Pergamon

Computers Math. Applic. Vol. 28, No. 4, pp. 131–145, 1994

0898-1221/94 $7.00 + 0.00

0898-1221(94)00134-0

Peristaltic Transport of a Particle-Fluid Suspension in a Cylindrical Tube

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(Received and accepted January 1993)

Abstract—Peristaltic pumping induced by a sinusoidal travelling wave of moderate amplitude is analysed in the axisymmetrical case for a viscous incompressible and Newtonian fluid mixed with rigid spherical particles which are of identical size. A perturbation method has been employed to find the solution of the problem, choosing the amplitude ratio (i.e., wave amplitude/tube radius) as a parameter. The analysis has been carried out by duly accounting for the nonlinear convective acceleration terms, and the nonslip condition for the fluid part on the wavy wall. The governing equations are developed up to the second order of the amplitude ratio. The zeroth order terms yield the Poiseuille flow and the first order terms give the Orr-Sommerfeld equation. In the absence of the pressure gradient and the wall motion, the mean flows (for the fluid and the solid particles) and the mean pressure gradient (averaged over time) are all found to be proportional to the square of the amplitude ratio. Numerical results are obtained for this simple case by approximating complicated groups of the products of Bessel functions by polynomials. It is observed that a reversal of flow occurs when the pressure gradient exceeds the critical value; this is favoured by the presence of the solid particles. The reversal of flow may take place near the boundaries also.

1. INTRODUCTION

Certain physiological phenomena, like the transportation of urine from the kidney to the bladder through the ureter, movement of chyme in gastrointestinal tract, and the vasomotion of some blood vessels involve a peculiar kind of motion that is caused by the movement of some progressive wave of contraction or expansion on the walls of the tube, which then relaxes and a lower portion becomes shortened and narrowed. This is termed as peristalsis. Moreover, the movement of spermatozoa in the *ductus efferentes* of the male reproductive tracts, that of the ovum in the fallopian tube, etc., are also of similar type, and the locomotion of some worms has also been found to be of peristaltic nature. Futhermore, by using the principle of peristalsis, some biomechanical instruments, e.g., heart-lung machine, have been fabricated.

Some of the aforementioned physiological phenomena, including the flow of diseased urine in the ureter are based on the flow of particle-fluid mixture instead of a pure fluid. Hung and Brown [1] initiated the study of the peristaltic transport of solid particles, which included an experimental work on the particle transport in two-dimensional vertical channels having various geometries. A two-dimensional analysis of the problem was subsequently carried out mathematically by Srivastava and Srivastava [2].

In this paper, we take up the axisymmetric flow of a suspension of solid particles in an incompressible Newtonian viscous fluid through a long flexible tube of uniform cross-sectional area. A sinusoidal wave is considered to be imposed along the walls of the tube.

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The problem is analysed theoretically. In order to make the problem more realistic, the contributions of the nonlinear convective acceleration terms in the flow equations have been duly accounted for. A perturbation method is used to analyse the problem. The analysis has been carried out for the situation in which the amplitude ratio (i.e., wave amplitude/tube ratio) is small. In order to illustrate the applicability of the theoretical analysis, numerical values of various physical quantities have been computed for a specific example, *viz.*, the flow of urine through the ureter.

It is known that urine from the kidneys passes through ureters and is trapped between the contracted segments of the ureter. When there is some obstruction in the ureter or in the ureter-bladder junction, the upstream ureter dilates. In such hydroureter cases, peristaltic motion becomes a travelling wave of relatively small amplitude over a cylindrical tube. It is evident that the efficiency of pumping is decreased in such cases and the quantity of urine pumped through the ureter is reduced. Ultimately, the urine is stored in the bladder, from which it is periodically delivered through the urethrae.

The roles of various parameters, like the Reynolds number, wave number, and the volume fraction of the particles have been examined in quantitative terms for the specific case, mentioned above, by using suitable data available in the literature.

