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Lyapunov–Schmidt Reduction and Melnikov Integrals for Bifurcation of Periodic Solutions in Coupled Oscillators*

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We consider a system of autonomous ordinary differential equations depending on a small parameter such that the unperturbed system has an invariant manifold of periodic solutions. The problem addressed here is the determination of sufficient geometric conditions for some of the periodic solutions on this invariant manifold to survive after perturbation. The main idea is to use a Lyapunov–Schmidt reduction for an appropriate displacement function in order to obtain the bifurcation function for the problem in a form which can be recognized as a generalization of the subharmonic Melnikov function. Thus, the multidimensional bifurcation problem can be cast in a form where the geometry of the problem is clearly incorporated. An important application can be made in case the uncoupled system of differential equations is a system of oscillators in resonance. In this case the invariant manifold of periodic solutions is just the product of the uncoupled oscillations. When each of the oscillators has one degree of freedom, the bifurcation function is computed by quadrature along the unperturbed oscillations. Additional applications include the computation of entrainment domains for a sinusoidally forced van der Pol oscillator and the computation of mutual synchronization domains for a system of inductively coupled van der Pol oscillators.

1. INTRODUCTION

We consider systems of autonomous differential equations depending on a small parameter such that the unperturbed system has an invariant manifold of periodic solutions and ask if any of these periodic solutions persists after perturbation. This is of course a classical problem to which a number of methods have been applied. The problem can be "solved" using perturbation expansion methods, the method of averaging, and various reduction methods. Notable examples of recent papers on the subject are

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[9, 10, 19]. Our intention is to show how the problem can be framed as a generalization of the geometric methods popularized in [12, 24, 25] and extended to the case of limit cycle oscillators in [5]. In particular, we obtain an appropriate bifurcation function as a natural generalization of the subharmonic Melnikov function familiar in forced oscillation problems. For this, the main idea is to apply the Lyapunov–Schmidt reduction to a displacement function whose zeros correspond to periodic solutions and then to identify the reduced bifurcation function as a generalization of the subharmonic Melnikov function. The implementation of these ideas not only provides a generalization of a successful geometric analytic method to a broad class of important multidimensional bifurcation problems, but also offers some new formulas for the reduced bifurcation functions which have proved useful in the applications. In fact, the specialization of the abstract bifurcation function to the bifurcation function for various applications is one of the main objectives of this paper.

In Section 2 we formulate the Lyapunov–Schmidt reduction as it will be applied in subsequent sections. In Section 3 the Lyapunov–Schmidt reduction is applied to the bifurcation of periodic solutions in a smooth system of autonomous differential equations of the form

$$\dot{x} = F(x, \varepsilon), \qquad x \in \mathbb{R}^n, \quad \varepsilon \in \mathbb{R},$$

where the unperturbed system ($\varepsilon = 0$) has a manifold of periodic solutions. In particular, we identify the bifurcation function and the geometric nondegeneracy conditions required to ensure that an unperturbed periodic solution, which corresponds to a simple zero of the bifurcation function on a nondegenerate manifold of unperturbed periodic solutions, persists after perturbation. The first nondegeneracy condition is intrinsic to the unperturbed flow on the invariant manifold and requires the periodic solutions on this manifold to satisfy a resonance condition. An invariant manifold satisfying this nondegeneracy condition is called a period manifold. The second nondegeneracy condition requires the invariant period manifold to be "normally nondegenerate" relative to the ambient unperturbed flow. This condition is satisfied if the period manifold is normally hyperbolic with respect to the unperturbed flow, but is in fact a weaker condition. The bifurcation function is defined as an integral over unperturbed periodic solutions of the system on the period manifold; it is the generalization of the Melnikov integral alluded to above. The final section, Section 4, gives several applications of the method. In particular, the method is applied to the bifurcation of subharmonics of forced oscillators with one degree of freedom along the lines of the more extensive investigation of forced oscillators presented in [5]. Here we also give an application of the results of this investigation to the computation of the tangents at the Arnold tongue of an entrainment domain. More generally, the method is also applied to systems of coupled oscillators. Although the formulation becomes quite complex, we obtain the bifurcation function for coupled oscillators with an arbitrary number of degrees of freedom. When these results are specialized to the case where each uncoupled oscillator has one degree of freedom, the formulas are especially appealing. In fact, immediately recognizable generalizations of the Melnikov integral arise in this case after applying Diliberto's geometric integration [5, 8] of the variational equation for a two dimensional system of differential equations. Specific applications of the resulting formula for the bifurcation function are made to systems of coupled "standard" limit cycle oscillators and to the case of a limit cycle oscillator coupled to an integrable oscillator. In the second case, the relationship of our geometric analysis with the classical perturbation expansion method of Poincaré–Lindstedt is discussed. A final application is made to the problem of mutual synchronization for two inductively coupled van der Pol oscillators in the presence of a detuning. For this example we show how to obtain the infinitesimal synchronization domains, i.e., the first order approximations to the synchronization domains in the frequency amplitude space for the case of van der Pol oscillators with moderate damping, small detuning, and weak coupling. Finally, we use the bifurcation function obtained for this application to numerically approximate the shape of the infinitesimal (1:1) synchronization domain.

2. Lyapunov–Schmidt Reduction

In this section the main features of the Lyapunov–Schmidt reduction are outlined in a form suitable for later application to the bifurcation of periodic solutions in systems of ordinary differential equations. Here we consider a function $\delta\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ given by $(\xi, \varepsilon) \mapsto \delta(\xi, \varepsilon)$ and we assume the *unperturbed function* $\xi \mapsto \delta(\xi, 0)$ has a zero. The bifurcation problem is to determine if the given zero persists for nonzero values of the *bifurcation parameter* ε. More precisely, we say $\xi = \xi_0$ is a *branch point of zeros* of δ if there is a curve $\varepsilon \mapsto \sigma(\varepsilon)$ in $\mathbb{R}^n$ and $\varepsilon_* > 0$ such that $\sigma(0) = \xi_0$ and $\delta(\sigma(\varepsilon), \varepsilon) \equiv 0$ for $|\varepsilon| < \varepsilon_*$. The bifurcation problem now takes the following form: to determine which zeros of the unperturbed function are branch points. The basic tool for the construction of a bifurcation theory to solve the bifurcation problem is the Implicit Function Theorem. In fact, the first proposition of the theory is just a special case of the Implicit Function Theorem. If *$\delta(\xi_0, 0) = 0$ and the partial derivative $\delta_\xi(\xi_0, 0)\colon \mathbb{R}^n \mapsto \mathbb{R}^n$ is an isomorphism, then $\xi = \xi_0$ is a branch point of zeros of δ.* However, in many important physical problems this proposition does not apply because the linear transformation given by $\mathbf{L} := \delta_\xi(\xi_0, 0)$ has a nontrivial kernel. In these cases the existence of branch points of zeros can be reduced to an application of the Implicit Function Theorem for a new bifurcation function obtained from the Lyapunov–Schmidt reduction. This method is explained in detail in [11]. Here we will fix notation and give the reduced bifurcation function in the precise form needed for the later applications where the function δ will be interpreted as the displacement function defined on a Poincaré section transverse to an unperturbed periodic solution of a system of differential equations. There, the zeros of the displacement function correspond to periodic solutions of the system of differential equations.

The Lyapunov–Schmidt reduction for the function δ defined above begins with the identification of the kernel of the Jacobian at $\xi = \xi_0$ given by $\mathbf{L}$. If $\mathbf{L}$ has kernel $\mathscr{K}$ with $\dim \mathscr{K} = k > 0$ and $\mathscr{K}^\perp$ denotes a complement of $\mathscr{K}$ in $\mathbb{R}^n$, then, from linear algebra, $\mathbf{L}$ has range $\mathscr{R}$ with $\dim \mathscr{R} = n - k$ and this range has a complement $\mathscr{R}^\perp$ in $\mathbb{R}^n$ with $\dim \mathscr{R}^\perp = k$. We let $\pi: \mathbb{R}^n \to \mathscr{R}$ denote the linear projection onto the range and $\pi^\perp: \mathbb{R}^n \to \mathscr{R}^\perp$ the complementary projection. Also, we choose coordinates on $\mathbb{R}^n$ so that the first k coordinates correspond to $\mathscr{K}$ and the remaining $n-k$ coordinates correspond to $\mathscr{K}^\perp$, and such that the origin of the coordinate system corresponds to ξ_0. Then, $\rho: \mathbb{R}^k \times \mathbb{R}^{n-k} \times \mathbb{R} \to \mathscr{R}$ defined by $\rho(\theta, \zeta, \varepsilon) := \pi\delta((\theta, \zeta), \varepsilon)$ has its partial derivative

$$\rho_\zeta(0, 0, 0) = \mathbf{L}|_{\mathscr{K}^\perp}: \mathscr{K}^\perp \to \mathscr{R}$$

an isomorphism. Hence, the Implicit Function Theorem applies to ρ and there is a function $h: \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}^{n-k}$ such that $h(0, 0) = 0$ and, for sufficiently small $|\theta|$ and $|\varepsilon|$, $\rho(\theta, h(\theta, \varepsilon), \varepsilon) \equiv 0$. The Lyapunov–Schmidt reduced function is the complementary function $\tau: \mathbb{R}^k \times \mathbb{R} \to \mathscr{R}^\perp$ defined by $\tau(\theta, \varepsilon) := \pi^\perp\delta((\theta, h(\theta, \varepsilon)), \varepsilon)$. It is easy to see that $\tau(0, 0) = 0$. If there is a curve $\varepsilon \mapsto \gamma(\varepsilon)$ in $\mathbb{R}^k$ such that $\gamma(0) = 0$ and $\tau(\gamma(\varepsilon), \varepsilon) \equiv 0$, then $(\theta, \zeta) = (0, 0)$ is a branch point of zeros of δ with the required curve of zeros $\varepsilon \mapsto \sigma(\varepsilon)$ in $\mathbb{R}^k \times \mathbb{R}^{n-k}$ given by $\sigma(\varepsilon) := (\gamma(\varepsilon), h(\gamma(\varepsilon), \varepsilon))$. Of course, we will not be able to determine the existence of the curve γ by a direct application of the Implicit Function Theorem because $\mathbf{L}$ has a nontrivial kernel. However, a further reduction is possible in many important applications. In particular, a reduction can be made when the zero set of the unperturbed function is a submanifold of $\mathbb{R}^n$ on which the Jacobian of the unperturbed function has maximal rank. More precisely, suppose

$$\mathscr{Z} \subseteq \{\xi \in \mathbb{R}^n \mid \delta(\xi, 0) = 0\}$$

is a k-dimensional submanifold of $\mathbb{R}^n$, with $0 < k < n$. In a neighborhood of each point of $\mathscr{Z}$, for example in a neighborhood of ξ_0, there are local coordinates $\Delta: \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^n$ given by $(\theta, \zeta) \mapsto \Delta(\theta, \zeta)$ such that $\mathbb{R}^k$ identified with $\{(\theta, \zeta) \in \mathbb{R}^k \times \mathbb{R}^{n-k} \mid \zeta = 0\}$ is the local representation of $\mathscr{Z}$. This means the corresponding local coordinate representation of δ, namely $\delta(\theta, \zeta, \varepsilon) := \delta(\Delta(\theta, \zeta), \varepsilon)$, satisfies $\delta(\theta, 0, 0) \equiv 0$. We say the bifurcation problem is *minimally degenerate* if the linear transformation given by the partial derivative $\delta_\zeta(\theta, 0, 0): \mathbb{R}^{n-k} \to \mathbb{R}^n$ has rank $n-k$ for each $\theta \in \mathbb{R}^k$. Under the assumption of minimal degeneracy there are parameterized families of subspaces (vector bundles over $\mathscr{Z}$) corresponding to the kernel $\mathscr{K}$ and range $\mathscr{R}$ of L together with their complements $\mathscr{K}^\perp$ and $\mathscr{R}^\perp$ such that each subspace is isomorphic to a Euclidean space of appropriate dimension. In particular, there are linear isomorphisms (local trivializations of the vector bundles)

$$r(\theta): \mathscr{R}(\theta) \to \mathbb{R}^{n-k}, \qquad r^\perp(\theta): \mathscr{R}^\perp(\theta) \to \mathbb{R}^k$$

and linear projections

$$\pi(\theta): \mathbb{R}^n \to \mathscr{R}(\theta), \qquad \pi^\perp(\theta): \mathbb{R}^n \to \mathscr{R}^\perp(\theta).$$

If we express $\rho: \mathbb{R}^k \times \mathbb{R}^{n-k} \times \mathbb{R} \to \mathbb{R}^{n-k}$ in coordinates by

$$\rho(\theta, \zeta, \varepsilon) = r(\theta)\, \pi(\theta)\, \delta(\theta, \zeta, \varepsilon),$$

then, for each $\theta \in \mathbb{R}^k$, the linear transformation $\rho_\zeta(\theta, 0, 0): \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$ is an isomorphism and there is a map $h: \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}^{n-k}$ such that for a perhaps smaller domain $\rho(\theta, h(\theta, \varepsilon), \varepsilon) \equiv 0$. Next, as before, we define $\tau: \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}^k$ by

$$\tau(\theta, \varepsilon) = r^\perp(\theta)\, \pi^\perp(\theta)\, \delta(\theta, h(\theta, \varepsilon), \varepsilon).$$

We note $\tau(\theta, 0) \equiv 0$. Thus, by Taylor's Theorem, we have

$$\tau(\theta, \varepsilon) = \varepsilon(\tau_\varepsilon(\theta, 0) + O(\varepsilon)),$$

where

$$\tau_\varepsilon(\theta, 0) = r^\perp(\theta)\, \pi^\perp(\theta)\, \delta_\zeta(\theta, 0, 0)\, h_\varepsilon(\theta, 0) + r^\perp(\theta)\, \pi^\perp(\theta)\, \delta_\varepsilon(\theta, 0, 0).$$

However, the range of $\delta_\zeta(\theta, 0, 0)$ is $\mathscr{R}(\theta)$, so the first term of the last formula vanishes and we obtain the *bifurcation function* $\mathscr{B}: \mathbb{R}^k \to \mathbb{R}^k$ given by

$$\mathscr{B}(\theta) = r^\perp(\theta)\, \pi^\perp(\theta)\, \delta_\varepsilon(\theta, 0, 0).$$

Strictly speaking, $\tau_\varepsilon(\theta, 0): \mathbb{R} \to \mathbb{R}^k$ is a linear map which we have identified as an element of $\mathbb{R}^k$.

