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The noncommutative geometry of graph C^*-algebras I: The index theorem ☆

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Abstract

We investigate conditions on a graph C^*-algebra for the existence of a faithful semifinite trace. Using such a trace and the natural gauge action of the circle on the graph algebra, we construct a smooth $(1, \infty)$-summable semi-finite spectral triple. The local index theorem allows us to compute the pairing with K-theory. This produces invariants in the K-theory of the fixed point algebra, and these are invariants for a finer structure than the isomorphism class of $C^*(E)$.
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1. Introduction

The aim of this paper, and the sequel [25], is to investigate the noncommutative geometry of graph C^*-algebras. In particular we construct finitely summable spectral triples to which we can apply the local index theorem. The motivation for this is the

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need for new examples in noncommutative geometry. Graph C^*-algebras allow us to treat a large family of algebras in a uniform manner.

Graph C^*-algebras have been widely studied, see [2,21,20,17,24,28,35] and the references therein. The freedom to use both graphical and analytical tools make them particularly tractable. In addition, there are many natural generalisations of this family to which our methods will apply, such as Cuntz–Krieger, Cuntz–Pimsner algebras, Exel–Laca algebras, k-graph algebras and so on; for more information on these classes of algebras see the above references and [27]. We expect these classes to yield similar examples.

One of the key features of this work is that the natural construction of a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ for a graph C^*-algebra is almost never a spectral triple in the original sense, [8, Chapter VI]. That is, the key requirement that for all $a \in \mathcal{A}$ the operator $a(1 + \mathcal{D}^2)^{-1/2}$ be a compact operator on the Hilbert space $\mathcal{H}$ is almost never true. However, if we broaden our point of view to consider semifinite spectral triples, where we require $a(1 + \mathcal{D}^2)^{-1/2}$ to be in the ideal of compact operators in a semifinite von Neumann algebra, we obtain many $(1, \infty)$-summable examples. The only connected $(1, \infty)$-summable example arising from our construction which satisfies the original definition of spectral triples is the Dirac triple for the circle.

The way we arrive at the correct notion of compactness is to regard the fixed point subalgebrat F for the S^1 gauge action on a graph algebra as the scalars. This provides a unifying point of view that will help the reader motivate the various constructions, and understand the results. For instance the C^*-bimodule we employ is a C^*-module over F, the range of the (C^*-) index pairing lies in $K_0(F)$, the 'differential' operator $\mathcal{D}$ is linear over F and it is the 'size' of F that forces us to use a general semifinite trace. The single $(1, \infty)$-summable example where the operator trace arises as the natural trace is the circle, and in this case $F = \mathbf{C}$.

The algebras which arise from our construction, despite naturally falling into the semifinite picture of spectral triples, are all type I algebras, [10]. Thus even when dealing with type I algebras there is a natural and important role for general semifinite traces.

Many of our examples arise from nonunital algebras. Fortunately, graph C^*-algebras (and their smooth subalgebras) are quasi-local in the sense of [13], and many of the results for smooth local algebras presented in [30,31] are valid for smooth quasi-local algebras. Here 'local' refers to the possibility of using a notion of 'compact support' to deal with analytical problems.

After some background material, we begin in Section 4 by constructing an odd Kasparov module (X, V) for $C^*(E)$-F, where F is the fixed point algebra. This part of the construction applies to any locally finite directed graph with no sources. The class (X, V) can be paired with $K_1(C^*(E))$ to obtain an index class in $K_0(F)$. This pairing is described in the appendix, and it is given in terms of the index of Toeplitz operators on the underlying C^*-module. We conjecture that this pairing is the Kasparov product.

When our graph C^*-algebra has a faithful (semifinite, lower-semicontinuous) gauge invariant trace τ, we can define a canonical faithful (semifinite, lower semicontinuous) trace $\tilde{\tau}$ on the endomorphism algebra of the C^*-F-module X. Using $\tilde{\tau}$, in Section 5 we construct a semifinite spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ for a smooth subalgebra $\mathcal{A} \subset C^*(E)$.

The numerical index pairing of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ with $K_1(C^*(E))$ can be computed using the semifinite local index theorem, [5], and we prove that

$$\langle K_1(C^*(E)), (\mathcal{A}, \mathcal{H}, \mathcal{D})\rangle = \tilde{\tau}_* \langle K_1(C^*(E)), (X, V)\rangle,$$

where $\langle K_1(C^*(E)), (X, V)\rangle \subset K_0(F)$ denotes the $K_0(F)$-valued index and $\tilde{\tau}_*$ is the map induced on K-theory by $\tilde{\tau}$. We show by an example that this pairing is an invariant of a finer structure than the isomorphism class of $C^*(E)$.

To ensure that readers without a background in graph C^*-algebras or a background in spectral triples can access the results in this paper, we have tried to make it self-contained. The organisation of the paper is as follows. Section 2 describes graph C^*-algebras and semifinite spectral triples, as well as quasilocal algebras and the local index theorem. Section 3 investigates which graph C^*-algebras have a faithful positive trace, and we provide some necessary and some sufficient conditions. In Section 4 we construct a C^*-module for any locally finite graph C^*-algebra. Using the generator of the gauge action on this C^*-module, we obtain a Kasparov module whenever the graph has no sources, and so a *KK*-class. In Section 5, we restrict to those graph C^*-algebras with a faithful gauge invariant trace, and construct a spectral triple from our Kasparov module. Section 6 describes our results pertaining to the index theorem.

In the sequel to this paper, [25], we identify a large subclass of our graph C^*-algebras with faithful trace which satisfy a natural semifinite and nonunital generalisation of Connes' axioms for noncommutative manifolds. These examples are all one-dimensional.

2. Graph C^*-algebras and semifinite spectral triples

2.1. The C^*-algebras of graphs

For a more detailed introduction to graph C^*-algebras we refer the reader to [2,21] and the references therein. A directed graph $E = (E^0, E^1, r, s)$ consists of countable sets E^0 of vertices and E^1 of edges, and maps $r, s : E^1 \to E^0$ identifying the range and source of each edge. We will always assume that the graph is *row-finite* which means that each vertex emits at most finitely many edges. Later we will also assume that the graph is *locally finite* which means it is row-finite and each vertex receives at most finitely many edges. We write E^n for the set of paths $\mu = \mu_1\mu_2\cdots\mu_n$ of length $|\mu| := n$; that is, sequences of edges μ_i such that $r(\mu_i) = s(\mu_{i+1})$ for $1 \leqslant i < n$. The maps r, s extend to $E^* := \bigcup_{n\geqslant 0} E^n$ in an obvious way. A *loop* in E is a path $L \in E^*$ with $s(L) = r(L)$, we say that a loop L has an exit if there is $v = s(L_i)$ for some i which emits more than one edge. If $V \subseteq E^0$ then we write $V \geqslant w$ if there is a path $\mu \in E^*$ with $s(\mu) \in V$ and $r(\mu) = w$ (we also sometimes say that w is downstream from *V*). A *sink* is a vertex $v \in E^0$ with $s^{-1}(v) = \emptyset$, a *source* is a vertex $w \in E^0$ with $r^{-1}(w) = \emptyset$.

A *Cuntz–Krieger E-family* in a C^*-algebra B consists of mutually orthogonal projections $\{p_v : v \in E^0\}$ and partial isometries $\{S_e : e \in E^1\}$ satisfying the *Cuntz–Krieger* *relations*

$$S_e^* S_e = p_{r(e)} \text{ for } e \in E^1 \text{ and } p_v = \sum_{\{e:s(e)=v\}} S_e S_e^* \text{ whenever } v \text{ is not a sink.}$$

It is proved in [21, Theorem 1.2] that there is a universal C^*-algebra $C^*(E)$ generated by a nonzero Cuntz–Krieger E-family $\{S_e, p_v\}$. A product $S_\mu := S_{\mu_1} S_{\mu_2} \cdots S_{\mu_n}$ is nonzero precisely when $\mu = \mu_1 \mu_2 \cdots \mu_n$ is a path in E^n. Since the Cuntz–Krieger relations imply that the projections $S_e S_e^*$ are also mutually orthogonal, we have $S_e^* S_f = 0$ unless $e = f$, and words in $\{S_e, S_f^*\}$ collapse to products of the form $S_\mu S_\nu^*$ for $\mu, \nu \in E^*$ satisfying $r(\mu) = r(\nu)$ (cf. [21, Lemma 1.1]). Indeed, because the family $\{S_\mu S_\nu^*\}$ is closed under multiplication and involution, we have

$$C^*(E) = \overline{\operatorname{span}}\{S_\mu S_\nu^* : \mu, \nu \in E^* \text{ and } r(\mu) = r(\nu)\}. \tag{1}$$

The algebraic relations and the density of $\operatorname{span}\{S_\mu S_\nu^*\}$ in $C^*(E)$ play a critical role throughout the paper. We adopt the conventions that vertices are paths of length 0 that $S_v := p_v$ for $v \in E^0$, and that all paths μ, ν appearing in (1) are nonempty; we recover S_μ, for example, by taking $\nu = r(\mu)$, so that $S_\mu S_\nu^* = S_\mu p_{r(\mu)} = S_\mu$.

If $z \in S^1$, then the family $\{z S_e, p_v\}$ is another Cuntz–Krieger E-family which generates $C^*(E)$, and the universal property gives a homomorphism $\gamma_z : C^*(E) \to C^*(E)$ such that $\gamma_z(S_e) = z S_e$ and $\gamma_z(p_v) = p_v$. The homomorphism $\gamma_{\overline{z}}$ is an inverse for γ_z, so $\gamma_z \in \operatorname{Aut} C^*(E)$, and a routine $\varepsilon/3$ argument using (1) shows that γ is a strongly continuous action of S^1 on $C^*(E)$. It is called the *gauge action*. Because S^1 is compact, averaging over γ with respect to normalised Haar measure gives an expectation Φ of $C^*(E)$ onto the fixed-point algebra $C^*(E)^\gamma$:

$$\Phi(a) := \frac{1}{2\pi} \int_{S^1} \gamma_z(a)\, d\theta \quad \text{for } a \in C^*(E), \quad z = e^{i\theta}.$$

The map Φ is positive, has norm 1, and is faithful in the sense that $\Phi(a^*a) = 0$ implies $a = 0$.

From Eq. (1), it is easy to see that a graph C^*-algebra is unital if and only if the underlying graph is finite. When we consider infinite graphs, formulas which involve sums of projections may contain infinite sums. To interpret these, we use strict convergence in the multiplier algebra of $C^*(E)$:

Lemma 2.1. *Let E be a row-finite graph, let A be a C^*-algebra generated by a Cuntz–Krieger E-family $\{T_e, q_v\}$, and let $\{p_n\}$ be a sequence of projections in A. If $p_n T_\mu T_\nu^*$ converges for every $\mu, \nu \in E^*$, then $\{p_n\}$ converges strictly to a projection $p \in M(A)$.*

Proof. Since we can approximate any $a \in A = \pi_{T,q}(C^*(E))$ by a linear combination of $T_\mu T_\nu^*$, an $\varepsilon/3$-argument shows that $\{p_n a\}$ is Cauchy for every $a \in A$. We define

$p : A \to A$ by $p(a) := \lim_{n\to\infty} p_n a$. Since

$$b^* p(a) = \lim_{n\to\infty} b^* p_n a = \lim_{n\to\infty} (p_n b)^* a = p(b)^* a,$$

the map p is an adjointable operator on the Hilbert C^*-module A_A, and hence defines (left multiplication by) a multiplier p of A [29, Theorem 2.47]. Taking adjoints shows that $ap_n \to ap$ for all a, so $p_n \to p$ strictly. It is easy to check that $p^2 = p = p^*$. $\square$

2.2. Semifinite spectral triples

We begin with some semifinite versions of standard definitions and results. Let τ be a fixed faithful, normal, semifinite trace on the von Neumann algebra $\mathcal{N}$. Let $\mathcal{K}_\mathcal{N}$ be the τ-compact operators in $\mathcal{N}$ (that is the norm closed ideal generated by the projections $E \in \mathcal{N}$ with $\tau(E) < \infty$).

Definition 2.2. A semifinite spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is given by a Hilbert space $\mathcal{H}$, a $*$-algebra $\mathcal{A} \subset \mathcal{N}$ where $\mathcal{N}$ is a semifinite von Neumann algebra acting on $\mathcal{H}$, and a densely defined unbounded self-adjoint operator $\mathcal{D}$ affiliated to $\mathcal{N}$ such that

(1) $[\mathcal{D}, a]$ is densely defined and extends to a bounded operator for all $a \in \mathcal{A}$.
(2) $a(\lambda - \mathcal{D})^{-1} \in \mathcal{K}_\mathcal{N}$ for all $\lambda \notin \mathbf{R}$ and all $a \in \mathcal{A}$.
(3) The triple is said to be even if there is $\Gamma \in \mathcal{N}$ such that $\Gamma^* = \Gamma$, $\Gamma^2 = 1$, $a\Gamma = \Gamma a$ for all $a \in \mathcal{A}$ and $\mathcal{D}\Gamma + \Gamma\mathcal{D} = 0$. Otherwise it is odd.

Definition 2.3. A semifinite spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is QC^k for $k \geqslant 1$ (Q for quantum) if for all $a \in \mathcal{A}$ the operators a and $[\mathcal{D}, a]$ are in the domain of δ^k, where $\delta(T) = [|\mathcal{D}|, T]$ is the partial derivation on $\mathcal{N}$ defined by $|\mathcal{D}|$. We say that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is QC^∞ if it is QC^k for all $k \geqslant 1$.

Note: The notation is meant to be analogous to the classical case, but we introduce the Q so that there is no confusion between quantum differentiability of $a \in \mathcal{A}$ and classical differentiability of functions.

Remarks concerning derivations and commutators. By partial derivation we mean that δ is defined on some subalgebra of $\mathcal{N}$ which need not be (weakly) dense in $\mathcal{N}$. More precisely, $\mathrm{dom}\,\delta = \{T \in \mathcal{N} : \delta(T) \text{ is bounded}\}$. We also note that if $T \in \mathcal{N}$, one can show that $[|\mathcal{D}|, T]$ is bounded if and only if $[(1+\mathcal{D}^2)^{1/2}, T]$ is bounded, by using the functional calculus to show that $|\mathcal{D}| - (1+\mathcal{D}^2)^{1/2}$ extends to a bounded operator in $\mathcal{N}$. In fact, writing $|\mathcal{D}|_1 = (1+\mathcal{D}^2)^{1/2}$ and $\delta_1(T) = [|\mathcal{D}|_1, T]$ we have

$$\mathrm{dom}\,\delta^n = \mathrm{dom}\,\delta_1^n \quad \text{for all } n.$$

We also observe that if $T \in \mathcal{N}$ and $[\mathcal{D}, T]$ is bounded, then $[\mathcal{D}, T] \in \mathcal{N}$. Similar comments apply to $[|\mathcal{D}|, T]$, $[(1+\mathcal{D}^2)^{1/2}, T]$. The proofs can be found in [5].

The QC^∞ condition places some restrictions on the algebras we consider. Recall that a topological algebra is Fréchet if it is locally convex, metrisable and complete, and that a subalgebra of a C^*-algebra is a pre-C^*-algebra if it is stable under the holomorphic functional calculus. For nonunital algebras, we consider only functions f with $f(0)=0$.

Definition 2.4. A $*$-algebra $\mathcal{A}$ is smooth if it is Fréchet and $*$-isomorphic to a proper dense subalgebra $i(\mathcal{A})$ of a C^*-algebra A which is a pre-C^*-algebra.

Asking for $i(\mathcal{A})$ to be a *proper* dense subalgebra of A immediately implies that the Fréchet topology of $\mathcal{A}$ is finer than the C^*-topology of A. We will denote the norm closure $\overline{\mathcal{A}}=A$, when the norm closure $\overline{\mathcal{A}}$ is unambiguous.