2. FORMULATION OF THE PROBLEM

We consider an axisymmetric flow of a mixture of small spherical solid particles and an incompressible Newtonian viscous fluid through a uniform circular cylindrical tube. We consider that the tube wall is subjected to sinusoidal waves. With the continuum mechanics approach, the equations governing the conservation of mass and linear momentum for both the fluid and the solid particle phases may be written in the following manner (cf. [3]):

FLUID PHASE.

$$(1-C)\,\rho_f\left[\frac{\partial}{\partial t}+v_f\,\frac{\partial}{\partial r}+u_f\,\frac{\partial}{\partial z}\right]v_f$$
$$=-(1-C)\,\frac{\partial p}{\partial r}+(1-C)\,\mu_s(C)\left[\frac{\partial^2}{\partial r^2}+\frac{\partial^2}{\partial z^2}+\frac{1}{r}\,\frac{\partial}{\partial r}-\frac{1}{r}\right]v_f+C\,S\,(V_p-V_f), \tag{2.1}$$

$$(1-C)\,\rho_f\left[\frac{\partial}{\partial t}+v_f\,\frac{\partial}{\partial r}+u_f\,\frac{\partial}{\partial z}\right]u_f$$
$$=-(1-C)\,\frac{\partial p}{\partial z}+(1-C)\,\mu_s(C)\left[\frac{\partial^2}{\partial r^2}+\frac{\partial^2}{\partial z^2}+\frac{1}{r}\,\frac{\partial}{\partial r}\right]u_f+C\,S\,(u_p-u_f), \tag{2.2}$$

$$\frac{\partial}{\partial r}(1-C)\,v_f+\frac{\partial}{\partial z}(1-C)\,u_f+\frac{1}{r}\,(1-C)\,v_f=0. \tag{2.3}$$

PARTICULATE PHASE.

$$C_{\rho p}\left[\frac{\partial}{\partial t}+v_p\,\frac{\partial}{\partial r}+u_p\,\frac{\partial}{\partial z}\right]v_p=-C\,\frac{\partial p}{\partial r}+C\,S\,(v_f-v_p), \tag{2.4}$$

$$C_{\rho p}\left[\frac{\partial}{\partial t}+v_p\,\frac{\partial}{\partial r}+u_p\,\frac{\partial}{\partial z}\right]u_p=-C\,\frac{\partial p}{\partial z}+C\,S\,(u_f-u_p), \tag{2.5}$$

$$\frac{\partial}{\partial r}C\,v_p+\frac{\partial}{\partial z}C\,u_p+\frac{1}{r}\,C\,v_p=0. \tag{2.6}$$

In (2.1)–(2.6), z represents the direction of the wave propagation, whereas r stands for the radial coordinate, (u_f, v_f) denote the axial and radial velocity components of the fluid phase, and (u_p, v_p) those of the particulate phase; ρ_f, ρ_p, $(1-C)\,\rho_f$, and $C\rho_p$ are, respectively, the actual densities of the materials consisting of fluid and the solid particle phases, the fluid-phase density, and particle-phase density, C being the volume fraction of the particles in the mixture. In addition, p is the pressure, $\mu_s(C)$ is the particle-fluid mixture viscosity (also called as the effective viscosity of the suspension), and S the drag coefficient of the interaction for the force exerted by one phase on the other. We may neglect the field interaction between particles and regard the volume fraction, C, as a constant, when the concentration is low. Under the assumption that the solid particles are very small in size, the diffusivity terms, representing the effects of particle-particle interaction owing to the Brownian motion, may be considered to be negligible. The expression of the drag coefficient of the present problem is selected as (cf. [4])

$$\begin{aligned} S &= \frac{9}{2}\frac{\mu_0}{a^2}\lambda'(C), \qquad \text{where} \\ \lambda'(C) &= \frac{4+3\,[8C-3C^2]^{1/2}+3C}{[2-3C]^2}, \end{aligned} \tag{2.7}$$