By the above reductions, if there is a curve $\varepsilon \mapsto \gamma(\varepsilon)$ in $\mathbb{R}^k$ such that $\tau(\gamma(\varepsilon), \varepsilon) \equiv 0$ and $\varepsilon \mapsto \sigma(\varepsilon)$ is the curve in $\mathbb{R}^k \times \mathbb{R}^{n-k}$ defined by $\sigma(\varepsilon) := (\gamma(\varepsilon), h(\gamma(\varepsilon), \varepsilon))$, then $\delta(\sigma(\varepsilon), \varepsilon) \equiv 0$ and $\sigma(0) = (\gamma(0), 0) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$ is a branch point of zeros of δ. Thus, by a final application of the Implicit Function Theorem, we obtain a useful result: *If $\theta \in \mathbb{R}^k$ is a simple zero of the bifurcation function $\mathscr{B}$ in the sense that $\mathscr{B}(\theta) = 0$ and $D\mathscr{B}(\theta)\colon \mathbb{R}^k \to \mathbb{R}^k$ is an isomorphism, then $(\theta, 0) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$ is a branch point of zeros of δ.*

3. Identification of the Bifurcation Function

Consider a smooth system of differential equations E_ε given by

$$\dot{x} = F(x, \varepsilon), \qquad x \in \mathbb{R}^{n+1}, \quad \varepsilon \in \mathbb{R}$$

where the unperturbed system E_0 has periodic solutions. We ask if any of these periodic solutions persists for $\varepsilon \neq 0$. This bifurcation problem is framed within the context of the last section by showing it is equivalent to the bifurcation of zeros of a displacement function. In this section a precise formulation of this equivalence will be given together with an identification of the resulting bifurcation function in terms of the system E_ε. The analysis to follow applies verbatim to the more general system of differential equations given by

$$\dot{x} = f(x) + \varepsilon g(x, \dot{x}, \varepsilon), \qquad x \in \mathbb{R}^{n+1}, \quad \varepsilon \in \mathbb{R}.$$

The system E_ε is used in the formal presentation for notational simplicity.

The fundamental geometric construction of the subject is the Poincaré section. For this we choose a periodic trajectory Γ of E_0 whose period is $\eta > 0$. If $y_* \in \Gamma$, there are local coordinates $\Delta\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{n+1}$ given by $(s, \xi) \mapsto \Delta(s, \xi)$ such that $\Delta(0, 0) = y_*$, such that the map $s \mapsto \Delta(s, 0)$ parameterizes Γ near y_* and the map $\xi \mapsto \Delta(0, \xi)$ parameterizes an n-dimensional submanifold $\Sigma \subseteq \mathbb{R}^{n+1}$ that is transversal to Γ at y_*. We define the coordinate projections π_1 and π_2 onto the first and second factors of $\mathbb{R} \times \mathbb{R}^n$ and let $t \mapsto x(t, \xi, \varepsilon)$ denote the solution of E_ε satisfying the initial condition $x(0, \xi, \varepsilon) = \Delta(0, \xi)$. Since

$$\pi_1 \Delta^{-1}(x(\eta, 0, 0)) = 0, \qquad \pi_1 [D\Delta(0, 0)]^{-1}\, \dot{x}(\eta, 0, 0) \neq 0,$$

an application of the Implicit Function Theorem shows there is a smooth *parameterized transit time function* $\mathscr{T}\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ such that

$$\pi_1 \Delta^{-1}(x(\mathscr{T}(\xi, \varepsilon), \xi, \varepsilon)) \equiv 0.$$

Of course, $\mathscr{T}$ may only be defined for $|\xi|$ and $|\varepsilon|$ sufficiently small. However, we continue to use Σ to denote the Poincaré section contained in the first factor of the product neighborhood of $(y_*, 0) \in \mathbb{R}^n \times \mathbb{R}$ where the transit time is defined. In a similar manner the *parameterized Poincaré map* $P: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is defined by

$$P(\xi, \varepsilon) = \pi_2 \Delta^{-1}(x(\mathscr{T}(\xi, \varepsilon), \xi, \varepsilon))$$

and the *parameterized displacement function* $\delta: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is defined by

$$\delta(\xi, \varepsilon) = P(\xi, \varepsilon) - \xi.$$

Since $\delta(0, 0) = 0$, the reduction method of the last section can be applied to the displacement function. A branch point of its zeros is called a *branch point of periodic solutions* of E_ε. If $\xi = 0$ is a branch point of periodic solutions we also say the unperturbed periodic solution Γ *persists*.

As we have seen in the previous section, the reduction method begins with the identification of the kernel and the range of the linear transformation on $\mathbb{R}^n$ defined by $\mathbf{L} := \delta_\xi(0, 0)$. Here, as in all subsequent identifications, the quantities of interest are identified in terms of the solutions of appropriate variational equations along the unperturbed solution Γ. In fact, we have

$$\mathbf{L} = \pi_2[D\Delta(0, 0)]^{-1} (\dot{x}(\eta, 0, 0)\, \mathscr{T}_\xi(0, 0) + x_\xi(\eta, 0, 0)) - I,$$

where I denotes the identity on $\mathbb{R}^n$. Since $s \mapsto \Delta(s, 0)$ parameterizes Γ and $\dot{x}(\eta, 0, 0) = \dot{x}(0, 0, 0)$ is tangent to Γ at y_*, this formula for $\mathbf{L}$ reduces to

$$\mathbf{L} = \pi_2[D\Delta(0, 0)]^{-1} (x_\xi(\eta, 0, 0)) - I$$

where $t \mapsto x_\xi(t, 0, 0)$ is the (matrix) solution of the homogeneous variational initial value problem

$$\dot{W} = F_x(x(t, 0, 0), 0)\, W, \qquad W(0) = \Delta_\xi(0, \xi).$$

Here, the initial condition is satisfied because $x(0, \xi, \varepsilon) = \Delta(0, \xi)$. In these coordinates the linear transformation $\mathbf{L} + I$ that represents the derivative of the Poincaré map is called a *monodromy transformation* on Γ. We have a basic fact: *If a monodromy transformation on the unperturbed periodic solution* Γ *does not have* 1 *as an eigenvalue or equivalently if* $\mathbf{L}$ *has trivial kernel, then* Γ *persists.* Of course, having 1 as an eigenvalue is independent of the choice of the monodromy transformation. Also, hyperbolic periodic solutions persist; their monodromy transformations have spectra off the unit circle in the complex plane.

If the kernel of $\mathbf{L}$ is nontrivial, we can use the Lyapunov–Schmidt reduction of Section 2. There are many situations that can be studied using the abstract reduction procedure. However, since our goal is to find some of these cases where there is a reasonable chance of analyzing the resulting bifurcation function, some choices must be made. Here we consider a case anticipated by our parameterized version of the Lyapunov–Schmidt reduction. In particular, we consider the bifurcation problem when Γ belongs to a submanifold of periodic solutions of E_0. More precisely, we assume $\Gamma \subset \mathscr{A}$ where $\mathscr{A}$ is a $(k+1)$-dimensional connected submanifold of $\mathbb{R}^{n+1}$ with $k \geqslant 1$ such that every point of $\mathscr{A}$ lies on a periodic solution of E_0 and such that the following condition holds. For each $y_* \in \mathscr{A}$ and any Poincaré section Σ for the flow of E_0 at y_* with corresponding parameterized transit time map $\mathscr{T}$, the period of the periodic solution of E_0 through each $y \in \mathscr{A} \cap \Sigma$ is given by its transit time, i.e., $x(\mathscr{T}(y, 0), y, 0) = y$. Equivalently, the displacement function vanishes identically on $\mathscr{A} \cap \Sigma$. If this condition is satisfied, $\mathscr{A}$ is called a *period manifold.*

For the remainder of this section we assume Γ is a periodic solution of E_0 which lies in a period manifold $\mathscr{A}$ whose dimension is $k+1$ with $k \geqslant 1$. In this case it is natural to decompose $\mathbb{R}^{n+1}$ along $\mathscr{A}$ into three families of parameterized subspaces: the one dimensional space $\mathscr{E}$ generated by the unperturbed vector field, a k-dimensional complementary space $\mathscr{E}^{\tan}$ in the tangent space of $\mathscr{A}$ and an $(n-k)$-dimensional space $\mathscr{E}^{\text{nor}}$ which is normal to $\mathscr{A}$ in $\mathbb{R}^{n+1}$. These spaces determine a (vector bundle) splitting of $\mathbb{R}^{n+1}$ by taking $\mathbb{R}^{n+1} = \mathscr{E}(y) \oplus \mathscr{E}^{\tan}(y) \oplus \mathscr{E}^{\text{nor}}(y)$ at each point $y \in \mathscr{A}$. Moreover, at each point of $\mathscr{A}$ the coordinate map Δ can be redefined with respect to the splitting as a map $\Delta: \mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^{n+1}$ given by $(s, \theta, \zeta) \mapsto \Delta(s, \theta, \zeta)$. Then, the derivative of the map $s \mapsto \Delta(s, \theta, 0)$ takes the vector field defined by $\partial/\partial s$ on $\mathbb{R}$ to the vector field defined by $F(\Delta(s, \theta, 0), 0)$, the map $\theta \mapsto \Delta(s, \theta, 0)$ has tangent $\mathscr{E}^{\tan}$ and the map $\zeta \mapsto \Delta(s, \theta, \zeta)$ has tangent $\mathscr{E}^{\text{nor}}$ when evaluated at $\zeta = 0$. Such coordinates are called *adapted* to $\mathscr{A}$. Of course, the summands $\mathscr{E}^{\tan}$ and $\mathscr{E}^{\text{nor}}$ are not unique. The appropriate choice for these summands in the applications is determined by special geometric features of the system E_ε. For example, there are often naturally defined frame fields on the period manifold which serve to define the splitting.

In adapted coordinates we define $\delta(\theta, \zeta, \varepsilon) := \delta(\Delta(0, \theta, \zeta), \varepsilon)$ and let $D\delta$ denote the Jacobian of the map $(\theta, \zeta) \mapsto \delta(\theta, \zeta, 0)$. Then, with the usual identification, we clearly have $\mathbb{R}^k \subseteq \text{kernel}\, D\delta(\theta, 0)$. The bifurcation problem is minimally degenerate (using the definition of Section 2) when this inclusion is an equality. Equivalently, the problem is minimally degenerate when the monodromy transformation for the periodic trajectory through the point $\Delta(0, \theta, 0)$ has 1 as an eigenvalue with geometric multiplicity k. When this holds for every point on $\mathscr{A}$ we say the period manifold is *normally nondegenerate*. Later we will see some examples where this condition can be verified. Here we note the obvious fact: *If* $\mathscr{A}$ *is normally* *hyperbolic in* $\mathbb{R}^{n+1}$ *with respect to the unperturbed flow, then* $\mathscr{A}$ *is normally nondegenerate.*

The bifurcation function is identified in terms of the components, with respect to an adapted coordinate system, of a fundamental matrix solution of the homogeneous variational equation for E_0 along the unperturbed periodic solutions in the period manifold $\mathscr{A}$. In order to make this identification precise we require a few auxiliary definitions. First, define $t \mapsto x(t, \theta, \zeta, \varepsilon)$ to be the solution of $\dot{x} = F(x, \varepsilon)$ with $x(0, \theta, \zeta, \varepsilon) = \Delta(0, \theta, \zeta)$. For notational convenience we write $\gamma(t, \theta) := x(t, \theta, 0, 0)$ and recall the homogeneous variational equation along the solution $t \mapsto \gamma(t, \theta)$ is given by

$$\dot{W} = F_x(\gamma(t, \theta), 0)\, W.$$

This variational equation has a fundamental matrix solution $t \mapsto \Phi(t, \theta)$ with initial value $\Phi(0, \theta) = I$. There are parameterized linear maps

$$a(t, \theta)\colon \mathscr{E}^{\mathrm{nor}}(\gamma(0, \theta)) \to \mathscr{E}^{\mathrm{tan}}(\gamma(t, \theta)), \qquad b(t, \theta)\colon \mathscr{E}^{\mathrm{nor}}(\gamma(0, \theta)) \to \mathscr{E}^{\mathrm{nor}}(\gamma(t, \theta)),$$
$$c(t, \theta)\colon \mathscr{E}^{\mathrm{tan}}(\gamma(0, \theta)) \to \mathscr{E}^{\mathrm{tan}}(\gamma(t, \theta)), \qquad d(t, \theta)\colon \mathscr{E}^{\mathrm{nor}}(\gamma(0, \theta)) \to \mathscr{E}(\gamma(t, \theta)),$$
$$e(t, \theta)\colon \mathscr{E}^{\mathrm{tan}}(\gamma(0, \theta)) \to \mathscr{E}(\gamma(t, \theta)),$$

such that the block form of $\Phi(t, \theta)$ with respect to the splitting is

$$\Phi(t, \theta) = \begin{pmatrix} 1 & e(t, \theta) & d(t, \theta) \\ 0 & c(t, \theta) & a(t, \theta) \\ 0 & 0 & b(t, \theta) \end{pmatrix}$$

and such that

$$e(0, \theta) = 0, \qquad d(0, \theta) = 0, \qquad c(0, \theta) = I, \qquad a(0, \theta) = 0, \qquad b(0, \theta) = I.$$

Moreover, if $\mathscr{T}(\theta, \zeta, \varepsilon)$ denotes the parameterized transit time map in adapted coordinates and $T(\theta) := \mathscr{T}(\theta, 0, 0)$, then, in view of the fact that $\mathscr{A}$ is a period manifold, we have $c(T(\theta), \theta) = I$. Define

$$H(\theta)\colon \mathscr{E}(\gamma(0, \theta)) \oplus \mathscr{E}^{\mathrm{tan}}(\gamma(0, \theta)) \oplus \mathscr{E}^{\mathrm{nor}}(\gamma(0, \theta)) \to \mathbb{R}^k$$

to be the linear projection onto the complement of the range of $D\delta(\theta, 0)$ and note that $H(\theta)$ can be viewed as the linear projection $\mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k$ in adapted coordinates. Also, let the components of F_ε with respect to the splitting be given by

$$F_\varepsilon(\gamma(t, \theta), 0) = \begin{pmatrix} F_\varepsilon^{\mathscr{E}}(t, \theta) \\ F_\varepsilon^{\mathrm{tan}}(t, \theta) \\ F_\varepsilon^{\mathrm{nor}}(t, \theta) \end{pmatrix}.$$

The *bifurcation function for the system E_ε adapted to the period manifold $\mathscr{A}$* is the function $\mathscr{B}: \mathbb{R}^k \to \mathbb{R}^k$ defined by

$$\mathscr{B}(\theta) = H(\theta)\begin{pmatrix} 0 \\ \mathscr{N}(0) \\ \mathscr{M}(\theta) \end{pmatrix},$$

where

$$\mathscr{M}(\theta) := \int_0^{T(\theta)} b^{-1}(s, \theta)\, F_\varepsilon^{\text{nor}}(s, \theta)\, ds,$$

$$\mathscr{N}(\theta) := \int_0^{T(\theta)} c^{-1}(s, \theta)\, F_\varepsilon^{\tan}(s, \theta) - c^{-1}(s, \theta)\, a(s, \theta)\, b^{-1}(s, \theta)\, F_\varepsilon^{\text{nor}}(s, \theta)\, ds.$$

THEOREM 3.1. *Suppose the system E_ε given by*

$$\dot{x} = F(x, \varepsilon), \qquad x \in \mathbb{R}^{n+1}, \quad \varepsilon \in \mathbb{R}$$

has a normally nondegenerate period manifold $\mathscr{A}$. If θ is a simple zero of the bifurcation function for E_ε adapted to $\mathscr{A}$, then θ is a branch point of periodic solutions of E_ε.