If $\mathcal{A}$ is smooth in A then $M_n(\mathcal{A})$ is smooth in $M_n(A)$, [14,33], so $K_*(\mathcal{A})\cong K_*(A)$, the isomorphism being induced by the inclusion map i. A smooth algebra has a sensible spectral theory which agrees with that defined using the C^*-closure, and the group of invertibles is open. The point of contact between smooth algebras and QC^∞ spectral triples is the following Lemma, proved in [30].

Lemma 2.5. *If $(\mathcal{A},\mathcal{H},\mathcal{D})$ is a QC^∞ spectral triple, then $(\mathcal{A}_\delta,\mathcal{H},\mathcal{D})$ is also a QC^∞ spectral triple, where $\mathcal{A}_\delta$ is the completion of $\mathcal{A}$ in the locally convex topology determined by the seminorms*

$$q_{n,i}(a)=\|\delta^n d^i(a)\|,\quad n\geqslant 0,\quad i=0,1,$$

where $d(a)=[\mathcal{D},a]$. Moreover, $\mathcal{A}_\delta$ is a smooth algebra.

We call the topology on $\mathcal{A}$ determined by the seminorms $q_{n,i}$ of Lemma 2.5 the δ-topology.

Whilst smoothness does not depend on whether $\mathcal{A}$ is unital or not, many analytical problems arise because of the lack of a unit. As in [13,30,31], we make two definitions to address these issues.

Definition 2.6. An algebra $\mathcal{A}$ has local units if for every finite subset of elements $\{a_i\}_{i=1}^n\subset\mathcal{A}$, there exists $\phi\in\mathcal{A}$ such that for each i

$$\phi a_i=a_i\phi=a_i.$$

Definition 2.7. Let $\mathcal{A}$ be a Fréchet algebra and $\mathcal{A}_c\subseteq\mathcal{A}$ be a dense subalgebra with local units. Then we call $\mathcal{A}$ a quasi-local algebra (when $\mathcal{A}_c$ is understood.) If $\mathcal{A}_c$ is a dense ideal with local units, we call $\mathcal{A}_c\subset\mathcal{A}$ local.

Quasi-local algebras have an approximate unit $\{\phi_n\}_{n\geqslant 1}\subset\mathcal{A}_c$ such that for all n, $\phi_{n+1}\phi_n=\phi_n$, [30]; we call this a local approximate unit.

Example. For a graph C^*-algebra $A = C^*(E)$, Eq. (1) shows that

$$A_c = \mathrm{span}\{S_\mu S_\nu^* : \mu, \nu \in E^* \text{ and } r(\mu) = r(\nu)\}$$

is a dense subalgebra. It has local units because

$$p_v S_\mu S_\nu^* = \begin{cases} S_\mu S_\nu^* & v = s(\mu), \\ 0 & \text{otherwise.} \end{cases}$$

Similar comments apply to right multiplication by $p_{s(\nu)}$. By summing the source and range projections (without repetitions) of all $S_{\mu_i} S_{\nu_i}^*$ appearing in a finite sum

$$a = \sum_i c_{\mu_i, \nu_i} S_{\mu_i} S_{\nu_i}^*$$

we obtain a local unit for $a \in A_c$. By repeating this process for any finite collection of such $a \in A_c$ we see that A_c has local units.

We also require that when we have a spectral triple the operator $\mathcal{D}$ is compatible with the quasi-local structure of the algebra, in the following sense.

Definition 2.8. If $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a spectral triple, then we define $\Omega^*_{\mathcal{D}}(\mathcal{A})$ to be the algebra generated by $\mathcal{A}$ and $[\mathcal{D}, \mathcal{A}]$.

Definition 2.9. A local spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a spectral triple with $\mathcal{A}$ quasi-local such that there exists an approximate unit $\{\phi_n\} \subset \mathcal{A}_c$ for $\mathcal{A}$ satisfying

$$\Omega^*_{\mathcal{D}}(\mathcal{A}_c) = \bigcup_n \Omega^*_{\mathcal{D}}(\mathcal{A})_n,$$

where

$$\Omega^*_{\mathcal{D}}(\mathcal{A})_n = \{\omega \in \Omega^*_{\mathcal{D}}(\mathcal{A}) : \phi_n \omega = \omega \phi_n = \omega\}.$$

Remark. A local spectral triple has a local approximate unit $\{\phi_n\}_{n \geqslant 1} \subset \mathcal{A}_c$ such that $\phi_{n+1}\phi_n = \phi_n \phi_{n+1} = \phi_n$ and $\phi_{n+1}[\mathcal{D}, \phi_n] = [\mathcal{D}, \phi_n]\phi_{n+1} = [\mathcal{D}, \phi_n]$, see [30,31]. We require this property to prove the summability results we require.

2.3. Summability and the local index theorem

In the following, let $\mathcal{N}$ be a semifinite von Neumann algebra with faithful normal trace τ. Recall from [12] that if $S \in \mathcal{N}$, the t-th *generalised singular value* of S for each real $t > 0$ is given by

$$\mu_t(S) = \inf\{||SE|| : E \text{ is a projection in } \mathcal{N} \text{ with } \tau(1-E) \leqslant t\}.$$

The ideal $\mathcal{L}^1(\mathcal{N})$ consists of those operators $T \in \mathcal{N}$ such that $\|T\|_1 := \tau(|T|) < \infty$ where $|T| = \sqrt{T^*T}$. In the Type I setting this is the usual trace class ideal. We will simply write $\mathcal{L}^1$ for this ideal in order to simplify the notation, and denote the norm on $\mathcal{L}^1$ by $\|\cdot\|_1$. An alternative definition in terms of singular values is that $T \in \mathcal{L}^1$ if $\|T\|_1 := \int_0^\infty \mu_t(T)\,dt < \infty$.

Note that in the case where $\mathcal{N} \neq \mathcal{B}(\mathcal{H})$, $\mathcal{L}^1$ is not complete in this norm but it is complete in the norm $||.||_1 + ||.||_\infty$. (where $||.||_\infty$ is the uniform norm). Another important ideal for us is the domain of the Dixmier trace:

$$\mathcal{L}^{(1,\infty)}(\mathcal{N}) = \left\{ T \in \mathcal{N} : \|T\|_{\mathcal{L}^{(1,\infty)}} := \sup_{t>0} \frac{1}{\log(1+t)} \int_0^t \mu_s(T)\,ds < \infty \right\}.$$

We will suppress the $(\mathcal{N})$ in our notation for these ideals, as $\mathcal{N}$ will always be clear from context. The reader should note that $\mathcal{L}^{(1,\infty)}$ is often taken to mean an ideal in the algebra $\widetilde{\mathcal{N}}$ of τ-measurable operators affiliated to $\mathcal{N}$, [12]. Our notation is however consistent with that of [8] in the special case $\mathcal{N} = \mathcal{B}(\mathcal{H})$. With this convention the ideal of τ-compact operators, $\mathcal{K}(\mathcal{N})$, consists of those $T \in \mathcal{N}$ (as opposed to $\widetilde{\mathcal{N}}$) such that

$$\mu_\infty(T) := \lim_{t\to\infty} \mu_t(T) = 0.$$

Definition 2.10. A semifinite local spectral triple is $(1,\infty)$-summable if

$$a(\mathcal{D}-\lambda)^{-1} \in \mathcal{L}^{(1,\infty)} \qquad \text{for all } a \in \mathcal{A}_c, \lambda \in \mathbf{C} \setminus \mathbf{R}.$$

Equivalently, $a(1+\mathcal{D}^2)^{-1/2} \in \mathcal{L}^{(1,\infty)}$ for all $a \in \mathcal{A}_c$.

Remark. If $\mathcal{A}$ is unital, $\ker \mathcal{D}$ is τ-finite dimensional. Note that the summability requirements are only for $a \in \mathcal{A}_c$. We do not assume that elements of the algebra $\mathcal{A}$ are all integrable in the nonunital case.

We need to briefly discuss the Dixmier trace, but fortunately we will usually be applying it in reasonably simple situations. For more information on semifinite Dixmier traces, see [7]. For $T \in \mathcal{L}^{(1,\infty)}$, $T \geqslant 0$, the function

$$F_T : t \to \frac{1}{\log(1+t)} \int_0^t \mu_s(T)\,ds$$

is bounded. For certain generalised limits $\omega \in L^\infty(\mathbf{R}_*^+)^*$, we obtain a positive functional on $\mathcal{L}^{(1,\infty)}$ by setting

$$\tau_\omega(T) = \omega(F_T).$$

This is the Dixmier trace associated to the semifinite normal trace τ, denoted τ_ω, and we extend it to all of $\mathcal{L}^{(1,\infty)}$ by linearity, where of course it is a trace. The Dixmier trace τ_ω is defined on the ideal $\mathcal{L}^{(1,\infty)}$, and vanishes on the ideal of trace class operators. Whenever the function F_T has a limit at infinity, all Dixmier traces return the value of the limit. We denote the common value of all Dixmier traces on measurable operators by $⨍$. So if $T \in \mathcal{L}^{(1,\infty)}$ is measurable, for any allowed functional $\omega \in L^\infty(\mathbf{R}_*^+)^*$ we have

$$\tau_\omega(T) = \omega(F_T) = ⨍ T.$$

Example. Let $\mathcal{D} = \frac{1}{i}\frac{d}{d\theta}$ act on $L^2(S^1)$. Then it is well known that the spectrum of $\mathcal{D}$ consists of eigenvalues $\{n \in \mathbf{Z}\}$, each with multiplicity one. So, using the standard operator trace, the function $F_{(1+\mathcal{D}^2)^{-1/2}}$ is

$$N \to \frac{1}{\log 2N + 1} \sum_{n=-N}^{N} (1 + n^2)^{-1/2},$$

which is bounded. So $(1 + \mathcal{D}^2)^{-1/2} \in \mathcal{L}^{(1,\infty)}$ and for any Dixmier trace Trace_ω

$$\mathrm{Trace}_\omega((1 + \mathcal{D}^2)^{-1/2}) = ⨍ (1 + \mathcal{D}^2)^{-1/2} = 2.$$

In [30,31] we proved numerous properties of local algebras. The introduction of quasi-local algebras in [13] led us to review the validity of many of these results for quasi-local algebras. Most of the summability results of [31] are valid in the quasi-local setting. In addition, the summability results of [31] are also valid for general semifinite spectral triples since they rely only on properties of the ideals $\mathcal{L}^{(p,\infty)}$, $p \geqslant 1$, [8,7], and the trace property. We quote the version of the summability results from [31] that we require below.

Proposition 2.11 (*Rennie [31]*). *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a QC^∞, local $(1,\infty)$-summable semifinite spectral triple relative to $(\mathcal{N}, \tau)$. Let $T \in \mathcal{N}$ satisfy $T\phi = \phi T = T$ for some $\phi \in \mathcal{A}_c$. Then*

$$T(1 + \mathcal{D}^2)^{-1/2} \in \mathcal{L}^{(1,\infty)}.$$

For $Re(s) > 1$, $T(1+\mathcal{D}^2)^{-s/2}$ is trace class. If the limit

$$\lim_{s\to 1/2^+}(s-1/2)\tau(T(1+\mathcal{D}^2)^{-s}) \tag{2}$$

exists, then it is equal to

$$\frac{1}{2}⨍ T(1+\mathcal{D}^2)^{-1/2}.$$

In addition, for any Dixmier trace τ_ω, the function

$$a \mapsto \tau_\omega(a(1+\mathcal{D}^2)^{-1/2})$$

defines a trace on $\mathcal{A}_c \subset \mathcal{A}$.

In [5], the noncommutative geometry local index theorem of [9] was extended to semifinite spectral triples. In the simplest terms, the local index theorem provides a formula for the pairing of a finitely summable spectral triple $(\mathcal{A},\mathcal{H},\mathcal{D})$ with the K-theory of $\overline{\mathcal{A}}$. The precise statement that we require is

Theorem 2.12 (*Carey et al. [5]*). *Let $(\mathcal{A},\mathcal{H},\mathcal{D})$ be an odd QC^∞ $(1,\infty)$-summable local semifinite spectral triple, relative to $(\mathcal{N},\tau)$. Then for $u \in \mathcal{A}$ unitary the pairing of $[u] \in K_1(\overline{\mathcal{A}})$ with $(\mathcal{A},\mathcal{H},\mathcal{D})$ is given by*

$$\langle [u], (\mathcal{A},\mathcal{H},\mathcal{D})\rangle = \mathrm{res}_{s=0}\tau(u[\mathcal{D},u^*](1+\mathcal{D}^2)^{-1/2-s}).$$

In particular, the residue on the right exists.

For more information on this result see [5–7,9].

3. Graph C^*-algebras with semifinite graph traces

This section considers the existence of (unbounded) traces on graph algebras. We denote by A^+ the positive cone in a C^*-algebra A, and we use extended arithmetic on $[0,\infty]$ so that $0\times\infty = 0$. From [26] we take the basic definition:

Definition 3.1. A trace on a C^*-algebra A is a map $\tau : A^+ \to [0,\infty]$ satisfying

(1) $\tau(a+b) = \tau(a)+\tau(b)$ for all $a, b \in A^+$,
(2) $\tau(\lambda a) = \lambda\tau(a)$ for all $a \in A^+$ and $\lambda \geqslant 0$,
(3) $\tau(a^*a) = \tau(aa^*)$ for all $a \in A$.

We say: that τ is faithful if $\tau(a^*a) = 0 \Rightarrow a = 0$; that τ is semifinite if $\{a \in A^+ : \tau(a) < \infty\}$ is norm dense in A^+ (or that τ is densely defined); that τ is lower semicontinuous if whenever $a = \lim_{n\to\infty} a_n$ in norm in A^+ we have $\tau(a) \leqslant \liminf_{n\to\infty} \tau(a_n)$.

We may extend a (semifinite) trace τ by linearity to a linear functional on (a dense subspace of) A. Observe that the domain of definition of a densely defined trace is a two-sided ideal $I_\tau \subset A$.

Lemma 3.2. *Let E be a row-finite directed graph and let $\tau : C^*(E) \to \mathbf{C}$ be a semifinite trace. Then the dense subalgebra*

$$A_c := \operatorname{span}\{S_\mu S_\nu^* : \mu, \nu \in E^*\}$$

is contained in the domain I_τ of τ.

Proof. Let $v \in E^0$ be a vertex, and let $p_v \in A_c$ be the corresponding projection. We claim that $p_v \in I_\tau$. Choose $a \in I_\tau$ positive, so $\tau(a) < \infty$, and with $\|p_v - a\| < 1$. Since p_v is a projection, we also have $\|p_v - p_v a p_v\| < 1$ and $p_v a p_v \in I_\tau$, so we have $\tau(p_v a p_v) < \infty$.

The subalgebra $p_v C^*(E) p_v$ has unit p_v, and as $\|p_v - p_v a p_v\| < 1$, $p_v a p_v$ is invertible. Thus there is some $b \in p_v C^*(E) p_v$ such that $b p_v a p_v = p_v$. Then, again since the trace class elements form an ideal, we have $\tau(p_v) < \infty$.

Now since $S_\mu S_\nu^* = p_{s(\mu)} S_\mu S_\nu^*$, it is easy to see that every element of A_c has finite trace. $\square$

It is convenient to denote by $A = C^*(E)$ and $A_c = \operatorname{span}\{S_\mu S_\nu^* : \mu, \nu \in E^*\}$.

Lemma 3.3. *Let E be a row-finite directed graph.*

(i) *If $C^*(E)$ has a faithful semifinite trace then no loop can have an exit.*
(ii) *If $C^*(E)$ has a gauge-invariant, semifinite, lower semicontinuous trace τ then $\tau \circ \Phi = \tau$ and*

$$\tau(S_\mu S_\nu^*) = \delta_{\mu,\nu} \tau(p_{r(\mu)}).$$

In particular, τ is supported on $C^*(\{S_\mu S_\mu^* : \mu \in E^*\})$.

