JOURNAL OF ALGEBRA 85, 535–564 (1983)

Weighted Partitions and Patterns*

GEORGE R. KEMPF

*Department of Mathematics,
The Johns Hopkins University, Baltimore, Maryland 21218*

Communicated by D. Buchsbaum

Received September 20, 1982

The knowledge of the Chow ring of algebraic cycles on the modular variety of curves is very limited at the present time. As our conception of the modular variety has become more and more concrete in recent years, we may hope to obtain further understanding of the global geometry on these varieties.

This work was initiated in an attempt to determine the rational equivalence class of the divisor on the modular variety which appears in the work of D. Mumford and J. Harris [2]. In their paper they used degeneration methods to compute this class over the complete modular variety of stable curves after they had noted that the divisor is a rational multiple of the λ-class over the modular variety of smooth curves. This paper began in an attempt to compute this rational multiple directly. After much trial and error I succeeded in doing this calculation.

The calculation involves intersecting (virtually) various obvious cycles on the relative product of the universal smooth curve with itself several times. Then one has to determine the image in the sense of intersection theory of these cycles in the modular variety. My results are essentially purely combinatorial in nature and hopefully will be useful in other calculations.

Most noteworthy was the discovery of patterns occurring in many of the calculations. There is an ∞^2-dimensional commutative formal Lie monoid of all patterns which one may make in series based on the monoid of weighted partitions. At present I have limited my investigations of this monoid of patterns to making the calculations necessary for determining the exact rational multiple of the λ-class.

* Partly supported by NSF Grant MCS75-05578.

0021-8693/83 $3.00

1. Graphs and Weighted Partitions

Let S be a fixed set. Given two graphs Γ_1 and Γ_2 with vertices S, we may define another such graph $\Gamma_1 * \Gamma_2$ called the conjunction of Γ_1 and Γ_2. By definition the edges of $\Gamma_1 * \Gamma_2$ connecting two points of S is the disjoint union of the edges in Γ_1 connecting those points and the edges in Γ_2 connecting them. Clearly the trivial graph with no edges is an identity for conjunction. Also the isomorphism class of $\Gamma_1 * \Gamma_2$ as a graph (with fixed vertices) depends only on the classes of Γ_1 and Γ_2. Clearly the set $G(S)$ of isomorphism classes of graphs with vertices S is a commutative and associative monoid under conjunction.

Here we will only consider graphs with a finite number of vertices and edges. Hence the isomorphism class of a graph Γ with vertices S is completely determined by the knowledge of the number $f_\Gamma(s, t)$ of edges in Γ connecting each point s of S to another point t. This function f_Γ on $S \times S$ has non-negative integral values and is symmetric, and any such function arises from some graph. For any two such graphs Γ_1 and Γ_2, $f_{\Gamma_1 * \Gamma_2} = f_{\Gamma_1} + f_{\Gamma_2}$. Hence f defines an isomorphism from $G(S)$ to the monoid of non-negative integral-valued symmetric functions on $S \times S$.

A graph Γ with vertices S has many interesting invariants. The most useful of these is the partition H_Γ of S defined by Γ. Recall that a partition of S is a collection H of non-empty subsets h of S, which are called compartments, such that $S = \coprod_{h \in H} h$. The partition H_Γ is the finest partition of S such that any two points of S, which are connected by an edge of Γ, are in the same compartment. In other words two points of S are in the same compartment of H_Γ if and only if they lie in the same connected component of the graph.

The other invariant which we use counts the number of edges of Γ which lie in each component. Formally, for any graph Γ with vertices S and any compartment h of H_Γ, we define

$$\alpha_\Gamma(h) \equiv \frac{1}{2} \left(\sum_{(s,t) \in h \times h} f_\Gamma(s, t) + \sum_{s \in h} f_\Gamma(s, s) \right). \tag{1.1}$$

As $\operatorname{order}(h) - 1$ is the minimal number of edges required to connect the $\operatorname{order}(h)$ points of h, we have

$$\alpha_\Gamma(h) \geqslant \operatorname{order}(h) - 1. \tag{1.2}$$

Conversely given any integral-valued function α on a partition H of S such that $\alpha(h) \geqslant \operatorname{order}(h) - 1$ for each h in H, there are graphs Γ with $H = H_\Gamma$ and $\alpha = \alpha_H$. The set of all pairs (H, α) satisfying the above inequality is denoted by $W(S)$, the set of weighted partitions of S.

One remarkable property of weighted partitons is that we may multiply them.

LEMMA 1.3. *Let $\psi: G(S) \to W(S)$ be the mapping which assigns the weighted partition $(H_\Gamma, \alpha_\Gamma) = \psi(\Gamma)$ to the graph Γ. Then*

(a) *$W(S)$ possesses a unique commutative monoid structure such that ψ is a homomorphism, and*

(b) *the product $(K, \beta) \times (L, \gamma) \equiv (H, \alpha)$ of two elements (K, β) and (L, γ) of $W(S)$ is given by*

(i) *H is the finest partition of S which is coarser than both K and L, and*

(ii) *$\alpha(h) = \sum_{k \in K,\, k \subset h} \beta(k) + \sum_{l \in L,\, l \subset h} \gamma(l)$ for any h in H.*

Proof. As we have already noted that ψ is surjective, we need only check the truth of the statements (i) and (ii) when $(H, \alpha) = \psi(\Gamma_1 * \Gamma_2)$, $(K, \beta) = \psi(\Gamma_1)$ and $(L, \gamma) = \psi(\Gamma_2)$ for Γ_1 and Γ_2 in $G(S)$. In the situation the statements express how the vertices perceive the following two self-evident facts about graphs. A connected component C of $\Gamma_1 * \Gamma_2$ is a minimal subset of $\Gamma_1 * \Gamma_2$ which is the union of connected components of Γ_1 and simultaneously those of Γ_2. The number of edges in C equals the number of edges in C from Γ_1 plus the number of those from Γ_2. Q.E.D.

Most of the difficulties in this paper are caused by the complexities of the multiplication in the monoid $W(S)$ of weighted partitions. We will begin with a few easy observations about this structure. The identity of $W(S)$ is the finest partition $S = \coprod_{s \in S} \{s\}$ with each compartment having weight zero.

The monoid $W(S)$ is graded by total weight. For any element (H, α), we define

$$\deg(H, \alpha) \equiv \sum_{h \in H} \alpha(h). \tag{1.4}$$

By definition of the $\times$-multiplication we immediately see that

$$\deg((K, \beta) \times (L, \gamma)) = \deg(K, \beta) + \deg(L, \gamma).$$

Therefore $W(S)$ is a graded monoid and $G(S)$ has a unique graded monoid structure so that the homomorphism ψ preserves degrees. In fact, for any Γ in $G(S)$

$$\deg(\Gamma) = \frac{1}{2}\left(\sum_{(s,t) \in S \times S} f_\Gamma(s, t) + \sum_{s \in S} f_\Gamma(s, s) \right) \tag{1.5}$$

is the number of edges in the graph Γ.

Both graded monoids $G(S)$ and $W(S)$ are generated by their elements of degree one, which in either case may be identified with the collection S' of all subsets of S with one or two elements. First we explicitly identify these generators. For any pair s and t of S, let $\{s, t\}$ be the graph with vertices S which has only one edge which connects s and t. As $\{\ ,\ \}$ is symmetric in its variables, it defines a unique element of $G(S)$ to each subset in S'. Clearly this establishes a one-to-one correspondence between S' and the degree one elements of $G(S)$ (i.e., isomorphism classes of graphs with one edge). For any pair s and t of S, let $[s, t]$ denote the element $\psi(\{s, t\})$ of $W(S)$. Using the weight inequality one immediately sees that this process $[\ ,\ ]$ defines a bijection between S' and the degree one elements of $W(S)$ (i.e., weighted partitions of total weight one). Moreover our two identifications are compatible with the homomorphism ψ between $G(S)$ and $W(S)$.

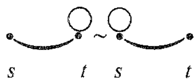
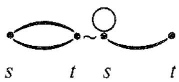
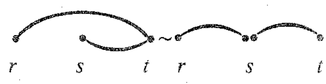
The monoid $G(S)$ has a very simple structure. In fact, $G(S)$ is free commutative graded monoid generated by its elements S' of degree one. As any graph is made up of edges, any element Γ of $G(S)$ may be written as $\Gamma = \prod_{S'} \{s, t\}^{n(\{s,t\})}$. As the exponent $n(\{s, t\})$ equals the previous $f_\Gamma(s, t)$, we see that the above expression for Γ is unique. Furthermore the degree of Γ is just its length as a word in the generators S'.

The monoid $W(S)$ is more complicated. As ψ is surjective, $W(S)$ is a commutative monoid generated by its elements S' of degree one. Still the degree of an element of $W(S)$ is the length of any word in the generators which express it. The difficulty is that an element may be expressed by many different words. In other terms, the surjection $\psi\colon G(S) \to W(S)$ is not injective and, hence, $W(S)$ is determined by the equivalence relation, $\Gamma_1 \sim \Gamma_2 \Leftrightarrow \psi(\Gamma_1) = \psi(\Gamma_2)$, on elements of $G(S)$. As ψ is a degree-preserving homomorphism of graded monoids, the equivalence relation $\sim$ automatically satisfies the formalisms; $\Gamma_1 \sim \Gamma_2 \Rightarrow \deg \Gamma_1 = \deg \Gamma_2$ and $\Gamma_1 * \Gamma_3 \sim \Gamma_2 * \Gamma_3$. In words the relation $\sim$ is homogeneous and stable under multiplication.

One easily determines all equivalences between graphs with two edges. I have listed the three types of these equivalences below (pictorially, verbally as equivalences in $G(S)$, and verbally as equalities in $W(S)$). Let r, s and t be distinct elements of S.

We will not have to look for any more equivalences of graphs. The equivalence relation $\sim$ on $G(S)$ is the smallest equivalence relation which is stable under multiplication and contains the second-degree relations of types A, B, and C listed in the table. The details of the proof of this fact will dramatize some important features of the monoid $W(S)$. In terms of $W(S)$ we intend to demonstrate the truth of

PROPOSITION 1.6. *The commutative monoid $W(S)$ of weighted partitions of S is generated by its first-degree elements S' modulo the quadratic relations* A, B, *and* C *in the below table.*

A	B	C
		
$\{s,t\} * \{t,t\} \sim \{s,s\} * \{s,t\}$	$\{s,t\}^2 \sim \{s,s\} * \{s,t\}$	$\{r,t\} * \{s,t\} \sim \{r,s\} * \{s,t\}$
$[s,t] \times [t,t] = [s,s] \times [s,t]$	$[s,t]^2 = [s,s] \times [s,t]$	$[r,t] \times [s,t] = [r,s] \times [s,t]$

To calculate systematically in the monoids we will eliminate the ambiguities due to the automorphisms of the set S by fixing a linear ordering of S. A graph Γ with vertices S is called tedious if each compartment h of H_Γ has the two properties: (i) two distinct points of h are connected by an edge of Γ if and only if they are neighbors in the induced partial ordering of h; (ii) no point of h is connected to itself by an edge of Γ unless it is the smallest point $\min(h)$ of h. Although this definition is quite long, the tedious graphs have very boring structure, which is completely determined by their weighted partition $(H_\Gamma, \alpha_\Gamma)$. In fact, the property (i) shows that the edges between distinct points in S are determined by the partition H_Γ and there are order $(h) - 1$ edges of this type in any compartment h of H_Γ. Combining this with property (ii) we see that the only other edges in Γ are $\alpha_\Gamma(h)$-order $(h) + 1$ edges connecting $\min(h)$ to itself as h runs through H_Γ. As any weighted partition may arise from some tedious graph, ψ induces a bijection from the tedious elements of $G(S)$ to the whole of $W(S)$. In other words any element of $G(S)$ is equivalent to one and only one tedious element of $G(S)$.