Proof. We show the bifurcation function for E_ε adapted to $\mathscr{A}$ is the identification in the context of bifurcation on a normally nondegenerate period manifold of the abstract bifurcation function derived from the Lyapunov–Schmidt reduction in Section 2. We use the notation developed above and assume adapted coordinates have been chosen. Consider the derivative of the displacement function at $(\theta, 0)$. In order to identify a complement to the range of the derivative $D\delta(\theta, 0)$ in adapted coordinates, recall π_2 denotes the projection of $\mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^{n-k}$ onto $\mathbb{R}^k \times \mathbb{R}^{n-k}$ and

$$\delta(\theta, \zeta, \varepsilon) = \pi_2 \Delta^{-1}(x(\mathscr{T}(\theta, \zeta, \varepsilon), \theta, \zeta, \varepsilon)) - (\theta, \zeta).$$

We already know the kernel of $D\delta(\theta, 0)$ is $\mathbb{R}^k$ so $\delta_\theta(\theta, 0, 0) = 0$. Thus, the range is obtained as the image of $\delta_\zeta(\theta, 0, 0)$. In order to determine this image, compute

$$\begin{aligned}\delta_\zeta(\theta, 0, 0) = {} & \pi_2[D\Delta(0, \theta, 0)]^{-1}\, \dot{x}(T(\theta), \theta, 0, 0)\, \mathscr{T}_\zeta(\theta, 0, 0) \\ & + \pi_2[D\Delta(0, \theta, 0)]^{-1}\, x_\zeta(T(\theta), \theta, 0, 0) - (0, I).\end{aligned}$$

Since $T(\theta)$ is the period of the periodic solution $t \mapsto \gamma(t, \theta)$, it follows that

$$\dot{x}(T(\theta), \theta, 0, 0) \in \mathscr{E}$$

and, therefore,

$$\pi_2[D\Delta(0, \theta, 0)]^{-1} \dot{x}(T(\theta), \theta, 0, 0) \mathscr{T}_\zeta(\theta, 0, 0) = 0.$$

Moreover, in view of the block form of Φ and the equality $x(0, \theta, \zeta, \varepsilon) = \Delta(0, \theta, \zeta)$, the partial derivative

$$x_\zeta(0, \theta, 0, 0) = \Delta_\zeta(0, \theta, 0)$$

is given by

$$\begin{aligned} x_\zeta(0, \theta, 0, 0) &= (0, a(0, \theta) \Delta_\zeta(0, \theta, 0), b(0, \theta) \Delta_\zeta(0, \theta, 0)) \\ &= (0, 0, \Delta_\zeta(0, \theta, 0)) \end{aligned}$$

with respect to the splitting and, in adapted coordinates,

$$D\Delta(0, \theta, 0)\colon \mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathscr{E} \oplus \mathscr{E}^{\tan} \oplus \mathscr{E}^{\mathrm{nor}}.$$

Thus,

$$\delta_\zeta(\theta, 0, 0) = (\alpha(\theta), \beta(\theta) - I),$$

where $\alpha(\theta)\colon \mathbb{R}^{n-k} \to \mathbb{R}^k$ is defined by

$$\alpha(\theta) := \Delta_\theta^{-1}(0, \theta, 0)\, a(T(\theta), \theta)\, \Delta_\zeta(0, \theta, 0)$$

and $\beta(\theta)\colon \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$ is defined by

$$\beta(\theta) := \Delta_\zeta^{-1}(0, \theta, 0)\, b(T(\theta), \theta)\, \Delta_\zeta(0, \theta, 0).$$

These linear maps have adjoints (given by matrix transpose with respect to the usual inner product)

$$\alpha^*(\theta)\colon \mathbb{R}^k \to \mathbb{R}^{n-k}, \qquad \beta^*(\theta)\colon \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$$

and, in terms of these maps, $\delta_\zeta(\theta, 0, 0)\colon \mathbb{R}^{n-k} \to \mathbb{R}^k \oplus \mathbb{R}^{n-k}$ has adjoint

$$\delta_\zeta^*(\theta, 0, 0)\colon \mathbb{R}^k \oplus \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$$

given by

$$(v, w) \mapsto \alpha^*(\theta) v + (\beta^*(\theta) - I) w.$$

A convenient choice for the complement $\mathscr{R}^\perp(\theta)$ to the range of $\delta_\zeta(\theta, 0, 0)$ required by the Lyapunov–Schmidt reduction is

$$\mathscr{R}^\perp(\theta) = (\text{range}\, \delta_\zeta(\theta, 0, 0))^\perp = \text{kernel}\, \delta_\zeta^*(\theta, 0, 0).$$

Once $\mathscr{R}^{\perp}(\theta)$ is chosen, there is a projection $\pi^{\perp}(\theta)\colon \mathbb{R}^n \to \mathscr{R}^{\perp}(\theta)$ and an isomorphism $r^{\perp}(\theta)\colon \mathscr{R}^{\perp}(\theta) \to \mathbb{R}^k$. Then, the bifurcation function is

$$\mathscr{B}(\theta) = r^{\perp}(\theta)\, \pi^{\perp}(\theta)\, \delta_{\varepsilon}(\theta, 0, 0).$$

To determine $\delta_{\varepsilon}(\theta, 0, 0)$ we use the definition of δ, hold ζ fixed at $\zeta = 0$, and differentiate and reduce as in the computation of $\delta_{\zeta}(\theta, 0, 0)$ to obtain

$$\delta_{\varepsilon}(\theta, 0, 0) = \pi_2[D\Delta(0, \theta, 0)]^{-1}\, x_{\varepsilon}(T(\theta), \theta, 0, 0).$$

Then $\mathscr{B}(\theta) := H(\theta)\, x_{\varepsilon}(T(\theta), \theta, 0, 0)$ where $H(\theta)\colon \mathscr{E} \oplus \mathscr{E}^{\tan} \oplus \mathscr{E}^{\mathrm{nor}} \to \mathbb{R}^k$ is the linear projection given by

$$H(\theta) := r^{\perp}(\theta)\, \pi^{\perp}(\theta)\, \pi_2[D\Delta(0, \theta, 0)]^{-1}$$

with

$$\pi_2[D\Delta(0, \theta, 0)]^{-1}\colon \mathscr{E} \oplus \mathscr{E}^{\tan} \oplus \mathscr{E}^{\mathrm{nor}} \to \mathbb{R}^k \times \mathbb{R}^{n-k}$$

the projection onto the coordinate representation of the second two summands of its domain.

The partial derivative x_{ε} is the solution of the inhomogeneous variational initial value problem

$$\dot{x}_{\varepsilon} = F_x(\gamma(t, \theta), 0)x_{\varepsilon} + F_{\varepsilon}(\gamma(t, \theta), 0), \qquad x_{\varepsilon}(0, \theta, 0, 0) = 0,$$

where the initial condition is obtained by differentiating the identity $x(0, \theta, \zeta, \varepsilon) = \Delta(0, \theta, \zeta)$. Using variation of parameters with respect to the fundamental matrix Φ, we find

$$\mathscr{B}(\theta) = H(\theta) \int_0^{T(\theta)} \Phi(T(\theta), \theta)\, \Phi^{-1}(s, \theta)\, F_{\varepsilon}(\gamma(s, \theta), 0)\, ds.$$

This formula can be simplified. For this we recall $c(T(\theta), \theta) = I$ and compute

$$\Phi^{-1}(s, \theta) = \begin{pmatrix} 1 & -e(s, \theta)\, c^{-1}(s, \theta) & e(s, \theta)\, c^{-1}(s, \theta)\, a(s, \theta)\, b^{-1}(s, \theta) - d(s, \theta)\, b^{-1}(s, \theta) \\ 0 & c^{-1}(s, \theta) & -c^{-1}(s, \theta)\, a(s, \theta)\, b^{-1}(s, \theta) \\ 0 & 0 & b^{-1}(s, \theta) \end{pmatrix}$$

in order to determine

$$\Phi(T(\theta), \theta)\, \Phi^{-1}(s, \theta)$$
$$= \begin{pmatrix} 1 & L_{12}(s, \theta) & L_{13}(s, \theta) \\ 0 & c^{-1}(s, \theta) & a(T(\theta), \theta)\, b^{-1}(s, \theta) - c^{-1}(s, \theta)\, a(s, \theta)\, b^{-1}(s, \theta) \\ 0 & 0 & b(T(\theta))\, b^{-1}(s, \theta) \end{pmatrix},$$

where

$$L_{12}(s,\theta) := (e(T(\theta),\theta) - e(s,\theta))\, c^{-1}(s,\theta),$$
$$L_{13}(s,\theta) := -(e(T(\theta),\theta) - e(s,\theta))\, c^{-1}(s,\theta)\, a(s,\theta)\, b^{-1}(s,\theta) + (d(T(\theta),\theta) - d(s,\theta))\, b^{-1}(s,\theta).$$

Then, with the decomposition

$$F_\varepsilon(\gamma(s,\theta),0) := F_\varepsilon^{\mathscr{E}}(s,\theta) + F_\varepsilon^{\tan}(s,\theta) + F_\varepsilon^{\mathrm{nor}}(s,\theta),$$

and the definitions

$$a(\theta) := a(T(\theta),\theta),$$
$$b(\theta) := b(T(\theta),\theta),$$
$$\mathscr{L}(\theta) := \int_0^{T(\theta)} F_\varepsilon^{\mathscr{E}}(s,\theta) + L_{12}(s,\theta)\, F_\varepsilon^{\tan}(s,\theta) + L_{13}(s,\theta)\, F_\varepsilon^{\mathrm{nor}}(s,\theta)\, ds,$$
$$\mathscr{M}(\theta) := \int_0^{T(\theta)} b^{-1}(s,\theta)\, F_\varepsilon^{\mathrm{nor}}(s,\theta)\, ds,$$
$$\mathscr{N}(\theta) := \int_0^{T(\theta)} c^{-1}(s,\theta)\, F_\varepsilon^{\tan}(s,\theta) - c^{-1}(s,\theta)\, a(s,\theta)\, b^{-1}(s,\theta)\, F_\varepsilon^{\mathrm{nor}}(s,\theta)\, ds,$$

we obtain

$$\mathscr{B}(\theta) = H(\theta) \begin{pmatrix} \mathscr{L}(\theta) \\ \mathscr{N}(\theta) + a(\theta)\,\mathscr{M}(\theta) \\ b(\theta)\,\mathscr{M}(\theta) \end{pmatrix}.$$

This formula can also be simplified. For this recall

$$H(\theta) := r^{\perp}(\theta)\, \pi^{\perp}(\theta)\, \pi_2[D\Delta(0,\theta,0)]^{-1},$$

where

$$\pi_2[D\Delta(0,\theta,0)]^{-1} : \mathscr{E} \oplus \mathscr{E}^{\tan} \oplus \mathscr{E}^{\mathrm{nor}} \to \mathbb{R}^k \times \mathbb{R}^{n-k}$$

represents projection onto the second two summands of its domain. It follows that $\mathscr{B}(\theta)$ does not depend on $\mathscr{L}(\theta)$. Next, note

$$\begin{pmatrix} \mathscr{N}(\theta) + a(\theta)\,\mathscr{M}(\theta) \\ b(\theta)\,\mathscr{M}(\theta) \end{pmatrix} = \begin{pmatrix} \mathscr{N}(\theta) + a(\theta)\,\mathscr{M}(\theta) \\ \mathscr{M}(\theta) + (b(\theta) - I)\,\mathscr{M}(\theta) \end{pmatrix}$$

and

$$\begin{pmatrix} 0 \\ a(\theta)\,\mathscr{M}(\theta) \\ (b(\theta) - I)\,\mathscr{M}(\theta) \end{pmatrix} \in \mathscr{R}(\theta).$$

Thus, since $H(\theta)$ is the coordinate representation of the projection onto $\mathscr{R}^{\perp}(\theta)$,

$$\mathscr{B}(\theta) = H(\theta) \begin{pmatrix} 0 \\ \mathcal{N}(\theta) \\ \mathcal{M}(\theta) \end{pmatrix}$$

as required. ∎

4. Applications

4.1. *The Single Forced Oscillator*

A prototypical application for our reduction method is the bifurcation of subharmonics in forced oscillators, cf. [5]. For this consider a differential equation

$$\dot{x} = f(x) + \varepsilon g(t, \omega), \qquad x \in \mathbb{R}^2, \quad \varepsilon \in \mathbb{R}, \quad \omega \in \mathbb{R},$$

where $t \mapsto g(t, \omega)$ is periodic with period $2\pi/\omega$ and such that the unforced system has a periodic solution Λ with period $\tau > 0$. To use our results, we study the forced oscillator as a first order system $\mathcal{O}_{\omega,\varepsilon}$

$$\dot{x} = f(x) + \varepsilon g(\varphi, \omega),$$
$$\dot{\varphi} = \omega \bmod 2\pi.$$

Here $\mathcal{O}_{\omega,\varepsilon}$ is defined on $\mathbb{R}^2 \times \mathbb{S}^1$ rather than on $\mathbb{R}^3$, where we view $\mathbb{S}^1$ as the interval $[0, 2\pi/\omega]$ with its end points identified. In this language, $\varphi \mapsto g(\varphi, \omega)$ is a map $\mathbb{S}^1 \to \mathbb{R}^2$, where the angular coordinate φ is related to the time variable via

$$e^{i\omega t} = e^{i\varphi}.$$

However, as is easily seen, the introduction of an angular variable does not change our reduction method. We also allow for the possibility that the frequency ω is a function of the amplitude ε. In particular, we assume the frequency of the forcing can be expressed as

$$\omega = \omega_0 + \omega_1 \varepsilon + O(\varepsilon^2)$$

and the period of the forcing as

$$\frac{2\pi}{\omega} = \eta + k\varepsilon + O(\varepsilon^2).$$

In particular, we have

$$\eta = \frac{2\pi}{\omega_0} \qquad k = -\frac{2\pi\omega_1}{\omega_0^2}.$$

We also assume the unforced oscillation Λ is in resonance with the forcing when $\varepsilon = 0$; i.e., there are relatively prime positive integers n and m such that $n\tau = m\eta$. In this case k is called a *detuning* parameter.

