omega} a(x)(|\nabla u_i|^{p(x)} - 2 \nabla u_i - |\nabla u_j|^{p(x)} - 2 \nabla u_j)(\nabla u_i - \nabla u_j) \\
+ b(x)(|u_i|^{p(x)} - |u_j|^{p(x)})(u_i - u_j) \, dx \\
\leq |J'(u_i)(u_i - u_j)| + |J'(u_j)(u_i - u_j)| \\
+ \left| \int_{\Omega} (f(x, u_i) - f(x, u_j))(u_i - u_j) \, dx \right| \\
\leq C \left( \left\| J'(u_i) \right\|_{-1, p'} + \left\| J'(u_j) \right\|_{-1, p'} + \left\| K'(u_i) - K'(u_j) \right\|_{-1, p'} \right) \\
\rightarrow 0.
\end{aligned}
\]
\[\tag{3.5}\]

On \(\Omega_1\) we have
\[
\begin{aligned}
\int_{\Omega_1} |\nabla u_i - \nabla u_j|^{p(x)} \, dx \\
\leq \int_{\Omega_1} \left( (|\nabla u_i|^{p(x)} - 2 \nabla u_i - |\nabla u_j|^{p(x)} - 2 \nabla u_j)(\nabla u_i - \nabla u_j) \right)^{\frac{p(x)}{2}} \\
\times \left( |\nabla u_i|^{p(x)} + |\nabla u_j|^{p(x)} \right)^{-\frac{p(x)}{2}} \, dx \\
\leq \left\| \left( (|\nabla u_i|^{p(x)} - 2 \nabla u_i - |\nabla u_j|^{p(x)} - 2 \nabla u_j)(\nabla u_i - \nabla u_j) \right)^{\frac{p(x)}{2}} \right\|_{\frac{2}{p}, \Omega_1} \\
\times \left\| \left( |\nabla u_i|^{p(x)} + |\nabla u_j|^{p(x)} \right)^{-\frac{p(x)}{2}} \right\|_{\frac{2}{p}, \Omega_1}.
\end{aligned}
\]
From (3.5) and Theorem 2.2 we get
\[
\left\| \left( (|\nabla u_i|^{p(x)} - 2 \nabla u_i - |\nabla u_j|^{p(x)} - 2 \nabla u_j)(\nabla u_i - \nabla u_j) \right)^{\frac{p(x)}{2}} \right\|_{\frac{2}{p}, \Omega_1} \rightarrow 0.
\]
As \(\int_{\Omega_1} (|\nabla u_i|^{p(x)} + |\nabla u_j|^{p(x)} \right)^{-\frac{p(x)}{2}} \, dx\) is bounded, by Theorem 2.2 and (3.6),
\[
\int_{\Omega_1} |\nabla u_i - \nabla u_j|^{p(x)} \, dx \rightarrow 0.
\]
\[\tag{3.7}\]
On \(\Omega_2\), by (3.5) we have
\[
\begin{aligned}
\int_{\Omega_2} |\nabla u_i - \nabla u_j|^{p(x)} \, dx \\
\leq C \int_{\Omega_2} \left( (|\nabla u_i|^{p(x)} - 2 \nabla u_i - |\nabla u_j|^{p(x)} - 2 \nabla u_j)(\nabla u_i - \nabla u_j) \right) \, dx \\
\rightarrow 0.
\end{aligned}
\]
\[\tag{3.8}\]
Combining (3.7) with (3.8) and by Theorem 2.2 we conclude \( \|\nabla u_i - \nabla u_j\|_{p} \to 0 \) and moreover \( \|u_i - u_j\|_{1,p} \to 0 \) by Theorem 2.8. Thus the (PS) condition holds.

The mountain pass theorem (see [21]) guarantees that \( J \) has a nontrivial critical point \( u \). Let \( \phi = \max\{-u(x), 0\} \) in (3.1) we arrive at the conclusion that \( u \geq 0 \) in \( \Omega \).