We will need a more simple-minded criterion for a graph Γ with vertices S to be tedious. The advantage of this less descriptive criterion is that it may be verified without determining the partition H_Γ. We define the graph Γ to be exciting at the point s of S if either (1) $\sum_{s<t} f_\Gamma(s,t) > 1$, or (2) $f_\Gamma(s,s) > 0$ and $\sum_{r<s} f_\Gamma(r,s) > 0$, or (3) $\sum_{r<s} f_\Gamma(r,s) > 1$. By this definition we may state our criterion.

LEMMA 1.7. *The graph Γ is tedious if and only if it is not exciting at any point of S.*

Proof. Assume that Γ is tedious. By (i) $\sum_{r<s} f_\Gamma(r,s) \leqslant 1$ and $\sum_{s<t} f_\Gamma(s,t) \leqslant 1$ for all points s of S and $\sum_{r<s} f_\Gamma(r,s) = 0$ if $s = \min(h)$ for some h in H_Γ. By (ii) $f_\Gamma(s,s) > 0$ only if $s = \min(h)$ for some h in H_Γ. Therefore Γ is never exciting at any point of S.

Assume that Γ is not exciting at any point of S. Let $s = \max(S)$ be the largest point of S and let $R = S - \{s\}$ be its complement. Let Δ be the graph with vertices R which is gotten by removing from Γ the vertex s and all edges which touch s. As Δ has fewer edges than Γ, Δ is not exciting at any point of R. Thus by induction we may assume that Δ is tedious. If $\sum_{r<s} f_\Gamma(r, s) = 0$, $\{s\}$ is an entire compartment of H_Γ and we immediately see that Γ is tedious. Otherwise the inequality $\sum_{r<s} f_\Gamma(r, s) \geqslant 1$ holds.

By (1) at the point s we may conclude that the above sum is one. Hence there is only one edge in Γ which connects s with a point, say, r_0, of R. By (3) applied a point r_0 and the tediousness of Δ, we conclude that r_0 must be the maximal element of the compartment, say, h_0, of H_Δ which contains it. As $h_0 \coprod \{s\}$ in the only compartment of H_Γ which is not a compartment of H_Δ, the condition (i) for the tediousness of Γ follows directly. By (2) at the point s the condition (ii) is also true for Γ. Thus Γ is tedious. Q.E.D.

The inductive technique exhibited in this proof of removing a point from S is also useful in many other situations. We will now use it to prove the proposition.

Proof of the Proposition 1.6. Let Γ be a graph with vertices S. We want to show that by using a sequence of the elementary edge switching operations A, B, and C that Γ is equivalent to a tedious graph. By induction we may assume that this procedure has already been delineated for a smaller set of vertices.

As in the last proof, let $s = \max(S)$, $R = S - \{s\}$ and Δ be the graph with vertices R gotten by removing the vertex s and all the edges touching it from Γ. If $\sum_{r<s} f_\Gamma(r, s) = 0$, we may modify Δ until it is tedious and we are done because Γ is tedious by the reasoning in the last proof.

Otherwise $\sum_{r<s} f_\Gamma(r, s) \geqslant 1$. Thus there is a greatest point r_0 of R such that there is an edge in Γ connecting r_0 and s. If $f_\Gamma(s, s) > 0$, we use the A-operation to move a loop at s to one at r_0 as to decrease $f_\Gamma(s, s) > 0$. Eventually we have $f_\Gamma(s, s) = 0$. If $f_\Gamma(r_0, s) > 1$, we may use the B-operation to replace an extra edge between r_0 and s by a loop at r_0 as to decrease $f_\Gamma(r_0, s)$ to one. Eventually we have $f_\Gamma(r_0, s) = 1$. If $\sum_{r<s} f_\Gamma(r, s) > 1$, then there is a greatest r_1 such that $r_1 < r_0$ and $f_\Gamma(r_1, s) > 0$. In this case we use the C-operation to move an edge connecting r_1 and s to an edge connecting r_1 and r_0 as to reduce $f_\Gamma(r_1, s)$ to zero. Repeating this procedure we eventually have $\sum_{r<s} f_\Gamma(r, s) = 1$. Reviewing our current position we now only have one edge touching s and it connects s and the point r_0 of R.

Now we modify Δ (and Γ similarly) until Δ is tedious. If r_0 is not the largest element in its compartment of H_Δ, we have an edge in Δ which connects r_0 and r_2 which is the next largest element of the compartment. Using the C-operation we may replace the edge between r_0 and s by one between r_2 and s. For the new graph r_0 equals the old r_2 and we may repeat this procedure until r_0 is the largest element of its compartment of H_Δ. In this situation the last proof has shown that Γ is tedious. Q.E.D.

This proof should give the reader a good taste of the flavor of actual calculations in the monoid of weighted partitions.

2. Patterns in Series Based on Weighted Partitions

Let R be a commutative ring and M be a graded commutative monoid. We will consider series $\sum_{m \in M} f_m \cdot m$, where the coefficient f_m of each element m of M is an element of R. One may add two such series coefficient-by-coefficient. Furthermore, if M has only a finite number of elements in each degree, one may multiply two such series as follows: $(\sum a_m \cdot m)(\sum b_m \cdot m) \equiv (\sum c_m \cdot m)$, where $c_m \equiv \sum_{r \cdot s = m} a_r \cdot b_s$. Immediately one verifies that the collection of such series forms a commutative ring which we will call the ring of series based on M with coefficients in R.

We will consider series which are based on the graded monoid $W(S)$ of weighted partitions of a fixed set S. Some of the most interesting of these series have definite "patterns" in their coefficients. I will first define "patterns" precisely.

Let

$$a \equiv \sum_{\substack{i \geqslant 1 \\ j \geqslant 0}} a_{i,j}\, y^i z^j$$

be a power series in the indeterminants y and z with coefficients $a_{i,j}$ in R. We may define a related series

$$A_S \equiv \sum_{(H,\alpha) \in W(S)} \prod_{h \in H} a_{\text{order}(h),\alpha(h)-\text{order}(h)+1} \cdot (H, \alpha) \tag{2.1}$$

based on the weighted partitions of S. In this situation we will say that the series a is the pattern in the coefficients of the series A_S, or the series A_S has pattern a. If a series based on $W(S)$ has a pattern, it will be easier simply to give its pattern. We will say that the pattern is strong if the coefficient $a_{1,0}$ of y in the series a is the unit 1 in R.

An example of a series (which we will later see arises in nature) with a strong pattern is the canonical polynomial

$$C_S \equiv \sum_{\alpha(h) \leqslant \text{order}(h)} \Gamma_H \cdot (H, \alpha) \tag{2.2}$$

where $\Gamma_H \equiv \prod_{h \in H} (\text{order}(h) - 1)!$ for any partition H of S. This series has the strong pattern $(\sum_{i \geqslant 1} (i-1)!\, y^i)(1 + z)$. This paper was motivated by a desire to compute the inverse series $1/C_S$ in the ring of series based on weighted partitions of S. To whet the reader's appetite for the abstract discussion of patterns I will mention that one of the achievements of the work is a demonstration that this inverse series has a strong pattern which is computable.

The essence of the formation of series with patterns is its functoriality as we change the set S. From this point of view, the pattern a assigns the series A_S to any finite set S. Thus we want to examine a whole family A of series A_S based on $W(S)$ for all finite sets S. Let $f: S \to T$ be a bijection between finite sets. Then we have an induced bijection $f_*: W(S) \to W(T)$ which induces a natural operation of the same name from series based on $W(S)$ to those based on $W(T)$. With this operation we define a functorial property of a family A of series A_S based on $W(S)$. Such a family is called consistent if we have

$$A_T = f_* A_S$$

for any bijection $f: T \to S$ between finite sets. The lowbrow interpretation of consistencies is that the coefficient of a weighted partition (H, α) of a set S in the series A_S is a function of the collection $\{(\text{order}(h), \alpha(h)) \in Z^2\}_{h \in H}$ up to order. Clearly the family given by a pattern is consistent.

The other characteristic functorial property of a family A of series will give a natural interpretation of the product which appears in the definition of the series A_S produced by a pattern. We will take some time to develop this property in detail. Let $S = \coprod_{i \in I} S_i$ be a fixed partition of a set S. This partition divides the set $W(S)$ into two pieces. We have the set X of intelligible weighted partitions defined by $X \equiv \{(H, \alpha) \in W(S) \mid \text{the partition } H \text{ is finer than the fixed partition } S = \coprod_{i \in I} S_i\}$. Its complement $W(S) - X$ is called the set Y of mysterious weighted partitions. Thus the monoid $W(S)$ has the decomposition $W(S) = X \coprod Y$. This decomposition has some monoidal properties listed below which are direct consequences of the definition of $\times$-multiplication and the lattice properties of partitions of S. The list is:

(a) X is closed under $\times$-multiplication and contains the identity of $W(S)$;

(b) Y is stable under $\times$-multiplication by any element of $W(S)$.

Next we will explain why the elements of X are intelligible. This is because they may be interpreted as words whose letters are weighted partitions of the smaller subsets S_i of S. More formally given a collection H_i of partitions of S_i for each i in I, we have a partition $\coprod_{i \in I} H_i$ given by $S = \coprod_{i \in I, h \in H_i} h$, which is finer than the fixed partition $S = \coprod_{i \in I} S_i$. Clearly any partition of S which is finer than the fixed one can be written uniquely in the form $\coprod_{i \in I} H_i$ as above. Furthermore a weight α on the partition $\coprod_{i \in I} H_i$ corresponds exactly to a collection of weights α_i on each of the partitions H_i. By consulting the definition of $\times$-multiplication, one may verify that the above decomposition gives a tautological identification $X \approx \times_{i \in I} W(S_i)$ where $\times$ denotes the product as commutative monoids.