From the resonance condition it follows that the torus $\mathscr{A} := \Lambda \times \mathbb{S}^1$ is a 2-dimensional period manifold for $\mathcal{O}_{\omega,0}$ in $\mathbb{R}^2 \times \mathbb{S}^1$. Let Γ denote a periodic solution on $\mathscr{A}$. To obtain appropriate coordinates, let θ denote an angular coordinate on Λ, let Σ denote the Poincaré section in $\mathbb{R}^2 \times \mathbb{S}^1$ given by

$$\Sigma := \mathbb{R}^2 \times \{0\},$$

and let

$$t \mapsto (x(t, \theta), \varphi(t))$$

denote the solution of $\mathcal{O}_{\omega,0}$ with initial condition $(x(0, \theta), \varphi(0)) = (\theta, 0)$. We will use the notation $f(t, \theta) := f(x(t, \theta))$. Also, for vectors $v = (v_1, v_2)$ and $w = (w_1, w_2)$ in $\mathbb{R}^2$, we define $v^\perp := (-v_2, v_1)$, $v \wedge w := v_1 w_2 - v_2 w_1$ and $< v, w > := v_1 w_1 + v_2 w_2$. Then, a (bundle) splitting along $\mathscr{A}$ is given by

$$\begin{aligned}(\mathscr{E} \oplus \mathscr{E}^{\tan} \oplus \mathscr{E}^{\text{nor}})(t, \theta) \\ = \left[\begin{pmatrix} f(t, \theta) \\ \omega_0 \end{pmatrix}\right] \oplus \left[\begin{pmatrix} f(t, \theta) \\ 0 \end{pmatrix}\right] \oplus \left[\begin{pmatrix} (\|f\|^{-2} f^\perp)(t, \theta) \\ 0 \end{pmatrix}\right],\end{aligned}$$

where the square bracket denotes the vector subspace generated in $\mathbb{R}^3$ by the enclosed vector. The normalization of the generator of $\mathscr{E}^{\text{nor}}$ is chosen to conform with the notation of [5]. This choice will simplify some of the formulas to follow.

The key result used to obtain the identification of the bifurcation function is

THEOREM 4.1 (Diliberto's Theorem [5, 8]). *If $\dot{x} = f(x)$, $x \in \mathbb{R}^2$, $f(p) \neq 0$, and $t \mapsto x(t, p)$ is the solution of the differential equation such that $x(0, p) = p$, then the homogeneous variational equation*

$$\dot{W} = Df(x(t, p))\, W$$

has a fundamental matrix solution $t \mapsto \Psi(t)$

$$\Psi(t) = \begin{pmatrix} 1 & \alpha(t, p) \\ 0 & \beta(t, p) \end{pmatrix}$$

with respect to the moving frame

$$\langle\langle f(t,p), \|f(t,p)\|^{-2} f^{\perp}(t,p)\rangle\rangle,$$

where

$$\beta(t,p) = \exp \int_0^t \operatorname{div} f(s,p)\, ds,$$

$$\alpha(t,p) = \int_0^t \left\{ \frac{1}{\|f\|^2} (2\kappa \|f\| - \operatorname{curl} f)\beta \right\} (s,p)\, ds$$

and κ denotes the signed scalar curvature

$$\kappa(t,p) := \frac{1}{\|f(t,p)\|^3} f(t,p) \wedge Df(t,p) f(t,p).$$

By an application of Diliberto's theorem we obtain the fundamental matrix solution of the homogeneous variational equation for $\mathscr{C}_{\omega,0}$ in the form

$$\Phi(t,\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \alpha(t,\theta) \\ 0 & 0 & \beta(t,\theta) \end{pmatrix}.$$

In the adapated coordinates the range of the derivative of the displacement is given by

$$\mathscr{R}(\theta) = \left[\alpha(m\eta,\theta) \begin{pmatrix} f(0,\theta) \\ 0 \end{pmatrix} + (\beta(m\eta,\theta) - 1) \begin{pmatrix} \|f(0,\theta)\|^{-2} f^{\perp}(0,\theta) \\ 0 \end{pmatrix} \right].$$

The period manifold $\mathscr{A}$ will be normally nondegenerate when this range is constantly one dimensional, i.e., when the function

$$\theta \mapsto (\alpha(m\eta,\theta))^2 + (\beta(m\eta,\theta) - 1)^2$$

never vanishes. If this is the case, a complement to $\mathscr{R}$ is given by

$$\mathscr{R}^{\perp}(\theta) = \left[(1 - \beta(m\eta,\theta)) \begin{pmatrix} f(0,\theta) \\ 0 \end{pmatrix} + \alpha(m\eta,\theta) \begin{pmatrix} \|f(0,\theta)\|^{-2} f^{\perp}(0,\theta) \\ 0 \end{pmatrix} \right].$$

For notational convenience we write $G(t) := g(\varphi(t), \omega_0)$. Then, in terms of the splitting, we compute

$$F_{\varepsilon}^{\tan}(t,\theta) = -\frac{\omega_1}{\omega_0} + \frac{1}{\|f(t,\theta)\|^2} \langle f(t,\theta), G(t)\rangle,$$

$$F_{\varepsilon}^{\mathrm{nor}}(t,\theta) = \langle f^{\perp}(t,\theta), G(t)\rangle = f(t,\theta) \wedge G(t),$$

$$\mathcal{N}(\theta) = \int_0^{m\eta} -\frac{\omega_1}{\omega_0} + \frac{1}{\|f(s,\theta)\|^2} \langle f(s,\theta), G(s) \rangle - \frac{\alpha(s,\theta)}{\beta(s,\theta)} f(s,\theta) \wedge G(s)\, ds$$

$$= mk + \int_0^{m\eta} \frac{1}{\|f(s,\theta)\|^2} \langle f(s,\theta), G(s) \rangle - \frac{\alpha(s,\theta)}{\beta(s,\theta)} f(s,\theta) \wedge G(s)\, ds,$$

$$\mathcal{M}(\theta) = \int_0^{m\eta} \frac{1}{\beta(s,\theta)} f(s,\theta) \wedge G(s)\, ds$$

and define

$$\mathcal{N}_0(\theta) := \int_0^{m\eta} \frac{1}{\|f(s,\theta)\|^2} \langle f(s,\theta), G(s) \rangle - \frac{\alpha(s,\theta)}{\beta(s,\theta)} f(s,\theta) \wedge G(s)\, ds.$$

Thus, the bifurcation function is given by

$$\mathcal{B}(\theta) = \frac{(1\text{-}\beta(m\eta,\theta))\,\mathcal{N}(\theta) + \alpha(m\eta,\theta)\,\mathcal{M}(\theta)}{(\alpha(m\eta,\theta))^2 + (1\text{-}\beta(m\eta,\theta))^2}.$$

Since $\mathcal{B}$ has the same simple zeros as the *normalized bifurcation function*

$$\mathcal{D}(\theta) = (1\text{-}\beta(m\eta,\theta))\,\mathcal{N}(\theta) + \alpha(m\eta,\theta)\,\mathcal{M}(\theta)$$

we have the following fact: *Consider the forced oscillator $\mathcal{O}_{\omega,\varepsilon}$ and suppose θ denotes an angular coordinate on an unperturbed limit cycle Λ whose period $\tau > 0$ is in $(m{:}n)$ resonance with the period $\eta > 0$ of the forcing. If the corresponding function $\theta \mapsto (1-\beta(m\eta,\theta))^2 + (\alpha(m\eta,\theta))^2$ never vanishes, then the simple zeros of the normalized bifurcation function on Λ are (subharmonic) branch points of periodic solutions; i.e., if $\mathcal{D}(\theta_0) = 0$ and $\mathcal{D}'(\theta_0) \neq 0$, then the point on Λ with angular coordinate θ_0 is a (subharmonic) branch point.* For additional results of this type and especially for the bifurcation analysis in more degenerate cases see [5].

The formula for the bifurcation function makes clear the roles of the functions α, β, $\mathcal{M}$, and $\mathcal{N}$. Here, $\mathcal{M}$ and $\mathcal{N}$ are the components of the variational derivative of the vector field in the direction of the bifurcation parameter. In the literature $\mathcal{M}$ is usually called the (subharmonic) Melnikov function after [16], but this variational derivative arises frequently in other work. The function β is the derivative of the section map of the unperturbed system defined on a section orthogonal to Λ at θ with range in an orthogonal section at $\gamma(t,\theta)$. In the resonant forced oscillator with $n\tau = m\eta$, we have

$$\beta(m\eta,\theta) = \beta(n\tau,\theta) = \beta^n(\tau,\theta),$$

where $\beta(\tau, \theta)$ is the derivative of the Poincaré map on the orthogonal section at θ. It is easy to see $\beta(\tau, \theta)$ is independent of θ. Therefore, we write

$$\beta_{\Lambda} := \beta(\tau, \theta).$$

Of course, β_{Λ} is just the characteristic multiplier of the unperturbed periodic solution. The function α is a normalized derivative of the transit time between the same orthogonal sections. More precisely, for the orthogonal Poincaré section at θ given by the integral curve of $f^{\perp}$, the derivative of the transit time is $-\|f\|\, \alpha(\tau, \theta)$, cf. [5, Theorem 2.2]. Also, we have

$$\alpha(m\eta, \theta) = \alpha(n\tau, \theta) = n\alpha(\tau, \theta).$$

In particular, if Λ is a member of a one parameter family of periodic solutions of the unperturbed system, then $-\|f\|\, \alpha(\tau, \theta)$ is the derivative of the period function for the family. With these identifications, we see that if Λ is hyperbolic, then $\beta_{\Lambda} \neq 1$ and $\mathscr{A}$ is normally nondegenerate (in fact it is normally hyperbolic). On the other hand, if Λ is not hyperbolic, then the condition $\alpha(\tau, \theta) \neq 0$ implies $\mathscr{A}$ is normally nondegenerate. In particular, when Λ is a member of a one parameter family of periodic solutions of the unperturbed system, the condition $\alpha(\tau, \theta) \neq 0$ is equivalent to the non-vanishing of the derivative of the period function at Λ. Thus, the normal nondegeneracy reduces to the the usual nondegeneracy condition for the subharmonic Melnikov theory [6, 12, 25].

As an application for the above results on the existence of subharmonic branch points consider the phenomenon of frequency entrainment. For this we study the forced oscillator $\mathcal{O}_{\omega,\varepsilon}$, but here we assume the unperturbed system, $\mathcal{O}_{\omega,0}$, has a hyperbolic limit cycle Λ with period $\tau := 2\pi/\Omega$ in $(m{:}n)$-resonance with the forcing, i.e., there are relatively prime positive integers n and m such that $n\tau = m\eta$. An entrainment domain for $\mathcal{O}_{\omega,\varepsilon}$ is the subset of the (ω, ε) parameter space corresponding to those oscillators which have a periodic solution in resonance with the forcing. To be more precise, note first that the period manifold $\mathscr{A} = \Lambda \times \mathbb{S}^1$ is an invariant torus in the extended phase space $\mathbb{R}^2 \times \mathbb{S}^1$ for the unperturbed system which inherits normal hyperbolicity from the limit cycle Λ. Thus, this invariant torus will persist for small values of the bifurcation parameter. Moreover, all the solutions of the unforced oscillator starting on Λ in the Poincaré section corresponding to $\varphi = 0$ will return to the same point of the section after winding around the invariant torus $\mathscr{A}$ exactly n times. The *entrainment domain* $\mathscr{F}_{m,n}$ is the set of all (ω, ε) such that $\mathcal{O}_{\omega,\varepsilon}$ has a periodic solution that has period $m\eta$ and that wraps around the perturbed invariant torus n times. Such a solution is said to be entrained (or phase locked) to the $(m{:}n)$-resonance. Equivalently stated in the language of circle maps, the solution is entrained if its return to the Poincaré section (modulo η) has rotation number n/m. The underlying theory for the existence of entrainment domains is well known [3, 4, 13, 17, 20] and many applications have been studied [2, 14, 15, 21, 22]. Our goal is to introduce some new exact formulas for the widths of these entrainment domains as they approach the ω-axis; i. e., we study the widths of the "Arnold tongues." We note that our formulas are valid for the case of fully nonlinear unperturbed systems.

In order to obtain the formulas for the tangents of the entrainment domains as they approach the ω-axis, consider the resonant point on this axis with coordinates $(\omega_0, 0)$ and a path $\gamma(\varepsilon) = (\omega(\varepsilon), \varepsilon)$ in the parameter space where, as above,

$$\omega(\varepsilon) = \omega_0 + \omega_1 \varepsilon + O(\varepsilon^2).$$

Entrainment will occur if, for sufficiently small $|\varepsilon|$, the image of γ lies in $\mathscr{F}_{m,n}$. However, we have just seen that this will occur when the normalized bifurcation function has a simple zero. Using this fact it is a simple matter to obtain the formulas for the tangents to the entrainment domain at the resonant point on the frequency axis. For this we define the *subharmonic bifurcation function* (cf. [5])

$$\mathscr{C}(\theta) := (1 - \beta(m\eta, \theta))\, \mathscr{N}_0(\theta) + \alpha \mathscr{M}(\theta)$$

and note that the normalized bifurcation function is given by

$$\mathscr{D}(\theta) = mk(1 - \beta(m\eta, \theta)) + \mathscr{C}(\theta).$$

Of course, $\mathscr{C}$ is just the normalized bifurcation function for $\mathscr{O}_{\omega_0, \varepsilon}$. Moreover, it is easy to see that $\mathscr{C}$ is a periodic function of θ with period corresponding to the choice of parameterization on Λ. To obtain a convenient parameterization, consider the flow $t \mapsto \phi_t$ of the unperturbed system and choose a point $\xi \in \Lambda$. Then $s \mapsto \mathscr{C}(\phi_s(\xi))$ is periodic of period τ. The subharmonic bifurcation function $\mathscr{C}$ does not depend on the detuning parameter k, but since $\mathscr{C}$ depends on the forcing function, it does depend implicitly on m and n. More importantly, detuning can be viewed geometrically as vertical translation of the graph of $u = \mathscr{C}(\phi_s(\xi))$. Thus, there are two critical values of the detuning given by

$$k_{\max} := -\frac{1}{m(1 - \beta_\Lambda^n)} \min_{\{s \mid 0 \leq s \leq \tau\}} \mathscr{C}(\phi_s(\xi)),$$

$$k_{\min} := -\frac{1}{m(1 - \beta_\Lambda^n)} \max_{\{s \mid 0 \leq s \leq \tau\}} \mathscr{C}(\phi_s(\xi)).$$

In effect, if $k_{\min} < k < k_{\max}$, then $\mathscr{D}$ has a zero along Λ. If this zero is simple, our path $\varepsilon \mapsto \gamma(\varepsilon)$ will lie in the entrainment domain. In terms of the frequency, we will have entrainment if the first order coefficient ω_1 of the forcing frequency satisfies the inequality

$$\frac{\omega_0^2}{2\pi m(1-\beta_\Lambda^n)} \mathscr{C}_{\min} < \omega_1 < \frac{\omega_0^2}{2\pi m(1-\beta_\Lambda^n)} \mathscr{C}_{\max}.$$

Since the slope of the tangent to our path at $\varepsilon = 0$ in the parameter space is just ω_1^{-1}, it follows that the tangents at the 'tongue' of the entrainment domain at the resonant point $(\omega_0, 0)$ are given by

$$\varepsilon = \frac{2\pi m(1-\beta_\Lambda^n)}{\mathscr{C}_{\min}} \omega_0^2(\omega - \omega_0)$$

$$\varepsilon = \frac{2\pi m(1-\beta_\Lambda^n)}{\mathscr{C}_{\max}} \omega_0^2(\omega - \omega_0).$$

In addition, if $\mathscr{C}$ has a simple zero, then the corresponding tongue opens with a nonzero angle between its tangents. On the other hand, if $\mathscr{C}(\theta) \equiv 0$, then these tangents coincide (they both have infinite slope) and the order of contact of the boundary curves must be determined from higher order methods. This is done in [17] for certain *weakly* nonlinear systems.