Second we study the prototype (1.4). To establish the existence of solutions, we suppose that \( f \) satisfies:

\[(H5) \quad f(x,t) \geq \tilde{c}_3|t|^\beta_0 \text{ as } t \to 0^+, \quad 0 \leq \beta_0 \leq \beta(x), \text{ where } \tilde{c}_3 > 0.\]

**Theorem 3.2.** Under conditions (H1), (H3) and (H5), the \( p(x) \)-Laplacian problem (1.1) has a nontrivial and nonnegative solution \( u \in W_0^{1,p(x)}(\Omega) \).

**Proof.** By condition (H3) the functional \( J \) is weakly continuous and differentiable.

Next we show that \( J \) is bounded below. In view of Theorem 2.8 it is easy to calculate that

\[
J(u) \geq \int_{\Omega} \frac{1}{p(x)} \left( a_0 |\nabla u|^{p(x)} + b_0 |u|^{p(x)} \right) dx - \int_{\Omega} \frac{\tilde{c}_2}{\beta(x)} |u|^{\beta(x)+1} dx \\
\geq \frac{1}{p_2} \int_{\Omega} a_0 |\nabla u|^{p(x)} + b_0 |u|^{p(x)} - p_2 \tilde{c}_1 |u| - p_2 \tilde{c}_2 |u|^{\beta(x)+1} dx \\
\geq \frac{1}{p_2} \int_{\Omega} |u| \left( \frac{C|u|^{p(x)-1}}{2} - p_2 \tilde{c}_1 \right) + |u|^{\beta(x)+1} \left( \frac{C|u|^{p(x)-\beta(x)-1}}{2} - p_2 \tilde{c}_2 \right) dx.
\]

Denote

\[L = \max \left\{ 1, \left( \frac{2p_2 \tilde{c}_1}{C} \right)^{\frac{1}{p(x)}}, \left( \frac{2p_2 \tilde{c}_2}{C} \right)^{\frac{1}{p(x)}} \right\}\]

and \( \Omega_1 = \{x \in \Omega : |u(x)| \geq L\} \), \( \Omega_2 = \{x \in \Omega : |u(x)| < L\} \). Then from

\[
\int_{\Omega_1} |u|^{p(x)} - p_2 \tilde{c}_1 |u| - p_2 \tilde{c}_2 |u|^{\beta(x)+1} dx \geq 0
\]

and

\[
\int_{\Omega_2} |u|^{p(x)} - p_2 \tilde{c}_1 |u| - p_2 \tilde{c}_2 |u|^{\beta(x)+1} dx \\
\leq \int_{\Omega_2} L^{p(x)} + p_2 \tilde{c}_1 L + p_2 \tilde{c}_2 L^{\beta(x)+1} dx \\
\leq 2(L^{p_2} + p_2 \tilde{c}_1 L + p_2 \tilde{c}_2 L^{p_2}) \text{ meas } \Omega,
\]

we conclude that \( J \) is bounded below. Thus \( J \) has a critical point \( u \): \( J(u) = \inf\{J(v) : v \in W_0^{1,p(x)}(\Omega)\} \) and \( u \) is a weak solution of (1.1).
At last we show $u$ is nontrivial and nonnegative. Pick $x_0 \in \Omega$ and $B_{2R}(x_0) \subset \Omega$ with $2R < 1$. Let $\phi \in C_0^\infty(B_{2R}(x_0))$ such that $\phi \equiv 1$, $x \in B_R(x_0)$; $0 \leqslant \phi(x) \leqslant 1$ and $|\nabla \phi| \leqslant \frac{1}{R}$.

Then for $s < 1$,

$$J(s\phi) \leqslant \int_{B_{2R}(x_0)} \frac{s^{p(x)}(x)}{p(x)} (a(x)|\nabla \phi|^{p(x)} + b(x)|\phi|^{p(x)}) \, dx - \int_{B_{2R}(x_0)} \frac{s^{\beta_0+1}c_3}{\beta_0 + 1} |\phi|^{\beta_0+1} \, dx$$

$$\leqslant \int_{B_{2R}(x_0)} s^{\beta_0+1} \left( \frac{C}{R^{p(x)}} + C \right) s^{\beta} \, dx < 0$$

if $s$ is sufficiently small. Therefore $u$ is nontrivial and similar to Theorem 3.1 we can conclude that $u$ is nonnegative. □

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