Using the $X \coprod Y$ decomposition of $W(S)$ a series $B = \sum B_{(H,\alpha)} \cdot (H, \alpha)$ based on $W(S)$ is defined to be intelligible (respectively mysterious) if $B_{(H,\alpha)} = 0$ unless $(H, \alpha) \in X$ (respectively Y). Thus we may decompose any such series B uniquely as the sum $B' + B''$, where B' is intelligible and B'' is mysterious. From the above-listed properties (a) and (b), we see that $(B + C)' = B' + C'$ and $(B \cdot C)' = B' \cdot C'$ for any two series B and C based on $W(S)$. Thus the operation of taking the intelligible part is an endomorphism of the ring of all series based on $W(S)$. In particuar the set of all intelligible series is a commutative ring. By the decomposition theory of the last paragraph we may (and will) identify the ring of intelligible series with the tensor product over i of the rings of series based on $W(S_i)$.

At last we may state the other functorial property which is possessed by the family A of series given by a fixed pattern. Given a family A of series A_S based on $W(S)$ for each finite set S we will say that the family A is compatible with decomposition if for any partition $S = \coprod_{i \in I} S_i$ we have

$$(A_S)' = \prod_{i \in I} A_{S_i}. \tag{2.3}$$

In terms of the coefficients $\psi_{(H,\alpha)}$ of the weight partitions (H, α) of sets T in the series A_T, the above equation is equivalent to

$$\psi_{\prod_{i \in I} (H_i, \alpha_i)} = \prod_{i \in I} \psi_{(H_i, \alpha_i)} \tag{2.4}$$

for all (H_i, α_i) in $W(S_i)$.

Using the last equation one immediately verifies that the family of series produced by a pattern a is compatible with respect to decomposition. In this case $\psi_{(H,\alpha)} = \prod_{h \in H} a_{h, \alpha(h)\text{-order}(h)+1}$ and the equation is obvious from the definitions.

Next we will see that these two functorial properties, which we have developed, characterize the families of series which are produced by pattern may be extracted from the family.

LEMMA 2.5. *A pattern a produces a family A of series A_S based on $W(S)$ for all finite sets such that*

(i) *A is consistent, and*

(ii) *A is compatible with decomposition.*

Conversely any family A with these properties is produced by a unique pattern a.

Proof. As we have verfied that the family of series produced by a pattern has the required properties, we need only verify the converse. Let $\psi_{(H,\alpha)}$ denote the coefficient of an element (H, α) of $W(S)$ in the series A_S. For any non-empty finite set T and non-negative integer j, let $\overline{(T, j)}$ denote the coarsest partition $T = T$ of T with weight $j + \text{order}(T) - 1$. By the consistence of our family, $\psi_{\overline{(T,j)}}$ is a function $a_{i,j}$ of the pair (i, j), where $i = \text{order}(T)$. Next let the fixed partition $S = \coprod S_i$ be the partition $S = \coprod_{h \in H} h$. By the decomposition rule we have

$$\psi_{(H,\alpha)} = \prod_{h \in H} \psi_{(h,\alpha(h))} = \prod_{h \in H} a_{\text{order}(h),\alpha(h)-\text{order}(h)+1}.$$

Therefore, if we set

$$a = \sum_{\substack{i \geqslant 1 \\ j \geqslant 0}} a_{i,j}\, y^i z^i,$$

the given family A is produced by the pattern a which is clearly uniquely determined by A. Q.E.D.

The only element of degree zero in the monoid $W(S)$ is its unit $(\coprod_{s \in S} \{s\}, 0) \equiv 1_S$. Thus for a series K based on $W(S)$ to have an inverse $1/K$ in the ring of series based on $W(S)$ it is necessary and sufficient that the coefficient of 1_S has an inverse in the ring R. We will say that the series K is obviously invertible if the coefficient of 1_S is equal to identity 1 of R. With this definition we can understand the significance of the adjective "strong" as applied to patterns.

COROLLARY 2.6. *Let A be the family of series produced by a pattern a.*

(i). *If the series A_S has an inverse for all finite sets S, the family $1/A_S$ is produced by a pattern.*

(ii) *The pattern a is strong if and only if the series A_S is obviously invertible for any finite set S.*

Proof. By the Lemma we need to see that the family $1/A_S$ has the two required properties if the family A_S does. As f_* is an isomorphism of rings, $f_*(1/A_S) = 1/f_* A_S$ and, hence, the inverse family $1/A_S$ is consistent because the family A_S is. As the operation of taking intelligible parts is an homomorphism, $(1/A_S)' = 1/A'_S$ and, hence, the inverse family $1/A_S$ is compatible with decomposition if the family A_S is. This proves the statement (i).

For (ii) just note that the coefficient of 1_S in A_S is $(a_{1,0})^{\text{order}(S)}$ by definition. If the pattern is strong (i.e., $a_{1,0} = 1$), then the coefficient of 1_S in A_S is 1 (i.e., A_S is obviously invertible). The converse is also trivial. Just take $\text{order}(S) = 1$. Q.E.D.

We can immediately apply this in a special case to deduce the previously advertised result about the inverse of the canonical polynomial C_S.

COROLLARY 2.7. *The family of inverse series $1/C_S$ produced a strong pattern.*

Our next objective is to compute the pattern for these inverse series $1/C_S$. This is a very isolated instance of the general problem of computing the pattern of an inverse series $1/A_S$ from the pattern for the series A_S. I have never thought about the general problem as I am presently interested in special results about a very special series. We will now see that the simplicity of the pattern $(\sum_{i\geqslant 1}(i-1)!\, y^i)(1+z)$ implies a simple interrelationship between two of the canonical polnomials C_S.

Let $S = R \coprod \{s\}$ be the fixed partition of S.

LEMMA 2.8. (a) *If R is empty, $C_S = 1 + [s, s]$.*

(b) *If R is not empty, $C_S = C_R \cdot (1 + \sum_{t\in S} [s, t])$.*

Proof. Recall that $C_S \equiv \sum' \Gamma_H \cdot (H, \alpha)$, where $\Gamma_H = \prod_{h\in H}(\text{order}(h) - 1)!$ and we sum over the subset $Z_S \equiv \{(H, \alpha) \mid \alpha(h) \leqslant \text{order}(h)\}$ of $W(S)$. If R is empty, $S = \{s\}$, $Z_{\{s\}} = \{1_S, [s, s]\}$ and $\Gamma_* = 1$. So the equation of (a) follows immediately from the definition. If R is not empty, the equation of (b) reduces to two equations between the intelligible and mysterious parts: $C'_S = C_R \cdot (1 + [s, s])$ and $C''_S = C_R \cdot (\sum_{r\in R} [s, r])$. As C_S is produced by a pattern, $C'_S = C_R \cdot C_{\{s\}}$ by the decomposition rule. Thus the intelligible equation follows from part (a).

It remains to prove the mysterious equation which we may write as

$$\sum_{(H,\alpha)\in Y\cap Z_S} \Gamma_H \cdot (H, \alpha) = \sum_{\substack{(H',\alpha')\in Z_R \\ r\in R}} \Gamma_{H'} \cdot (H', \alpha') \times [s, r].$$

Now given (H', α') in Z_R and r in R, the product $(H', \alpha') \times [s, r]$ is an element of $Y \cap Z_S$. A given element (H, α) of $Y \cap Z_S$ can be factored as $(H', \alpha') \times [s, r]$, where $(H', \alpha') \in W(R)$ and $r \in R$. Furthermore the factor (H', α') must lie in Z_R and is uniquely determined by (H, α), and the element r may be any element of $R \cap h_0$, where h_0 is the compartment of H containing s. One verifies that $\Gamma_H = \Gamma_{H'} \cdot \{\text{number of such factorizations}\}$. Thus our mysterious equation follows and, hence, (b) is true. Q.E.D.

We may immediately translate the last lemma into properties of the inverse series $1/C_S$.

COROLLARY 2.9. (a) *If R is empty,* $1/C_S=\sum_{i\geqslant 0}^{\infty}(-1)^i[s,s]^i$.

(b) *If R is not empty,*

$$1/C_S=1/C_R-\left(\sum_{t\in S}[s,t]\right)1/C_S.$$

Proof. For (a), we have $1/C_{\{s\}}=1/1+[s,s]=\sum_{i\geqslant 0}^{\infty}(-1)^i[s,s]^i$. For (b) just rewrite the equation $(1/C_S)(1+\sum_{t\in S}[s,t])=1/C_R$ appropriately.

Q.E.D.

Now we begin determining the pattern

$$b=\sum_{\substack{i\geqslant 1\\ j\geqslant 0}} b_{i,j}\,y^iz^j$$

for the inverse family $1/C_S=\sum_{(H,\alpha)\in W(S)}\Phi_{(H,\alpha)}\cdot(H,\alpha)$. Recall from the proof of Lemma that we have a coarse weighted partition $\overline{(S,j)}$ of any finite set S, where j is a non-negative integer and the pattern is determined by the rule $b_{i,j}=\Phi_{\overline{(S,j)}}$, where S is any finite set of order i.

Thus the statement (a) of the Corollary is equivalent to the equations

$$b_{1,j}=(-1)^j \qquad \text{for} \quad j=1.$$

If $\operatorname{order}(S)=i>0$, we may use the decomposition $S=R\coprod\{s\}$ as the weighted partition $\overline{(S,j)}$ is then always mysterious. Thus statement (b) of the Corollary gives the equations

$$b_{i,j}=\Phi_{\overline{(S,j)}}=-\left(\sum_{t\in S}\sum_{(H,\alpha)\in M_j(t)}\phi_{(H,\alpha)}\right)$$

where $M_j(t)=\{(H,\alpha)\in W(S)\mid (H,\alpha)\times[t,s]=\overline{(S,j)}\}$. Here $M_j(s)=\{\overline{(S,j-1)}\}$ if $j>0$ and empty if $j=0$. Also, if $t\neq s$, $M_j(t)=M_j(s)\coprod N_j(t)$, where $N_j(t)=\overline{(S_1,j_1)}\times\overline{(S_2,j_2)}$ where $S=S_1\coprod S_2$, $t\in S_1$, $s\in S_2$ and $j_1+j_2=j$ with $j_*\geqslant 0$. Therefore using the decomposition rule and Corollary 2.9 we get if $i>0$, $b_{i,j}=-ib_{i,j-1}-\sum_{1\leqslant p<i}p^{I(p)}\sum_{0\leqslant q\leqslant j}b_{p,q}\cdot b_{i-p,j-q}$, where $b_{i,-1}\equiv 0$ and $I(p)$ is the number of subsets S_1 of R such that $\operatorname{order}(S_1)=p$. So

$$I(p)=\frac{\operatorname{order}(R)!}{p!\,(\operatorname{order}(R)-p)!}=\frac{(i-1)!}{p!\,(i-1-p)!}.$$

Summarizing the contents of the above material setting $b_{0,*}\equiv 0$ we surmise

$$b_{1,0} = 1$$

$$b_{i,j} = -ib_{i,j-1} - \sum_{\substack{i_1+i_2=i\\ j_1+j_2=j\\ i_*, j_* \geqslant 0}} \frac{(i-1)!}{(i_1-1)!\,(i_2-1)!} b_{i_1,j_1} \cdot b_{i_2,j_2} \qquad \text{if } (i,j) \neq (1,0). \tag{2.10}$$

This gives a recursive relation for finding the pattern

$$b = \sum_{\substack{i \geqslant 1\\ j \geqslant 0}} b_{i,j}\, y^i z^j.$$

From the form of these equations one is led to consider another series

$$d = \sum_{\substack{i \geqslant 1\\ j \geqslant 0}} d_{i,j}\, y^i z^i,$$

where

$$b_{i,j} = (-1)^{i+j-1}(i-1)!\, d_{i,j}. \tag{2.11}$$

The relation satisfied by the new series is

$$\begin{aligned} &d_{1,0} = 1\\ &d_{i,j} = id_{i,j-1} + \sum_{\substack{i_1+i_2=i\\ j_1+j_2=j}} d_{i_1,j_1} \cdot d_{i_2,j_2} \qquad \text{if} \quad (i,j) \neq (1,0). \end{aligned} \tag{2.12}$$

In other words the series d satisfies the differential equation

$$-d^2 + d - y = zy \frac{\partial d}{\partial y} \tag{2.13}$$

with the initial condition $d(0, z) = 0$.