Proof. Suppose E has a loop $L = e_1 \ldots e_n$ which has an exit. Let $v_i = s(e_i)$ for $i = 1, \ldots, n$ so that $r(e_n) = v_1$. Without loss of generality suppose that v_1 emits an edge f which is not part of L. If $w = r(f)$ then we have

$$\tau(p_{v_1}) \geqslant \tau(S_{e_1} S_{e_1}^* + S_f S_f^*) = \tau(S_{e_1}^* S_{e_1}) + \tau(S_f^* S_f) = \tau(p_{v_2}) + \tau(p_w).$$

Similarly we may show that $\tau(p_{v_i}) \geqslant \tau(p_{v_{i+1}})$ for $i = 1, \ldots, n-1$ and so $\tau(p_{v_1}) \geqslant \tau(p_{v_1}) + \tau(p_w)$ which means, by Lemma 3.2, that we must have $\tau(p_w) = 0$. Since p_w is positive, this implies that τ is not faithful. Now suppose the trace τ is gauge-invariant. Then

$$\tau(S_\mu S_\nu^*) = \tau(\gamma_z S_\mu S_\nu^*) = \tau(z^{|\mu|-|\nu|} S_\mu S_\nu^*) = z^{|\mu|-|\nu|}\tau(S_\mu S_\nu^*)$$

for all $z \in S^1$, and so $\tau(S_\mu S_\nu^*)$ is zero unless $|\mu| = |\nu|$. Hence $\tau \circ \Phi = \tau$ on A_c. Moreover, if $|\mu| = |\nu|$ then

$$\tau(S_\mu S_\nu^*) = \tau(S_\nu^* S_\mu) = \tau(\delta_{\mu,\nu} p_{r(\mu)}) = \delta_{\mu,\nu}\tau(p_{r(\mu)}),$$

so the restriction of τ to A_c is supported on $\mathrm{span}\{S_\mu S_\mu^* : \mu \in E^*\}$. To extend these conclusions to the C^* completions, let $\{\phi_n\} \subset \Phi(A)$ be an approximate unit for A consisting of an increasing sequence of projections. Then for each n, the restriction of τ to $A_n := \phi_n A \phi_n$ is a finite trace, and so norm continuous. Observe also that $\phi_n A_c \phi_n$ is dense in A_n and $\phi_n A_c \phi_n \subseteq A_c$. We claim that

$$\text{when restricted to } A_n,\ \ \tau \text{ satisfies } \tau \circ \Phi = \tau. \tag{3}$$

To see this we make two observations, namely that

$$\Phi(A_n) = \Phi(\phi_n A \phi_n) = \phi_n \Phi(A) \phi_n \subseteq \phi_n A \phi_n = A_n$$

and that on $\phi_n A_c \phi_n \subseteq A_c$ we have $\tau \circ \Phi = \tau$. The norm continuity of τ on A_n now completes the proof of the claim. Now let $a \in A^+$, and let $a_n = a^{1/2}\phi_n a^{1/2}$ so that $a_n \leqslant a_{n+1} \leqslant \cdots \leqslant a$ and $\|a_n - a\| \to 0$. Then

$$\tau(a) \geqslant \limsup \tau(a_n) \geqslant \liminf \tau(a_n) \geqslant \tau(a),$$

the first inequality coming from the positivity of τ, and the last inequality from lower semicontinuity. Since τ is a trace and $\phi_n^2 = \phi_n$ we have

$$\tau(a) = \lim_{n\to\infty} \tau(a_n) = \lim_{n\to\infty} \tau(\phi_n a \phi_n). \tag{4}$$

Similarly, let $b_n = \Phi(a)^{1/2}\phi_n\Phi(a)^{1/2}$ so that $b_n \leqslant b_{n+1} \leqslant \cdots \leqslant \Phi(a)$ and $\|b_n - \Phi(a)\| \to 0$. Then

$$\tau(\Phi(a)) = \lim_{n\to\infty} \tau(b_n) = \lim_{n\to\infty} \tau(\phi_n \Phi(a) \phi_n) = \lim_{n\to\infty} \tau(\Phi(\phi_n a \phi_n)). \tag{5}$$

However $\phi_n a \phi_n \in A_n$ so by (3) we have $(\tau \circ \Phi)(\phi_n a \phi_n) = \tau(\phi_n a \phi_n)$. Then by Eqs. (4) and (5) we have $\tau(a) = (\tau \circ \Phi)(a)$ for all $a \in A^+$. By linearity this is true for all

$a \in A$, so $\tau = \tau \circ \Phi$ on all of A. Finally,

$$\phi_n \mathrm{span}\{S_\mu S_\mu^* : \mu \in E^*\}\phi_n \subseteq \mathrm{span}\{S_\mu S_\mu^* : \mu \in E^*\},$$

so by the arguments above τ is supported on $C^*(\{S_\mu S_\mu^* : \mu \in E^*\})$. $\square$

Whilst the condition that no loop has an exit is necessary for the existence of a faithful semifinite trace, it is not sufficient.

One of the advantages of graph C^*-algebras is the ability to use both graphical and analytical techniques. There is an analogue of the above discussion of traces in terms of the graph.

Definition 3.4 (*cf. Tomforde [35]*). If E is a row-finite directed graph, then a graph trace on E is a function $g : E^0 \to \mathbf{R}^+$ such that for any $v \in E^0$ we have

$$g(v) = \sum_{s(e)=v} g(r(e)). \tag{6}$$

If $g(v) \neq 0$ for all $v \in E^0$ we say that g is faithful.

Remark. One can show by induction that if g is a graph trace on a directed graph with no sinks, and $n \geqslant 1$

$$g(v) = \sum_{s(\mu)=v,\ |\mu|=n} g(r(\mu)). \tag{7}$$

For graphs with sinks, we must also count paths of length at most n which end on sinks. To deal with this more general case we write

$$g(v) = \sum_{s(\mu)=v,\ |\mu|\preccurlyeq n} g(r(\mu)) \geqslant \sum_{s(\mu)=v,\ |\mu|=n} g(r(\mu)), \tag{8}$$

where $|\mu| \preccurlyeq n$ means that μ is of length n or is of length less than n and terminates on a sink.

As with traces on $C^*(E)$, it is easy to see that a necessary condition for E to have a faithful graph trace is that no loop has an exit.

Lemma 3.5. *Suppose that E is a row-finite directed graph and there exist vertices $v, w \in E^0$ with an infinite number of paths from v to w. Then there is no faithful graph trace on E^0.*

Proof. First suppose that there are an infinite number of paths from v to w of the same length, k say. Then for any $N \in \mathbf{N}$ and any graph trace $g : E^0 \to \mathbf{R}^+$

$$g(v) = \sum_{s(\mu)=v,\ |\mu| \preccurlyeq k} g(r(\mu)) \geqslant \sum^{N} g(w) = Ng(w).$$

So to assign a finite value to $g(v)$ we require $g(w) = 0$.

Thus we may suppose that there are infinitely many paths of different length from v to w, and without loss of generality that all the paths have different length. Choose the shortest path μ_1 of length k_1, say. Then, with $E^m(v) = \{\mu \in E^* : s(\mu) = v,\ |\mu| \preccurlyeq m\}$, we have

$$g(v) = \sum_{\mu \in E^{k_1}(v)} g(r(\mu)) = g(w) + \sum_{\mu \in E^{k_1}(v),\ r(\mu) \neq w} g(r(\mu)). \tag{9}$$

Observe that at least one of the paths, call it μ_2, in the rightmost sum can be extended until it reaches w. Choose the shortest such extension from $r(\mu_2)$ to w, and denote the length by k_2. So

$$\begin{aligned}\sum_{\mu \in E^{k_1}(v),\ \mu \neq \mu_1} g(r(\mu)) &= g(r(\mu_2)) + \sum_{\mu \in E^{k_1}(v),\ \mu \neq \mu_1, \mu_2} g(r(\mu)) \\ &= \sum_{\mu \in E^{k_2}(r(\mu_2))} g(r(\mu)) + \sum_{\mu \in E^{k_1}(v),\ \mu \neq \mu_1, \mu_2} g(r(\mu)) \\ &= g(w) + \sum_{\mu \in E^{k_2}(r(\mu_2)),\ \mu \neq \mu_2} g(r(\mu)) + \sum_{\mu \in E^{k_1}(v),\ \mu \neq \mu_1, \mu_2} g(r(\mu)). \end{aligned} \tag{10}$$

So by Eq. (9) we have

$$g(v) = 2g(w) + \ \mathrm{sum}_1 + \mathrm{sum}_2.$$

The two sums on the right contain at least one path which can be extended to w, and so choosing the shortest,

$$g(v) = 3g(w) + \ \mathrm{sum}_1 + \mathrm{sum}_2 + \mathrm{sum}_3.$$

It is now clear how to proceed, and we deduce as before that for all $N \in \mathbf{N}$, $g(v) \geqslant Ng(w)$. $\square$

Definition 3.6. Let E be a row-finite directed graph. An *end* will mean a sink, a loop without exit or an infinite path with no exits.

Remark. We shall identify an end with the vertices which comprise it. Once on an end (of any sort) the graph trace remains constant.

Corollary 3.7. *Suppose that E is a row-finite directed graph and there exists a vertex $v \in E^0$ with an infinite number of paths from v to an end. Then there is no faithful graph trace on E^0.*

Proof. Because the value of the graph trace is constant on an end Ω, say g_Ω, we have, as in Lemma 3.5,

$$g(v) \geqslant N g_\Omega$$

for all $N \in \mathbf{N}$. Hence there can be no faithful graph trace. $\square$

Thus if a row-finite directed graph E is to have a faithful graph trace, it is necessary that no vertex connects infinitely often to any other vertex or to an end, and that no loop has an exit.

Proposition 3.8. *Let E be a row-finite directed graph and suppose there exists $N \in \mathbf{N}$ such that for all vertices v and w and for all ends Ω,*

(1) *the number of paths from v to w, and*
(2) *the number of paths from v to Ω*

is less than or equal to N. If in addition the only infinite paths in E are eventually in ends, then E has a faithful graph trace.

Proof. First observe that our hypotheses on E rule out loops with exit, since we can define infinite paths using such loops, but they are not ends.

Label the set of ends by $i = 1, 2, \ldots$. Assign a positive number g_i to each end, and define $g(v) = g_i$ for all v in the ith end. If there are infinitely many ends, choose the g_i so that $\sum_i g_i < \infty$.

For each end, choose a vertex v_i on the end. For $v \in E^0$ not on an end, define

$$g(v) = \sum_i \sum_{s(\mu)=v,\ r(\mu)=v_i} g_i. \tag{11}$$

Then the conditions on the graph ensure this sum is finite. Using Eq. (8), one can check that $g : E^0 \to \mathbf{R}^+$ is a faithful graph trace. $\square$

There are many directed graphs with much more complicated structure than those described in Proposition 3.8 which possess faithful graph traces. The difficulty in defining a graph trace is going 'forward', and this is what prevents us giving a concise sufficiency condition. Extending a graph trace 'backward' from a given set of values can always be handled as in Eq. (11).

Proposition 3.9. *Let E be a row-finite directed graph. Then there is a one-to-one correspondence between faithful graph traces on E and faithful, semifinite, lower semicontinuous, gauge invariant traces on $C^*(E)$.*

Proof. Given a faithful graph trace g on E we define τ_g on A_c by

$$\tau_g(S_\mu S_\nu^*) := \delta_{\mu,\nu} g(r(\mu)). \tag{12}$$

One checks that τ_g is a gauge invariant trace on A_c, and is faithful because for $a = \sum_{i=1}^n c_{\mu_i,\nu_i} S_{\mu_i} S_{\nu_i}^* \in A_c$ we have $a^*a \geqslant \sum_{i=1}^n |c_{\mu_i,\nu_i}|^2 S_{\nu_i} S_{\nu_i}^*$ and then

$$\begin{aligned}\langle a, a\rangle_g := \tau_g(a^*a) &\geqslant \tau_g\left(\sum_{i=1}^n |c_{\mu_i,\nu_i}|^2 S_{\nu_i} S_{\nu_i}^*\right)\\ &= \sum_{i=1}^n |c_{\mu_i,\nu_i}|^2 \tau_g(S_{\nu_i} S_{\nu_i}^*) = \sum_{i=1}^n |c_{\mu_i,\nu_i}|^2 g(r(\nu_i)) > 0.\end{aligned} \tag{13}$$

Then $\langle a, b\rangle_g = \tau_g(b^*a)$ defines a positive definite inner product on A_c which makes it a Hilbert algebra (that the left regular representation of A_c is nondegenerate follows from $A_c^2 = A_c$).

Let $\mathcal{H}_g$ be the Hilbert space completion of A_c. Then defining $\pi : A_c \to \mathcal{B}(\mathcal{H}_g)$ by $\pi(a)b = ab$ for $a, b \in A_c$ yields a faithful $*$-representation. Thus $\{\pi(S_e), \pi(p_v) : e \in E^1,\ v \in E^0\}$ is a Cuntz–Krieger E family in $\mathcal{B}(\mathcal{H}_g)$. The gauge invariance of τ_g shows that for each $z \in S^1$ the map $\gamma_z : A_c \to A_c$ extends to a unitary $U_z : \mathcal{H}_g \to \mathcal{H}_g$. Then for $a, b \in A_c$ we compute

$$(U_z \pi(a) U_{\bar z})(b) = U_z a \gamma_{\bar z}(b) = \gamma_z(a\gamma_{\bar z}(b)) = \gamma_z(a)b = \pi(\gamma_z(a))(b).$$

Hence $U_z\pi(a)U_{\bar z} = \pi(\gamma_z(a))$ and defining $\alpha_z(\pi(a)) := U_z\pi(a)U_{\bar z}$ gives a point norm continuous action of S^1 on $\pi(A_c)$ implementing the gauge action. Since for all $v \in E^0$, $\pi(p_v)p_v = p_v$, $\pi(p_v) \neq 0$. Thus we can invoke the gauge invariant uniqueness theorem, [2, Theorem 2.1], and the map $\pi : A_c \to \mathcal{B}(\mathcal{H}_g)$ extends by continuity to $\pi : C^*(E) \to \mathcal{B}(\mathcal{H}_g)$ and $\pi(C^*(E)) = \overline{\pi(A_c)}^{\|\cdot\|}$ in $\mathcal{B}(\mathcal{H}_g)$. In particular the representation is faithful on $C^*(E)$.

Now, $\pi(C^*(E)) \subseteq \pi(A_c)'' = \overline{\pi(A_c)}^{u.w.}$, where $u.w.$ denotes the ultra-weak closure. The general theory of Hilbert algebras, see for example [11, Theorem 1, Section 2, Chapter 6, Part I], now shows that the trace τ_g extends to an ultra weakly lower semicontinuous, faithful, (ultra weakly) semifinite trace $\bar\tau_g$ on $\pi(A_c)''$. Trivially, the restriction of this extension to $\pi(C^*(E))$ is faithful. It is semifinite in the norm sense on $C^*(E)$ since $\pi(A_c)$ is norm dense in $\pi(C^*(E))$ and τ_g is finite on $\pi(A_c)$. To see that this last statement is true, let $a \in A_c$, choose any local unit $\phi \in A_c$ for a and then

$$\infty > \tau_g(a) = \tau_g(\phi a) = \langle a, \phi\rangle_g =: \bar\tau_g(\phi a) = \bar\tau_g(a).$$

It is norm lower semicontinuous on $\pi(C^*(E))$ because if $\pi(a) \in C^*(E)^+$ and $\pi(a_n) \in C^*(E)^+$ with $\pi(a_n) \to \pi(a)$ in norm, then $\pi(a_n) \to \pi(a)$ ultra weakly and so $\bar{\tau}_g(\pi(a)) \leqslant \liminf \bar{\tau}_g(\pi(a_n))$.

