Generalized Jordan derivation on nest algebras ⚫

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Abstract

Let $\mathcal{N}$ be a nest on a Banach space $X$, and $\text{Alg} \mathcal{N}$ be the associated nest algebra. In this paper, we prove that, if there is a nontrivial element in $\mathcal{N}$ which is complemented in $X$, then every additive generalized Jordan derivation from $\text{Alg} \mathcal{N}$ into itself is an additive generalized derivation. Moreover, we give a characterization of linear generalized Jordan derivations of nest algebras on complex separable Hilbert spaces.

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1. Introduction

Let $\mathcal{A}$ be an algebra and $\mathcal{M}$ be an $\mathcal{A}$-bimodule. Recall that a linear (additive) map $\delta$ from $\mathcal{A}$ into $\mathcal{M}$ is called a linear (additive) derivation if $\delta(AB) = \delta(A)B + A\delta(B)$ for all $A, B \in \mathcal{A}$. Linear derivations are very important maps both in theory and applications, and studied intensively. More generally, a linear (additive) map $\delta$ from $\mathcal{A}$ into $\mathcal{M}$ is called a linear (additive) Jordan derivation if $\delta(A^2) = \delta(A)A + A\delta(A)$ for each $A \in \mathcal{A}$; if there is a linear (additive) derivation $\tau : \mathcal{A} \rightarrow \mathcal{A}$ such that $\delta(AB) = \delta(A)B + A\tau(B)$ for all $A, B \in \mathcal{A}$, then $\delta$ is called a linear (additive) generalized derivation and $\tau$ is the relating derivation [5]; similarly, if there is a linear...
(additive) Jordan derivation \( \tau : \mathcal{A} \to \mathcal{A} \) such that \( \delta(A^2) = \delta(A)A + A\tau(A) \) for all \( A \in \mathcal{A} \), then \( \delta \) is called a linear (additive) generalized Jordan derivation and \( \tau \) is the relating Jordan derivation. In [7,11], authors gave another definition for generalized Jordan derivation: \( \delta \) is called a generalized Jordan derivation if \( \delta(A^2) = \delta(A)A + A\delta(A) - A\delta(I)A \) for all \( A \in \mathcal{A} \). It is easy to prove that their definition is a special case of ours.

The structures of derivations, Jordan derivations, generalized derivations and generalized Jordan derivations were systematically studied. Especially, the question when a Jordan derivation is a derivation was discussed by several authors. Herstein in [4] proved that every Jordan derivation from a prime ring of characteristic not 2 into itself is a derivation and that there are no nonzero antiderivations on a prime ring. Brešar in [1] showed that every Jordan derivation from a 2-torsion free semiprime ring into itself is a derivation. Recall that a ring or module is said to be 2-torsion free if \( 2a \) is nonzero for any nonzero element \( a \) in it. Zhang in [9] proved that every linear Jordan derivation on a nest algebra is a linear derivation, and further it is an inner derivation. Recently, Lu in [6] proved that every additive Jordan derivation on reflexive algebra is an additive derivation, which generalized the result in [9]. Similarly, it is also an interesting question when a generalized Jordan derivation is a generalized derivation. Zhu in [11] showed that every generalized Jordan derivation from a 2-torsion free semiprime ring with identity into itself is a generalized derivation. Ji in [7] proved that every generalized Jordan derivation from the algebra of all upper triangular matrices over a commutative ring with identity into its bimodule is the sum of a generalized derivation and an antiderivation. In the present paper we consider the above question for the nest algebra case. We prove that if there is a nontrivial element in a nest \( \mathcal{N} \) on a Banach space \( X \) which is complemented in \( X \), then every additive generalized Jordan derivation of the corresponding nest algebra is an additive generalized derivation. Moreover, we give a characterization of linear generalized Jordan derivations of nest algebras on complex separable Hilbert spaces.