This differential equation may be solved by expanding d in powers of z as

$$d = \sum_{j=0}^{\infty} d_j(y) z^i \qquad \text{where} \quad d_j(y) = \sum_{i \geqslant 1} d_{i,j}\, y^i$$

and solving for $d_0, d_1, \ldots$. Thus d_0 satisfies the quadratic equation $-d_0^2 + d_0 - y = 0$ and $d_0(0) = 0$. Therefore d_0 is the algebraic function of y given by

$$d_0 = \tfrac{1}{2}(1 - \sqrt{1-4y}). \tag{2.14}$$

Proceeding further let $d_0 = d_0 + zl$, where $l = \sum_{i \geqslant 0}^{\infty} d_{i+1} z^i$. Then we must solve the equation

$$l = \frac{1}{1-2d_0} \left\{ y \frac{\partial d_0}{\partial y} + z \left(y \frac{\partial l}{\partial y} + l^2 \right) \right\}$$

and hence by (2.14) we have

$$l = \frac{y}{1-4y} + \frac{z}{\sqrt{1-4y}}\left(y\frac{\partial l}{\partial y} + l^2\right) \tag{2.15}$$

which gives a reasonable inductive formula for the coefficients $d_1, d_2, \ldots$ of l.

Doing the first few terms by hand we get

$$d_1 = y(1-4y)^{-1}$$

and

$$d_2 = (y+y^2)(1-4y)^{-5/2}. \tag{2.16}$$

At this point one may make the educated guess that we should write

$$d_{i+1} = e_i(y)(1-4y)^{-1-(3/2)i} \qquad \text{when} \quad i \geqslant 0 \tag{2.17}$$

and hope that $e_i(y)$ turns out to be a polynomial of degree $i+1$ in y with $e_i(0)=0$. This actually works well and we find the recursion formulae for the polynomials e_j as

$$\begin{aligned} e_0 &= y \\ e_j &= y(1-4y)\frac{\partial e_{j-1}}{\partial y} + 2(3j-1)\,e_{j-1}\,y + \sum_{i_1+i_2=j-1} e_{i_1}e_{i_2} \qquad \text{if} \quad j \geqslant 1. \end{aligned} \tag{2.18}$$

3. A Linear Functional Applied to Series Based on Weighted Partitions

Assume that our ring R of coefficients is a graded ring. A series $A_S = \sum_{W(S)} A_{(H,\alpha)} \cdot (H,\alpha)$ based on weight partitions of a set S is homogeneous of degree p with a shift q if the coefficient $A_{(H,\alpha)}$ is always a homogeneous element of R which has degree equal to $p \cdot \deg(H,\alpha) + q \cdot \operatorname{order}(S)$.

We may complete the graded ring R in a natural way. The completion $\hat{R}$ consists of infinite sums $\sum_{i \geqslant 0}^{\infty} r_i$ of homogeneous elements r_i in R of degree i. As there are only a finite number of weighted partitions of given degree, given any series A_S which is homogeneous of positive degree p with a shift of q the expression

$$\int A_S = \sum_{W(S)} A_{(H,\alpha)} \tag{3.1}$$

defines an element of $\hat{R}$. The ith homogeneous component of $\int A_S$ equals

$$\sum_{p\cdot\deg(H,\alpha)+q\,\operatorname{order}(S)=i} A_{(H,\alpha)}.$$

Consider the case where the series A_S is produced by the pattern

$$a = \sum_{\substack{i \geqslant 1 \\ j \geqslant 0}} a_{i,j}\, y^i z^j.$$

Clearly A_S is homogeneous of degree p with a shift q if the pattern a satisfies the condition: for all $i \geqslant 1$ and $j \geqslant 0$, $a_{i,j}$ is a homogeneous element of R of degree $p \cdot (i + j - 1) + q \cdot i$. Conversely, if for all S the series A_S have that property, the pattern must satisfy those conditions.

Before considering the question of computing the integrals $\int A_S$, we will develop some particular examples of homogeneous series given by patterns.

EXAMPLE 1. Let R be any graded ring with a particular element x of degree one. The strong pattern $y + xyz$ produces the family A of series $A_S = \sum_{T \subset S} x^{\text{order}(T)} \prod_{t \in T} [t, t]$, which is homogeneous of degree one with zero shift.

EXAMPLE 2. Let

$$b = \sum_{\substack{i \geqslant 1 \\ j \geqslant 0}} b_{i,j}\, y^i z^j$$

be a pattern which produces the family B of series B_S which is homogeneous of degree p and shift q. By general reasoning the product family $A \cdot B$ of series $A_S \cdot B_S$ is produced by a pattern

$$c = \sum_{\substack{i \geqslant 1 \\ j \geqslant 0}} c_{i,j}\, y^i z^j$$

and it is homogeneous of degree $p + 1$ with a shift q. Here the remarkable fact is that there is an easy rule for determining the pattern c from the pattern b as the family A is so simple. In fact

$$c_{i,j} = \sum_{0 \leqslant k \leqslant i} \frac{i!}{k!\,(i-k)!}\, b_{i,j-k} x^k \tag{3.2}$$

as the coefficient of $\overline{(S, j)}$ in the product $A_S \cdot B_S$ is the sum of terms x^k (coefficient of $\overline{(S, j-k)}$ in B_S) (number of subsets T of S with $\text{order}(T) = k$) as k goes from 0 to order $S = i$.

EXAMPLE 3. The ring of coefficients is the graded ring $\mathbf{Z}[\kappa_0, \kappa_1, \ldots]$ which is freely generated by elements κ_i of degree i. For formal reasons let $\kappa_{-1} \equiv 0$.

Consider the pattern

$$\sum_{\substack{i\geqslant 1\\ j\geqslant 1}} \mathscr{K}_{j-1}\, y^i z^j = \left(\sum_{j=1}^{\infty} \mathscr{K}_{j-1} z^j\right)(y/1-y).$$

This pattern produces the family $\mathscr{K}$ of series

$$\mathscr{K}_S = \sum_{W(S)} \prod_{h\in H} \mathscr{K}_{\alpha(h)\text{-order}(h)} \cdot (H,\alpha) \tag{3.3}$$

which is homogeneous of degree one with a shift minus one.

Let $F_S = \sum F_{(H,\alpha)} \cdot (H,\alpha)$ and $G_S = \sum G_{(H,\alpha)} \cdot (H,\alpha)$ be two series based on $W(S)$. We may form another series $F_S * G_S = \sum (F_S * G_S)_{(H,\alpha)} \cdot (H,\alpha)$ based on $W(S)$ by the formula $(F_S * G_S)_{(H,\alpha)} = F_{(H,\alpha)} \cdot G_{(H,\alpha)}$ for all (H,α) in $W(S)$. Evidently if we have two families F and G of series F_S and G_S which are produced by patterns, then the family $F * G$ of product series $F_S * G_S$ is produced by a pattern. In fact, if $f = \sum f_{i,j}\, y^i z^j$ and $g = \sum g_{i,j}\, y^i z^j$ are the patterns of the families F and G, then $\sum (f_{i,j}\, g_{i,j})\, y^i z^j$ is the pattern of the family $F * G$.

Given two families F and G of series F_S and G_S which are homogeneous of degree p_F and p_G with shifts q_F and q_G, the family $F * G$ of series $F_S * G_S$ is homogeneous of degree $p_E + p_G$ with a shift $q_F + q_G$. In this situation we obtain the element

$$\langle F, G\rangle_S = \int F_S * G_S \tag{3.4}$$

of the completion $\hat{R}$ for each finite set S.

LEMMA 3.5. *Assume that F and G are produced by patterns $f = \sum f_{i,j}\, y^i z^j$ and $g = \sum g_{i,j}\, y^i z^j$ and $s \equiv$ order(S). Then*

$$\langle F, G\rangle_S = \sum_{\substack{k_{i,j}\geqslant 0,\ \sum ik_{i,j}=s\\ i\geqslant 1 \text{ and } j\geqslant 0}} \frac{s!\prod_{i,j}(f_{i,j}\, g_{i,j})^{k_{i,j}}}{\prod_{i,j} k_{i,j}!\prod_{i,j}(i!)^{k_{i,j}}}.$$

Proof. By definition $\langle F, G\rangle_S = \sum_{(H,\alpha)\in W(S)} F_{(H,\alpha)} \cdot G_{(H,\alpha)}$. Using the action of the automorphism group $\mathrm{Aut}(S)$ on $W(S)$ and the invariance of $F_S * G_S$, we may conclude that

$$\langle F, G\rangle_S = \sum_{\mathrm{Aut}(S)\backslash W(S)} F_{(H,\alpha)} \cdot G_{(H,\alpha)} \cdot \mathrm{order}\{\mathrm{Aut}(S) \cdot (H,\alpha)\}.$$

Given any (H,α) in $W(S)$ we have a double sequence $(k_{i,j})_{i\geqslant 1, j\geqslant 0}$, where $k_{i,j} \equiv \mathrm{order}(H_{i,j})$ and $H_{i,j} \equiv \{h \in H \mid \mathrm{order}(h) = i$ and $\alpha(h) - \mathrm{order}(h) + 1 = j\}$. As $S = \prod_{h\in H} h$ and $H = \prod_{i,j} H_{i,j}$, $\mathrm{order}(S) \equiv$

$s=\sum_{i,j} ik_{i,j}$ and $k_{i,j}\geqslant 0$. Clearly the double sequence $(k_{i,j})$ depends exactly on the Aut(S)-orbit of (H,α) and any double sequence $(k_{i,j})_{i\geqslant 1, j\geqslant 0}$ such that $\sum_{i,j} ik_{i,j}=s$ and $k_{i,j}\geqslant 0$ arises this way. Furthermore $F_{(H,\alpha)}\cdot G_{(H,\alpha)}\equiv \prod_{i,j}(f_{i,j}\,g_{i,j})^{k_{i,j}}$, where (H,α) has the double sequence $(k_{i,j})$. Thus from the above we will have proven this lemma once we show that Order(Aut$(S)\cdot(H,\alpha))=s!/\prod_{i,j}k_{i,j}!\prod_{i,j}(i!)^{k_{i,j}}$, or equivalently, order (Aut(S))/order(Aut $(H_{i,j}))\prod_{h\in H}$ order(Aut(h)). This way the formula is obvious because the denominator is the order of the stabilizer of (H,α) in Aut(S). Q.E.D.