We have seen that the gauge action of S^1 on $C^*(E)$ is implemented in the representation π by the unitary representation $S^1 \ni z \to U_z \in \mathcal{B}(\mathcal{H}_g)$. We wish to show that $\bar{\tau}_g$ is invariant under this action, but since the U_z do not lie in $\pi(A_c)''$, we cannot use the tracial property directly. Now $T \in \pi(A_c)''$ is in the domain of definition of $\bar{\tau}_g$ if and only if $T = \pi(\xi)\pi(\eta)^*$ for left bounded elements $\xi, \eta \in \mathcal{H}_g$. Then $\bar{\tau}_g(T) = \bar{\tau}_g(\pi(\xi)\pi(\eta)^*) := \langle \xi, \eta \rangle_g$. Since $U_z(\xi)$ and $U_z(\eta)$ are also left bounded elements of $\mathcal{H}_g$ we have

$$\begin{aligned}\bar{\tau}_g(U_z T U_{\bar{z}}) &= \bar{\tau}_g(U_z \pi(\xi)\pi(\eta)^* U_{\bar{z}}) = \bar{\tau}_g(U_z \pi(\xi)[U_z \pi(\eta)]^*) \\ &= \bar{\tau}_g(\pi(\gamma_z(\xi))[\pi(\gamma_z(\eta))]^*) = \langle U_z(\xi), U_z(\eta) \rangle_g \\ &= \langle \xi, \eta \rangle_g = \bar{\tau}_g(T).\end{aligned}$$

That is, $\bar{\tau}_g(\alpha_z(T)) = \bar{\tau}_g(T)$, and $\bar{\tau}_g$ is α_z-invariant. Thus $a \to \bar{\tau}_g(\pi(a))$ defines a faithful, semifinite, lower semicontinuous, gauge invariant trace on $C^*(E)$.

Conversely, given a faithful, semifinite, lower semicontinuous and gauge invariant trace τ on $C^*(E)$, we know by Lemma 3.2 that τ is finite on A_c and so we define $g(v) := \tau(p_v)$. It is easy to check that this is a faithful graph trace. $\square$

4. Constructing a C^*- and Kasparov module

There are several steps in the construction of a spectral triple. We begin in Section 4.1 by constructing a C^*-module. We define an unbounded operator $\mathcal{D}$ on this C^*-module as the generator of the gauge action of S^1 on the graph algebra. We show in Section 4.2 that $\mathcal{D}$ is a regular self-adjoint operator on the C^*-module. We use the phase of $\mathcal{D}$ to construct a Kasparov module.

4.1. Building a C^*-module

The constructions of this subsection work for any locally finite graph. Let $A = C^*(E)$ where E is any locally finite directed graph. Let $F = C^*(E)^\gamma$ be the fixed point subalgebra for the gauge action. Finally, let A_c, F_c be the dense subalgebras of A, F given by the (finite) linear span of the generators.

We make A a right inner product F-module. The right action of F on A is by right multiplication. The inner product is defined by

$$(x|y)_R := \Phi(x^*y) \in F.$$

Here Φ is the canonical expectation. It is simple to check the requirements that $(\cdot|\cdot)_R$ defines an F-valued inner product on A. The requirement $(x|x)_R = 0 \Rightarrow x = 0$ follows from the faithfulness of Φ.

Definition 4.1. Define X to be the C^*-F-module completion of A for the C^*-module norm

$$\|x\|_X^2 := \|(x|x)_R\|_A = \|(x|x)_R\|_F = \|\Phi(x^*x)\|_F.$$

Define X_c to be the pre-C^*-F_c-module with linear space A_c and the inner product $(\cdot|\cdot)_R$.

Remark. Typically, the action of F does not map X_c to itself, so we may only consider X_c as an F_c module. This is a reflection of the fact that F_c and A_c are quasilocal not local.

The inclusion map $\iota : A \to X$ is continuous since

$$\|a\|_X^2 = \|\Phi(a^*a)\|_F \leqslant \|a^*a\|_A = \|a\|_A^2.$$

We can also define the gauge action γ on $A \subset X$, and as

$$\begin{aligned}\|\gamma_z(a)\|_X^2 &= \|\Phi((\gamma_z(a))^*(\gamma_z(a)))\|_F = \|\Phi(\gamma_z(a^*)\gamma_z(a))\|_F \\ &= \|\Phi(\gamma_z(a^*a))\|_F = \|\Phi(a^*a)\|_F = \|a\|_X^2,\end{aligned}$$

for each $z \in S^1$, the action of γ_z is isometric on $A \subset X$ and so extends to a unitary U_z on X. This unitary is F linear, adjointable, and we obtain a strongly continuous action of S^1 on X, which we still denote by γ.

For each $k \in \mathbf{Z}$, the projection onto the kth spectral subspace for the gauge action defines an operator Φ_k on X by

$$\Phi_k(x) = \frac{1}{2\pi}\int_{S^1} z^{-k}\gamma_z(x)\, d\theta, \quad z = e^{i\theta}, \quad x \in X.$$

Observe that on generators we have $\Phi_k(S_\alpha S_\beta^*) = S_\alpha S_\beta^*$ when $|\alpha| - |\beta| = k$ and is zero when $|\alpha| - |\beta| \neq k$. The range of Φ_k is

$$\text{Range } \Phi_k = \{x \in X : \gamma_z(x) = z^k x \quad \text{for all } z \in S^1\}. \tag{14}$$

These ranges give us a natural $\mathbf{Z}$-grading of X.

Remark. If E is a finite graph with no loops, then for k sufficiently large there are no paths of length k and so $\Phi_k = 0$. This will obviously simplify many of the convergence issues below.

Lemma 4.2. *The operators Φ_k are adjointable endomorphisms of the F-module X such that $\Phi_k^* = \Phi_k = \Phi_k^2$ and $\Phi_k\Phi_l = \delta_{k,l}\Phi_k$. If $K \subset \mathbf{Z}$ then the sum $\sum_{k\in K}\Phi_k$ converges* *strictly to a projection in the endomorphism algebra. The sum* $\sum_{k\in\mathbf{Z}}\Phi_k$ *converges to the identity operator on X.*

Proof. It is clear from the definition that each Φ_k defines an F-linear map on X. First, we show that Φ_k is bounded:

$$\|\Phi_k(x)\|_X \leqslant \frac{1}{2\pi}\int_{S^1}\|\gamma_z(x)\|_X\,d\theta \leqslant \frac{1}{2\pi}\int_{S^1}\|x\|_X\,d\theta = \|x\|_X.$$

So $\|\Phi_k\|\leqslant 1$. Since $\Phi_k S_\mu = S_\mu$ whenever μ is a path of length k, $\|\Phi_k\| = 1$. On the subspace X_c of finite linear combinations of generators, one can use Eq. (14) to see that $\Phi_k\Phi_l = \delta_{k,l}\Phi_k$ since

$$\Phi_k\Phi_l S_\alpha S_\beta^* = \Phi_k\delta_{|\alpha|-|\beta|,l}S_\alpha S_\beta^* = \delta_{|\alpha|-|\beta|,k}\delta_{|\alpha|-|\beta|,l}S_\alpha S_\beta^*.$$

For general $x\in X$, we approximate x by a sequence $\{x_m\}\subset X_c$, and the continuity of the Φ_k then shows that the relation $\Phi_k\Phi_l = \delta_{k,l}\Phi_k$ holds on all of X. Again using the continuity of Φ_k, the following computation allows us to show that for all k, Φ_k is adjointable with adjoint Φ_k:

$$\begin{aligned}(\Phi_k S_\alpha S_\beta^*|S_\rho S_\sigma^*)_R &= \Phi(\delta_{|\alpha|-|\beta|,k}S_\beta S_\alpha^* S_\rho S_\sigma^*)\\ &= \delta_{|\alpha|-|\beta|,k}\delta_{|\beta|-|\alpha|+|\rho|-|\sigma|,0}S_\beta S_\alpha^* S_\rho S_\sigma^*\\ &= \Phi\left(\delta_{|\rho|-|\sigma|,k}S_\beta S_\alpha^* S_\rho S_\sigma^*\right) = (S_\alpha S_\beta^*|\Phi_k S_\rho S_\sigma^*)_R.\end{aligned}$$

To address the last two statements of the Lemma, we observe that the set $\{\Phi_k\}_{k\in\mathbf{Z}}$ is norm bounded in $End_F(X)$, so the strict topology on this set coincides with the $*$-strong topology, [29, Lemma C.6]. First, if $K\subset\mathbf{Z}$ is a finite set, the sum

$$\sum_{k\in K}\Phi_k$$

is finite, and defines a projection in $End_F(X)$ by the results above. So assume K is infinite and let $\{K_i\}$ be an increasing sequence of finite subsets of K with $K = \cup_i K_i$. For $x\in X$, let

$$T_i x = \sum_{k\in K_i}\Phi_k x.$$

Choose a sequence $\{x_m\}\subset X_c$ with $x_m\to x$. Let $\varepsilon>0$ and choose m so that $\|x_m - x\|_X < \varepsilon/2$. Since x_m has finite support, for i, j sufficiently large we have

$T_i x_m - T_j x_m = 0$, and so for sufficiently large i, j

$$\begin{aligned}\|T_i x - T_j x\|_X &= \|T_i x - T_i x_m + T_i x_m - T_j x_m + T_j x_m - T_j x_m\|_X \\ &\leqslant \|T_i(x - x_m)\|_X + \|T_j(x - x_m)\|_X + \|T_i x_m - T_j x_m\|_X \\ &< \varepsilon.\end{aligned}$$

This proves the strict convergence, since the Φ_k are all self-adjoint. To prove the final statement, let $x, \{x_m\}$ be as above, $\varepsilon > 0$, and choose m so that $\|x - x_m\|_X < \varepsilon/2$. Then

$$\begin{aligned}\left\| x - \sum_{k\in\mathbf{Z}} \Phi_k x \right\|_X &= \left\| x - \sum \Phi_k x_m + \sum \Phi_k x_m - \sum \Phi_k x \right\|_X \\ &\leqslant \|x - x_m\|_X + \left\| \sum \Phi_k (x - x_m) \right\|_X < \varepsilon. \quad \square\end{aligned}$$

Corollary 4.3. *Let $x \in X$. Then with $x_k = \Phi_k x$ the sum $\sum_{k\in\mathbf{Z}} x_k$ converges in X to x.*

4.2. The Kasparov module

In this subsection we assume that E is locally finite and furthermore has no sources. That is, every vertex receives at least one edge.

Since we have the gauge action defined on X, we may use the generator of this action to define an unbounded operator $\mathcal{D}$. We will not define or study $\mathcal{D}$ from the generator point of view, rather taking a more bare-hands approach. It is easy to check that $\mathcal{D}$ as defined below is the generator of the S^1 action.

The theory of unbounded operators on C^*-modules that we require is all contained in Lance's book, [22, Chapters 9,10]. We quote the following definitions (adapted to our situation).

Definition 4.4. *Let Y be a right C^*-B-*module. A densely defined unbounded operator $\mathcal{D} : \mathrm{dom}\ \mathcal{D} \subset Y \to Y$ is a B-linear operator defined on a dense B-submodule $\mathrm{dom}\ \mathcal{D} \subset Y$. The operator $\mathcal{D}$ is closed if the graph

$$G(\mathcal{D}) = \{(x|\mathcal{D}x)_R : x \in \mathrm{dom}\ \mathcal{D}\}$$

is a closed submodule of $Y \oplus Y$.

If $\mathcal{D} : \mathrm{dom}\ \mathcal{D} \subset Y \to Y$ is densely defined and unbounded, define a submodule

$$\mathrm{dom}\ \mathcal{D}^* := \{y \in Y : \exists z \in Y \quad \text{such that } \forall x \in \mathrm{dom}\ \mathcal{D}, (\mathcal{D}x|y)_R = (x|z)_R\}.$$

Then for $y \in \operatorname{dom} \mathcal{D}^*$ define $\mathcal{D}^* y = z$. Given $y \in \operatorname{dom} \mathcal{D}^*$, the element z is unique, so $\mathcal{D}^* : \operatorname{dom} \mathcal{D}^* \to Y$, $\mathcal{D}^* y = z$ is well-defined, and moreover is closed.

Definition 4.5. Let Y be a right C^*-B-module. A densely defined unbounded operator $\mathcal{D} : \operatorname{dom} \mathcal{D} \subset Y \to Y$ is symmetric if for all $x, y \in \operatorname{dom} \mathcal{D}$

$$(\mathcal{D}x|y)_R = (x|\mathcal{D}y)_R.$$

A symmetric operator $\mathcal{D}$ is self-adjoint if $\operatorname{dom} \mathcal{D} = \operatorname{dom} \mathcal{D}^*$ (and so $\mathcal{D}$ is necessarily closed). A densely defined unbounded operator $\mathcal{D}$ is regular if $\mathcal{D}$ is closed, $\mathcal{D}^*$ is densely defined, and $(1 + \mathcal{D}^*\mathcal{D})$ has dense range.

The extra requirement of regularity is necessary in the C^*-module context for the continuous functional calculus, and is not automatic, [22, Chapter 9].

With these definitions in hand, we return to our C^*-module X.

Proposition 4.6. *Let X be the right C^*-F-module of Definition* 4.1. *Define $X_{\mathcal{D}} \subset X$ to be the linear space*

$$X_{\mathcal{D}} = \left\{ x = \sum_{k\in\mathbf{Z}} x_k \in X : \left\| \sum_{k\in\mathbf{Z}} k^2 (x_k|x_k)_R \right\| < \infty \right\}.$$

For $x = \sum_{k\in\mathbf{Z}} x_k \in X_{\mathcal{D}}$ define

$$\mathcal{D}x = \sum_{k\in\mathbf{Z}} k x_k.$$

Then $\mathcal{D} : X_{\mathcal{D}} \to X$ is a self-adjoint regular operator on X.

Remark. Any $S_\alpha S_\beta^* \in A_c$ is in $X_{\mathcal{D}}$ and

$$\mathcal{D} S_\alpha S_\beta^* = (|\alpha| - |\beta|) S_\alpha S_\beta^*.$$

Proof. First we show that $X_{\mathcal{D}}$ is a submodule. If $x \in X_{\mathcal{D}}$ and $f \in F$, in the C^*-algebra F we have

$$\begin{aligned}\sum_{k\in\mathbf{Z}} k^2 (x_k f|x_k f)_R &= \sum_{k\in\mathbf{Z}} k^2 f^*(x_k|x_k)_R f = f^* \sum_{k\in\mathbf{Z}} k^2 (x_k|x_k)_R f \\ &\leqslant f^* f \left\| \sum_{k\in\mathbf{Z}} k^2 (x_k|x_k)_R \right\|.\end{aligned}$$

So

$$\left\|\sum_{k\in\mathbf{Z}} k^2(x_k f|x_k f)_R\right\| \leqslant \|f^* f\| \left\|\sum_{k\in\mathbf{Z}} k^2(x_k|x_k)_R\right\| < \infty.$$

Observe that if $x \in X$ is a finite sum of graded components,

$$x = \sum_{k=-N}^{M} x_k,$$

then $x \in X_{\mathcal{D}}$. In particular if $P = \sum_{\text{finite}} \Phi_k$ is a finite sum of the projections Φ_k, $Px \in X_{\mathcal{D}}$ for any $x \in X$.