Let \( X \) be a Banach space, and \( \mathcal{B}(X) \) denote the algebra of all bounded linear operators on \( X \). A nest \( \mathcal{N} \) on \( X \) is a chain of closed (under norm topology) subspaces of \( X \) which is closed under the formation of arbitrary closed linear span (denote by \( \overline{\bigvee} \) and intersection (denote by \( \overline{\bigwedge} \), and which includes \( \{0\} \) and \( X \). The nest algebra associated to the nest \( \mathcal{N} \), denoted by \( \text{Alg} \mathcal{N} \), is the weak closed operator algebra consisting of all operators that leave \( \mathcal{N} \) invariant, i.e.,

\[
\text{Alg} \mathcal{N} = \{ T \in \mathcal{B}(X) : TN \subseteq N \text{ for all } N \in \mathcal{N} \}.
\]

When \( \mathcal{N} \neq \{0, X\} \), we say that \( \mathcal{N} \) is non-trivial. If \( \mathcal{N} \) is trivial, then \( \text{Alg} \mathcal{N} = \mathcal{B}(X) \). We refer the reader to [3] for the theory of nest algebras.

2. Results and proofs

In this section, we discuss the additive generalized Jordan derivations on nest algebras. The following is our main result:

**Theorem 2.1.** Let \( \mathcal{N} \) be a nest on a Banach space \( X \), and \( \delta \) be an additive generalized Jordan derivation from \( \text{Alg} \mathcal{N} \) into itself. If there exists a non-trivial element in \( \mathcal{N} \) which is complemented in \( X \), then \( \delta \) is an additive generalized derivation.

We will complete the proof of above theorem by proving several lemmas. In the sequel, we always assume that \( \mathcal{N} \neq \{0, X\} \), and that \( N_1 \in \mathcal{N} \) is complemented (\( \{0\} \neq N_1 \neq X \)). Thus there exists an idempotent \( E \in \text{Alg} \mathcal{N} \) such that \( \text{ran} E = N_1 \). We denote the idempotent \( I - E \)
by $E^\perp$. As a notational convenience, we denote $\mathcal{A} = \text{Alg } \mathcal{N}$, $\mathcal{A}_{11} = E \mathcal{A} E$, $\mathcal{A}_{12} = E \mathcal{A} (I - E)$ and $\mathcal{A}_{22} = (I - E) \mathcal{A} (I - E)$. Thus $\mathcal{A} = \mathcal{A}_{11} + \mathcal{A}_{12} + \mathcal{A}_{22}$. For $S_{ij}$, $(i, j) \in \{1, 2\}$, we always mean $S_{ij} \in \mathcal{A}_{ij}$. Assume that $\delta : \mathcal{A} \rightarrow \mathcal{A}$ is an additive generalized Jordan derivation, and $\tau : \mathcal{A} \rightarrow \mathcal{A}$ the additive Jordan derivation such that $\delta(A^2) = \delta(A)A + A\tau(A)$ for all $A \in \mathcal{A}$.

For the additive Jordan derivation $\tau$, we have the following lemma.

**Lemma 2.1.** $\tau(E) = [E, S]$ for some $S \in \mathcal{A}$.

**Proof.** Write $\tau(E) = S_{11} + S_{12} + S_{22}$. Since $\tau(E) = \tau(E)E + E\tau(E)$, we have $S_{11} + S_{12} + S_{22} = 2S_{11} + S_{12}$, which implies that $S_{11} = S_{22} = 0$ and $\tau(E) = S_{12}$. Let $S = S_{12}$. It is obvious that $\tau(E) = [E, S]$. □

Now define $\delta'$ by $\delta' = \delta(A) - [A, S]$, for each $A \in \mathcal{A}$. Clearly, $\delta'$ is also an additive generalized Jordan derivation from $\mathcal{A}$ into itself, and $\tau' : \mathcal{A} \rightarrow \mathcal{A}$ defined by $\tau'(A) = \tau(A) - [A, S]$ for each $A \in \text{Alg } \mathcal{N}$ is the relating additive Jordan derivation, which is in fact an additive derivation by [6]. Note that $\tau'(E) = 0$. Moreover, we have $\tau'(I - E) = 0$. $\tau'(\mathcal{A}_{12}) \subset \mathcal{A}_{12}$, and $\tau'(\mathcal{A}_{ii}) \subset \mathcal{A}_{ii}$ for each $i = 1, 2$ by [10].