This analysis also gives a clear geometric picture of the bifurcation at the crossings of the boundaries of the entrainment domain. As we have just seen, the detuning translates the graph of the subharmonic bifurcation function in the vertical direction. When a boundary of the entrainment domain is crossed near the resonant point on the frequency axis, the translated graph of the subharmonic bifurcation function will be tangent to the frequency axis at the θ-coordinate of either its maximum or minimum value. Thus, the position of θ giving the subharmonic branch point on Λ will be the θ-coordinate of the minimum of $\mathscr{C}$ at the left boundary of the tongue and the θ-coordinate of the maximum of $\mathscr{C}$ at the right boundary. In case the limit cycle can be parametrized by the phase angle in the phase plane, the phase of the entrained periodic solution will shift from the phase angle of the minimum of $\mathscr{C}$ at the left boundary of the tongue to the phase angle of the maximum of $\mathscr{C}$ at the right boundary of the tongue. For example, this would be observable by fixing the amplitude of the forcing and then adjusting the frequency of the forcing so as to cross the tongue in the direction of increasing frequency. Of course, since crossing a boundary of the tongue corresponds to the passage of the minimum or the maximum of the graph of $u = \mathscr{C}(\phi_s(\theta))$ by vertical translation past the s-axis, we see that the graph crosses the s-axis twice near its maximum or its minimum. As the detuning changes in the appropriate direction, these two crossings coalesce and then disappear. In other words, there is (generically) a saddle node bifurcation as the boundaries of the tongue are crossed by changing the frequency parameter.

We now illustrate typical computations of the infinitesimal boundaries of the entrainment domains. Our first example is provided by the system

$$\dot{x} = -y + x(1 - x^2 - y^2)$$
$$\dot{y} = x + y(1 - x^2 - y^2) + \varepsilon \cos(\omega t),$$

which is chosen so that explicit formulas can be obtained. Here, the unperturbed system has the unit circle as a limit cycle. In fact, the solution starting at $(\cos\theta, \sin\theta)$ is given by

$$x(t) = \cos(t + \theta), \qquad y(t) = \sin(t + \theta).$$

If we take $\omega(\varepsilon) = m/n + \omega_1 \varepsilon$, then the period of the forcing will be

$$\frac{2\pi}{\omega(\varepsilon)} = \frac{2\pi n}{m} - 2\pi \left(\frac{n}{m}\right)^2 \omega_1 \varepsilon + O(\varepsilon^2).$$

We compute $\alpha \equiv 0$,

$$\begin{aligned}\mathscr{D}(\theta) &= (1 - \beta(2\pi n))mk + (1 - \beta(2\pi n))\, \mathscr{N}_0(\theta, 2\pi n) \\ &= (1 - e^{-4\pi n})mk + (1 - e^{-4\pi n}) \int_0^{2\pi n} \cos(t + \theta) \cos(\omega t)\, dt\end{aligned}$$

and, for $\omega > 0$,

$$\begin{aligned}\mathscr{C}(\theta) &= (1 - e^{-4\pi n}) \int_0^{2\pi n} \cos(t + \theta) \cos(\omega t)\, dt \\ &= (1 - e^{-4\pi n}) \\ &\times \begin{cases} \pi n \cos\theta, & \omega = 1 \\ \dfrac{1}{\omega^2 - 1}\left(\sin\theta + \dfrac{\omega + 1}{2}\sin(2\pi n\omega - \theta) + \dfrac{\omega - 1}{2}\sin(2\pi n\omega + \theta)\right), & \omega \neq 1. \end{cases}\end{aligned}$$

At resonance $\omega = m/n$. So

$$\mathscr{C}(\theta) = \begin{cases} \pi n \cos\theta, & m = n \\ 0, & m \neq n. \end{cases}$$

Thus, for $m = n$, the $(1{:}1)$-resonance, we compute the tangents at the resonant point $(\omega, \varepsilon) = (1, 0)$ to be

$$\varepsilon = \pm 2(\omega - 1)$$

while, for the case $m \neq n$, the tangents have infinite slope.

The phase shift mentioned above is also easy to see in this example. The phase angle is θ and, for $m=n$, the detuning simply translates the graph of $\pi n \cos(\theta)$. Thus, at the left boundary of the tongue, the phase of the entrained solution will be near $\theta=\pi$ while it will be near $\theta=0$ at the right hand boundary.

The next example shows our formulas can be used in conjunction with standard perturbation techniques to obtain estimates of the width of tongues when the unperturbed system is not explicitly integrable. For this we consider the forced van der Pol oscillator in the form

$$\ddot{x}+\delta(x^2-1)\dot{x}+x=\varepsilon\cos(\omega t).$$

For small δ, the second order Poincaré–Lindstedt approximation for the unperturbed limit cycle is given by [23]

$$\begin{aligned} x(t) &= 2\cos s+(\tfrac{3}{4}\sin s-\tfrac{1}{4}\sin 3s)\delta \\ &\quad +(-\tfrac{1}{8}\cos s+\tfrac{3}{16}\cos 3s-\tfrac{5}{96}\cos 5s)\delta^2+O(\delta^3), \end{aligned}$$

where $s=(1-\delta^2/16+O(\delta^4))t$. Also, the approximate period of the limit cycle is given by $\tau:=2\pi(1+\delta^2/16)+O(\delta^4)$. These approximations are valid on the time scale of one period of the limit cycle. To obtain an approximation of $\mathscr{C}$, we approximate the phase plane parameterization of Λ by

$$\theta\longmapsto(x(t+\theta),\dot{x}(t+\theta)),$$

insert the resulting formulas into $\mathscr{C}$, and, using an algebraic processor, compute an expression of the form

$$\mathscr{C}(\theta)=c_1(\theta)\delta+c_2(\theta)\delta^2+O(\delta^3).$$

This approximation vanishes unless $m=n$ or $m=3n$ reflecting the resonances that appear in the approximation of the limit cycle and the order of the approximation. For these resonances, we find

$$\beta(n\tau)=1-2n\pi\delta+2n^2\pi^2\delta^2+O(\delta^3).$$

Also, for $m=n$

$$\begin{aligned} \mathscr{C}(\theta) = &-(n^2\pi^2\cos\theta)\delta+\tfrac{1}{8}n^2\pi^2(\sin 3\theta-3\sin 5\theta+\sin\theta \\ &+8n\pi\cos\theta+4\sin\theta\cos 2\theta+6\sin\theta\cos 4\theta)\delta^2+O(\delta^3) \end{aligned}$$

and for $m=3n$

$$\mathscr{C}(\theta)=-\tfrac{1}{8}(n^2\pi^2\sin 3\theta)\delta^2.$$

In order to approximate the extrema of $\mathscr{C}$ in case $n=m$, we note that the extrema of $\theta \mapsto \mathscr{C}(\theta)/\delta$ at $\delta=0$ occur at $\theta=0$ and $\theta=\pi$. The perturbed extrema are then found from $\mathscr{C}'(\theta)=0$ by series expansion. We find for $m=n$

$$\mathscr{C}_{\min} = -n^2\pi^2\delta + n^3\pi^3\delta^2 + O(\delta^3), \qquad \mathscr{C}_{\max} = n^2\pi^2\delta - n^3\pi^3\delta^2 + O(\delta^3),$$

while for $m=3n$

$$\mathscr{C}_{\min} = -\tfrac{1}{8}n^2\pi^2\delta^2 + O(\delta^3), \qquad \mathscr{C}_{\max} = \tfrac{1}{8}n^2\pi^2\delta^2 + O(\delta^3).$$

Inserting these expressions into the formulas for the tangent lines of the tongues we obtain a $O(\delta^4)$ approximation for $m=n$

$$\varepsilon = \pm(4+\tfrac{1}{2}\delta^2)(\omega-(1-\tfrac{1}{16}\delta^2)),$$

while for $m=3n$ the approximation is $O(\delta^3)$

$$\varepsilon = \pm\left(\frac{32}{3}\delta^{-1} - \frac{32n\pi}{3} + \frac{4}{3}\delta - \frac{4n\pi}{3}\delta^2\right)\left(\omega - 3\left(1-\frac{1}{16}\delta^2\right)\right).$$

It is interesting to compare these approximations with numerical computations of $\mathscr{C}$. Our numerical algorithm uses a variable step variable order Adams ODE solver and Simpson's rule. For $\delta=0.1$ and $m=n$ the tangents (rounded to three decimal places) obtained from this numerical method are

$$\varepsilon = -3.996(\omega-1.000)$$
$$\varepsilon = 3.995(\omega-1.000).$$

Substitution in the series gives the tangents

$$\varepsilon = \pm 4.005(\omega-0.999).$$

For $m=3n$ the tangents obtained from the numerical algorithm are

$$\varepsilon = -73.206(\omega-3.000)$$
$$\varepsilon = 73.255(\omega-3.000).$$

Substitution in the series gives the tangents

$$\varepsilon = \pm 73.248(\omega-2.998).$$

Of course, the accuracy of these computations can be improved and higher order resonances can be studied by starting with a higher order Poincaré–Lindstedt approximation. For such approximations see [1, 7].

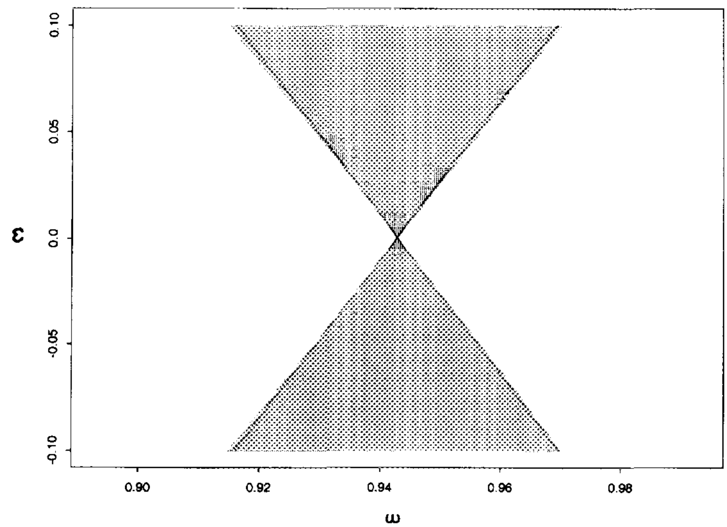

FIG. 1. The entrainment domain for the (1:1)-lock in the system $\ddot{x}+(x^2-1)+x=\varepsilon\cos(\omega t)$. The horizontal axis represents the frequency and the vertical axis the amplitude (for the range $[-0.1, 0.1]$ of the forcing. The shaded region is the computed entrainment domain while the solid lines are the tangents to the entrainment domain at zero amplitude computed from the bifurcation function. The equations for these tangents are given approximately by $\varepsilon=3.7400(\omega-0.9430)$ and $\varepsilon=-3.7402(\omega-0.9430)$.

Finally, we have conducted numerical experiments to obtain some indications of how well the tangents to the tongues computed from the formulas of this paper compare with the actual shape of the tongues. It turns out that the agreement is quite good. A typical example is illustrated in Fig. 1.

4.2. *Coupled Oscillators*

A natural and important generalization of the single forced oscillator is a system of coupled oscillators. For this application consider the system E_ε of differential equations given by

$$\dot{x}_k=f_k(x_k)+\varepsilon g_k(x_1, x_2, ..., x_N, \varepsilon), \qquad x_k\in\mathbb{R}^{m_k}, \quad \varepsilon\in\mathbb{R}, \quad k=1, ..., N.$$

Here $f_k\colon \mathbb{R}^{m_k}\to\mathbb{R}^{m_k}$ and $g_k\colon\mathbb{R}^{M+1}\to\mathbb{R}^{m_k}$ are smooth functions and $M:=\sum_{k=1}^N m_k$. Let E_ε^k denote the kth equation of the system and let ϕ_t^k denote the flow of E_0^k. If each E_0^k, $k=1, ..., N$ has a periodic trajectory Γ^k with period $\eta_k>0$ and frequency $\omega_k:=2\pi/\eta_k$, we say E_ε^k is a *system of coupled oscillators.* If, in addition, there are relatively prime positive integers $K_1, ..., K_N$ and a number $\eta>0$ such that $\eta=K_k\eta_k$ for $k=1, ..., N$

we say the system of coupled oscillators is in *resonance*. If E_ε is in resonance and if $\xi_k \in \Gamma_k$ for each $k = 1, ..., N$, then

$$t \mapsto (\phi_t^1(\xi_1), ..., \phi_t^N(\xi_N))$$

defines a periodic solution Γ of the unperturbed system E_0 whose period is η. This periodic solution lies in the period manifold $\mathscr{A} := \Gamma_1 \times \cdots \times \Gamma_N$, an N dimensional torus in $\mathbb{R}^M$. The function $\phi\colon \mathbb{R} \times \mathbb{R}^{N-1} \to \mathscr{A}$ given by

$$\phi(s, \theta_2, ..., \theta_N) = (\phi_s^1(\xi_1), \phi_{s+\theta_2}^2(\xi_2), ..., \phi_{s+\theta_N}^N(\xi_N))$$

defines an adapted angular coordinate system on $\mathscr{A}$. The distinguished first coordinate serves to define a Poincaré section for the unperturbed flow on $\mathscr{A}$, viz.,

$$\Sigma^{\tan} := \{\phi(0, \theta_2, ..., \theta_N) \mid (\theta_2, ..., \theta_N) \in \mathbb{R}^{N-1}\}.$$

Using the definition $f_k(s, \theta) := f_k(\phi(s, \theta_2, ..., \theta_N))$, we have the first two summands of the (bundle) splitting for $\mathbb{R}^M$ given by

$$\mathscr{E}(s, \theta) := \left[\begin{pmatrix} f_1(s, \theta) \\ \vdots \\ f_N(s, \theta) \end{pmatrix}\right],$$

$$\mathscr{E}^{\tan}(s, \theta) := \left[\begin{pmatrix} 0 \\ f_2(s, \theta) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \cdots \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f_N(s, \theta) \end{pmatrix}\right].$$

Of course, $\mathscr{E}^{\tan}$ is just the image of $\mathbb{R}^N$ generated by

$$\{\partial/\partial\theta_2, ... \partial/\partial\theta_N\}$$

under the linear transformation given by the derivative $D\phi(s, \theta)\colon \mathbb{R}^N \to T_\theta\mathscr{A}$, where $T_\theta\mathscr{A}$ denotes the tangent space of $\mathscr{A}$ at θ.