The following calculation shows that $\mathcal{D}$ is symmetric on its domain, so that the adjoint is densely defined. Let $x, y \in \operatorname{dom} \mathcal{D}$ and use Corollary 4.3 to write $x = \sum_k x_k$ and $y = \sum_k y_k$. Then

$$\begin{aligned}(\mathcal{D}x|y)_R &= \left(\sum_k kx_k | \sum_m y_m\right)_R = \Phi\left(\left(\sum_k kx_k\right)^*\left(\sum_m y_m\right)\right) = \Phi\left(\sum_{k,m} kx_k^* y_m\right)\\ &= \sum_k kx_k^* y_k = \Phi\left(\sum_{k,m} x_m^* k y_k\right) = \Phi\left(\left(\sum_m x_m\right)^*\left(\sum_k ky_k\right)\right)\\ &= (x|\mathcal{D}y)_R.\end{aligned}$$

Thus $\operatorname{dom} \mathcal{D} \subseteq \operatorname{dom} \mathcal{D}^*$, and so $\mathcal{D}^*$ is densely defined, and of course closed. Now choose any $x \in X$ and any $y \in \operatorname{dom}\mathcal{D}^*$. Let $P_{N,M} = \sum_{k=-N}^{M} \Phi_k$, and recall that $P_{N,M}x \in \operatorname{dom} \mathcal{D}$ for all $x \in X$. Then

$$\begin{aligned}(x|P_{N,M}\mathcal{D}^* y)_R &= (P_{N,M}x|\mathcal{D}^* y)_R = (\mathcal{D}P_{N,M}x|y)_R\\ &= \left(\sum_{k=-N}^{M} kx_k|y\right)_R = \left(x\,\middle|\,\sum_{k=-N}^{M} ky_k\right)_R.\end{aligned}$$

Since this is true for all $x \in X$ we have

$$P_{N,M}\mathcal{D}^* y = \sum_{k=-N}^{M} ky_k.$$

Letting $N, M \to \infty$, the limit on the left hand side exists by Corollary 4.3, and so the limit on the right exists, and so $y \in \operatorname{dom}\mathcal{D}$. Hence $\mathcal{D}$ is self-adjoint.

Finally, we need to show that $\mathcal{D}$ is regular. By [22, Lemma 9.8], $\mathcal{D}$ is regular if and only if the operators $\mathcal{D} \pm iId_X$ are surjective. This is straightforward though, for if $x = \sum_k x_k$ we have

$$x = \sum_{k\in\mathbf{Z}} \frac{(k \pm i)}{(k \pm i)} x_k = (\mathcal{D} \pm iId_X) \sum_{k\in\mathbf{Z}} \frac{1}{(k \pm i)} x_k.$$

The convergence of $\sum_k x_k$ ensures the convergence of $\sum_k (k \pm i)^{-1} x_k$. $\square$

There is a continuous functional calculus for self-adjoint regular operators, [22, Theorem 10.9], and we use this to obtain spectral projections for $\mathcal{D}$ at the C^*-module level. Let $f_k \in C_c(\mathbf{R})$ be 1 in a small neighbourhood of $k \in \mathbf{Z}$ and zero on $(-\infty, k - 1/2] \cup [k + 1/2, \infty)$. Then it is clear that

$$\Phi_k = f_k(\mathcal{D}).$$

That is the spectral projections of $\mathcal{D}$ are the same as the projections onto the spectral subspaces of the gauge action.

The next Lemma is the first place where we need our graph to be locally finite and have no sources.

Lemma 4.7. *Assume that the directed graph E is locally finite and has no sources. For all $a \in A$ and $k \in \mathbf{Z}$, $a\Phi_k \in End_F^0(X)$, the compact endomorphisms of the right F-module X. If $a \in A_c$ then $a\Phi_k$ is finite rank.*

Remark. The proof actually shows that for $k > 0$

$$\Phi_k = \sum_{|\rho|=k} \Theta^R_{S_\rho, S_\rho},$$

where the sum converges in the strict topology.

Proof. We will prove the lemma by first showing that for each $v \in E^0$ and $k \geqslant 0$

$$p_v \Phi_k = \sum_{s(\rho)=v,\ |\rho|=k} \Theta^R_{S_\rho, S_\rho}.$$

This is a finite sum, by the row-finiteness of E. For $k < 0$ the situation is more complicated, but a similar formula holds in that case also.

First suppose that $k \geqslant 0$ and $a = p_v \in A_c$ is the projection corresponding to a vertex $v \in E^0$. For α with $|\alpha| \geqslant k$ denote by $\underline{\alpha} = \alpha_1 \cdots \alpha_k$ and $\overline{\alpha} = \alpha_{k+1} \cdots \alpha_{|\alpha|}$. With this notation we compute the action of p_v times the rank one endomorphism $\Theta^R_{S_\rho, S_\rho}$,

$|\rho| = k$, on $S_\alpha S_\beta^*$. We find

$$p_v \Theta^R_{S_\rho, S_\rho} S_\alpha S_\beta^* = p_v S_\rho (S_\rho | S_\alpha S_\beta^*)_R = \delta_{v,s(\rho)} p_v S_\rho \Phi(S_\rho^* S_\alpha S_\beta^*)$$
$$= \delta_{v,s(\rho)} p_v S_\rho \delta_{|\alpha|-|\beta|,k} \delta_{\rho,\underline{\alpha}} S_{\overline{\alpha}} S_\beta^* = \delta_{|\alpha|-|\beta|,k} \delta_{\rho,\underline{\alpha}} \delta_{v,s(\rho)} S_\alpha S_\beta^*.$$

Of course if $|\alpha| < |\rho|$ we have

$$p_v \Theta^R_{S_\rho, S_\rho} S_\alpha S_\beta^* = p_v S_\rho \Phi(S_\rho^* S_\alpha S_\beta^*) = 0.$$

This too is $\delta_{|\alpha|-|\beta|,k} p_v S_\alpha S_\beta^*$. Thus for any α we have

$$\sum_{|\rho|=k} p_v \Theta^R_{S_\rho, S_\rho} S_\alpha S_\beta^* = \sum_{|\rho|=k, s(\rho)=v} \delta_{v,s(\rho)} \delta_{|\alpha|-|\beta|,k} \delta_{\rho,\underline{\alpha}} p_v S_\alpha S_\beta^* = \delta_{v,s(\alpha)} \delta_{|\alpha|-|\beta|,k} S_\alpha S_\beta^*.$$

This is of course the action of $p_v \Phi_k$ on $S_\alpha S_\beta^*$, and if v is a sink, $p_v \Phi_k = 0$, as it must. Since E is locally finite, the number of paths of length k starting at v is finite, and we have a finite sum. For general $a \in A_c$ we may write

$$a = \sum_{i=1}^n c_{\mu_i, v_i} S_{\mu_i} S_{v_i}^*$$

for some paths μ_i, v_i. Then $S_{\mu_i} S_{v_i}^* = S_{\mu_i} S_{v_i}^* p_{s(v_i)}$, and we may apply the above reasoning to each term in the sum defining a to get a finite sum again. Thus $a\Phi_k$ is finite rank.

Now we consider $k < 0$. Given $v \in E^0$, let $|v|_k$ denote the number of paths ρ of length $|k|$ ending at v, i.e. $r(\rho) = v$. Since we assume that E is locally finite and has no sources, $\infty > |v|_k > 0$ for each $v \in E^0$. We consider the action of the finite rank operator

$$\frac{1}{|v|_k} \sum_{|\rho|=|k|, r(\rho)=v} p_v \Theta^R_{S_\rho^*, S_\rho^*}.$$

For $S_\alpha S_\beta^* \in X$ we find

$$\frac{1}{|v|_k} \sum_{|\rho|=|k|, r(\rho)=v} p_v \Theta^R_{S_\rho^*, S_\rho^*} S_\alpha S_\beta^* = \frac{1}{|v|_k} \sum_{|\rho|=|k|, r(\rho)=v} p_v S_\rho^* \Phi(S_\rho S_\alpha S_\beta^*)$$
$$= \frac{1}{|v|_k} \sum_{|\rho|=|k|, r(\rho)=v} \delta_{|\alpha|-|\beta|,-|k|} p_v S_\rho^* S_\rho S_\alpha S_\beta^*$$
$$= \delta_{|\alpha|-|\beta|,-|k|} \delta_{v,s(\alpha)} p_v S_\alpha S_\beta^* = p_v \Phi_k S_\alpha S_\beta^*.$$

Thus $p_v\Phi_{-|k|}$ is a finite rank endomorphism, and by the argument above, we have $a\Phi_{-|k|}$ finite rank for all $a \in A_c$. To see that $a\Phi_k$ is compact for all $a \in A$, recall that every $a \in A$ is a norm limit of a sequence $\{a_i\}_{i\geqslant 0} \subset A_c$. Thus for any $k \in \mathbf{Z}$ $a\Phi_k = \lim_{i\to\infty} a_i\Phi_k$ and so is compact. $\square$

Lemma 4.8. *Let E be a locally finite directed graph with no sources. For all $a \in A$, $a(1+\mathcal{D}^2)^{-1/2}$ is a compact endomorphism of the F-module X.*

Proof. First let $a = p_v$ for $v \in E^0$. Then the sum

$$R_{v,N} := p_v \sum_{k=-N}^{N} \Phi_k(1+k^2)^{-1/2}$$

is finite rank, by Lemma 4.7. We will show that the sequence $\{R_{v,N}\}_{N\geqslant 0}$ is convergent with respect to the operator norm $\|\cdot\|_{End}$ of endomorphisms of X. Indeed, assuming that $M > N$,

$$\begin{aligned} \|R_{v,N} - R_{v,M}\|_{End} &= \left\| p_v \sum_{k=-M}^{-N} \Phi_k(1+k^2)^{-1/2} + p_v \sum_{k=N}^{M} \Phi_k(1+k^2)^{-1/2} \right\|_{End} \\ &\leqslant 2(1+N^2)^{-1/2} \to 0, \end{aligned} \tag{15}$$

since the ranges of the $p_v\Phi_k$ are orthogonal for different k. Thus, using the argument from Lemma 4.7, $a(1+\mathcal{D}^2)^{-1/2} \in End_F^0(X)$. Letting $\{a_i\}$ be a Cauchy sequence from A_c, we have

$$\|a_i(1+\mathcal{D}^2)^{-1/2} - a_j(1+\mathcal{D}^2)^{-1/2}\|_{End} \leqslant \|a_i - a_j\|_{End} = \|a_i - a_j\|_A \to 0,$$

since $\|(1+\mathcal{D}^2)^{-1/2}\| \leqslant 1$. Thus the sequence $a_i(1+\mathcal{D}^2)^{-1/2}$ is Cauchy in norm and we see that $a(1+\mathcal{D}^2)^{-1/2}$ is compact for all $a \in A$. $\square$

Proposition 4.9. *Assume that the directed graph E is locally finite and has no sources. Let $V = \mathcal{D}(1+\mathcal{D}^2)^{-1/2}$. Then (X, V) defines a class in $KK^1(A, F)$.*

Proof. We will use the approach of [19, Section 4]. We need to show that various operators belong to $End_F^0(X)$. First, $V - V^* = 0$, so $a(V - V^*)$ is compact for all $a \in A$. Also $a(1 - V^2) = a(1+\mathcal{D}^2)^{-1}$ which is compact from Lemma 4.8 and the boundedness of $(1+\mathcal{D}^2)^{-1/2}$. Finally, we need to show that $[V, a]$ is compact for all $a \in A$. First we suppose that $a \in A_c$. Then

$$\begin{aligned} [V, a] &= [\mathcal{D}, a](1+\mathcal{D}^2)^{-1/2} - \mathcal{D}(1+\mathcal{D}^2)^{-1/2}[(1+\mathcal{D}^2)^{1/2}, a](1+\mathcal{D}^2)^{-1/2} \\ &= b_1(1+\mathcal{D}^2)^{-1/2} + Vb_2(1+\mathcal{D}^2)^{-1/2}, \end{aligned}$$

where $b_1 = [\mathcal{D}, a] \in A_c$ and $b_2 = [(1+\mathcal{D}^2)^{1/2}, a]$. Provided that $b_2(1+\mathcal{D}^2)^{-1/2}$ is a compact endomorphism, Lemma 4.8 will show that $[V, a]$ is compact for all $a \in A_c$. So consider the action of $[(1+\mathcal{D}^2)^{1/2}, S_\mu S_\nu^*](1+\mathcal{D}^2)^{-1/2}$ on $x = \sum_{k\in\mathbf{Z}} x_k$. We find

$$\begin{aligned}\sum_{k\in\mathbf{Z}} &[(1+\mathcal{D}^2)^{1/2}, S_\mu S_\nu^*](1+\mathcal{D}^2)^{-1/2} x_k \\ &= \sum_{k\in\mathbf{Z}} ((1+(|\mu|-|\nu|+k)^2)^{1/2} - (1+k^2)^{1/2})(1+k^2)^{-1/2} S_\mu S_\nu^* x_k \\ &= \sum_{k\in\mathbf{Z}} f_{\mu,\nu}(k) S_\mu S_\nu^* \Phi_k x. \qquad (16)\end{aligned}$$

The function

$$f_{\mu,\nu}(k) = ((1+(|\mu|-|\nu|+k)^2)^{1/2} - (1+k^2)^{1/2})(1+k^2)^{-1/2}$$

goes to 0 as $k \to \pm\infty$, and as the $S_\mu S_\nu^* \Phi_k$ are finite rank with orthogonal ranges, the sum in (16) converges in the endomorphism norm, and so converges to a compact endomorphism. For $a \in A_c$ we write a as a finite linear combination of generators $S_\mu S_\nu^*$, and apply the above reasoning to each term in the sum to find that $[(1+\mathcal{D}^2)^{1/2}, a](1+\mathcal{D}^2)^{-1/2}$ is a compact endomorphism. Now let $a \in A$ be the norm limit of a Cauchy sequence $\{a_i\}_{i\geqslant 0} \subset A_c$. Then

$$\|[V, a_i - a_j]\|_{End} \leqslant 2\|a_i - a_j\|_{End} \to 0,$$

so the sequence $[V, a_i]$ is also Cauchy in norm, and so the limit is compact. $\square$

5. The gauge spectral triple of a graph algebra

In this section we will construct a semifinite spectral triple for those graph C^*-algebras which possess a faithful gauge invariant trace, τ. Recall from Proposition 3.9 that such traces arise from faithful graph traces.

We will begin with the right F_c module X_c. In order to deal with the spectral projections of $\mathcal{D}$ we will also assume throughout this section that E is locally finite and has no sources. This ensures, by Lemma 4.7 that for all $a \in A$ the endomorphisms $a\Phi_k$ of X are compact endomorphisms.

As in the proof of Proposition 3.9, we define a $\mathbf{C}$-valued inner product on X_c:

$$\langle x, y\rangle := \tau((x|y)_R) = \tau(\Phi(x^* y)) = \tau(x^* y).$$

This inner product is linear in the second variable. We define the Hilbert space $\mathcal{H} = L^2(X, \tau)$ to be the completion of X_c for $\langle\cdot,\cdot\rangle$. We need a few lemmas in order to obtain the ingredients of our spectral triple.