μ_0 being the fluid viscosity, and a the radius of each solid particle suspended in the fluid. The above expression for the drag coefficient bears the potential to account for the finite particulate fractional volume through the function $\lambda'(C)$. The following empirical relation, suggested by Charm and Kurland [5], will be used in the foregoing analysis, for the viscosity of the suspension:

$$\mu_s(C) = \mu_0 \frac{1}{1-q\,C}, \tag{2.8}$$

where

$$q = 0.070\ \exp\left[2.49\,C + \frac{1107}{T}\ \exp(-1.69\,C)\right],$$

in which T represents the absolute temperature ($^\circ K$). This formula has been tested by Charm and Kurland [5] by using a cone and a plate viscometer, and it has been proclaimed that it is reasonably accurate up to $C = 0.6$. Nonslip and impermeability conditions constitute the boundary conditions of the problem. The tube wall is assumed to be flexible but inextensible. It is also assumed that the displacement of the tube wall takes place in the radial direction only. Thus, the boundary conditions may be put as

$$\left.\begin{aligned} u_f &= 0, \\ v_f &= \frac{\partial \eta}{\partial t}, \\ v_p &= \frac{\partial \eta}{\partial t}, \end{aligned}\right\} \qquad \text{on} \quad r = R + \eta. \tag{2.9}$$

We also introduce the stream functions ψ_f and ψ_p such that

$$\begin{aligned} u_f &= -\frac{1}{r}\frac{\partial \psi_f}{\partial r}, & u_p &= -\frac{1}{r}\frac{\partial \psi_p}{\partial r}, \\ v_f &= \frac{1}{r}\frac{\partial \psi_f}{\partial z}, & v_p &= \frac{1}{r}\frac{\partial \psi_p}{\partial z}, \end{aligned} \tag{2.10}$$

The transverse displacement, η, is given by

$$\eta = b\ \cos\frac{2\pi}{\lambda}\,(z - c\,t), \tag{2.11}$$

where b is the amplitude, λ is the wave-length, and c is the wave-speed.

Furthermore, we introduce the following nondimensional variables based on R and c:

$$\begin{aligned} &z' = \frac{z}{R}, \quad r' = \frac{r}{R}, \quad u_f' = \frac{u_f}{c}, \quad v_f' = \frac{v_f}{c}, \quad u_p' = \frac{u_p}{c}, \quad v_p' = \frac{v_p}{c}, \\ &\eta' = \frac{\eta}{R}, \quad \psi_f' = \frac{\psi_f}{c\,R^2}, \quad \psi_p' = \frac{\psi_p}{c\,R^2}, \quad t' = \frac{c\,t}{R}, \quad p' = \frac{p}{\rho_f\,c^2}, \quad \nu = \frac{\mu_0}{\rho_f}, \end{aligned} \tag{2.12}$$

with

$$\begin{aligned}
\mathrm{Re} &= \frac{c\,R\,\rho_f}{(1-C)\,\mu_s} && \text{(suspension Reynolds number)},\\
\alpha &= \frac{2\pi\,r}{\lambda} && \text{(the wave number)},\\
\varepsilon &= \frac{b}{R} && \text{(the amplitude ratio)},\\
&\left.\begin{aligned} M &= \tfrac{S\,R^2}{(1-C)\,\mu_s}\\ N &= \tfrac{S\,R^2\,\rho_f}{(1-C)\,\rho_p\,\mu_s}\end{aligned}\right\} && \text{(suspension parameters)}.
\end{aligned}$$

In terms of these, the equations (2.1)–(2.6), (2.9), and (2.11) become (after the primes are dropped):

$$(1-C)\,\mathrm{Re}\left[\frac{\partial}{\partial t}\nabla^2\psi_f + \frac{1}{r}\frac{\partial}{\partial z}\psi_f\left\{\nabla^2\frac{\partial}{\partial r} - \frac{2}{r}\nabla^2 + \frac{1}{r^2}\frac{\partial}{\partial r}\right\}\psi_f - \frac{1}{r}\frac{\partial}{\partial r}\psi_f\nabla^2\frac{\partial}{\partial z}\psi_f\right] = \nabla^2\nabla^2\psi_f + C\,M\,(\nabla^2\psi_p - \nabla^2\psi_f), \quad (2.13)$$