To obtain $\mathscr{E}^{\text{nor}}$, note for each $k = 1, ..., N$ there are $m_k - 1$ mutually orthogonal vector fields each orthogonal to f_k and each defined in a neighborhood of Γ_k which form an ordered normalized frame

$$f_k^\perp := \langle\langle \|f_{k1}\|^{-2} f_{k1}^\perp, ..., \|f_{k,m_k-1}\|^{-2} f_{k,m_k-1}^\perp \rangle\rangle.$$

Using this collection of frames we define $\mathscr{E}^{\text{nor}}$ as the family of $M - N$ dimensional subspaces given by

$$\mathscr{E}^{\text{nor}}(s, \theta) := [f_1^\perp(s, \theta), ..., f_N^\perp(s, \theta)].$$

Each normal field has an associated flow, ψ_t^{kj} in $\mathbb{R}^{m_k}$. These serve to define for each $k=1, ..., N$ a map $\Psi^k: \mathbb{R}^{m_k-1} \times \Gamma_k \to \mathbb{R}^{m_k}$ given by

$$\Psi^k(\zeta_k, \xi_k) := (\psi_{\zeta_{k1}}^{k1}(\xi_k), ..., \psi_{\zeta_{k,m_k-1}}^{k,m_k-1}(\xi_k)).$$

Then, for $\xi_k \in \Gamma_k$, $k=1, ..., N$, adapted coordinates

$$\Delta: \mathbb{R} \times \mathbb{R}^{N-1} \times \mathbb{R}^{M-N} \to \mathbb{R}^M$$

are given by

$$\Delta(s, \theta, \zeta) := (\Psi^1(\zeta_1, \phi_s^1(\xi_1)), \Psi^2(\zeta_2, \phi_{s+\theta_2}^2(\xi_2)), ..., \Psi^N(\zeta_N, \phi_{s+\theta_N}^N(\xi_N))).$$

The identification of the bifurcation function depends on the matrix representation, with respect to the bases defined above, of the fundamental matrix solution $\Phi(t, \theta)$ of the homogeneous variational equation for the unperturbed system. This matrix has the block form

$$\Phi(t, \theta) = \begin{pmatrix} 1 & 0 & d(t, \theta) \\ 0 & I & \alpha(t, \theta) \\ 0 & 0 & \beta(t, \theta) \end{pmatrix},$$

where I is the $(N-1) \times (N-1)$ identity, $\alpha(t, \theta)$ is $(N-1) \times (M-N)$, $\beta(t, \theta)$ is $(M-N) \times (M-N)$, and $d(t, \theta)$ is $1 \times (M-N)$, where these blocks can be partitioned as

$$d(t, \theta) = (\alpha_{11}(t, \theta), 0, ..., 0)$$

with $\alpha_{11}(t, \theta)$ a $1 \times (m_k - 1)$ matrix,

$$\alpha(t, \theta) = \begin{pmatrix} -\alpha_{11}(t, \theta) & \alpha_{22}(t, \theta) & 0 & 0 & \cdots & 0 & 0 \\ -\alpha_{11}(t, \theta) & 0 & \alpha_{33}(t, \theta) & 0 & \cdots & 0 & 0 \\ \vdots & & & & & & \\ -\alpha_{11}(t, \theta) & 0 & & 0 & \cdots & 0 & \alpha_{NN}(t, \theta) \end{pmatrix},$$

with $\alpha_{kj}(t, \theta)$ a $1 \times (m_k - 1)$ matrix and

$$\beta(t, \theta) = \begin{pmatrix} \beta_{11}(t, \theta) & & \\ & \ddots & \\ & & \beta_{NN}(t, \theta) \end{pmatrix}$$

a block diagonal matrix with $\beta_{kk}(t, \theta)$ an $(m_k - 1) \times (m_k - 1)$ matrix.

For general coupled oscillators it is not clear how to obtain useful representations of the components of α and β. However, as we have seen in the case of the periodically forced oscillator, reasonable formulas can be obtained if each uncoupled oscillator has a two dimensional phase space. We will restrict ourselves to the case below. Here we make general remarks about the meaning of these operators. In particular, we know the period manifold $\mathscr{A}$ is normally nondegenerate when the range of the map $V: \mathbb{R}^{M-N} \to \mathbb{R}^{N-1} \times \mathbb{R}^{M-N}$ given by

$$V(u) = (\alpha(\eta, \theta)u, (\beta(\eta, \theta) - I)u)$$

has dimension $M-N$. The obvious sufficient condition for normal nondegeneracy is the requirement that $\beta(\eta, \theta): \mathbb{R}^{M-N} \to \mathbb{R}^{M-N}$ does not have 1 as an eigenvalue. For example, this is the case if Γ_k is a hyperbolic limit cycle for each $k = 1, ..., N$. Of course, as is easily seen from the structure of V, it is possible for the period manifold to be normally nondegenerate when some of the individual oscillators are not hyperbolic. This occurs, as we have seen, for the periodically forced oscillator. In general, the lack of hyperbolicity can be compensated by the nonvanishing of appropriate derivatives of the transit time maps. Specifically, consider the action of the monodromy transformation $\Phi(\eta, \theta)$ on the oscillator E_0^k. If w is orthogonal to $f_k(\theta_k)$ in $\mathbb{R}^{m_k - 1}$ at $\theta_k \in \Gamma_k$, then w determines an element in $\mathscr{E}(0, \theta) \oplus \mathscr{E}^{\tan}(0, \theta) \oplus \mathscr{E}^{\mathrm{nor}}(0, \theta)$ given by $\mathscr{W}_0 := (0, 0, W)$. Here $W \in \mathbb{R}^{M-N}$, where $W := (w_1, ..., w_N)$ with $w_j \in \mathbb{R}^{m_k - 1}$ defined by $w_j = 0$ if $j \neq k$ and $w_k = w$. The monodromy transformation applied to $\mathscr{W}$ has image $\mathscr{W} := (X, Y, Z)$, where $X \in \mathbb{R}$, $Y = (y_1, ..., y_N)$, $y_k \in \mathbb{R}^{m_k - 1}$, and $Z = (z_1, ..., z_N)$, $z_k \in \mathbb{R}^{m_k - 1}$. In fact, for $k \geqslant 2$ all of these components vanish except $y_k = \alpha_{kk}(\eta, \theta)w$ and $z_k = \beta_{kk}(\eta, \theta)w$. For $k = 1$, we have $X = d(\eta, \theta)w$, $y_j = -\alpha_{11}(\eta, \theta)w$ for $j = 2, ..., N$ and $Z = \beta_{11}(\eta, \theta)w$. Here, $\beta_{kk}(\eta, \theta)$ gives the matrix representation of the derivative of the Poincaré map on the section orthogonal to Γ_k in $\mathbb{R}^{m_k - 1}$ after K_k iterations while $\alpha_{kk}(\eta, \theta)$ gives the projection of $\mathscr{W}$ onto $f_k(\theta)$. Finally, we indicate how $\alpha_{kk}(\eta, \theta)$ can be interpreted as the derivative of a transit time map. For this general fact let $\dot{x} = f(x)$, $x \in \mathbb{R}^{n+1}$ be a differential equation that has a periodic solution Γ. Choose a vector v tangent to an orthogonal Poincaré section Σ at $\theta \in \Gamma$ and let $h: \Sigma \to \Sigma$ denote the Poincaré map. There is a curve $s \mapsto \sigma(s)$ in Σ with $\sigma(0) = \theta$ and $\dot{\sigma}(0) = v$. If the flow of the differential equation is given by $t \mapsto \phi_t$, then an application of the Implicit Function Theorem shows there is a transit time function $T: \Sigma \to \mathbb{R}$ such that

$$\phi_{T(\sigma(s))}(\sigma(s)) \equiv h(\sigma(s)).$$

After differentiation with respect to s, we obtain

$$(dT(\theta)v)\, f(\theta) + D_{\phi_{T(\theta)}}(\theta)v = h'(\theta)v.$$

Also, there are scalars a and $b_1, ..., b_n$ such that

$$D_{\phi_{T(\theta)}}(\theta)v = af(\theta) + \sum_{i=1}^{n} b_i f_i^{\perp}(\theta),$$

where $\{f_1^{\perp}, ..., f_n^{\perp}\}$ is a basis for the orthogonal vector fields defined in a neighborhood of Γ. Hence,

$$(dT(\theta)v + a)\, f(\theta) + \sum_{i=1}^{n} b_i f_i^{\perp}(\theta) = h'(\theta)v$$

and this implies

$$dT(\theta)v = -a, \qquad h'(\theta)\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}.$$

In other words, the projection onto $f(\theta)$ of the image of v under the monodromy transformation is the directional derivative of the transit time map in the direction v. In case $n = 1$, Diliberto's Theorem applies. With $v = f^{\perp}/\|f\|$ we find $a = \|f\|\,\alpha$. Thus, for a plane vector field $dT(\theta)(f^{\perp}/\|f\|) = -\|f\|\,\alpha$.

Let us now specialize to the case $m_k = 2$, $k = 1, ..., N$. Under the hypothesis of normal nondegeneracy, the range of the operator $V(\theta)$: $\mathbb{R}^N \to \mathbb{R}^{N-1} \times \mathbb{R}^N$ given by

$$V(\theta)u = (\alpha(\eta, \theta)u, (\beta(\eta, \theta) - I)u)$$

has dimension N. Of course, this is just the statement that the column space of the $(2N-1) \times N$ matrix

$$\begin{pmatrix} -\alpha_{11} & \alpha_{22} & 0 & \cdots & 0 \\ -\alpha_{11} & 0 & \alpha_{33} & \cdots & 0 \\ \vdots & & & & \vdots \\ -\alpha_{11} & 0 & 0 & \cdots & \alpha_{NN} \\ \beta_{11} - 1 & & & & \\ & \beta_{22} - 1 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \beta_{NN} - 1 \end{pmatrix}$$

has rank N. We require a projection onto the complement of the range. In case $\beta_{kk} \neq 1$ for each $k = 1, .,.., N$, a complement is just $\mathbb{R}^{N-1} \times \{0\}$. Thus, the projection $\pi^{\perp}(\theta)\colon \mathbb{R}^{2N-1} \to \mathbb{R}^{N-1}$ is represented (in adapted coordinates) by the $(N-1)\times(2N-1)$ partitioned matrix $(I_{N-1}|0)$, where I_{N-1} denotes the $(N-1)\times(N-1)$ identity. In case $\beta_{kk}-1=0$ for some $k=1, ..., N$, we have the following result: *If $\beta_{kk}-1=0$ for at most $N-1$ of the indices $k=1, ..., N$ and if for each such index $\alpha_{kk}\neq 0$, then the range of $V(\theta)$ has dimension N.* This follows easily by showing the (column) rank of the matrix representation of $V(\theta)$ is N. To see this, note the rank of the first N rows of the matrix is $N-1$ when $\alpha_{kk}\neq 0$ for each $k=1, ..., N$ and that any proper subset of columns of this submatrix has rank equal to the number of nonzero columns. To obtain a projection to a complement for this range one can take a sum of the projections onto $N-1$ unit vectors corresponding to the usual basis vectors $e_1, e_2, ..., e_{N-1}$ with e_k replaced by e_{k+N-1} whenever $\beta_{kk}-1=0$.

In case $\beta_{kk}\neq 1$ for each $k=1, ..., N$, the bifurcation function relative to the projection defined in the last paragraph is just

$$\mathscr{B}(\theta)=H(\theta)\begin{pmatrix}0\\ \mathscr{N}(\theta)\\ \mathscr{M}(\theta)\end{pmatrix}=\mathscr{N}(\theta),$$

where

$$\mathscr{N}(\theta)=\int_0^{T(\theta)} F_\varepsilon^{\tan}(s,\theta)-a(s,\theta)\,b^{-1}(s,\theta)\,F_\varepsilon^{\text{nor}}(s,\theta)\,d\theta.$$

To obtain the coordinate representation of this formula, define $G_k(s,\theta):=g_k(\phi(s,\theta),0)$ and note the bases for the summands of the splitting $\mathscr{E}\oplus\mathscr{E}^{\tan}\oplus\mathscr{E}^{\text{nor}}$ are

$$\mathscr{E}:=\left[\begin{pmatrix}f_1\\ \vdots\\ f_N\end{pmatrix}\right],$$

$$\mathscr{E}^{\tan}:=\left[\begin{pmatrix}0\\ f_2\\ 0\\ \vdots\\ 0\end{pmatrix}\cdots\begin{pmatrix}0\\ 0\\ \vdots\\ 0\\ f_N\end{pmatrix}\right],$$

$$\mathscr{E}^{\text{nor}}:=\left[\begin{pmatrix}f_1^{\perp}\\ 0\\ \vdots\\ 0\end{pmatrix}\cdots\begin{pmatrix}0\\ \vdots\\ 0\\ f_N^{\perp}\end{pmatrix}\right].$$

It follows that

$$F_\varepsilon^{\tan} = \begin{pmatrix} \|f_2\|^{-2} \langle G_2, f_2\rangle - \|f_1\|^{-2} \langle G_1, f_1\rangle \\ \vdots \\ \|f_N\|^{-2} \langle G_N, f_N\rangle - \|f_1\|^{-2} \langle G_1, f_1\rangle \end{pmatrix},$$

$$F_\varepsilon^{\mathrm{nor}} = \begin{pmatrix} \|f_1\|^{-2} f_1 \wedge G_1 \\ \vdots \\ \|f_N\|^{-2} f_N \wedge G_N \end{pmatrix},$$

and therefore

$$\mathcal{N}(\theta) = \begin{pmatrix} \mathcal{N}_2(\theta) \\ \vdots \\ \mathcal{N}_N(\theta) \end{pmatrix},$$

where, for $k = 2, ..., N$ and $j = 1, ..., N$,

$$\beta_{jj}(t, \theta) = \exp \int_0^t \operatorname{div} f_j(s, \theta)\, ds,$$

$$\alpha_{jj}(t, \theta) = \int_0^t \left(\frac{1}{\|f_j\|^2} (2\kappa \|f_j\| - \operatorname{curl} f_j)\beta \right)(s, \theta)\, ds,$$

$$\mathcal{N}_k(\theta) = \int_0^{T(\theta)} \left(\frac{1}{\|f_k\|^2} \langle G_k, f_k\rangle - \frac{1}{\|f_1\|^2} \langle G_1, f_1\rangle \right)(s, \theta)\, ds$$

$$+ \int_0^{T(\theta)} \left(\frac{\alpha_{11}}{\beta_{11}} f_1 \wedge G_1 - \frac{\alpha_{kk}}{\beta_{kk}} f_k \wedge G_k \right)(s, \theta)\, ds.$$