Lemma 5.1. *The C^*-algebra $A = C^*(E)$ acts on $\mathcal{H}$ by an extension of left multiplication. This defines a faithful nondegenerate $*$-representation of A. Moreover, any endomorphism of X leaving X_c invariant extends uniquely to a bounded linear operator on $\mathcal{H}$.*

Proof. The first statement follows from the proof of Proposition 3.9. Now let T be an endomorphism of X leaving X_c invariant. Then [29, Corollary 2.22],

$$(Tx|Ty)_R \leqslant \|T\|_{End}^2 (x|y)_R$$

in the algebra F. Now the norm of T as an operator on $\mathcal{H}$, denoted $\|T\|_\infty$, can be computed in terms of the endomorphism norm of T by

$$\begin{aligned} \|T\|_\infty^2 &:= \sup_{\|x\|_{\mathcal{H}} \leqslant 1} \langle Tx, Tx\rangle = \sup_{\|x\|_{\mathcal{H}} \leqslant 1} \tau((Tx|Tx)_R) \\ &\leqslant \sup_{\|x\|_{\mathcal{H}} \leqslant 1} \|T\|_{End}^2 \tau((x|x)_R) = \|T\|_{End}^2. \qquad \square \end{aligned} \tag{17}$$

Corollary 5.2. *The endomorphisms $\{\Phi_k\}_{k\in\mathbf{Z}}$ define mutually orthogonal projections on $\mathcal{H}$. For any $K \subset \mathbf{Z}$ the sum $\sum_{k\in K} \Phi_k$ converges strongly to a projection in $\mathcal{B}(\mathcal{H})$. In particular, $\sum_{k\in\mathbf{Z}} \Phi_k = Id_{\mathcal{H}}$, and for all $x \in \mathcal{H}$ the sum $\sum_k \Phi_k x$ converges in norm to x.*

Proof. As in Lemma 4.2, we can use the continuity of the Φ_k on $\mathcal{H}$, which follows from Corollary 5.1, to see that the relation $\Phi_k\Phi_l = \delta_{k,l}\Phi_k$ extends from $X_c \subset \mathcal{H}$ to $\mathcal{H}$. The strong convergence of sums of Φ_k's is just as in Lemma 4.2 after replacing the C^*-module norm with the Hilbert space norm. $\square$

Lemma 5.3. *The operator $\mathcal{D}$ restricted to X_c extends to a closed self-adjoint operator on $\mathcal{H}$.*

Proof. The proof is essentially the same as Proposition 4.6. $\square$

Lemma 5.4. *Let $\mathcal{H}, \mathcal{D}$ be as above and let $|\mathcal{D}| = \sqrt{\mathcal{D}^*\mathcal{D}} = \sqrt{\mathcal{D}^2}$ be the absolute value of $\mathcal{D}$. Then for $S_\alpha S_\beta^* \in A_c$, the operator $[|\mathcal{D}|, S_\alpha S_\beta^*]$ is well-defined on X_c, and extends to a bounded operator on $\mathcal{H}$ with*

$$\|[|\mathcal{D}|, S_\alpha S_\beta^*]\|_\infty \leqslant \Big||\alpha| - |\beta|\Big|.$$

Similarly, $\|[\mathcal{D}, S_\alpha S_\beta^*]\|_\infty = \Big||\alpha| - |\beta|\Big|.$

Proof. It is clear that $S_\alpha S_\beta^* X_c \subset X_c$, so we may define the action of the commutator on elements of X_c. Now let $x = \sum_k x_k \in \mathcal{H}$ and consider the action of $[|\mathcal{D}|, S_\alpha S_\beta^*]$ on x_k. We have

$$[|\mathcal{D}|, S_\alpha S_\beta^*]x_k = \left(\Big||\alpha| - |\beta| + k\Big| - \Big|k\Big|\right) S_\alpha S_\beta^* x_k$$

and so, by the triangle inequality,

$$\|[|\mathcal{D}|, S_\alpha S_\beta^*]x_k\|_\infty \leqslant \Big||\alpha| - |\beta|\Big| \|x_k\|_\infty,$$

since $\|S_\alpha S_\beta^*\|_\infty = 1$. As the x_k are mutually orthogonal, $\|[|\mathcal{D}|, S_\alpha S_\beta^*]\|_\infty \leqslant \Big||\alpha| - |\beta|\Big|$. The statements about $[\mathcal{D}, S_\alpha S_\beta^*] = (|\alpha| - |\beta|) S_\alpha S_\beta^*$ are easier. □

Corollary 5.5. *The algebra A_c is contained in the smooth domain of the derivation δ where for $T \in \mathcal{B}(\mathcal{H})$, $\delta(T) = [|\mathcal{D}|, T]$. That is*

$$A_c \subseteq \bigcap_{n \geqslant 0} \operatorname{dom} \delta^n.$$

Definition 5.6. Define the $*$-algebra $\mathcal{A} \subset A$ to be the completion of A_c in the δ-topology. By Lemma 2.5, $\mathcal{A}$ is Fréchet and stable under the holomorphic functional calculus.

Lemma 5.7. *If $a \in \mathcal{A}$ then $[\mathcal{D}, a] \in \mathcal{A}$ and the operators $\delta^k(a)$, $\delta^k([\mathcal{D}, a])$ are bounded for all $k \geqslant 0$. If $\phi \in F \subset \mathcal{A}$ and $a \in \mathcal{A}$ satisfy $\phi a = a = a\phi$, then $\phi[\mathcal{D}, a] = [\mathcal{D}, a] = [\mathcal{D}, a]\phi$. The norm closed algebra generated by $\mathcal{A}$ and $[\mathcal{D}, \mathcal{A}]$ is A. In particular, $\mathcal{A}$ is quasi-local.*

We leave the straightforward proofs of these statements to the reader.

5.1. Traces and compactness criteria

We still assume that E is a locally finite graph with no sources and that τ is a faithful semifinite lower semicontinuous gauge invariant trace on $C^*(E)$. We will define a von Neumann algebra $\mathcal{N}$ with a faithful semifinite normal trace $\tilde{\tau}$ so that $\mathcal{A} \subset \mathcal{N} \subset \mathcal{B}(\mathcal{H})$, where $\mathcal{A}$ and $\mathcal{H}$ are as defined in the last subsection. Moreover the operator $\mathcal{D}$ will be affiliated to $\mathcal{N}$. The aim of this subsection will then be to prove the following result.

Theorem 5.8. *Let E be a locally finite graph with no sources, and let τ be a faithful, semifinite, gauge invariant, lower semicontinuous trace on $C^*(E)$. Then $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a QC^∞, $(1, \infty)$-summable, odd, local, semifinite spectral triple (relative to $(\mathcal{N}, \tilde{\tau})$).*

For all $a \in \mathcal{A}$, the operator $a(1+\mathcal{D}^2)^{-1/2}$ is not trace class. If $v \in E^0$ has no sinks downstream

$$\tilde{\tau}_\omega(p_v(1+\mathcal{D}^2)^{-1/2}) = 2\tau(p_v),$$

where $\tilde{\tau}_\omega$ is any Dixmier trace associated to $\tilde{\tau}$.

We require the definitions of $\mathcal{N}$ and $\tilde{\tau}$, along with some preliminary results.

Definition 5.9. Let $End_F^{00}(X_c)$ denote the algebra of finite rank operators on X_c acting on $\mathcal{H}$. Define $\mathcal{N} = (End_F^{00}(X_c))''$, and let $\mathcal{N}_+$ denote the positive cone in $\mathcal{N}$.

Definition 5.10. Let $T \in \mathcal{N}$ and $\mu \in E^*$. Let $|v|_k =$ the number of paths of length k with range v, and define for $|\mu| \neq 0$

$$\omega_\mu(T) = \langle S_\mu, TS_\mu\rangle + \frac{1}{|r(\mu)|_{|\mu|}}\langle S_\mu^*, TS_\mu^*\rangle.$$

For $|\mu| = 0$, $S_\mu = p_v$, for some $v \in E^0$, set $\omega_\mu(T) = \langle S_\mu, TS_\mu\rangle$. Define

$$\tilde{\tau}\colon \mathcal{N}_+ \to [0,\infty], \quad \text{by} \quad \tilde{\tau}(T) = \lim_{L\uparrow} \sum_{\mu\in L\subset E^*} \omega_\mu(T),$$

where L is in the net of finite subsets of E^*.

Remark. For $T, S \in \mathcal{N}_+$ and $\lambda \geqslant 0$ we have

$$\tilde{\tau}(T+S) = \tilde{\tau}(T) + \tilde{\tau}(S) \quad \text{and} \quad \tilde{\tau}(\lambda T) = \lambda\tilde{\tau}(T) \text{ where } 0\times\infty = 0.$$

Proposition 5.11. *The function $\tilde{\tau} : \mathcal{N}_+ \to [0,\infty]$ defines a faithful normal semifinite trace on $\mathcal{N}$. Moreover,*

$$End_F^{00}(X_c) \subset \mathcal{N}_{\tilde{\tau}} := \mathrm{span}\{T \in \mathcal{N}_+ : \tilde{\tau}(T) < \infty\},$$

the domain of definition of $\tilde{\tau}$, and

$$\tilde{\tau}(\Theta^R_{x,y}) = \langle y, x\rangle = \tau(y^*x), \quad x, y \in X_c.$$

Proof. First, since $\tilde{\tau}$ is defined as the limit of an increasing net of sums of positive vector functionals, $\tilde{\tau}$ is a positive ultra-weakly lower semicontinuous weight on $\mathcal{N}_+$, [18], that is a normal weight. Now observe (using the fact that $p_v\Phi_k$ is a projection for all $k \in \mathbf{Z}$ and $v \in E^0$) that for any vertex $v \in E^0$, $k \in \mathbf{Z}$ and $T \in \mathcal{N}_+$

$$\tilde{\tau}(p_v\Phi_k T p_v \Phi_k) = \langle \Phi_k p_v, T\Phi_k p_v\rangle + \sum_{s(\mu)=v} \langle \Phi_k S_\mu, T\Phi_k S_\mu\rangle + \sum_{r(\mu)=v} \frac{1}{|r(\mu)|_{|\mu|}} \langle \Phi_k S_\mu^*, T\Phi_k S_\mu^*\rangle.$$

If $k = 0$ this is equal to $\langle p_v, Tp_v\rangle < \infty$. If $k > 0$ we find

$$\begin{aligned}\tilde{\tau}(p_v\Phi_k T p_v \Phi_k) &= \sum_{s(\mu)=v,|\mu|=k} \langle S_\mu, TS_\mu\rangle \leqslant \|T\| \sum_{s(\mu)=v,|\mu|=k} \tau(S_\mu^* S_\mu) \\ &= \|T\| \sum_{s(\mu)=v,|\mu|=k} \tau(p_{r(\mu)}) \leqslant \|T\|\tau(p_v) < \infty,\end{aligned}$$

the last inequality following from the fact that τ arises from a graph trace, by Proposition 3.9, and Eqs. (7) and (8). Similarly, if $k < 0$

$$\begin{aligned}\tilde{\tau}(p_v\Phi_k T p_v \Phi_k) &= \sum_{r(\mu)=v,|\mu|=|k|} \frac{1}{|v|_{|k|}} \langle S_\mu^*, TS_\mu^*\rangle \leqslant \|T\| \sum_{r(\mu)=v,|\mu|=|k|} \frac{1}{|v|_{|k|}} \tau(S_\mu^* S_\mu) \\ &= \|T\| \sum_{r(\mu)=v,|\mu|=k} \frac{1}{|v|_{|k|}} \tau(p_{r(\mu)}) = \|T\|\tau(p_v) < \infty.\end{aligned}$$

Hence $\tilde{\tau}$ is a finite positive function on each $p_v\Phi_k\mathcal{N}p_v\Phi_k$. Taking limits over finite sums of vertex projections, $p = p_{v_1} + \cdots + p_{v_n}$, converging to the identity, and finite sums $P = \Phi_{k_1} + \cdots + \Phi_{k_m}$, we have for $T \in \mathcal{N}_+$

$$\lim_{pP\nearrow 1} \sup \tilde{\tau}(pPTpP) \leqslant \tilde{\tau}(T) \leqslant \lim_{pP\nearrow 1} \inf \tilde{\tau}(pPTpP),$$

the first inequality following from the definition of $\tilde{\tau}$, and the latter from the ultra-weak lower semicontinuity of $\tilde{\tau}$, so for $T \in \mathcal{N}_+$

$$\lim_{pP\nearrow 1} \tilde{\tau}(pPTpP) = \tilde{\tau}(T). \tag{18}$$

For $x \in X_c \subset \mathcal{H}$, $\Theta^R_{x,x} \geqslant 0$ and so we compute

$$\begin{aligned}\tilde{\tau}(\Theta^R_{x,x}) &= \sup_F \sum_{\mu\in F} \langle S_\mu, x(x|S_\mu)_R\rangle + \frac{1}{|r(\mu)|_{|\mu|}} \langle S_\mu^*, x(x|S_\mu^*)_R\rangle \\ &= \sup_F \sum_{\mu\in F} \tau(\Phi(S_\mu^* x\Phi(x^* S_\mu))) + \frac{1}{|r(\mu)|_{|\mu|}} \tau(\Phi(S_\mu x\Phi(x^* S_\mu^*))).\end{aligned}$$

Now since $x \in X_c$, there are only finitely many ω_μ which are nonzero on $\Theta^R_{x,x}$, so this is always a finite sum, and $\tilde{\tau}(\Theta^R_{x,x}) < \infty$.

To compute $\Theta^R_{x,y}$, suppose that $x = S_\alpha S_\beta^*$ and $y = S_\sigma S_\rho^*$. Then $(y|S_\mu)_R = \Phi(S_\rho S_\sigma^* S_\mu)$ and this is zero unless $|\sigma| = |\mu| + |\rho|$. In this case, $|\sigma| \geqslant |\mu|$ and we write $\sigma = \underline{\sigma}\overline{\sigma}$ where $|\underline{\sigma}| = |\mu|$. Similarly, $(y|S_\mu^*)_R = \Phi(S_\rho S_\sigma^* S_\mu^*)$ is zero unless $|\rho| = |\sigma| + |\mu|$. We also require the computation

$$S_\alpha S_\beta^* S_\rho S_\sigma^* S_\mu S_\mu^* = S_\alpha S_\beta^* S_\rho S_\sigma^* \delta_{\underline{\sigma},\mu}, \quad |\sigma| \geqslant |\mu|,$$

$$S_\alpha S_\beta^* S_\rho S_\sigma^* S_\mu^* S_\mu = S_\alpha S_\beta^* S_\rho S_\sigma^* \delta_{r(\mu),s(\sigma)} \quad |\mu| \geqslant |\sigma|.$$

Now we can compute for $|\rho| \neq |\sigma|$, so that only one of the sums over $|\mu| = \pm(|\sigma| - |\rho|)$ in the next calculation is nonempty:

$$\begin{aligned}
\tilde{\tau}(\Theta^R_{x,y}) &= \sum_\mu \tau(S_\mu^* x \Phi(y^* S_\mu)) + \sum_\mu \frac{1}{|r(\mu)|_{|\mu|}} \tau(S_\mu x \Phi(y^* S_\mu^*)) \\
&= \sum_{|\mu|=|\sigma|-|\rho|} \tau(xy^* S_\mu S_\mu^*) + \sum_{|\mu|=|\rho|-|\sigma|} \frac{1}{|r(\mu)|_{|\mu|}} \tau(xy^* S_\mu^* S_\mu) \\
&= \sum_{|\mu|=|\sigma|-|\rho|} \tau(xy^* \delta_{\underline{\sigma},\mu}) + \sum_{|\mu|=|\rho|-|\sigma|, r(\mu)=s(\sigma)} \frac{1}{|r(\mu)|_{|\mu|}} \tau(xy^*) \\
&= \tau(xy^*) = \tau(y^* x) = \tau((y|x)_R) = \langle y, x \rangle.
\end{aligned}$$

When $|\sigma| = |\rho|$, we have

$$\tilde{\tau}(\Theta^R_{x,y}) = \sum_{v \in E^0} \tau(\Phi(p_v x y^* p_v)) = \sum_{v \in E^0} \tau(y^* p_v x)$$

and the same conclusion is obtained as above. By linearity, whenever $x, y \in X_c$, $\tilde{\tau}(\Theta^R_{x,y}) = \tau((y|x)_R)$. For any two $\Theta^R_{x,y}$, $\Theta^R_{w,z} \in End^{00}_F(X_c)$ we find

$$\begin{aligned}
\tilde{\tau}(\Theta^R_{w,z}\Theta^R_{x,y}) &= \tilde{\tau}(\Theta^R_{w(z|x)_R,y}) = \tau((y|w(z|x)_R)_R) = \tau((y|w)_R(z|x)_R) \\
&= \tau((z|x)_R(y|w)_R) = \tilde{\tau}(\Theta^R_{x(y|w)_R,z}) = \tilde{\tau}(\Theta^R_{x,y}\Theta^R_{w,z}).
\end{aligned}$$

Hence by linearity, $\tilde{\tau}$ is a trace on $End^{00}_F(X_c) \subset \mathcal{N}$.