Thus the number $\langle F,G\rangle_S$ just depends on the number $s=\text{order}(S)$ and we may denote it by $\langle F,G\rangle_s$. To simplify writing the equation of the lemma, we will introduce the series

$$\langle F,G\rangle=\sum_{s=1}^{\infty}\frac{\langle F,G\rangle_s}{s!}\,w^s \tag{3.6}$$

where w is a new bookkeeping variable. Then our lemma is equivalent to the equation

$$\begin{aligned}\langle F,G\rangle&=\sum_{k\geqslant 0}^{\infty}\frac{1}{k!}\left(\sum_{\substack{i\geqslant 1\\ j\geqslant 0}}\frac{f_{i,j}\,g_{i,j}w^i}{i!}\right)^k\\ &=\exp\left\{\sum_{\substack{i\geqslant 1\\ j\geqslant 0}}\frac{f_{i,j}\,g_{i,j}w^i}{i!}\right\}.\end{aligned} \tag{3.7}$$

Now we are well positioned to attack the problem which motivated this paper, i.e., we want to compute

$$\langle\mathscr{K},A/C\rangle_S$$

where $\mathscr{K}$ is defined in Example 3, A is defined in Example 1 and C is the canonical polynomial introduced in Section 2 where we found an expression for its inverse family $1/C$. We will recall some of the relevant details while we study the homogeneity of the families C and $1/C$.

EXAMPLE 4. Let coefficient ring be $\mathbf{Z}$ with the trivial grading with all elements of degree zero. Both the series C_S and $1/C_S$ are homogeneous of degree zero with zero shifts. The pattern producing the inverse family $1/C$ is

$$\sum_{\substack{i\geqslant 0\\ j\geqslant 1}}(-1)^{i+j-1}(i-1)!\,d_{i,j}\,y^iz^j$$

where the double sequence $(d_{i,j})$ of integers is defined near the end of Section 2.

For the rest of this calculation the coefficient ring will be the graded ring $\mathbf{Z}[x, \kappa_0, \kappa_1, \ldots]$ which is freely generated by the homogeneous element x of degree one and the homogeneous elements κ_i of degree i. When it is necessary, $\kappa_{-1} \equiv 0$. By definition

$$\langle \mathscr{K}, A/C\rangle_S = \int \mathscr{K}_S * (A_S/C_S)$$

where the integrand $\mathscr{K} * (A/C)$ is a member of a patterned homogeneous family of positive degree 2 $(= 1 + 1 - 0)$ with shift minus s $(= -s + 0 - 0)$, where $s \equiv \operatorname{order}(S)$. Thus the integral converges and the integrand is produced by the pattern

$$\sum_{\substack{i \geqslant 1 \\ j \geqslant 1}} g_{i,j}\, y^i z^j,$$

where

$$g_{i,j} = \kappa_{j-1}\, i! \left(\sum_{0 \leqslant k \leqslant i} \frac{(i-1)!}{k!\,(i-k)!} (-1)^{i+j+k+1} x^k d_{i,j-k} \right)$$

by Examples 2, 3, and 4. We will apply our last lemma using the notation which was introduced after it. Thus in the present situation we get

$$\begin{aligned}
\langle \mathscr{K}, A/C\rangle &= \exp\left(\sum_{\substack{i \geqslant 0 \\ j \geqslant 0 \\ i \geqslant k \geqslant 0}} \mathscr{K}_{j-1} \frac{(i-1!\,(-1)^{i+j+k+1}}{k!\,(i-k)!} x^k w^i d_{i,j-k} \right) \\
&= \exp\left(\sum_{\substack{i \geqslant 0 \\ m \geqslant 0 \\ i \geqslant k \geqslant 0}} \mathscr{K}_{m+k-1} \frac{(i-1)!\,(-1)^{i+m+1}}{k!\,(i-k)!} x^k w_i\, d_{i,m} \right) \\
&= \exp\left(\sum_{\substack{m \geqslant 0 \\ k \geqslant 0 \\ n \geqslant 0}} \mathscr{K}_{m+k-1} \frac{(k+n-1)!\,(-1)^{n+k+m+1}}{k!\,n!} x^k w^{k+n}\, d_{k+n,m} \right) \\
&= \exp\left(\sum_{k \geqslant 0} (-1)^{k+1} \frac{x^k w^k}{k!} \sum_{m \geqslant 0} \mathscr{K}_{m+k-1} (-1)^m \right. \\
&\qquad \left. \times \sum_{n \geqslant 0} \frac{(k+n-1)!\,(-1)^n}{n!} w^n\, d_{k+n,n} \right)
\end{aligned}$$

by taking $m = j - k$, $n = i - k$ and rewriting. If we define

$$I_{k,m}(w) = \sum_{n \geqslant 0} \frac{(k+n-1)!\,(-1)^n}{n!} d_{k+n,m}\, w^n \tag{3.8}$$

we rewrite our formula as

$$\langle \mathscr{K}, A/C\rangle = \exp\left(\sum_{\substack{m\geqslant 0\\ k\geqslant 0}} \mathscr{K}_{m+k-1}(-1)^{m+1+k}\frac{x^k w^k}{k!} I_{k,m}\right). \tag{3.9}$$

Thus we need to clarify the expression $I_{k,m}(w)$. For instance, $k=0$,

$$\begin{aligned} I_{0,m}(w) &= \sum_{n\geqslant 1} (-1)^n \frac{d_{n,m}}{n} w^n \\ &= \int_0^{-w} \frac{d_m(y)}{y}\, dy \end{aligned} \tag{3.10}$$

where $d_m(y)=\sum_{n\geqslant 1}^{\infty} d_{n,m}\, y^n$ was studied in Section 2. When $k=1$, we have

$$I_{1,m}(w) = \sum_{n\geqslant 0} (-1)^n d_{n+1,m} w^n = + \left.\frac{d_m(y)}{y}\right|_{y=-w}$$

or

$$= -\frac{\partial}{\partial w} I_{0,m} \qquad \text{if you please.} \tag{3.11}$$

More generally

$$I_{k,m} = (-1)^k \frac{\partial^k}{\partial w^k}(I_{0,m})$$

and our big formula becomes

$$\begin{aligned} \langle \mathscr{K}, A/C\rangle &= \exp\left(\sum_{\substack{m\geqslant 0\\ k\geqslant 0}} \mathscr{K}_{m+k-1}(-1)^{m+1}\frac{x^k w^k}{k!}\frac{\partial^k}{\partial w^k}(I_{0,m})\right) \\ &= \exp\left(-\mathrm{Res}_{z=0}\left(\left(\sum_{p=0}^{\infty} \mathscr{K}_p z^p\right)\right.\right. \\ &\qquad \left.\left.\times\left(\sum_{q=0}^{\infty} \frac{z^q x^q w^q}{q!}\frac{\partial^q}{\partial w^q}\left[\sum_{p=0} I_{0,p}(-1)^p z^p\right]\right)\right)\right). \end{aligned} \tag{3.12}$$

I will not need this full formula as I want to write $\langle \mathscr{K}, A/C\rangle = c_0 + c_1 x +$ remaining terms when c_0 and c_1 are constant in x and have degree $\leqslant 1$. It turns out that c_0 and c_1 can be easily computed.

So

$$\begin{aligned}\langle \mathscr{K}, A/C\rangle &= \exp(\mathscr{K}_0 I_{0,1} - \mathscr{K}_1 I_{0,2} + xw\mathscr{K}_0 I_{1,0} - xw\mathscr{K}_1 I_{1,1} + \text{rest})\\ &= \exp(\mathscr{K}_0 I_{0,1})(1 - \mathscr{K}_1 I_{0,2})(1 - xw(\mathscr{K}_0 I_{1,0} - \mathscr{K}_1 I_{1,1})) + \text{rest}\\ &= \exp(\mathscr{K}_0 I_{0,1})(1 - \mathscr{K}_1 I_{1,2})\\ &\quad - xw\exp(\mathscr{K}_0 I_{0,1})\left\{\begin{matrix}\mathscr{K}_0 I_{1,0} - \mathscr{K}_1 I_{1,1}\\ -\mathscr{K}_0 I_{1,0}\mathscr{K}_1 I_{0,2}\end{matrix}\right\} + \text{rest}.\end{aligned}$$

Thus

$$c_0 = \exp(\mathscr{K}_0 I_{0,1})(1 - \mathscr{K}_1 I_{1,2})$$

and $$\tag{3.13}$$

$$c_1 = -w\exp(\mathscr{K}_0 I_{0,1})(\mathscr{K}_0 I_{1,0} - \mathscr{K}_1 I_{1,1} - \mathscr{K}_0\mathscr{K}_1 I_{1,0} I_{0,2}).$$

We will continue giving more and more explicit expressions for the power series c_0 and c_1. Recall from (2.14) and (2.16) that $d_0 = \frac{1}{2}(1 - \sqrt{1-4y})$, $d_1 = y(1-4y)^{-1}$ and $d_2 = (y + y^2)(1-4y)^{-5/2}$. By (3.10) we have

$$I_{0,1} = \int_0^{-w} 1/(1-4y)\,dy = -\tfrac{1}{4}\log(1+4w). \tag{3.14}$$

Also from (3.11), we have $I_{1,*} = -d_*(-w)w^{-1}$. Therefore using the formulas from (3.13) we have

$$c_0 = \exp\left(-\frac{\mathscr{K}_0}{4}\log(1+4w)\right)(1 + \mathscr{K}_1(-1+w)(1+4w)^{-5/2})$$

and

$$c_1 = \exp\left(-\frac{\mathscr{K}_0}{4}\log(1+4w)\right) \times \left\{\begin{matrix}-\dfrac{\mathscr{K}_0}{2}(1-\sqrt{1+4w})\\ +\mathscr{K}_1(w/1+4w)\\ +\dfrac{\mathscr{K}_0\mathscr{K}_1}{2}(1-\sqrt{1+4w})\left(\displaystyle\int_0^{-w}\frac{1+y}{\sqrt{1-4y^5}}\,dy\right)\end{matrix}\right\}. \tag{3.15}$$