As an application consider the *standard limit cycle oscillator* given by the planar system

$$\dot{x} = -\mu y + x(\lambda^2 - x^2 - y^2), \qquad \dot{y} = \mu x + y(\lambda^2 - x^2 - y^2), \quad \lambda > 0, \quad \mu > 0.$$

It has a stable hyperbolic limit cycle on the circle centered at the origin with radius λ. This limit cycle is given by the family of solutions

$$x(t, \theta) = \lambda \cos(\mu(t + \theta)), \qquad y(t, \theta) = \lambda \sin(\mu(t + \theta)),$$

where $0 \leqslant \theta < 2\pi/\mu$ is an angular coordinate on the limit cycle that corresponds to the initial value. Using Diliberto's Theorem, it is easy to compute $\alpha(t, \theta) \equiv 0$ and $\beta(t, \theta) = \exp(-2\lambda^2 t)$. This means the derivative of the transit time map between orthogonal sections on the limit cycle vanishes at the point of intersection with the limit cycle and that the characteristic multiplier of the limit cycle is given by

$$\beta(2\pi/\mu, \theta) = \exp(-4\lambda^2\pi/\mu) < 1.$$

Now consider a system of N coupled standard limit cycle oscillators where the kth oscillator has the form

$$\dot{x}_k = -\mu y_k + x_k(\lambda^2 - x_k^2 - y_k^2) + \varepsilon g_{1k}(x_1, ..., x_N, y_1, ..., y_N),$$
$$\dot{y}_k = \mu x_k + y_k(\lambda^2 - x_k^2 - y_k^2) + \varepsilon g_{2k}(x_1, ..., x_N, y_1, ..., y_N).$$

Let θ_k denote the angular coordinate for the kth limit cycle, define $\theta := (\theta_2, ..., \theta_N)$, and define

$$g_{jk}(s, \theta) := g_{jk}(\lambda_1 \cos(\mu_1 s), \lambda_2 \cos(\mu_2(s+\theta_2)), ..., \lambda_N \cos(\mu_N(s+\theta_N)),$$
$$\lambda_1 \sin(\mu_1 s), \lambda_2 \sin(\mu_2(s+\theta_2)), ..., \lambda_N \sin(\mu_N(s+\theta_N))),$$

for $j=1, 2$, and $k=1, ..., N$. We assume the limit cycle oscillators are in resonance so there are relatively prime positive integers $K_1, ..., K_N$ and a number $\eta > 0$ such that $2\pi K_k/\mu_k = \eta$, for $k=1, ..., N$. Then, the bifurcation function is given by

$$\mathscr{B}(\theta) = (\mathscr{N}_2(\theta), ..., \mathscr{N}_N(\theta)),$$

where

$$\mathscr{N}_k = \frac{1}{\mu_k^2 \lambda_k} \int_0^\eta \cos(\mu_k(s+\theta_k))\, g_{2k}(s, \theta) - \sin(\mu_k(s+\theta_k))\, g_{1k}(s, \theta)\, ds$$
$$- \frac{1}{\mu_1^2 \lambda_1} \int_0^\eta \cos(\mu_1 s)\, g_{21}(s, \theta) - \sin(\mu_1 s)\, g_{11}(s, \theta)\, ds.$$

As a special case suppose $N=2$, $\mu := \mu_1 = \mu_2$, and

$$g_{11}(x_1, x_2, y_1, y_2) = 2(x_2 - x_1), \qquad g_{21}(x_1, x_2, y_1, y_2) = 2(1-2c)(y_2 - y_1),$$
$$g_{12}(x_1, x_2, y_1, y_2) = 2(x_1 - x_2), \qquad g_{22}(x_1, x_2, y_1, y_2) = 2(1-2c)(y_1 - y_2).$$

Then,

$$\mathscr{B}(\theta) = -\frac{8\pi(1-c)}{\mu^3} \sin(\mu\theta)$$

and there are branch points at $\theta = 0$ and $\theta = \pi/\mu$, cf. [19, p. 416].

As an illustration of a normally nondegenerate but normally nonhyperbolic period manifold in a coupled oscillator, consider a standard limit cycle oscillator coupled with an integrable system containing a period annulus. For example, consider the system

$$\begin{aligned}
\dot{x}_1 &= -y_1(k+x_1^2+y_1^2)+\varepsilon g_{11}(x_1, x_2, y_1, y_2),\\
\dot{y}_1 &= x_1(k+x_1^2+y_1^2)+\varepsilon g_{12}(x_1, x_2, y_1, y_2),\\
\dot{x}_2 &= -\mu y_2+x_2(\lambda^2-x_2^2-y_2^2)+\varepsilon g_{21}(x_1, x_2, y_1, y_2),\\
\dot{y}_2 &= \mu x_2+y_2(\lambda^2-x_2^2-y_2^2)+\varepsilon g_{22}(x_1, x_2, y_1, y_2),
\end{aligned}$$

with $k>0$, $u>0$, and $\lambda>0$. The periodic solution of the integrable system with initial value $(r\cos\theta_1, r\sin\theta_1)$ is given by

$$x_1(t, r, \theta)=r\cos((k+r^2)t+\theta_1), \qquad y_1(t, r, \theta)=r\sin((k+r^2)t+\theta_1).$$

This solution has period $2\pi/(k+r^2)$. The system is in resonance provided there are relatively prime positive integers p and q such that $\eta := 2\pi p/(k+r^2)=2\pi q/\mu$. As usual we let $\theta := \theta_2$ denote an angular coordinate for the limit cycle of the second oscillator. As before we must compute the range of the operator V with matrix

$$\begin{pmatrix} -\alpha_{11}(\eta, 0) & \alpha_{22}(\eta, 0) \\ \beta_{11}(\eta, 0)-1 & 0 \\ 0 & \beta_{22}(\eta, \theta)-1 \end{pmatrix}.$$

Here, we already know the standard limit cycle oscillator has $\alpha_{22}=0$ and

$$\beta_{22}(\eta,\theta)=\exp\left(\frac{-4\pi\lambda^2 q}{\mu}\right)-1\neq 0.$$

A computation shows $\beta_{11}(t, 0)-1\equiv 0$ and

$$\alpha_{11}(t, 0)=\frac{-2t}{(k+r^2)^2}.$$

As an internal check we compute the derivative of the transit time map directly. The periodic solution through the point $(r\cos\theta_1, r\sin\theta_1)$ for the first oscillator has period $2\pi/(k+r^2)$. Thus, the derivative of the period function with respect to r is $-4\pi r/(k+r^2)^2$. As mentioned above, the derivative of the transit time on the orthogonal section defined by $f^\perp$ is $-\|f\|\alpha_{11}$, where f denotes the vector field corresponding to the oscillator. Here, $\|f\|=k+r^2$ and $f^\perp$ is oriented inward on the periodic solution. Thus we obtain

$$-\|f(r\cos(\theta_1), r\sin(\theta_1))\|\,\alpha_{11}\left(\frac{2\pi}{k+r^2}, \theta_1\right)=\frac{4\pi r}{(k+r^2)^2}$$

as required.

Returning to the operator V we find a complement to the range is given by

$$\mathscr{R}^{\perp}(\theta) := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Thus, the required projection is just projection onto the first component of $\mathscr{E}^{\mathrm{nor}}$. In other words

$$\mathscr{B}(\theta) = H(\theta) \begin{pmatrix} 0 \\ \mathscr{N}(\theta) \\ \mathscr{M}_1(\theta) \\ \mathscr{M}_2(\theta) \end{pmatrix} = \mathscr{M}_1(\theta).$$

From this we compute

$$\begin{aligned}\mathscr{B}(\theta) &= \int_0^{\eta} \frac{1}{\beta_{11}(s,0)\,\|f(s,0)\|^2} f_1(s,0) \wedge G_1(s,\theta)\,ds \\ &= \frac{-1}{r(k+r^2)} \int_0^{\eta} \cos((k+r^2)s)\, g_{12}(x,y) + \sin((k+r^2)s)\, g_{11}(x,y)\,ds,\end{aligned}$$

where we have used $x = (x_1, x_2)$ and $y = (y_1, y_2)$. As a simple example take $\mu = 2$, $r = 1$, $k = 1$ so that $2\pi/(k+r^2) = 2\pi/\mu$. Also, for the coupling take $g_{11} = 0$ and $g_{21}(x_1, x_2, y_1, y_2) = y_2$. Then,

$$\mathscr{B}(\theta) = \frac{\lambda}{2} \int_0^{\pi} \cos(2s) \sin(2(s+\theta))\,ds = -\frac{\lambda\pi}{4} \sin(2\theta)$$

and there are branch points at $\theta = 0$ and $\theta = \pi/2$ along the limit cycle which has been parameterized by $0 \leqslant \theta < \pi$. This may appear somewhat surprising since the position of the *branch points* is not affected by the coupling functions g_{21} and g_{22}. Of course, the *curve of initial positions* for the perturbed periodic solutions will be dependent on the full coupling.

We now show how our geometric analysis can be used in concert with a classical expansion method, in this case the Poincaré–Lindstedt method, to help obtain a perturbation expansion for the branching periodic solutions. For notational convenience and to avoid double subscripts we consider the system in the form

$$\begin{aligned}\dot{u} &= -v(k + u^2 + v^2), \\ \dot{v} &= u(k + u^2 + v^2) + \varepsilon y, \\ \dot{x} &= -\mu y + x(\lambda^2 - x^2 - y^2) + \varepsilon g(u, v, x, y), \\ \dot{y} &= \mu x + y(\lambda^2 - x^2 - y^2) + \varepsilon h(u, v, x, y).\end{aligned}$$

The idea is to expand each of the variables u, v, x, and y, the frequency, and the initial conditions in power series with respect to the perturbation parameter. We take the perturbation series in the form

$$u(t) := u_0(\omega t) + u_1(\omega t)\varepsilon + O(\varepsilon^2),$$
$$v(t) := v_0(\omega t) + v_1(\omega t)\varepsilon + O(\varepsilon^2),$$
$$x(t) := x_0(\omega t) + x_1(\omega t)\varepsilon + O(\varepsilon^2),$$
$$y(t) := y_0(\omega t) + y_1(\omega t)\varepsilon + O(\varepsilon^2),$$
$$\frac{1}{\omega} := \frac{1}{2 + \omega_1 \varepsilon + O(\varepsilon^2)} = \frac{1}{2} - \frac{1}{4}\omega_1 \varepsilon + O(\varepsilon^2),$$
$$g := g_0 + g_1 \varepsilon + O(\varepsilon^2),$$
$$h := h_0 + h_1 \varepsilon + O(\varepsilon^2).$$

Guided by our geometric analysis and using the angular coordinate $\theta \in [0, 2\pi)$ on the unperturbed limit cycle, the appropriate initial conditions are given by

$$u(0) = 1, \qquad v(0) = 0, \qquad x(0) = \cos\theta + \xi_1 \varepsilon + O(\varepsilon^2),$$
$$y(0) = \sin(\theta) + \eta_1 \varepsilon + O(\varepsilon^2).$$

After the change of variables $\varphi := \omega t$, substitution into the differential equations and the equating of terms of like order we find with $' := d/d\varphi$ that

$$u_0' = -\tfrac{1}{2} v_0(1 + u_0^2 + v_0^2),$$
$$v_0' = \tfrac{1}{2} u_0(1 + u_0^2 + v_0^2),$$
$$x_0' = -y_0 + \tfrac{1}{2} x_0(1 - x_0^2 - y_0^2),$$
$$y_0' = x_0 + \tfrac{1}{2} y_0(1 - x_0^2 - y_0^2).$$

Thus, assuming the existence of a solution with initial values

$$u_0(0) = 1, \qquad v_0(0) = 0, \qquad x_0(0) = \cos\theta, \qquad y_0(0) = \sin\theta,$$

we find

$$u_0(\varphi) = \cos\varphi, \qquad v_0(\varphi) = \sin\varphi,$$
$$x_0(\varphi) = \cos(\varphi + \theta), \qquad y_0(\varphi) = \sin(\varphi + \theta).$$

Using these substitutions, we find the differential equations for the $O(\varepsilon)$ terms to be

$$\begin{pmatrix} u_1' \\ v_1' \end{pmatrix} = \begin{pmatrix} -\sin\varphi\cos\varphi & \cos^2\varphi - 2 \\ \cos^2\varphi + 1 & \cos\varphi\sin\varphi \end{pmatrix}\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} + \begin{pmatrix} \frac{1}{2}w_1\sin\varphi \\ \frac{1}{2}\sin(\varphi+\theta) - \frac{1}{2}w_1\cos\varphi \end{pmatrix},$$

$$\begin{pmatrix} x_1' \\ y_1' \end{pmatrix} = \begin{pmatrix} -\cos^2(\varphi+\theta) & -1-\cos(\varphi+\theta)\sin(\varphi+\theta) \\ 1-\sin(\varphi+\theta)\cos(\varphi+\theta) & \cos^2(\varphi+\theta)-1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$
$$+ \begin{pmatrix} \frac{1}{2}g_0 + \frac{1}{2}w_1\sin(\varphi+\theta) \\ \frac{1}{2}h_0 - \frac{1}{2}w_1\cos(\varphi+\theta) \end{pmatrix}$$

with the initial conditions

$$u_1(0) = 0, \qquad v_1(0) = 0, \qquad x_1(0) = \xi_1, \qquad y_1(0) = \eta_1.$$

A fundamental matrix for the first two equations (using Diliberto's Theorem) is

$$\Phi(\varphi) = \begin{pmatrix} -\frac{1}{2}\sin\varphi & 2\varphi\sin\varphi - 2\cos\varphi \\ \frac{1}{2}\cos\varphi & -2\varphi\cos\varphi - 2\sin\varphi \end{pmatrix}.$$

Thus, using variation of parameters, we find

$$\begin{pmatrix} u_1(2\pi) \\ v_1(2\pi) \end{pmatrix} = \int_0^{2\pi} \Phi(2\pi)\,\Phi^{-1}(s)\begin{pmatrix} \frac{1}{2}w_1\sin s \\ (\sin(s+\theta) - \frac{1}{2}w_1\cos s) \end{pmatrix} ds$$
$$= \begin{pmatrix} -\dfrac{\pi}{2}\sin\theta \\ -\pi w_1 - \dfrac{1}{2}\pi^2\sin\theta + \dfrac{3\pi}{4}\cos\theta \end{pmatrix}.$$

To obtain a periodic solution we must have $u_1(2\pi) = u_1(0) = 0$. This implies $\theta = 0$ or $\theta = \pi$. The result agrees with the previous analysis since here the angular variable was taken on the interval $[0, 2\pi)$, whereas before the angular variable was taken on the interval $[0, \pi)$. Asymptotic approximations to the perturbed periodic solution starting at one of the two branch points can be found by imposing the remaining periodicity conditions and by continuing in the same manner to higher order.