$$C\,\mathrm{Re}\left[\frac{\partial}{\partial t}\nabla^2\psi_p + \frac{1}{r}\frac{\partial}{\partial z}\psi_p\left\{\nabla^2\frac{\partial}{\partial r} - \frac{2}{r}\nabla^2 + \frac{1}{r^2}\frac{\partial}{\partial r}\right\}\psi_p - \frac{1}{r}\frac{\partial}{\partial r}\psi_p\nabla^2\frac{\partial}{\partial z}\psi_p\right] = C\,N\,(\nabla^2\psi_f - \nabla^2\psi_p), \quad (2.14)$$

$$\eta = \varepsilon\,\cos\alpha\,(z - c\,t), \quad \nabla^2 \equiv \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r}, \quad (2.15)$$

and

$$\left.\begin{aligned} \tfrac{\partial}{\partial r}\psi_f &= 0\\ \tfrac{\partial}{\partial z}\psi_f &= \alpha\,\varepsilon\,r\sin\alpha(z-t)\\ \tfrac{\partial}{\partial z}\psi_p &= \alpha\,\varepsilon\,r\sin\alpha(z-t)\end{aligned}\right\} \quad \text{at } r = (1+\eta), \quad (2.16)$$

where ∇^2 denotes the Laplace operator.

3. METHOD OF SOLUTION

Taking the amplitude ratio, ε, of the wave to be small, let us consider the following series expansions of the stream functions and the pressure gradient (in powers of ε).

$$\psi_f = \psi_{f_0} + \varepsilon\,\psi_{f_1} + \varepsilon^2\,\psi_{f_2} + \cdots \quad (3.1)$$

$$\psi_p = \psi_{p_0} + \varepsilon\,\psi_{p_1} + \varepsilon^2\,\psi_{p_2} + \cdots \quad (3.2)$$

$$\frac{\partial p}{\partial z} = \left(\frac{\partial p}{\partial z}\right)_0 + \varepsilon\left(\frac{\partial p}{\partial z}\right)_1 + \varepsilon^2\left(\frac{\partial p}{\partial z}\right)_2 + \cdots \quad (3.3)$$

In the equation (3.3), the first term on the right hand side corresponds to the imposed pressure gradient associated with the primary flow, and the other terms associated with the peristaltic motion.

Substituting (3.1) and (3.2) in (2.13) and (2.14), and collecting terms of like powers of ε, we obtain

$$\nabla^2\nabla^2\psi_{f_0} = \mathrm{Re}\,(1-C)\left[\frac{\partial}{\partial t}\nabla^2\psi_{f_0} + \frac{1}{r}\frac{\partial}{\partial z}\psi_{f_0}\left\{\nabla^2\frac{\partial}{\partial r} - \frac{2}{r}\nabla^2 + \frac{1}{r^2}\frac{\partial}{\partial r}\right\}\psi_{f_0} - \frac{1}{r}\frac{\partial}{\partial r}\psi_{f_0}\nabla^2\frac{\partial}{\partial z}\psi_{f_0}\right] - C\,M\,\nabla^2(\psi_{p_0} - \psi_{f_0}), \quad (3.4)$$

$$\nabla^2\nabla^2\psi_{f_1} = \mathrm{Re}\,(1-C)\Bigg[\frac{\partial}{\partial t}\nabla^2\psi_{f_1} + \frac{1}{r}\frac{\partial}{\partial z}\psi_{f_1}\left\{\nabla^2\frac{\partial}{\partial r} - \frac{2}{r}\nabla^2 + \frac{1}{r^2}\frac{\partial}{\partial r}\right\}\psi_{f_0}$$
$$- \frac{1}{r}\frac{\partial}{\partial r}\psi_{f_1}\nabla^2\frac{\partial}{\partial z}\psi_{f_0} + \frac{1}{r}\frac{\partial}{\partial z}\psi_{f_0}\left\{\nabla^2\frac{\partial}{\partial r} - \frac{2}{r}\nabla^2 + \frac{1}{r}\frac{\partial}{\partial r}\right\}\psi_{f_1}$$
$$- \frac{1}{r}\frac{\partial}{\partial r}\psi_{f_0}\nabla^2\frac{\partial}{\partial z}\psi_{f_1}\Bigg] - C\,M\,\nabla^2(\psi_{p_1} - \psi_{f_1}), \tag{3.5}$$