We will show that $\delta'$ is an additive generalized derivation by Lemma 2.2–2.6.

**Lemma 2.2.** For all $A, B, C \in \mathcal{A}$, the following statements hold:

1. $\delta'(AB + BA) = \delta'(A)B + A\tau'(B) + \delta'(B)A + B\tau'(A)$,
2. $\delta'(ABA) = \delta'(A)BA + A\tau'(B)A + AB\tau'(A)$,
3. $\delta'(ABC + CBA) = \delta'(A)BC + A\tau'(B)C + AB\tau'(C) + \delta'(C)BA + C\tau'(B)A + C\tau'(B)A$.

**Proof.** (1) On the one hand, $\delta'((A + B)^2) = \delta'(A + B)(A + B) + (A + B)\tau'(A + B)$, and on the other hand, $\delta'((A + B)^2) = \delta'(A^2 + AB + BA + B^2) = \delta'(A)A + A\tau'(A) + \delta'(AB + BA) + \delta'(B)B + B\tau'(B)$.

Comparing these two expressions we obtain that $\delta'(AB + BA) = \delta'(A)B + A\tau'(B) + \delta'(B)A + B\tau'(A)$.

(2) Let $S = \delta'(A(AB + BA) + (AB + BA)A)$. Using (1) we have $S = \delta'(A)(AB + BA) + A\tau'(AB + BA) + \delta'(AB + BA)A + (AB + BA)\tau'(A)$.

On the other hand, we also have $S = \delta'(A^2B + 2ABA + BA^2) = \delta'(A^2)B + A^2\tau'(B) + \delta'(B)A^2 + B\tau'(A^2) + 2\delta'(ABA)$.

Since $\tau'$ is a Jordan derivation, it is known that $\tau'(AB + BA) = \tau'(A)B + A\tau'(B) + \tau'(B)A + B\tau'(A)$.
So we get
\[ \delta'(ABA) = \delta'(A)BA + A\tau'(B)A + AB\tau'(A). \]

(3) Replacing \( A \) by \( A + C \) in (2), one obtains (3). \( \square \)

**Lemma 2.3.** \( \delta'(\mathcal{A}_{12}) \subset \mathcal{A}_{12}. \)

**Proof.** Firstly, we prove that \( \delta'(E) \in \mathcal{A}_{11}. \) Let \( \delta'(E) = S_{11} + S_{12} + S_{22}. \) Since \( \delta'(E) = \delta'(E)E + E\tau'(E) = \delta'(E)E, \) we see that \( S_{11} + S_{12} + S_{22} = S_{11}, \) which implies that \( S_{12} = S_{22} = 0 \) and \( \delta'(E) = S_{11} \in \mathcal{A}_{11}. \)

Now let \( A_{12} \in \mathcal{A}_{12} \) and \( \delta'(A_{12}) = S_{11} + S_{12} + S_{22}. \) Then
\[ S_{11} + S_{12} + S_{22} = \delta'(A_{12}) \]
\[ = \delta'(EA_{12} + A_{12}E) \]
\[ = \delta'(E)A_{12} + E\tau'(A_{12}) + \delta'(A_{12})E + A_{12}\tau'(E) \]
\[ = \delta'(E)A_{12} + \tau'(A_{12}) + S_{11}. \]

Hence \( S_{12} + S_{22} = \delta'(E)A_{12} + \tau'(A_{12}) \in \mathcal{A}_{12}, \) and \( S_{22} = 0. \)

On the other hand,
\[ S_{11} + S_{12} = \delta'(A_{12}) \]
\[ = \delta'(A_{12}E^\perp + E^\perp A_{12}) \]
\[ = \delta'(A_{12})E^\perp + A_{12}\tau'(E^\perp) \]
\[ + \delta'(E^\perp)A_{12} + E^\perp\tau'(A_{12}) \]
\[ = \delta'(A_{12})E^\perp + \delta'(E^\perp)A_{12} \]
\[ = (S_{11} + S_{12})E^\perp + \delta'(E^\perp)A_{12}. \]