We can simplify the formula for c_1 slighly. After integrating

$$\int_0^{-w}\frac{1+y}{\sqrt{1-4y^5}}\,dy = \frac{1}{24}[5(1+4w)^{-3/2} - 3(1+4w)^{-1/2} - 2],$$

substituting and collecting terms we have

$$c_1 = \exp\left(-\frac{\mathscr{K}_0}{4}\log(1+4w)\right) \times \left\{ \begin{array}{l} \mathscr{K}_1 w(1+4w)^{-1} \\ -\dfrac{\mathscr{K}_0}{4}\left[2-2(1+4w)^{-1/2}+\dfrac{\mathscr{K}_1}{12}\left\{\begin{array}{l}2-3(1+4w)^{-1}+5(1+4w)^{-2}\\ +(1+4w)^{-1/2}-5(1+4w)^{-3/2}\end{array}\right.\right] \end{array} \right\}. \tag{3.16}$$

4. Application to the Moduli of Curves

Let $\sigma: \mathscr{C} \to \mathscr{M}$ be a smooth projective morphism between quasi-projective smooth varieties where the fibers of σ are irreducible curves of genus g. For any finite set S we may deform the relative product $p_S: \mathscr{C}^S \to \mathscr{M}$ where a point of $\mathscr{C}^S$ is a collection $(c_s)_{s\in S}$ of points of $\mathscr{C}$ and a point $p_S((c_s)_{s\in S})$ of $\mathscr{M}$ which equals $\sigma(c_s)$ for all s in S. We also have the relative symmetric product $\pi_S: \mathscr{C}^{(S)} \to \mathscr{M}$, where $\mathscr{C}^{(S)}$ is the quotient of $\mathscr{C}^S$ under the action of the automorphism group $\mathrm{Aut}(S)$ of the set S of indices. We have a commutative diagram

$$\begin{tikzcd} \mathscr{C}^S \arrow[rr, "q"] \arrow[dr, "p_S"'] & & \mathscr{C}^{(S)} \arrow[dl, "\pi_S"] \\ & \mathscr{M} & \end{tikzcd}$$

where q is the quotient morphism. The morphisms p_S, π_S and q are projective, and p_S and π_S are smooth with relative dimension order(S), and q is a finite flat morphism.

Furthermore for each subset R of S we have the projection $\rho_S^R: \mathscr{C}^S \to \mathscr{C}^R$ which forgets the coordinates outside of R. If the set S has exactly one point, then we have an unequivocal identification between $\mathscr{C}$, $\mathscr{C}^S$, and $\mathscr{C}^{(S)}$ which will be taken for granted.

We will define the pluri-diagonal cycles in the product $\mathscr{C}^S$. Let H be any partition $S = \coprod_{h\in H} h$ of S. Then we have the closed subvariety Δ_H of $\mathscr{C}^S$ which consists of the points $(c_s)_{s\in S}$ such that $c_s = c_t$ if s and t are contained in the same compatment of H. Clearly we have a natural isomorphism $x_X: \mathscr{C}^H \xrightarrow{\sim} \Delta_H$ of $\mathscr{M}$-schemes. Also let $y_H: \mathscr{C}^H \to \mathscr{C}^S$ be the closed morphism gotten by composing x_H with the inclusion $\Delta_H \subset \mathscr{C}^S$. Finally, for any selection γ of a point $\gamma(h)$ in each component h of H, let T_γ be the subset $\{\gamma(h) \mid h \in H\}$ of S. Then the composition $\sigma(\gamma) \equiv \rho_S^{T_\gamma} \circ y_H: \mathscr{C}^H \to \mathscr{C}^{T_\gamma}$ is an isomorphism of $\mathscr{M}$-schemes.

The free abelian group generated by the classes $\underline{\Delta}_H$ of the pluri-diagonal cycles in the Chow ring $\mathrm{Chow}(\mathscr{C}^S)$ of $\mathscr{C}^S$ is not closed under the intersection-product. We will need to introduce more auxiliary virtual cycles which will span the smallest subring of $\mathrm{Chow}(\mathscr{C}^S)$ which contains the pluri-diagonal cycles.

Let θ denote the divisor class of the sheaf of relative vector fields $\Theta_{\mathscr{C}/\mathscr{M}} \approx (\Omega_{\mathscr{C}/\mathscr{M}})^{\otimes -1}$. Thus θ is an element of $\mathrm{Chow}(\mathscr{C})$ and it is homogeneous of degree (i.e., codimension) one. For any integral-valued function α on H such that $\alpha(h) \geqslant \mathrm{order}(h) - 1$ for each h in H (i.e., (H, α) is a weighted partition of S), we may define the cycle class

$$Z(H, \alpha) \equiv y_{H^*}\left(\prod_{h \in H} \sigma_H^{\{h\}}(\theta^{\alpha(h)\text{-order}(h)+1})\right) \tag{4.1}$$

in $\mathrm{Chow}(\mathscr{C}^S)$. Clearly $Z(H, \alpha)$ has codimension $\sum_{h \in H} \alpha(h) - \mathrm{order}(S) + \mathrm{order}(H)$ in the variety $Z(H, 0) = \underline{\Delta}_H$ of codimension $\mathrm{order}(S) - \mathrm{order}(H)$ in $\mathscr{C}^S$. Therefore $Z(H, \alpha)$ is a homogeneous element of $\mathrm{Chow}(\mathscr{C}^S)$ of degree $\sum_{h \in H} \alpha(h)$.

We may now show that the set $\{Z(H, \alpha)\}$ spans a subring of $\mathrm{Chow}(\mathscr{C}^S)$ by determining the virtual intersections of the $Z(H, \alpha)$ and showing that the set is closed under multiplication in $\mathrm{Chow}(\mathscr{C}^S)$.

LEMMA 4.2. *The mapping* $Z: W(S) \to \mathrm{Chow}(\mathscr{C}^S)$ *is a homomorphism of graded monoids.*

Proof. By the definition of the degree of a weighted partition (H, α), $\deg(H, \alpha) = \deg(Z(H, \alpha))$ by the above remarks. Hence Z preserves degrees. By Proposition 1.6, we know that the monoid $W(S)$ is generated by its first-degree elements $[s, t]$ for $(s, t) \in S^2$ modulo the quadratic relations

(A) $[s, s] \times [s, t] = [s, t] \times [t, t]$,

(B) $[s, t]^2 = [s, s] \times [s, t]$, and

(C) $[s, t] \times [t, r] = [r, s] \times [s, t]$,

for distinct elements r, s, and t of S. Thus we need to verify the following intersection relations for divisors on $\mathscr{C}^S$:

(a) $Z([s, s]) \cdot Z([s, t]) = Z([s, t]) \cdot Z([t, t])$,

(b) $Z([s, t])^2 = Z([s, s]) \cdot Z([s, t])$, and

(c) $Z([s, t]) \cdot Z([t, r]) = Z([r, s]) \cdot Z([s, t])$.

The relations (a) and (c) are trivial because one checks directly from the definition that they are equal to Z applied to the weighted partition given by the common value of the products in (A) and (C). The second relation is a virtual computation of the self-intersection of the diagonal $Z([s,t])$ as Z applied to the common value in (A). This follows directly from the adjunction equation $\Omega^2_{C^2}(\Delta)|_\Delta \approx \Omega^1_\Delta$, where Δ is the ordinary diagonal in the product C^2. Q.E.D.

Using the homomorphism Z, we define a ring homomorphism $Z(\sum M_{(H,\alpha)} \cdot (H,\alpha)) = \sum M_{(H,\alpha)} \cdot Z(H,\alpha)$ from polynomials (or series) based on $W(S)$ to the Chow ring (or its completion) as long as the coefficients $M_{(H,\alpha)}$ are good enough to make sense of the resulting expressions.

Let $\mathscr{F}$ be any locally free $\mathscr{O}_X$-module on a quasi-projective smooth variety X. We have the Chern class $c(\mathscr{F})$ in the Chow ring $\mathrm{Chow}(X)$ of X, which has homogeneous decomposition $c(\mathscr{F}) = 1 + c_1(\mathscr{F}) + \cdots + c_f(\mathscr{F})$, where f is the rank of $\mathscr{F}$. We will be interested in the Chern class $c(\Theta_{\mathscr{C}^{(S)}/\mathscr{M}})$, where $\Theta_{\mathscr{C}^{(S)}/\mathscr{M}} \approx (\Omega^1_{\mathscr{C}^{(S)}/\mathscr{M}})^{\mathrm{dual}}$ is the sheaf of regular vector fields on the symmetric product $\mathscr{C}^{(S)}$ over $\mathscr{M}$.

First I will explain how the sheaf $\Theta_{\mathscr{C}^{(S)}/\mathscr{M}}$ arises in many calculations. Consider the universal effective divisor $\mathrm{Div}_S \subset \mathscr{C}^{(S)} \times_{\mathscr{M}} \mathscr{C}$ of degree order(S). Set-theoretically, if C denotes the fiber $\mathscr{C}_x = \sigma^{-1}(x)$ over a point x in $\mathscr{M}$ the points of Div_S over x are pairs $(\sum_{s\in S} d_s, c)$ of $C^{(S)} \times C$, where c equals d_s for at least one s in S. Algebraically $\mathrm{Div}._S$ is a smooth irreducible divisor defined locally in $\mathscr{C}^{(S)} \times_{\mathscr{M}} \mathscr{C}$ by one equation. In many situations the expression $\pi_{\mathscr{C}^{(S)}*}(\mathscr{O}_{\mathscr{C}^{(S)} \times_{\mathscr{M}} \mathscr{C}}(\mathrm{Div}_S)|_{\mathrm{Div}_S})$ arises where π_- denotes projection onto the factor $-$ of a product. It is worthwhile knowing that by deformation theory (see [3], for instance, for a sketch for a constant curve) we have a canonical isomorphism from

$$\Theta_{\mathscr{C}^{(S)}/\mathscr{M}} \text{ to the above expression.}$$

We may specialize this isomorphism from the universal case.