We saw previously that $\tilde{\tau}$ is finite on $pP\mathcal{N}pP$ whenever p is a finite sum of vertex projections p_v and P is a finite sum of the spectral projections Φ_k.

Since $\tilde{\tau}$ is ultra-weakly lower semicontinuous on $pP\mathcal{N}_+pP$, it is completely additive in the sense of [18, Definition 7.1.1], and therefore is normal by [18, Theorem 7.1.12], which is to say, ultra-weakly continuous.

The algebra $End_F^{00}(X_c)$ is strongly dense in $\mathcal{N}$, so $pPEnd_F^{00}(X_c)pP$ is strongly dense in $pP\mathcal{N}pP$. Let $T \in pP\mathcal{N}pP$, and choose a bounded net T_i, converging $*$-strongly to T, with $T_i \in pPEnd_F^{00}(X_c)pP$. Then, since multiplication is jointly continuous on bounded sets in the $*$-strong topology,

$$\tilde{\tau}(TT^*) = \lim_i \tilde{\tau}(T_i T_i^*) = \lim_i \tilde{\tau}(T_i^* T_i) = \tilde{\tau}(T^*T).$$

Hence $\tilde{\tau}$ is a trace on each $pP\mathcal{N}pP$ and so on $\cup_{pP} pP\mathcal{N}pP$, where the union is over all finite sums p of vertex projections and finite sums P of the Φ_k.

Next we want to show that $\tilde{\tau}$ is semifinite, so for all $T \in \mathcal{N}$ we want to find a net $R_i \geqslant 0$ with $R_i \leqslant T^*T$ and $\tilde{\tau}(R_i) < \infty$. Now

$$\lim_{pP\nearrow 1} T^*pPT = T^*T, \quad T^*pPT \leqslant T^*T$$

and we just need to show that $\tilde{\tau}(T^*pPT) < \infty$. It suffices to show this for $pP = p_v\Phi_k$, $v \in E^0$, $k \in \mathbf{Z}$. In this case we have (with q a finite sum of vertex projections and Q a finite sum of Φ_k)

$$\begin{aligned}
\tilde{\tau}(T^*p_v\Phi_k T) &= \lim_{qQ\nearrow 1} \tilde{\tau}(qQT^*p_v\Phi_k TqQ) \quad \text{by Eq.(18)} \\
&= \lim_{qQ\nearrow 1} \tilde{\tau}(qQT^*qQp_v\Phi_k TqQ) \quad \text{eventually } qQp_v\Phi_k = p_v\Phi_k \\
&= \lim_{qQ\nearrow 1} \tilde{\tau}(qQp_v\Phi_k TqQT^*qQp_v\Phi_k) \quad \tilde{\tau} \text{ is a trace on } qQ\mathcal{N}qQ \\
&= \lim_{qQ\nearrow 1} \tilde{\tau}(p_v\Phi_k TqQT^*p_v\Phi_k) = \tilde{\tau}(p_v\Phi_k TT^*p_v\Phi_k) < \infty.
\end{aligned}$$

Thus $\tilde{\tau}$ is semifinite, normal weight on $\mathcal{N}_+$, and is a trace on a dense subalgebra. Now let $T \in \mathcal{N}$. By the above

$$\tilde{\tau}(T^*pPT) = \tilde{\tau}(pPTT^*pP). \tag{19}$$

By lower semicontinuity and the fact that $T^*pPT \leqslant T^*T$, the limit of the left-hand side of Eq. (19) as $pP \to 1$ is $\tilde{\tau}(T^*T)$. By Eq. (18), the limit of the right-hand side is $\tilde{\tau}(TT^*)$. Hence $\tilde{\tau}(T^*T) = \tilde{\tau}(TT^*)$ for all $T \in \mathcal{N}$, and $\tilde{\tau}$ is a normal, semifinite trace on $\mathcal{N}$. $\square$

Notation: If $g : E^0 \to \mathbf{R}_+$ is a faithful graph trace, we shall write τ_g for the associated semifinite trace on $C^*(E)$, and $\tilde{\tau}_g$ for the associated faithful, semifinite, normal trace on $\mathcal{N}$ constructed above.

Lemma 5.12. *Let E be a locally finite graph with no sources and a faithful graph trace g. Let $v \in E^0$ and $k \in \mathbf{Z}$. Then*

$$\tilde{\tau}_g(p_v\Phi_k) \leqslant \tau_g(p_v)$$

with equality when $k \leqslant 0$ or when $k > 0$ and there are no sinks within k vertices of v.

Proof. Let $k \geqslant 0$. Then, by Lemma 4.7 we have

$$\begin{aligned}\tilde{\tau}_g(p_v\Phi_k) &= \tilde{\tau}_g\left(p_v \sum_{|\rho|=k} \Theta^R_{S_\rho,S_\rho}\right) = \tilde{\tau}_g\left(\sum_{|\rho|=k} \Theta^R_{p_vS_\rho,S_\rho}\right)\\ &= \tau_g\left(\sum_{|\rho|=k} (S_\rho|p_vS_\rho)_R\right) = \tau_g\left(\sum_{|\rho|=k} \Phi(S_\rho^* p_v S_\rho)\right)\\ &= \tau_g\left(\sum_{|\rho|=k,s(\rho)=v} S_\rho^* S_\rho\right) = \tau_g\left(\sum_{|\rho|=k,s(\rho)=v} p_{r(\rho)}\right).\end{aligned}$$

Now $\tau_g(p_v) = g(v)$ where g is the graph trace associated to τ_g, Proposition 3.9, and Eq. (8) shows that

$$g(v) = \sum_{|\rho| \preccurlyeq k,s(\rho)=v} g(r(\rho)) \geqslant \sum_{|\rho|=k,s(\rho)=v} g(r(\rho)) \tag{20}$$

with equality provided there are no sinks within k vertices of v (always true for $k = 0$). Hence for $k \geqslant 0$ we have $\tilde{\tau}_g(p_v\Phi_k) \leqslant \tau_g(p_v)$, with equality when there are no sinks within k vertices of v. For $k < 0$ we proceed as above and observe that there is at least one path of length $|k|$ ending at v since E has no sources. Then

$$\begin{aligned}\tilde{\tau}_g(p_v\Phi_k) &= \frac{1}{|v|_k} \sum_{|\rho|=|k|,\ r(\rho)=v} \tau_g(S_\rho p_v S_\rho^*) = \frac{1}{|v|_k} \sum_{|\rho|=|k|,\ r(\rho)=v} \tau_g(S_\rho^* S_\rho p_v)\\ &= \frac{1}{|v|_k} \sum_{|\rho|=|k|,\ r(\rho)=v} \tau_g(p_v) = \tau_g(p_v). \qquad \Box\end{aligned} \tag{21}$$

Proposition 5.13. *Assume that the directed graph E is locally finite, has no sources and has a faithful graph trace g. For all $a \in A_c$ the operator $a(1+\mathcal{D}^2)^{-1/2}$ is in the ideal $\mathcal{L}^{(1,\infty)}(\mathcal{N}, \tilde{\tau}_g)$.*

Proof. It suffices to show that $a(1+\mathcal{D}^2)^{-1/2} \in \mathcal{L}^{(1,\infty)}(\mathcal{N}, \tilde{\tau}_g)$ for a vertex projection $a = p_v$ for $v \in E^0$, and extending to more general $a \in A_c$ using the arguments of

Lemma 4.7. Since $p_v\Phi_k$ is a projection for all $v \in E^0$ and $k \in \mathbf{Z}$, we may compute the Dixmier trace using the partial sums (over $k \in \mathbf{Z}$) defining the trace of $p_v(1+\mathcal{D}^2)^{-1/2}$. For the partial sums with $k \geqslant 0$, Lemma 5.12 gives us

$$\tilde{\tau}_g\left(p_v \sum_0^N (1+k^2)^{-1/2}\Phi_k\right) \leqslant \sum_{k=0}^N (1+k^2)^{-1/2}\tau_g(p_v). \tag{22}$$

We have equality when there are no sinks within N vertices of v. For the partial sums with $k < 0$ Lemma 5.12 gives

$$\sum_{k=-N}^{-1} (1+k^2)^{-1/2}\tilde{\tau}_g(p_v\Phi_k) = \sum_{k=-N}^{-1} (1+k^2)^{-1/2}\tau_g(p_v),$$

and the sequence

$$\frac{1}{\log 2N+1} \sum_{k=-N}^{N} (1+k^2)^{-1/2}\tilde{\tau}_g(p_v\Phi_k)$$

is bounded. Hence $p_v(1+\mathcal{D}^2)^{-1/2} \in \mathcal{L}^{(1,\infty)}$ and for any ω-limit we have

$$\tilde{\tau}_{g\omega}(p_v(1+\mathcal{D}^2)^{-1/2}) = \omega\text{-lim}\, \frac{1}{\log 2N+1} \sum_{k=-N}^{N} (1+k^2)^{-1/2}\tilde{\tau}_g(p_v\Phi_k).$$

When there are no sinks downstream from v, we have equality in Eq. (22) for any $v \in E^0$ and so

$$\tilde{\tau}_{g\omega}(p_v(1+\mathcal{D}^2)^{-1/2}) = 2\tau_g(p_v). \qquad \square$$

Remark. Using Proposition 2.11, one can check that

$$res_{s=0}\tilde{\tau}_g(p_v(1+\mathcal{D}^2)^{-1/2-s}) = \tfrac{1}{2}\,\tilde{\tau}_{g\omega}(p_v(1+\mathcal{D}^2)^{-1/2}). \tag{23}$$

We will require this formula when we apply the local index theorem.

Corollary 5.14. *Assume E is locally finite, has no sources and has a faithful graph trace g. Then for all $a \in A$, $a(1+\mathcal{D}^2)^{-1/2} \in \mathcal{K}_{\mathcal{N}}$.*

Proof. (of Theorem 5.8.) That we have a QC^∞ spectral triple follows from Corollary 5.5, Lemma 5.7 and Corollary 5.14. The properties of the von Neumann algebra $\mathcal{N}$

and the trace $\tilde{\tau}$ follow from Proposition 5.11. The $(1, \infty)$-summability and the value of the Dixmier trace comes from Proposition 5.13. The locality of the spectral triple follows from Lemma 5.7. $\square$

6. The index pairing

Having constructed semifinite spectral triples for graph C^*-algebras arising from locally finite graphs with no sources and a faithful graph trace, we can apply the semifinite local index theorem described in [5]. See also [6,9,15].

There is a C^*-module index, which takes its values in the K-theory of the core which is described in the appendix. The numerical index is obtained by applying the trace $\tilde{\tau}$ to the difference of projections representing the K-theory class. Thus for any unitary u in a matrix algebra over the graph algebra A

$$\langle [u], [(\mathcal{A}, \mathcal{H}, \mathcal{D})] \rangle \in \tilde{\tau}_*(K_0(F)).$$

We compute this pairing for unitaries arising from loops (with no exit), which provide a set of generators of $K_1(\mathcal{A})$. To describe the K-theory of the graphs we are considering, recall the notion of ends introduced in Definition 3.6.

Lemma 6.1. *Let $C^*(E)$ be a graph C^*-algebra such that no loop in the locally finite graph E has an exit. Then,*

$$K_0(C^*(E)) = \mathbf{Z}^{\#ends}, \quad K_1(C^*(E)) = \mathbf{Z}^{\#loops}.$$

Proof. This follows from the continuity of K_* and [28, Corollary 5.3]. $\square$

If $A = C^*(E)$ is nonunital, we will denote by A^+ the algebra obtained by adjoining a unit to A; otherwise we let A^+ denote A.

Definition 6.2. Let E be a locally finite graph such that $C^*(E)$ has a faithful graph trace g. Let L be a loop in E, and denote by $p_1, \dots, p_n$ the projections associated to the vertices of L and $S_1, \dots, S_n$ the partial isometries associated to the edges of L, labelled so that $S_n^* S_n = p_1$ and

$$S_i^* S_i = p_{i+1}, \; i = 1, \dots, n-1, \quad S_i S_i^* = p_i, \; i = 1, \dots, n.$$

Lemma 6.3. *Let $A = C^*(E)$ be a graph C^*-algebra with faithful graph trace g. For each loop L in E we obtain a unitary in A^+,*

$$u = 1 + S_1 + S_2 + \cdots + S_n - (p_1 + p_2 + \cdots + p_n),$$

whose K_1 class does not vanish. Moreover, distinct loops give rise to distinct K_1 classes, and we obtain a complete set of generators of K_1 in this way.

Proof. The proof that u is unitary is a simple computation. The K_1 class of u is the generator of a copy of $K_1(S^1)$ in $K_1(C^*(E))$, as follows from [28]. Distinct loops give rise to distinct copies of $K_1(S^1)$, since no loop has an exit. □

Proposition 6.4. *Let E be a locally finite graph with no sources and a faithful graph trace g and $A = C^*(E)$. The pairing between the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ of Theorem* 5.8 *with $K_1(A)$ is given on the generators of Lemma* 6.3 *by*

$$\langle [u], [(\mathcal{A}, \mathcal{H}, \mathcal{D})] \rangle = -\sum_{i=1}^{n} \tau_g(p_i) = -n\tau_g(p_1).$$

Proof. The semifinite local index theorem, [5] provides a general formula for the Chern character of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. In our setting it is given by a one-cochain

$$\phi_1(a_0, a_1) = res_{s=0}\sqrt{2\pi i}\,\tilde{\tau}_g(a_0[\mathcal{D}, a_1](1+\mathcal{D}^2)^{-1/2-s})$$

and the pairing (spectral flow) is given by

$$sf(\mathcal{D}, u\mathcal{D}u^*) = \langle [u], (\mathcal{A}, \mathcal{H}, \mathcal{D}) \rangle = \frac{1}{\sqrt{2\pi i}}\,\phi_1(u, u^*).$$

Now $[\mathcal{D}, u^*] = -\sum S_i^*$ and $u[\mathcal{D}, u^*] = -\sum_{i=1}^n p_i$. Using Eq. (23) and Proposition 5.13,

$$sf(\mathcal{D}, u\mathcal{D}u^*) = -res_{s=0}\tilde{\tau}_g\left(\sum_{i=1}^{n} p_i(1+\mathcal{D}^2)^{-1/2-s}\right) = -\sum_{i=1}^{n}\tau_g(p_i) = -n\tau_g(p_1),$$

the last equalities following since all the p_i have equal trace and there are no sinks 'downstream' from any p_i, since no loop has an exit. □

Remark. The C^*-algebra of the graph consisting of a single edge and single vertex is $C(S^1)$ (we choose Lebesgue measure as our trace, normalised so that $\tau(1) = 1$). For this example, the spectral triple we have constructed is the Dirac triple of the circle, $(C^\infty(S^1), L^2(S^1), \frac{1}{i}\frac{d}{d\theta})$, (as can be seen from Corollary 6.6.) The index theorem above gives the correct normalisation for the index pairing on the circle. That is, if we denote by z the unitary coming from the construction of Lemma 6.3 applied to this graph, then $\langle [\bar{z}], (\mathcal{A}, \mathcal{H}, \mathcal{D}) \rangle = 1$.