Thus we get \( S_{11} = \delta'(E^\perp)A_{12}, \) which implies that \( S_{11} = S_{11}E = \delta'(E^\perp)A_{12}E = 0. \) Therefore \( \delta'(A_{12}) = S_{12} \in \mathcal{A}_{12}. \) \( \square \)

**Lemma 2.4.** \( \delta'(\mathcal{A}_{11}) \subset \mathcal{A}_{11} \) and \( \delta'(\mathcal{A}_{22}) \subset \mathcal{A}_{22}. \)

**Proof.** Let \( A_{11} \in \mathcal{A}_{11}. \) By Lemma 2.2(2), we have
\[ \delta'(A_{11}) = \delta'(EA_{11}E) \]
\[ = \delta'(E)A_{11}E + E\tau'(A_{11})E + EA_{11}\tau'(E) \]
\[ = \delta'(E)A_{11}E + \tau'(A_{11})E. \]

Since \( \delta'(E)A_{11} \in \mathcal{A}_{11} \) and \( E\tau'(A_{11})E \in \mathcal{A}_{11}, \) we get \( \delta'(A_{11}) \in \mathcal{A}_{11}. \)

Similarly, one can check that \( \delta'(\mathcal{A}_{22}) \subset \mathcal{A}_{22}. \) \( \square \)

**Lemma 2.5.** \( \delta' \) has the following properties:

1. \( \delta'(A_{11}B_{12}) = \delta'(A_{11})B_{12} + A_{11}\tau'(B_{12}) \) holds for all \( A_{11} \in \mathcal{A}_{11} \) and \( B_{12} \in \mathcal{A}_{12}. \)
2. \( \delta'(A_{12}B_{22}) = \delta'(A_{12})B_{22} + A_{12}\tau'(B_{22}) \) holds for all \( A_{12} \in \mathcal{A}_{12} \) and \( B_{22} \in \mathcal{A}_{22}. \)
3. \( \delta'(A_{22}B_{22}) = \delta'(A_{22})B_{22} + A_{22}\tau'(B_{22}) \) holds for all \( A_{22}, B_{22} \in \mathcal{A}_{22}. \)
Lemma 2.6. \( \delta \) we get that
\[
\delta'(A_{11}B_{12}) = \delta'(A_{11}B_{12} + B_{12}A_{11}) \\
= \delta'(A_{11})B_{12} + A_{11}\tau'(B_{12}) + \delta'(B_{12})A_{11} + B_{12}\tau'(A_{11}) \\
= \delta'(A_{11})B_{12} + A_{11}\tau'(B_{12}).
\]
Similarly, (2) is true for all \( A_{12} \in \mathcal{A}_{12} \) and \( B_{22} \in \mathcal{A}_{22} \).

For any \( A_{22} \in \mathcal{A}_{22} \), by Lemma 2.2(2), we have
\[
\delta'(A_{22}) = \delta'(E^\bot A_{22}E^\bot) \\
= \delta'(E^\bot)A_{22}E^\bot + E^\bot \tau'(A_{22})E^\bot + E^\bot A_{22}\tau'(E^\bot) \\
= \delta'(E^\bot)A_{22} + \tau'(A_{22}),
\]
and hence
\[
\delta'(A_{22}B_{22}) = \delta'(E^\bot)A_{22}B_{22} + \tau'(A_{22}B_{22})
\]
holds for all \( A_{22}, B_{22} \in \mathcal{A}_{22} \). Since
\[
\delta'(A_{22})B_{22} + A_{22}\tau'(B_{22}) = \delta'(E^\bot)A_{22}B_{22} + \tau'(A_{22})B_{22} + A_{22}\tau'(B_{22}) \\
= \delta'(E^\bot)A_{22}B_{22} + \tau'(A_{22}B_{22}),
\]
we get that \( \delta'(A_{22}B_{22}) = \delta'(A_{22})B_{22} + A_{22}\tau'(B_{22}). \)

Lemma 2.6. \( \delta'(AB) = \delta'(A)B + A\tau'(B) \) for all \( A, B \in \mathcal{A} \), that is, \( \delta' \) is an additive generalized derivation.