$$\nabla^2\nabla^2\psi_{f_2} = \mathrm{Re}\,(1-C)\Bigg[\frac{\partial}{\partial t}\nabla^2\psi_{f_2} + \frac{1}{r}\frac{\partial}{\partial z}\psi_{f_2}\left\{\nabla^2\frac{\partial}{\partial r} - \frac{2}{r}\nabla^2 + \frac{1}{r^2}\frac{\partial}{\partial r}\right\}\psi_{f_0}$$
$$- \frac{1}{r}\frac{\partial}{\partial r}\psi_{f_2}\nabla^2\frac{\partial}{\partial z}\psi_{f_0} + \frac{1}{r}\frac{\partial}{\partial z}\psi_{f_1}\left\{\nabla^2\frac{\partial}{\partial r} - \frac{2}{r}\nabla^2 + \frac{1}{r^2}\frac{\partial}{\partial r}\right\}\psi_{f_1}$$
$$- \frac{1}{r}\frac{\partial}{\partial r}\psi_{f_1}\nabla^2\frac{\partial}{\partial z}\psi_{f_1} + \frac{1}{r}\frac{\partial}{\partial z}\psi_{f_0}\left\{\nabla^2\frac{\partial}{\partial r} - \frac{2}{r}\nabla^2 + \frac{1}{r^2}\frac{\partial}{\partial r}\right\}\psi_{f_2}$$
$$- \frac{1}{r}\psi_{f_0}\nabla^2\frac{\partial}{\partial z}\psi_{f_2}\Bigg] - C\,M\,\nabla^2(\psi_{p_2} - \psi_{f_2}), \tag{3.6}$$

$$C\,\mathrm{Re}\Bigg[\frac{\partial}{\partial t}\nabla^2\psi_{p_0} + \frac{1}{r}\frac{\partial}{\partial z}\psi_{p_0}\left\{\nabla^2\frac{\partial}{\partial r} - \frac{2}{r}\nabla^2 + \frac{1}{r^2}\frac{\partial}{\partial r}\right\}\psi_{p_0}$$
$$- \frac{1}{r}\frac{\partial}{\partial r}\psi_{p_0}\nabla^2\frac{\partial}{\partial z}\psi_{p_0}\Bigg] = C\,N\,\nabla^2(\psi_{f_0} - \psi_{p_0}), \tag{3.7}$$

$$C\,\mathrm{Re}\Bigg[\frac{\partial}{\partial t}\nabla^2\psi_{p_1} + \frac{1}{r}\frac{\partial}{\partial z}\psi_{p_0}\left\{\nabla^2\frac{\partial}{\partial r} - \frac{2}{r}\nabla^2 + \frac{1}{r^2}\frac{\partial}{\partial r}\right\}\psi_{p_1}$$
$$- \frac{1}{r}\frac{\partial}{\partial r}\psi_{p_0}\nabla^2\frac{\partial}{\partial z}\psi_{p_1} + \frac{1}{r}\frac{\partial}{\partial z}\psi_{p_1}\left\{\nabla^2\frac{\partial}{\partial r} - \frac{2}{r}\nabla^2 + \frac{1}{r^2}\frac{\partial}{\partial r}\right\}\psi_{p_0}$$
$$- \frac{1}{r}\frac{\partial}{\partial r}\psi_{p_1}\nabla^2\frac{\partial}{\partial z}\psi_{p_0}\Bigg] = C\,N\,\nabla^2(\psi_{f_1} - \psi_{p_1}), \tag{3.8}$$