Assume that we have a family $D \subset X \times_{\mathscr{M}} \mathscr{C}$ of effective divisors of degree order(S) parameterized by a variety X over $\mathscr{M}$. Then one has a natural isomorphism

$$\pi_{X_*}(\mathscr{O}_{X \times_{\mathscr{M}} \mathscr{C}}(D)|_D) \xrightarrow{\approx} \psi_D^*(\Theta_{\mathscr{C}^{(S)}/\mathscr{M}}) \tag{4.3}$$

where $\psi_D : X \to \mathscr{C}^{(S)}$ is the classifying $\mathscr{M}$-morphism of the family D which is characterized by the equation $(\psi_D \times_{\mathscr{M}} \mathscr{C})^{-1}(\mathrm{Div}) = D$. In this situation the above isomorphism gives a relation between Chern classes:

$$\psi_D^{-1}(c(\Theta_{\mathscr{C}^{(S)}/\mathscr{M}})) = c(\pi_{X_*}(\mathscr{O}_{X \times_{\mathscr{M}} \mathscr{C}}(D)|_D)). \tag{4.4}$$

We will next use this last equation to compute the left side in a particular case where the right side may be easily computed because the divisor D is the sum of families D_s for s in S which has degree one. Consider the quotient morphism $q_S : \mathscr{C}^S \to \mathscr{C}^{(S)}$. Let D be the family $(q_S \times_{\mathscr{M}} \mathscr{C})^{-1}(\mathrm{Div})$ of divisors on $\mathscr{C}$ parameterized by the full product $\mathscr{C}^S$. Here the classifying morphism ψ_D is q itself and our above formula tells us how to compute the inverse image, i.e.,

$$q_S^{-1}(c(\Theta_{\mathscr{C}(S)/\mathscr{M}})) = c(\pi_{X_*}(\mathscr{O}_{X \times_{\mathscr{M}} \mathscr{C}}(D)|_D)). \tag{4.5}$$

Now D is a divisor on the product $\mathscr{C}^S \times_{\mathscr{M}} \mathscr{C}$, which we can identify with the product $\mathscr{C}^{S \amalg \{\infty\}}$, where ∞ is some new index. Roughly $D = \{((c_s)_{s\in S}, c_\infty) \mid c_\infty = c_s \text{ for some } s \text{ in } S\}$. Correctly $D = \sum_{s\in S} D_s$, where $D_s = Z([s, \infty])$ as a divisor on $\mathscr{C}^{S\amalg\{\infty\}}$. Thus we want to compute the Chern class of the sheaf, $\pi_{\mathscr{C}^S *}(\mathscr{O}_{\mathscr{C}^S \times_{\mathscr{M}} \mathscr{C}}(\sum_{s\in S} D_s)|_{\sum_{s\in S} D_s})$, where each of the divisors D_s are isomorphic to $\mathscr{C}^S$ via the projection. This calculation may be done inductively using the correct Noetherian isomorphisms.

Let $S = R \coprod \{0\}$ be a fixed partition of the index set S. Then we have an exact sequence of sheaves on $Y \equiv \mathscr{C}^S \times_{\mathscr{M}} \mathscr{C} = \mathscr{C}^R \times_{\mathscr{M}} \mathscr{C} \times_{\mathscr{M}} \mathscr{C}$,

$$\begin{aligned} 0 \to \mathscr{O}_Y\left(\sum_{r\in R} D_r\right)\Big|_{\sum_{r\in R} D_r} \\ \to \mathscr{O}_Y\left(\sum_{s\in S} D_s\right)\Big|_{\sum_{s\in S} D_s} \to \mathscr{O}_Y\left(\sum_{s\in S} D_s\right)\Big| D_0 \to 0. \end{aligned}$$

The first sheaf is naturally isomorphic to $(\rho_S^R \times \mathscr{C})^* \mathscr{O}_{\mathscr{C}^R \times_{\mathscr{M}} \mathscr{C}}(D')|_{D'}$, where D' is the same kind of divisor as D but for the smaller index set R. The second sheaf is the sheaf on D_0 which is isomorphic under projection $\mathscr{C}^S$ to $\mathscr{O}_{\mathscr{C}^S}(\sum_{s\in S} Z([s, 0]))$. As the divisor D is finite over $\mathscr{C}^S$, we have an exact sequence

$$\begin{aligned} 0 \to \rho_S^{R^*}(\pi_{\mathscr{C}^R *}(\mathscr{O}_{\mathscr{C}^{R} * \times \mathscr{C}}(D')|_{D'})) \to \pi_{\mathscr{C}^S *}(\mathscr{O}_{\mathscr{C}^S \times \mathscr{C}}(D)|_D) \\ \to \mathscr{O}_{\mathscr{C}^S}\left(\sum_{s\in S} Z([s, 0])\right) \to 0. \end{aligned}$$

So for Chern classes we have the relation

$$\begin{aligned} &c(\pi_{\mathscr{C}^S *}(\mathscr{O}_{\mathscr{C}^S \times \mathscr{C}}(D)|_D)) \\ &\quad = (\rho_S^R)^{-1}(c(\pi_{\mathscr{C}^R *}(\mathscr{O}_{\mathscr{C}^R \times \mathscr{C}}(D')|_{D'}))) \cdot \left\{1 + \sum_{s\in S} Z([s, 0])\right\}. \end{aligned} \tag{4.6}$$

Comparing this formula with Lemma 2.8, we may easily conclude that

$$c(q^{-1}\Theta_{\mathscr{C}(S)/\mathscr{M}}) = c(\pi_{\mathscr{C}^S *}(\mathscr{O}_{\mathscr{C}^S \times_{\mathscr{M}} \mathscr{C}}(D)|_D)) = Z(C_S) \tag{4.7}$$

where C_S is the expression studied extensively in Section 2.

Next we will try to understand a little about the push-down operation p_{S_*}: $\mathrm{Chow}(\mathscr{C}^S) \to \mathrm{Chow}(\mathscr{M})$. As the structure morphism p_S has relative dimension order (S) p_{S_*} is homogeneous of degree one with a shift minus order(S); i.e., if M is a cycle class of codimension i on $\mathscr{C}^S$, then $p_{S_*}M$ is a cycle class of codimension i-order(S). We shall restrict our attention to finding the first facts about the cycle classes $p_{S_*}(Z(H, \alpha))$ for weighted partitions (H, α) of the index set S.

The most obvious examples are when the index set S has only one element and $\mathscr{C}^S = \mathscr{C}$. The possible cycles of interest are the powers θ^α of the relative anti-canonical divisor θ on $\mathscr{C}$, where α is a nonnegative integer. Here we define

$$\kappa_{\alpha-1} \equiv \sigma_*(\theta^\alpha). \tag{4.8}$$

As the structure morphism $\sigma: \mathscr{C} \to S$ has relative dimension one, κ_i is a homogeneous element of $\mathrm{Chow}(\mathscr{M})$ of degree i.

These particular examples may be used to determine all the others using the multiplication in the ring $\mathrm{Chow}(\mathscr{M})$. Specifically we have

LEMMA 4.9. *For any weighted partition (H, α) of S,*

$$p_{S_*}(Z(H, \alpha)) = \prod_{h \in H} \kappa_{\alpha(h)\text{-order}(h)}.$$

Proof. We will first prove the case when H is the finest partition $S = \coprod_{s \in S} \{s\}$ of S. Thus the weight α is an arbitrary non-negative integral-valued function on S. Let $\times_S X$ denote the actual product of a variety X with itself S-times and let π_s denote the projection onto the sth factor for s in S. Clearly $\times_S \sigma_*(\prod_{s \in S} \pi_s^{-1} \theta^{\alpha(s)}) = \prod_{s \in S} \pi_s^{-1}(\sigma_*(\theta^{\alpha(s)}) = \prod_{s \in S} \pi_s^{-1}(\kappa_{\alpha(s)-1})$, where $\times_S \sigma: \times_S \mathscr{C} \to \times_S \mathscr{M}$ is the product morphism. If $\Delta: \mathscr{M} \to \times_S \mathscr{M}$ is the diagonal inclusion, we have $\Delta^{-1}(\times_S \sigma_*(\prod_{s \in S} \pi_s^{-1} \theta^{\alpha(s)})) = \prod_{s \in S} \kappa_{\alpha(s)-1}$. As $\times_S \sigma$ is a smooth morphism and $\Delta^*(\times_S \sigma) = \sigma_S: \mathscr{C}^S \to \mathscr{M}$, we have

$$\Delta^{-1}\left(\times_S \sigma_*\left(\prod_{s \in S} \pi_s^{-1} \theta^{\alpha(s)}\right)\right) = \sigma_{S_*}\left(\prod_{s \in S} \rho_S^{\{s\}-1}(\theta^{\alpha(s)})\right) = \sigma_{S_*}(Z(S, \alpha)).$$

Putting the two equations together we have proven our result in the finest case.

For the general case, let $\beta(h) = \alpha(h) - \mathrm{order}(h) + 1$ define a weight function on the coarsest partition of H. We have $Z(H, \alpha) = y_{H_*}(Z'(H, \beta))$, where the prime (′) denotes the analogous cycle on the relative product $\mathscr{C}^H$. As $p_S \cdot y_H = p_H$, the formula for the special case (H, β) implies the general formula for (H, α). Q.E.D.

With this lemma we see from 3.3 that $p_{S_*}(Z(H, \alpha)) =$ coefficient of (H, α)

in the series $\mathscr{K}_S$ based on (H, α) which previously defined with indeterminants $\kappa_0, \kappa_1, \ldots$. Therefore we may pleasantly write the series

$$\mathscr{K}_S = \sum_{W(S)} p_{S_*}(Z(H, \alpha)) \cdot (H, \alpha) \tag{4.10}$$

whose coefficients are in the Chow ring Chow($\mathscr{M}$). This series is a convenient form to display our calculation of the $p_{S_*}(Z(H, \alpha)) \cdot (H, \alpha)$ and the last lemma simply asserts that $\mathscr{K}_S$ is given by the pattern forming the family $\mathscr{K}$.

Given a cycle $\sum_{\text{finite } W(S)} \beta_{(H,\alpha)} \cdot Z(H, \alpha) \equiv (\beta \cdot Z)$ in Chow($\mathscr{C}^S$), where the coefficients $\beta_{(H,\alpha)} = p_S^*(\beta_{(H,\alpha)})$ for elements $\beta_{(H,\alpha)}$ of Chow($\mathscr{M}$), we may easily check by the projection formula that

$$p_{S_*}(\beta \cdot Z) = \sum \beta_{(H,\alpha)} \cdot p_{S_*}(Z(H, \alpha)).$$

With our integral notation this equation may be written as

$$p_{S_*}\left(\sum \beta_{(H,\alpha)} \cdot Z(H, \alpha)\right) = I(\beta_S^* Z_S) = \langle Z_S, \beta_S \rangle_S \tag{4.11}$$

where $\beta_S = \sum_{\text{finite}} \beta_{(H,\alpha)} \cdot (H, \alpha)$ has coefficients in Chow($\mathscr{M}$).

Now that we have made the formal connections between the present material and the previous abstract material, we can remark that the first few cycles κ_i in Chow($\mathscr{M}$) are well known:

$$\begin{aligned} \kappa_0 &= \deg(\theta) \cdot [\mathscr{M}] = -2(g-1) \\ \kappa_1 &= 12\lambda \qquad \text{where} \quad \lambda = c_1({}_*\Omega_{\mathscr{C}/\mathscr{M}}) = -c_1(R^1\sigma_* \mathscr{O}_{\mathscr{C}}). \end{aligned} \tag{4.12}$$

The first formula is trivial as it only gives the fiber degree of the relative anti-canonical class θ. The proof of the second equation uses the Grithendieck–Riemann–Roch theorem for the morphism σ and it appears in [5].