Proof. For any \( A, B \in \mathcal{A} \) and \( S_{12} \in \mathcal{A}_{12} \), by Lemma 2.2–2.5, we have
\[
\delta'(AB S_{12}) = \delta'(A_{11}B_{11}S_{12}) \\
= \delta'(A_{11})B_{11}S_{12} + A_{11}B_{11}\tau'(S_{12}) \\
= \delta'(A_{11}B_{11} + A_{11}B_{12} + A_{12}B_{22} + A_{22}B_{22})S_{12} \\
+ (A_{11}B_{11} + A_{11}B_{12} + A_{12}B_{22} + A_{22}B_{22})\tau'(S_{12}) \\
= \delta'(AB)S_{12} + AB\tau'(S_{12}).
\]

On the other hand,
\[
\delta'(AB S_{12}) = \delta'(A_{11}B_{11}S_{12}) \\
= \delta'(A_{11})B_{11}S_{12} + A_{11}\tau'(B_{11}S_{12}) \\
= \delta'(A)BS_{12} + A_{11}\tau'(B_{11})S_{12} + A_{11}B_{11}\tau'(S_{12}) \\
= \delta'(A)BS_{12} + A\tau'(B)S_{12} + AB\tau'(S_{12}).
\]
So we get \( \delta'(AB)S_{12} = (\delta'(A)B + A\tau'(B))S_{12} \), that is \( [\delta'(AB) - (\delta'(A)B + A\tau'(B))]S_{12} = 0 \) for any \( S_{12} \in \mathcal{A}_{12} \). Hence \( E[\delta'(AB) - (\delta'(A)B + A\tau'(B))]E = 0. \)

Similarly, for any \( S_{22} \in \mathcal{A}_{22} \), we compute \( \delta'(AB S_{22}) \) in two ways. On the one hand,
\[
\delta'(AB S_{22}) = \delta'(A_{11}B_{12}S_{22}) + \delta'(A_{12}B_{22}S_{22}) + \delta'(A_{22}B_{22}S_{22}) \\
= \delta'(A_{11}B_{12})S_{22} + A_{11}B_{12}\tau'(S_{22}) + \delta'(A_{12}B_{22})S_{22}
\]
Lemma 2.7. Let derivation from be a linear generalized derivation.

Proof of Theorem 2.1. From the above lemmas, we have proved that \( \delta = \delta(A_1 B_{12} S_{22}) + \delta'(A_2 B_{22} S_{22}) + \delta'(A_2 B_{22} S_{22}) \)

On the other hand,

\[
\delta'(AB S_{22}) = \delta'(A_1 B_{12} S_{22}) + \delta'(A_2 B_{22} S_{22}) + \delta'(A_2 B_{22} S_{22}) \\
= \delta'(A_1 B_{12} S_{22}) + A_1 \tau'(B_{12} S_{22}) + \delta'(A_2 B_{22} S_{22}) \\
+ A_2 \tau'(B_{22} S_{22}) + \delta'(A_2 B_{22} S_{22}) \\
= \delta'(A) B S_{22} - \delta'(A_2) B_{22} S_{22} + A_1 \tau'(B_{12} S_{22}) \\
+ A_2 \tau'(B_{22} S_{22}) + \delta'(A_2 B_{22} S_{22}) \\
= \delta'(A) B S_{22} - \delta'(A_2) B_{22} S_{22} + \delta'(A) B_{22} S_{22} \\
+ A_1 B_{12} \tau'(S_{22}) + A_2 \tau'(B_{22} S_{22}) + \delta'(A_2 B_{22} S_{22}) \\
= \delta'(A) B S_{22} - \delta'(A_2) B_{22} S_{22} + \delta'(A) B_{22} S_{22} \\
+ A_1 B_{12} \tau'(S_{22}) + A_2 \tau'(B_{22} S_{22}) + \delta'(A_2 B_{22} S_{22}) \\
= \delta'(A) B S_{22} - \delta'(A_2) B_{22} S_{22} + \delta'(A) B_{22} S_{22} \\
+ A_1 B_{12} \tau'(S_{22}) + A_2 \tau'(B_{22} S_{22}) + \delta'(A_2 B_{22} S_{22}).
\]