4.3. *Mutual Synchronization*

Consider the problem of mutual synchronization for the coupled system of van der Pol oscillators

$$\ddot{u} + (u^2 - 1)\dot{u} + u = \varepsilon Q_1 \ddot{x},$$
$$\ddot{x} + (x^2 - 1)\dot{x} + \omega^2 x = \varepsilon Q_2 \ddot{u},$$

considered in [18, p. 448]. We assume the unperturbed systems are nearly in (1:1) resonance; i.e., the frequency ω is detuned as a function of the perturbation parameter but in tune at $\varepsilon = 0$. In particular we assume $\omega = 1 - k\varepsilon + O(\varepsilon^2)$ and we call k the detuning parameter. We are interested in the (1:1) synchronization domain. To give a precise definition of this set note that each unperturbed oscillator has a limit cycle. The cross product of these limit cycles is an invariant torus for the unperturbed system in the product of the phase spaces for the oscillators. This invariant torus persists for small perturbations and the branching periodic solutions will lie on it. In fact, the flow of the coupled system restricted to the invariant torus has no fixed points. Thus, a periodic solution can be classified by its (nonzero) number of meridianal and longitudinal wraps on the torus before closing. The (1:1) *synchronization domain* is, for fixed Q_1 and Q_2, the set of points in the (ω, ε) space such that the corresponding system has a periodic solution with exactly one meridianal and exactly one longitudinal wrap. Our bifurcation analysis is applicable near the point where the synchronization domain meets the frequency axis. Of course, for the (1:1) synchronization this is the point $(\omega, \varepsilon) = (1, 0)$. In order to apply the bifurcation analysis we consider the phase coordinate system given by

$$\dot{u} = v, \qquad \dot{v} = -u + (1 - u^2)v - \varepsilon Q_1(x + (x^2 - 1)y),$$

$$\dot{x} = y, \qquad \dot{y} = -x + (1 - x^2)y + \varepsilon(2kx - Q_2(u + (u^2 - 1)v)),$$

that is $O(\varepsilon)$ equivalent to the coupled oscillator. Let $\eta > 0$ denote the common period of the limit cycles of the unperturbed tuned van der Pol oscillators and let θ denote the arc length variable on the limit cycle of the first oscillator. By our previous analysis, the bifurcation function is

$$\mathscr{B}(\theta) = \mathscr{N}_2(\theta) = \int_0^\eta \frac{1}{\|f_2\|^2} \langle G_2, f_2 \rangle - \frac{1}{\|f_1\|^2} \langle G_1, f_1 \rangle \, ds$$

$$+ \int_0^\eta \frac{\alpha_{11}}{\beta_{11}} f_1 \wedge G_1 - \frac{\alpha_{22}}{\beta_{22}} f_2 \wedge G_2 \, ds,$$

where the subscripted variables have the obvious identifications. However, since the unperturbed van der Pol oscillators are identical, we define $f := f_1 = f_2$, $\alpha := \alpha_{11} = \alpha_{22}$, and $\beta := \beta_{11} = \beta_{22}$. Then, using these definitions and a computation, we find

$$\mathscr{B}(\theta) = k\mathscr{B}_1(\theta) + (Q_2 - Q_1)\mathscr{B}_2(\theta),$$

where

$$\mathscr{B}_1(\theta) = \int_0^\eta 2z \left(\frac{\ddot{z}}{\dot{z}^2 + \ddot{z}^2} - \frac{\alpha}{\beta} \dot{z} \right) dt,$$

$$\mathscr{B}_2(\theta) = \int_0^\eta \ddot{z} \left(\frac{\ddot{z}}{\dot{z}^2 + \ddot{z}^2} - \frac{\alpha}{\beta} \dot{z} \right) dt,$$

and $t \mapsto z(t)$ is the solution of

$$\ddot{z} + (z^2 - 1)\dot{z} + z = 0$$

starting on the limit cycle at the point with angular coordinate θ.

The synchronization problem must be considered in the three dimensional parameter space given by the frequency, the amplitude, and the difference of the coupling strengths, i.e., in the coordinates $(\omega, \varepsilon, Q_1 - Q_2)$. However, since the natural control parameters are (ω, ε) and since two dimensional bifurcation diagrams are easier to draw, we will consider the synchronization domain in the (ω, ε) space as a function of the difference of the coupling strengths. For the remainder of the discussion of the synchronization problem we proceed to interpret the results of *numerical*

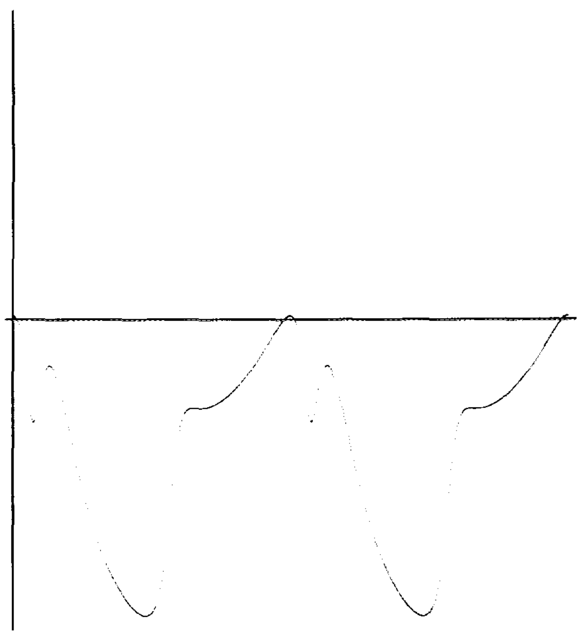

FIG. 2. The graph of the computed values of $\theta \mapsto \mathscr{B}_1(\theta)$ for $0 \leqslant \theta \leqslant L$, where L is the arc length of the limit cycle for the unperturbed van der Pol oscillator. The horizontal θ-axis is depicted for the interval $[-.2, 14.5]$, while the vertical axis is for the interval $[-900, 900]$.

experiments with the bifurcation function; at present we do not know how to obtain the simple zeros of $\mathscr{B}$ analytically. First, it appears that, while (Fig. 2) $\mathscr{B}_1$ has four simple zeros, (Fig. 3) $\mathscr{B}_2$ has only positive values. Thus, to obtain the zeros of the bifurcation function there are three cases determined by the sign of $Q_1 - Q_2$. If $Q_1 - Q_2 = 0$ there are four simple zeros of the bifurcation function provided $k \neq 0$. This means the synchronization domain contains a disk centered at $(\omega, \varepsilon) = (1, 0)$ with the line given by $\omega = 1$ removed. Since the bifurcation function vanishes identically for the systems corresponding to this line, a higher order method would ordinarily be required to determine if this line is in the synchronization domain. However, in the example, if $Q_1 - Q_2 = 0$ and $k = 0$, then by the symmetry we see there are periodic solutions for sufficiently small ε. In fact, two of these periodic solutions are given by $t \mapsto (u(t), \pm u(t))$, where $t \mapsto u(t)$ is the periodic solution of

$$(1 \mp \varepsilon Q_1)\ddot{u} + (u^2 - 1)\dot{u} + u = 0.$$

Thus, the line $\omega = 1$ is actually in the synchronization domain. If $Q_1 - Q_2 \neq 0$, the ratio $\mathscr{B}^*(\theta) := \mathscr{B}_2(\theta)/\mathscr{B}_1(\theta)$ defined on the complement of

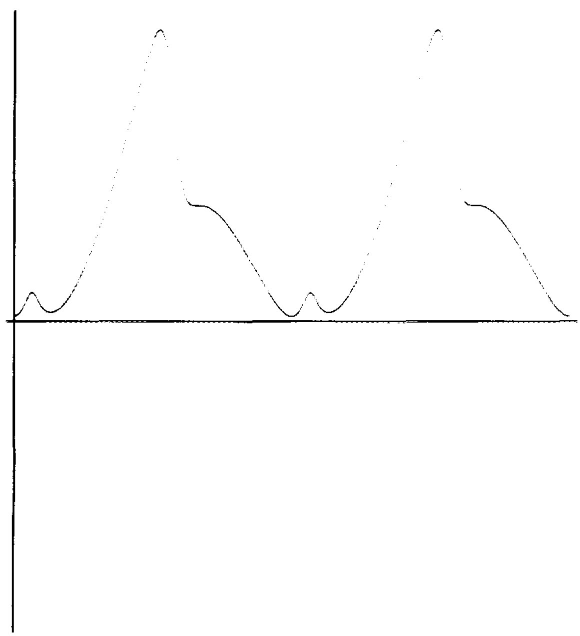

FIG. 3. The graph of the computed values of $\theta \mapsto \mathscr{B}_2(\theta)$ for $0 \leqslant \theta \leqslant L$, where L is the arc length of the limit cycle for the unperturbed van der Pol oscillator. The horizontal θ-axis is depicted for the interval $[-.2, 14.5]$, while the vertical axis is for the interval $[-850, 850]$.

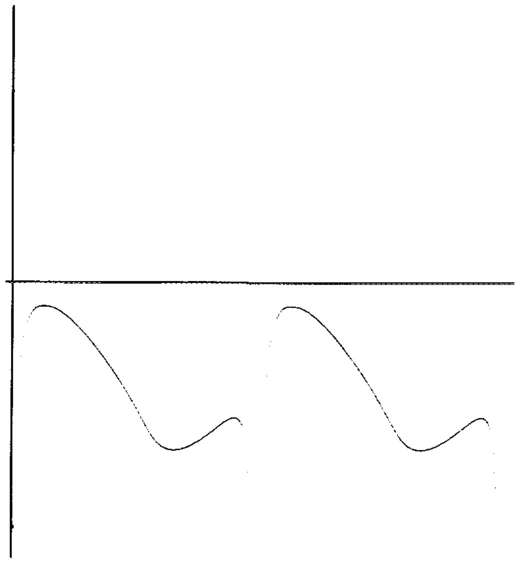

FIG. 4. The graph of the computed values of $\theta \mapsto \mathscr{B}^*(\theta)$ for $0 \leqslant \theta \leqslant L$, where L is the arc length of the limit cycle for the unperturbed van der Pol oscillator. The horizontal θ-axis is depicted for the interval $[-.2, 14.5]$, while the vertical axis is for the interval $[-2, 2]$.

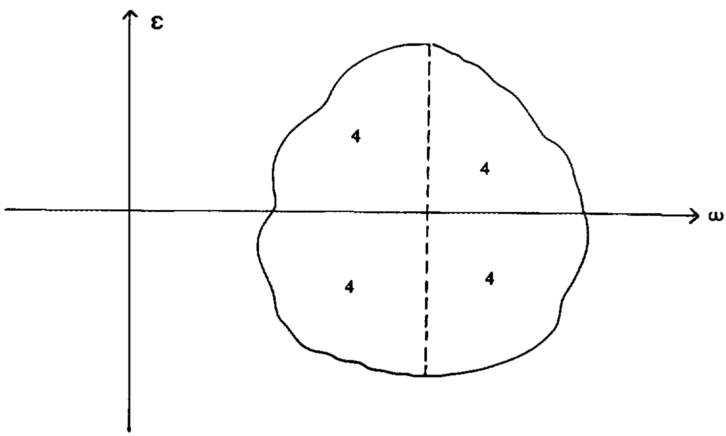

FIG. 5. Infinitesimal qualitative bifurcation diagram for entrainment domain in case $Q_1 - Q_2 = 0$. The number of periodic solutions for sufficiently small ε is indicated by the numerals between the lines bounding the regions.

the zero set of $\mathscr{B}_2$ has range (Fig. 4) covering the complement of an open interval $(a, b) \approx (-0.17, 1.07)$ in $\mathbb{R}$. Thus, there are branch points of periodic solutions provided the ratio $k/(Q_1 - Q_2)$ is in the range of $\mathscr{B}^*$. In other words, the curve of detuned frequencies given by $\omega(\varepsilon) = 1 - k\varepsilon$ lies in the synchronization domain for small values of ε when this line lies in the region bounded by the lines

$$\varepsilon = -\frac{\omega - 1}{a(Q_1 - Q_2)}, \qquad \varepsilon = -\frac{\omega - 1}{b(Q_1 - Q_2)}$$

and containing the ω-axis. These lines are the tangents to the synchronization domain at the point $(\omega, \varepsilon) = (1, 0)$. By our definition, the frequency axis is in the synchronization domain. Of course, the corresponding uncoupled system has a period torus and no attracting synchronous solutions.

There is one other interesting phenomenon in examples of mutual synchronization, namely, the number of stable synchronous solutions. For sufficiently small $\varepsilon \neq 0$ these correspond to the number of zeros of the bifurcation function. In case there is an invariant torus, the periodic solutions will generically occur in pairs with adjacent periodic solutions having opposite stability. In the example, the number of periodic solutions depends on the parameters. In fact, the numerical experiments suggest there are either 4 or 8 periodic solutions. Qualitative pictures of typical bifurcation diagrams are given in Figs. 5 and 6.

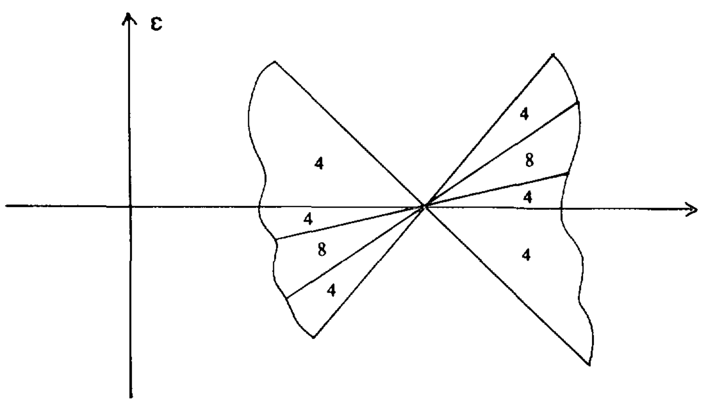

FIG. 6. Qualitative bifurcation diagram for entrainment domain in case $Q_1 - Q_2 > 0$. The number of periodic solutions for sufficiently small ε is indicated by the numerals between the lines bounding the regions.

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