$$C\,\mathrm{Re}\Bigg[\frac{\partial}{\partial t}\nabla^2\psi_{p_2} + \frac{1}{r}\frac{\partial}{\partial z}\psi_{p_0}\left\{\nabla^2\frac{\partial}{\partial r} - \frac{2}{r}\nabla^2 + \frac{1}{r^2}\frac{\partial}{\partial r}\right\}\psi_{p_2}$$
$$- \frac{1}{r}\frac{\partial}{\partial r}\psi_{p_0}\nabla^2\frac{\partial}{\partial z}\psi_{p_2} + \frac{1}{r}\frac{\partial}{\partial z}\psi_{p_1}\left\{\nabla^2\frac{\partial}{\partial r} - \frac{2}{r}\nabla^2 + \frac{1}{r^2}\frac{\partial}{\partial r}\right\}\psi_{p_1}$$
$$- \frac{1}{r}\frac{\partial}{\partial r}\psi_{p_1}\nabla^2\frac{\partial}{\partial z}\psi_{p_1} + \frac{1}{r}\frac{\partial}{\partial z}\psi_{p_2}\left\{\nabla^2\frac{\partial}{\partial r} - \frac{2}{r}\nabla^2 + \frac{1}{r^2}\frac{\partial}{\partial r}\right\}\psi_{p_0}$$
$$- \frac{1}{r}\frac{\partial}{\partial r}\psi_{p_2}\nabla^2\frac{\partial}{\partial z}\psi_{p_0}\Bigg] = C\,N\,\nabla^2(\psi_{f_2} - \psi_{p_2}). \tag{3.9}$$

In a similar manner, from the boundary conditions (16), we have

$$\frac{\partial}{\partial r}\psi_{f_0}(1) = 0, \tag{3.10}$$

$$\frac{\partial}{\partial r}\psi_{f_1}(1) + \frac{\partial^2}{\partial r^2}\psi_{f_0}(1)\,\cos\alpha(z-t) = 0, \tag{3.11}$$

$$\frac{\partial}{\partial r}\psi_{f_2}(1)+\frac{\partial^2}{\partial r^2}\psi_{f_1}(1)\,\cos\alpha(z-t)+\frac{1}{2}\,\frac{\partial^3}{\partial r^3}\psi_{f_1}(1)\,\cos\alpha(z-t)=0, \tag{3.12}$$

$$\frac{\partial}{\partial z}\psi_{f_0}(1)=0, \tag{3.13}$$

$$\frac{\partial}{\partial z}\psi_{f_1}(1)+\frac{\partial^2}{\partial r\,\partial z}\psi_{f_0}(1)\,\cos\alpha(z-t)=\alpha\sin\alpha(z-t), \tag{3.14}$$

$$\frac{\partial}{\partial z}\psi_{f_2}(1)+\frac{\partial^2}{\partial r\,\partial z}\psi_{f_1}\,\cos\alpha(z-t)+\frac{1}{2}\,\frac{\partial^3}{\partial z\,\partial r^2}\psi_{f_0}(1)\,\cos^2\alpha(z-t)=0, \tag{3.15}$$

$$\frac{\partial}{\partial z}\psi_{p_0}(1)=0, \tag{3.16}$$

$$\frac{\partial}{\partial z}\psi_{p_1}(1)+\frac{\partial^2}{\partial r\,\partial z}\psi_{p_0}(1)\,\cos\alpha(z-t)=\alpha\sin\alpha(z-t), \tag{3.17}$$

$$\frac{\partial}{\partial z}\psi_{p_2}(1)+\frac{\partial^2}{\partial r\,\partial z}\psi_{p_1}(1)\,\cos\alpha(z-t)+\frac{1}{2}\,\frac{\partial^3}{\partial r^2\,\partial z}\psi_{p_0}(1)\cos^2\alpha(z-t)=0. \tag{3.18}$$

The differential equations (3.4), (3.7), (3.10), (3.13), and (3.16), subject to the steady parallel flow and transverse symmetry assumption for a constant pressure gradient in the z-direction, give the classical Poiseuille flow for the fluid and the particle phases for which the stream functions are given by

$$\psi_{f_0}=K\left(r^2-\frac{r^4}{2}\right),\qquad \text{and} \tag{3.19}$$

$$\psi_{p_0}=K\left\{r^2+\frac{4}{M}r^2-\frac{r^4}{2}\right\}, \tag{3.20}$$

where

$$K=\frac{\mathrm{Re}}{8}\left(\frac{\partial p}{\partial z}\right)_0 \tag{3.21}$$

is the Poiseuille parameter.