Comparing the above two equations, we get \([\delta'(AB) - (\delta'(A) B + A \tau'(B))] S_{22} = 0\), that is \(E[\delta'(AB) - (\delta'(A) B + A \tau'(B))] E^\perp + E^\perp[\delta'(AB) - (\delta'(A) B + A \tau'(B))] E^\perp = 0\). So \(E[\delta'(AB) - (\delta'(A) B + A \tau'(B))] E^\perp = 0\) and \(E^\perp[\delta'(AB) - (\delta'(A) B + A \tau'(B))] E^\perp = 0\). Therefore \(\delta'(AB) = \delta'(A) B + A \tau'(B)\).

Proof of Theorem 2.1. From the above lemmas, we have proved that \(\delta' : \mathcal{A} \to \mathcal{A}\) is an additive generalized derivation. Since \(\delta'(A) = \delta(A) - [A, S]\) for each \(A \in \mathcal{A}\), by a simple calculation, we see that \(\delta\) is also an additive generalized derivation. The proof is completed.

From Theorem 2.1, one gets the following corollary immediately.

Corollary 2.2. Let \(\mathcal{N}\) be a nest on a Hilbert space \(H\), and \(\delta\) be an additive generalized Jordan derivation from \(\text{Alg} \mathcal{N}\) into itself. Then \(\delta\) is an additive generalized derivation.

By a result of Christensen in \([2]\), the following lemma is obvious.

Lemma 2.7. Let \(\mathcal{N}\) be a nest on a complex separable Hilbert space \(H\), and \(\delta : \text{Alg} \mathcal{N} \to \text{Alg} \mathcal{N}\) be a linear generalized derivation. Then there exist \(T, S \in \text{Alg} \mathcal{N}\) such that \(\delta(A) = TA + AS\) for all \(A \in \text{Alg} \mathcal{N}\).

Hence, for linear generalized Jordan derivation, the following result is true.

Theorem 2.3. Let \(\mathcal{N}\) be a nest on a complex separable Hilbert space \(H\), and \(\delta : \text{Alg} \mathcal{N} \to \text{Alg} \mathcal{N}\) be a linear generalized Jordan derivation. Then there exist \(T, S \in \text{Alg} \mathcal{N}\) such that \(\delta(A) = TA + AS\) for all \(A \in \text{Alg} \mathcal{N}\).
In [8], the authors introduced the concept of generalized Jordan triple derivation. Let \( R \) be a ring and \( \delta : R \rightarrow R \) an additive map. If there is a Jordan triple derivation \( \tau : R \rightarrow R \) such that
\[
\delta(ABA) = \delta(A)BA + A\tau(B)A + AB\tau(A)
\]
for every \( A, B \in R \), then \( \delta \) is called a generalized Jordan triple derivation, and \( \tau \) is the relating Jordan triple derivation. Recall that \( \tau \) is a Jordan triple derivation if \( \tau(ABA) = \tau(A)BA + A\tau(B)A + AB\tau(A) \) for any \( A, B \in R \).

Let \( \tau : \text{Alg} \mathcal{N} \rightarrow \text{Alg} \mathcal{N} \) be a Jordan triple derivation. Note that \( \tau(I) = 0 \), so \( \tau \) is in fact a Jordan derivation. Now it is easy to check that a generalized Jordan triple derivation on a nest algebra is a generalized Jordan derivation. Hence we get the following corollary.

**Corollary 2.4.** Let \( \mathcal{N} \) be a nest on a Hilbert space \( H \), and \( \delta : \text{Alg} \mathcal{N} \rightarrow \text{Alg} \mathcal{N} \) be a generalized Jordan triple derivation. Then \( \delta \) is a generalized derivation.

**References**