Now that we are approaching the main course of this feast, we will have to be prepared to appreciate fully the delicacies on which we will dine by first being oriented to their general species. Let $|D|$ be the complete linear system of effective divisors containing a given effective divisor D on some particular curve C of genus g. As we vary the curve C and the effective divisor D, the dimension of the linear system $|D|$ varies quite dramatically but in an organized way. The function $\dim |D|$ is an upper-semi-continuous function of the pair (D, C). A point x of $\mathscr{C}^{(S)}$ should be regarded as an effective divisor $\sum_{s \in S} d_s$ of degree order(S) on the particular curve $\sigma^{-1}(\sigma(x))$. By general principles one may define a natural closed subscheme of $\mathscr{C}^{(S)}$ whose points correspond to pairs (D, C) satisfying the inequality $\dim |D| \geqslant d$ for any non-negative integer d.

I will explain these general principles explicitly. Recall that we have previously introduced the universal effective divisor Div on $\mathscr{C}^{(S)} \times_{\mathscr{M}} \mathscr{C}$. For a fixed effective divisor D on C, the study of the linear system $|D|$ is equivalent to the study of the global sections of the sheaf $\mathscr{O}_C(D)$. One naturally tires to extend this gambit to study the variational problem by replacing the fixed divisor D by the universal one Div. Thus one is misled to studying the coherent sheaf $\pi_{\mathscr{C}(S)*}(\mathscr{O}_{\mathscr{C}(S) \times_{\mathscr{M}} \mathscr{C}}(\mathrm{Div})) \equiv \mathscr{F}^0$ on $\mathscr{C}^{(S)}$ which turns out not be functorial as we specialize the inversal family to a particular one and even worse does not contain the information that we want about $\dim |D|$. To finesse this difficulty one notes that the first cohomology group $H^1(C, \mathscr{O}_C(D))$ contains the information implicitly about $\dim |D|$ as the curve C has dimension one. Thus one examines the coherent sheaf $R^1\pi_{\mathscr{C}(S)*}(\mathscr{O}_{\mathscr{C}(S) \times_{\mathscr{M}} \mathscr{C}}(\mathrm{Div})) \equiv \mathscr{F}^1$ on $\mathscr{C}^{(S)}$ which turns out to be functorial and contains all the information that we need. For instance the Fitting subschemes of $\mathscr{F}^1$ provide the correct closed subschemes of $\mathscr{C}^{(S)}$ for studying the $\dim |D|$ properly.

To use these ideas effectively one must make some attempt to compute $\mathscr{F}^1$. The obvious approach is to try the long exact sequence of direct images under $\pi_{\mathscr{C}(S)}$ of the sequence

$$0 \to \mathscr{O}_{\mathscr{C}(S) \times_{\mathscr{M}} \mathscr{C}} \to \mathscr{O}_{\mathscr{C}(S) \times_{\mathscr{M}} \mathscr{C}}(\mathrm{Div}) \to \mathscr{O}_{\mathscr{C}(S) \times_{\mathscr{M}} \mathscr{C}}(\mathrm{Div})|_{\mathrm{Div}} \to 0$$

which gives

$$\begin{array}{l} 0 \to \mathscr{O}_{\mathscr{C}(S)} \to \mathscr{F}^0 \\ \quad \to \pi_{\mathscr{C}(S)*}(\mathscr{O}_{\mathscr{C}(S) \times_{\mathscr{M}} \mathscr{C}}(\mathrm{Div})|_{\mathrm{Div}}) \xrightarrow{\delta} R^1\pi_{\mathscr{C}(S)*}\mathscr{O}_{\mathscr{C}(S) \times_{\mathscr{M}} \mathscr{C}} \to \mathscr{F}^1 \to 0 \\ \qquad\qquad \wr\wr \qquad\qquad\qquad\qquad\qquad\qquad\qquad \wr\wr \\ \qquad\qquad \Theta_{\mathscr{C}(S)/\mathscr{M}} \qquad\qquad\qquad\qquad\quad p_S^*(R^1\sigma_* \mathscr{O}_{\mathscr{C}}). \end{array} \tag{4.13}$$

Thus the homomorphism δ is a stronger invariant than the sheaf $\mathscr{F}^1$ and its formation is clearly functorial. The Fitting schemes for $\mathscr{F}^1$ are the determinantal subschemes defined by putting rank conditions on the homomorphism δ between locally free sheaves.

Recall that Porteous (for instance, [4]) has given formulas describing the Chern classes of determinantal subscheme in terms of the quotient Chern classes (which in this case amount to $p_S^* c(R^1\sigma_* \mathscr{O}_{\mathscr{C}}/c(\theta_{\mathscr{C}(S)})))$ provided we have the favorable transversality assumptions. For our present purposes $c(R^1\sigma_* \mathscr{O}_{\mathscr{C}})$ may be assumed to be known. Thus in the favorable circumstances the inverse $1/c(\Theta_{\mathscr{C}(S)/\mathscr{M}})$ provided the most visible obstruction to using this machinery to compute the cycle classes in $\mathrm{Chow}(\mathscr{C}^{(S)})$ of the varieties of special divisors. I have been rather vague here because we are only going to consider a very special case.

Assume that the genus g is an odd number $2k-1 \geqslant 3$ and order$(K)=k$. Consider the locus H in $\mathscr{C}^{(K)}$ representing divisors which move in linear systems along the fibers of $\sigma: \mathscr{C} \to \mathscr{M}$ (i.e., pairs (D, C) such that $|D| \geqslant 1$). We will assume that enough transversality [1] is present to ensure that the class of the cycle $[H]=$ homogeneous component of degree k in $c(R^1\sigma_* \mathscr{O}_{\mathscr{C}})/c(\Theta_{\mathscr{C}^{(K)}/\mathscr{M}})$ and $p_{(K)}$ maps H onto a divisor D in $\mathscr{M}$. We want to compute the cycle class $[D]$ in Chow$(\mathscr{M})$.

There are two problems in finding $[D]$.

(1) I don't know how to compute in Chow$(\mathscr{C}^{(K)})$ well enough.

(2) In the Chow ring of $\mathscr{M}$, $p_{(K)_*}[H]=0$ not $[D]$ because $p_{(K)}$ is constant on the $\mathbf{P}^1$'s passing through each point of H.

The first difficulty is circumvented by computing the class of $q^{-1}[H]$ in Chow$(\mathscr{C}^K)$. The second difficulty may be end-played by intersecting $q^{-1}[H]$ with the divisor $\sum_{k\in K}[k, k]$ and then computing p_{K_*} of the intersection. This works well enough up to torsion as order$(K)!\deg\theta[D]=p_{K_*}(q^{-1}[H]\cdot(\sum_{k\in K}[k,k]))$. Therefore we have modulo torsion in the Chow ring of

$$[D]=\frac{\text{degree one term}}{k!\deg\theta}$$

$$\text{of } p_{K_*}\left(\sum_{k\in K} Z[k,k]/q^*(c(\Theta_{\mathscr{C}^{(K)}/\mathscr{M}}))\right)\cdot(c(R^1\sigma_*\mathscr{O}_{\mathscr{C}})). \tag{4.14}$$

By (4.11) and (4.7) as $c(R^1\sigma_*\mathscr{O}_{\mathscr{C}})=1-\lambda+$ higher terms, we have

$$[D]=\frac{\text{degree one term}}{k!\deg\theta}\text{ of }(1-\lambda)\left\langle \mathscr{K}_K, \sum_{k\in K}[k,k]/C_K\right\rangle_K.$$

By the definition of the family A in Example 1 of Section 3, $\sum_{k\in K}[k,k]$ is the coefficient of x in A_K. Hence

$$\begin{aligned}[D] &= \text{degree one term of the coefficient of } x \text{ in } \frac{(1-\lambda)}{\deg\theta}\frac{\langle \mathscr{K}_K, A_K/C_K\rangle_K}{k!}\\ &= \text{degree one term of the coefficient of } xw^k \text{ in } \frac{(1-\lambda)}{\deg\theta}\langle \mathscr{K}, A/C\rangle\\ &= \text{degree one term of the coefficient of } w^k \text{ in } \frac{(1-\lambda)}{\deg\theta}c_1\end{aligned}$$

from the definition (3.7) and the definition c_1, prior to (3.13). Now if we apply (3.16) which computes c_1, and use the relations (4.12) together with $g-1=2(k-1)$, we arrive at

$[D] =$ degree one term in the coefficient of w^k of

$$\frac{(1-\lambda)}{-4(k-1)} \exp((k-1)\log(1+4w))$$

$$\left\{ \begin{array}{l} 12\lambda w(1+4w)^{-1} \\ +(k-1) \left[\begin{array}{l} +2-2(1+4w)^{-1/2} \\ +\lambda \left\{ \begin{array}{c} -3(1+4w)^{-1} - 2 + (1+4w)^{-1/2} \\ +5(1+4w)^{-2} - 5(1+4w)^{-3/2} \end{array} \right\} \end{array} \right] \end{array} \right\}.$$

As $\exp((k-1)\log(1+4w)) = (1+4w)^{k-1}$, we may conclude that

$$[D] = \frac{-\lambda}{4} \text{ times the coefficient of } w^k$$

$$\text{in} \left\{ \begin{array}{c} 5(1+4w)^{k-3} - 3(1+4w)^{k-2} + (12w/(k-1))(1+4w)^{k-2} \\ +3(1+4w)^{(2k-3)/2} - 5(1+4w)^{(2k-5)/2} \end{array} \right\}.$$

If $k \geqslant 3$ and the first three terms are polynomials of degree $<k$. So

$$[D] = \frac{-\lambda}{4} \text{ times the coefficient of } w^k \text{ in } 3(1+4w)^{(2k-3)/2} - 5(1+4w)^{(2k-5)/2}$$

$$= \frac{-\lambda}{4} \left[3 \cdot \frac{2^k}{k!} \prod_{p=-1}^{k-2} 2p+1 - 5 \cdot \frac{2^k}{k!} \prod_{p=-2}^{k-3} 2p+1 \right]$$

$$= \frac{-\lambda 2^{k-2}}{k!} \prod_{0=p}^{k-3} 2p+1 \left[\begin{array}{c} 3(2k-3)(-1) \\ -5(-1)(-3) \end{array} \right]$$

$$= \frac{+\lambda(2k-2)!}{k!\,(k-1)!} [6(k+1)].$$

Acknowledgments

I would like to thank The Institute for Advanced Study at Princeton for providing me with the opportunity to pursue this research. Also I am indebted to D. Mumford and M. Teicher for their interest which encouraged me to finish this project.

References

1. H. Farkas, Special divisors and analytic subloci of Teichmueller space, *Amer. J. Math.* **88** (1966), 881–901.
2. J. Harris and D. Mumford, On the Kodaira dimension of the moduli space of curves, to appear.

3. G. KEMPF, Deformations of symmetric products, Riemann surfaces and related topics, Proceedings of 1978 Stony Brook Conference, *Ann. of Math. Stud.* **97** (1981), 319–341.
4. G. KEMPF AND D. LAKSOV, The determinal formula of Schubert calculus, *Acta Math.* **132** (1974), 153–162.
5. D. MUMFORD, Stability of projective varieties, *Enseign. Math.* **23** (1977), 39–110.