Proposition 6.5. *Let E be a locally finite graph with no sources and a faithful graph trace g, and $A = C^*(E)$. The pairing between the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ of Theorem*

5.8 *with* $K_1(A)$ *can be computed as follows. Let* P *be the positive spectral projection for* $\mathcal{D}$, *and perform the* C^* *index pairing of Proposition* A.1:

$$K_1(A) \times KK^1(A, F) \to K_0(F), \quad [u] \times [(X, P)] \to [\ker PuP] - [\operatorname{coker} PuP].$$

Then we have

$$sf(\mathcal{D}, u\mathcal{D}u^*) = \tilde{\tau}_g(\ker PuP) - \tilde{\tau}_g(\operatorname{coker} PuP) = \tilde{\tau}_{g*}([\ker PuP] - [\operatorname{coker} PuP]).$$

Proof. It suffices to prove this on the generators of K_1 arising from loops L in E. Let $u = 1 + \sum_i S_i - \sum_i p_i$ be the corresponding unitary in A^+ defined in Lemma 6.3. We will show that $\ker PuP = \{0\}$ and that $\operatorname{coker} PuP = \sum_{i=1}^n p_i \Phi_1$. For $a \in PX$ write $a = \sum_{m \geqslant 1} a_m$. For each $m \geqslant 1$ write $a_m = \sum_{i=1}^n p_i a_m + (1 - \sum_{i=1}^n p_i) a_m$. Then

$$\begin{aligned}
PuPa_m &= P\left(1 - \sum_{i=1}^n p_i + \sum_{i=1}^n S_i\right) a_m \\
&= P\left(1 - \sum^n p_i + \sum^n S_i\right)\left(\sum^n p_i a_m\right) \\
&\quad + P\left(1 - \sum^n p_i + \sum^n S_i\right)\left(1 - \sum^n p_i\right) a_m \\
&= P\sum^n S_i a_m + P\left(1 - \sum^n p_i\right) a_m \\
&= \sum^n S_i a_m + \left(1 - \sum^n p_i\right) a_m.
\end{aligned}$$

It is clear from this computation that $PuPa_m \neq 0$ for $a_m \neq 0$.

Now suppose $m \geqslant 2$. If $\sum_{i=1}^n p_i a_m = a_m$ then $a_m = \lim_N \sum_{k=1}^N S_{\mu_k} S_{\nu_k}^*$ with $|\mu_k| - |\nu_k| = m \geqslant 2$ and $S_{\mu_{k1}} = S_i$ for some i. So we can construct b_{m-1} from a_m by removing the initial S_i's. Then $a_m = \sum_{i=1}^n S_i b_{m-1}$, and $\sum_{i=1}^n p_i b_{m-1} = b_{m-1}$. For arbitrary a_m, $m \geqslant 2$, we can write $a_m = \sum_i p_i a_m + (1 - \sum_i p_i) a_m$, and so

$$\begin{aligned}
a_m &= \sum^n p_i a_m + \left(1 - \sum^n p_i\right) a_m \\
&= \sum^n S_i b_{m-1} + \left(1 - \sum^n p_i\right) a_m \quad \text{and by adding zero} \\
&= \sum^n S_i b_{m-1} + \left(1 - \sum^n p_i\right) b_{m-1} + \left(\sum^n S_i + \left(1 - \sum^n p_i\right)\right)\left(1 - \sum^n p_i\right) a_m
\end{aligned}$$

$$= ub_{m-1} + u\left(1 - \sum^{n} p_i\right) a_m$$

$$= PuPb_{m-1} + PuP\left(1 - \sum^{n} p_i\right) a_m.$$

Thus PuP maps onto $\sum_{m\geqslant 2} \Phi_m X$.

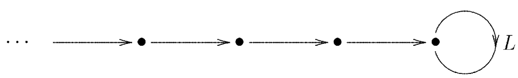
For $m = 1$, if we try to construct b_0 from $\sum_{i=1}^{n} p_i a_1$ as above, we find $PuPb_0 = 0$ since $Pb_0 = 0$. Thus $\mathrm{coker} PuP = \sum^{n} p_i \Phi_1 X$. By Proposition 6.4, the pairing is then

$$sf(\mathcal{D}, u\mathcal{D}u^*) = -\sum^{n} \tau_g(p_i) = -\tilde{\tau}_g\left(\sum^{n} p_i \Phi_1\right)$$
$$= -\tilde{\tau}_{g*}([\mathrm{coker}\, PuP]) = -\tilde{\tau}_g(\mathrm{coker}\, PuP). \tag{24}$$

Thus we can recover the numerical index using $\tilde{\tau}_g$ and the C^*-index. $\square$

The following example shows that the semifinite index provides finer invariants of directed graphs than those obtained from the ordinary index. The ordinary index computes the pairing between the K-theory and K-homology of $C^*(E)$, while the semifinite index also depends on the core and the gauge action.

Corollary 6.6 (*Example*). *Let $C^*(E_n)$ be the algebra determined by the graph*



where the loop L has n edges. Then $C^*(E_n) \cong C(S^1) \otimes \mathcal{K}$ for all n, but n is an invariant of the pair of algebras $(C^*(E_n), F_n)$ where F_n is the core of $C^*(E_n)$.

Proof. Observe that the graph E_n has a one parameter family of faithful graph traces, specified by $g(v) = r \in \mathbf{R}_+$ for all $v \in E^0$.

First consider the case where the graph consists only of the loop L. The C^*-algebra A of this graph is isomorphic to $M_n(C(S^1))$, via

$$S_i \to e_{i,i+1}, \ \ i = 1, \ldots, n-1, \quad S_n \to id_{S^1} e_{n,1},$$

where the $e_{i,j}$ are the standard matrix units for $M_n(\mathbf{C})$, [1]. The unitary

$$S_1 S_2 \cdots S_n + S_2 S_3 \cdots S_1 + \cdots + S_n S_1 \cdots S_{n-1}$$

is mapped to the orthogonal sum $id_{S^1}e_{1,1} \oplus id_{S^1}e_{2,2} \oplus \cdots \oplus id_{S^1}e_{n,n}$. The core F of A is $\mathbf{C}^n = \mathbf{C}[p_1, \ldots, p_n]$. Since $KK^1(A, F)$ is equal to

$$\oplus^n KK^1(A, \mathbf{C}) = \oplus^n KK^1(M_n(C(S^1)), \mathbf{C}) = \oplus^n K^1(C(S^1)) = \mathbf{Z}^n$$

we see that n is the rank of $KK^1(A, F)$ and so an invariant, but let us link this to the index computed in Propositions 6.4 and 6.5 more explicitly. Let $\phi : C(S^1) \to A$ be given by $\phi(id_{S^1}) = S_1S_2 \cdots S_n \oplus \sum_{i=2}^n e_{i,i}$. We observe that $\mathcal{D} = \sum_{i=1}^n p_i\mathcal{D}$ because the 'off-diagonal' terms are $p_i\mathcal{D}p_j = \mathcal{D}p_ip_j = 0$. Since $S_1S_1^* = S_n^*S_n = p_1$, we find (with P the positive spectral projection of $\mathcal{D}$)

$$\phi^*(X, P) = (p_1X, p_1Pp_1) \oplus \text{ degenerate module } \in KK^1(C(S^1), F).$$

Now let $\psi : F \to \mathbf{C}^n$ be given by $\psi(\sum_j z_jp_j) = (z_1, z_2, \ldots, z_n)$. Then

$$\psi_*\phi^*(X, P) = \oplus_{j=1}^n (p_1Xp_j, p_1Pp_1) \in \oplus^n K^1(C(S^1)).$$

Now $X \cong M_n(C(S^1))$, so $p_1Xp_j \cong C(S^1)$ for each $j = 1, \ldots, n$. It is easy to check that $p_1\mathcal{D}p_1$ acts by $\frac{1}{i}\frac{d}{d\theta}$ on each p_1Xp_j, and so our Kasparov module maps to

$$\psi_*\phi^*(X, P) = \oplus^n (C(S^1), P_{\frac{1}{i}\frac{d}{d\theta}}) \in \oplus^n K^1(C(S^1)),$$

where $P_{\frac{1}{i}\frac{d}{d\theta}}$ is the positive spectral projection of $\frac{1}{i}\frac{d}{d\theta}$. The pairing with id_{S^1} is nontrivial on each summand, since $\phi(id_{S^1}) = S_1 \cdots S_n \oplus \sum_{i=2}^n e_{i,i}$ is a unitary mapping p_1Xp_j to itself for each j. So we have, [16],

$$\begin{aligned} id_{S^1} \times \psi_*\phi^*(X, P) &= \sum_{j=1}^n \mathit{Index}(Pid_{S^1}P \colon p_1PXp_j \to p_1PXp_j) \\ &= -\sum_{j=1}^n [p_j] \in K_0(\mathbf{C}^n). \end{aligned} \tag{25}$$

By Proposition 6.5, applying the trace to this index gives $-n\tau_g(p_1)$. Of course in Proposition 6.5 we used the unitary $S_1 + S_2 + \cdots + S_n$, however in $K_1(A)$

$$[S_1S_2 \cdots S_n] = [S_1 + S_2 + \cdots + S_n] = [id_{S^1}].$$

To see this, observe that

$$(S_1 + \cdots + S_n)^n = S_1S_2 \cdots S_n + S_2S_3 \cdots S_1 + \cdots + S_nS_1 \cdots S_{n-1}.$$

This is the orthogonal sum of n copies of id_{S^1}, which is equivalent in K_1 to $n[id_{S^1}]$. Finally, $[S_1 + \cdots + S_n] = [id_{S^1}]$ and so

$$[(S_1 + \cdots + S_n)^n] = n[S_1 + \cdots + S_n] = n[id_{S^1}].$$

Since we have cancellation in K_1, this implies that the class of $S_1 + \cdots + S_n$ coincides with the class of $S_1 S_2 \cdots S_n$.

Having seen what is involved, we now add the infinite path on the left. The core becomes $\mathcal{K} \oplus \mathcal{K} \oplus \cdots \oplus \mathcal{K}$ (n copies). Since $A = C(S^1) \otimes \mathcal{K} = M_n(C(S^1)) \otimes \mathcal{K}$, the intrepid reader can go through the details of an argument like the one above, with entirely analogous results. $\square$

Since the invariants obtained from the semifinite index are finer than the isomorphism class of $C^*(E)$, depending as they do on $C^*(E)$ and the gauge action, they can be regarded as invariants of the differential structure. That is, the core F can be recovered from the gauge action, and we regard these invariants as arising from the differential structure defined by $\mathcal{D}$. Thus in this case, the semifinite index produces invariants of the differential topology of the noncommutative space $C^*(E)$.

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Appendix A. Toeplitz operators on C^*-modules

In this appendix we define a bilinear product

$$K_1(A) \times KK^1(A, B) \to K_0(B).$$

Here we suppose that A, B are ungraded C^*-algebras. This product should be the Kasparov product, though it is difficult to compare the two (see the footnote to Proposition A.1 below).

We denote by A^+ the minimal (one-point) unitisation if A is nonunital. Otherwise A^+ will mean A. To deal with unitaries in matrix algebras over A, we recall that $K_1(A)$ may be defined by considering unitaries in matrix algebras over A^+ which are equal to 1_n mod A (for some n), [16, p. 107].

We consider odd Kasparov A-B-modules. So let E be a fixed countably generated ungraded B-C^*-module, with $\phi : A \to End_B(E)$ a $*$-homomorphism, and let $P \in End_B(E)$ be such that $a(P - P^*), a(P^2 - P), [P, a]$ are all compact endomorphisms.

Then by [19, Lemma 2, Section 7], the pair (ϕ, P) determines a $KK^1(A, B)$ class, and every class has such a representative. The equivalence relations on pairs (ϕ, P) that give KK^1 classes are unitary equivalence $(\phi, P) \sim (U\phi U^*, UPU^*)$ and homology, $P_1 \sim P_2$ if $P_1\phi_1(a) - P_2\phi_2(a)$ is a compact endomorphism for all $a \in A$.

Now let $u \in M_m(A^+)$ be a unitary, and (ϕ, P) a representative of a $KK^1(A, B)$ class. Observe that $(P \otimes 1_m)E \otimes \mathbf{C}^m$ is a B-module, and so can be extended to a B^+ module. Writing $P_m = P \otimes 1_m$, the operator $P_m\phi(u)P_m$ is Fredholm, since (dropping the ϕ for now)

$$P_m u P_m P_m u^* P_m = P_m[u, P_m]u^* P_m + P_m$$

and this is P_m modulo compact endomorphisms. To ensure that $\ker P_m u P_m$ and $\ker P_m u^* P_m$ are closed submodules, we need to know that $P_m u P_m$ is regular, but by [14, Lemma 4.10], we can always replace $P_m u P_m$ by a regular operator on a larger module. Then the index of $P_m u P_m$ is defined as the index of this regular operator, so there is no loss of generality in supposing that $P_m u P_m$ is regular. Then we can define

$$\mathit{Index}(P_m u P_m) = [\ker P_m u P_m] - [\operatorname{coker} P_m u P_m] \in K_0(B).$$

This index lies in $K_0(B)$ rather than $K_0(B^+)$ by [14, Proposition 4.11]. So given u and (ϕ, P) we define a $K_0(B)$ class by setting

$$u \times (\phi, P) \to [\ker P_m u P_m] - [\operatorname{coker} P_m u P_m].$$

Observe the following. If $u = 1_m$ then $1_m \times (\phi, P) \to \mathit{Index}(P_m) = 0$ so for any (ϕ, P) the map defined on unitaries sends the identity to zero. Given the unitary $u \oplus v \in M_{2m}(A^+)$ (say) then

$$u \oplus v \times (\phi, P) \to \mathit{Index}(P_{2m}(u \oplus v)P_{2m}) = \mathit{Index}(P_m u P_m) + \mathit{Index}(P_m v P_m),$$

so for each (ϕ, P) the map respects direct sums. Finally, if u is homotopic through unitaries to v, then $P_m u P_m$ is norm homotopic to $P_m v P_m$, so

$$\mathit{Index}(P_m u P_m) = \mathit{Index}(P_m v P_m).$$

By the universal property of K_1, [32, Proposition 8.1.5], for each (ϕ, P) as above there exists a unique homomorphism $H_P : K_1(A) \to K_0(B)$ such that

$$H_P([u]) = \mathit{Index}(P_m u P_m).$$

Now observe that $H_{UPU^*, U\phi(\cdot)U^*} = H_{P,\phi}$ since

$$\mathit{Index}(UPU^*(U\phi(u)U^*)UPU^*) = \mathit{Index}(UPuPU^*) = \mathit{Index}(PuP).$$

The homomorphisms H_P are bilinear, since

$$\begin{aligned} H_{P\oplus Q}([u]) &= \mathit{Index}((P\oplus Q)(\phi(u)\oplus\psi(u))(P\oplus Q)) \\ &= \mathit{Index}(P\phi(u)P) + \mathit{Index}(Q\psi(u)Q) = H_P([u]) + H_Q([u]). \end{aligned}$$

Finally, if (ϕ_1, P_1) and (ϕ_2, P_2) are homological, the classes defined by $(\phi_1\oplus\phi_2, P_1\oplus 0)$ and $(\phi_1\oplus\phi_2, 0\oplus P_2)$ are operator homotopic, [19, p 562], so

$$\begin{aligned} \mathit{Index}(P_1\phi_1(u)P_1) &= \mathit{Index}((P_1\oplus 0)(\phi_1(u)\oplus\phi_2(u))(P_1\oplus 0)) \\ &= \mathit{Index}((0\oplus P_2)(\phi_1(u)\oplus\phi_2(u))(0\oplus P_2)) \\ &= \mathit{Index}(P_2\phi_2(u)P_2). \end{aligned}$$

So H_P depends only on the *KK*-equivalence class of (ϕ, P). Thus

Proposition A.1. *With the notation above, the map*[2]

$$H : K_1(A)\times KK^1(A,B)\to K_0(B)$$

$$H([u],[(\phi,P)]) := [\ker(PuP)] - [\operatorname{coker} PuP]$$

is bilinear.

This is a kind of spectral flow, where we are counting the net number of eigen-B-modules which cross zero along any path from P to uPu^*.

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