Thus, the effect of the particles on the fluid velocity profile is to cause an increase in the viscosity, i.e., the fluid viscosity, μ_0, is replaced by the suspension viscosity $\mu_s=\mu_0/(1-q\,C)$, and thus for a given pressure difference less fluid will flow through the tube.

The above differential equations in ψ_{f_1}, ψ_{p_1}, ψ_{f_2}, and ψ_{p_2}, together with the corresponding boundary conditions are satisfied if

$$\begin{aligned}\psi_{f_1}&=\phi_{f_1}(r)\,\exp\{i\alpha\,(z-t)\}+\phi^*_{f_1}(r)\,\exp\{-i\alpha\,(z-t)\}\\ \psi_{p_1}&=\phi_{p_1}(r)\,\exp\{i\alpha\,(z-t)\}+\phi^*_{p_1}(r)\,\exp\{-i\alpha\,(z-t)\},\end{aligned} \tag{3.22}$$

$$\begin{aligned}\psi_{f_2}&=\phi_{f_{20}}(r)+\phi_{f_{22}}(r)\,\exp\{2i\alpha\,(z-t)\}+\phi^*_{f_{22}}(r)\,\exp\{-2i\alpha\,(z-t)\},\\ \psi_{p_2}&=\phi_{f_{20}}(r)+\phi_{p_{22}}(r)\,\exp\{2i\alpha\,(z-t)\}+\phi^*_{p_{22}}(r)\,\exp\{-2i\alpha\,(z-t)\},\end{aligned} \tag{3.23}$$

where the ϕ's are arbitrary functions of r alone and $*$ denotes the complex conjugate.

Substituting (3.22) and (3.23) into the differential equations and their corresponding boundary conditions in ψ_{f_1}, ψ_{p_1}, ψ_{f_2}, and ψ_{p_2}, we obtain the following set of differential equations:

$$\left[\frac{d^2}{dr^2}-\frac{1}{r}\,\frac{d}{dr}-\alpha^2+i\alpha\,\mathrm{Re}\,(1-C)\,\{1+2K\,(1-r^2)\}\right]\left\{\frac{d^2}{dr^2}-\frac{1}{r}\,\frac{d}{dr}-\alpha^2\right\}\phi_{f_1}$$
$$=C\,M\left[\frac{d^2}{dr^2}-\frac{1}{r}\,\frac{d}{dr}-\alpha^2\right](\phi_{f_1}-\phi_{p_1}), \tag{3.24}$$

$$i\alpha\,\mathrm{Re}\,C\left[1+2K\left\{1+\frac{4}{M}-r^2\right\}\left\{\frac{d^2}{dr^2}-\frac{1}{r}\,\frac{d}{dr}-\alpha^2\right\}\phi_{p_1}\right]$$
$$=C\,N\left[\frac{d^2}{dr^2}-\frac{1}{r}\,\frac{d}{dr}-\alpha^2\right](\phi_{p_1}-\phi_{f_1}), \tag{3.25}$$

$$\frac{d}{dr}\phi_{f_1}(1) = 2K, \qquad \phi_{f_1}(1) = -\frac{1}{2}, \qquad \phi_{p_1}(1) = -\frac{1}{2}, \tag{3.26}$$

$$\begin{aligned}
&\left[\left\{\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right\}\left\{\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right\}\right]\phi_{f_{20}} \\
&\quad = i\alpha \operatorname{Re}(1-C)\left[\frac{1}{r}\left\{(\phi_{f_1}\phi_{f_1}^{*\prime\prime} - \phi_{f_1}^{\prime\prime}\phi_{f_1}^{*}) - \frac{1}{r}(\phi_{f_1}\phi_{f_1}^{*\prime} - \phi_{f_1}^{\prime}\phi_{f_1}^{*})\right\}\right]' \\
&\quad - i\alpha \operatorname{Re}\frac{1-C}{r^2}\left[(\phi_{f_1}\phi_{f_1}^{*\prime\prime} - \phi_{f_1}^{\prime\prime}\phi_{f_1}^{*}) - \frac{1}{r}(\phi_{f_1}\phi_{f_1}^{*\prime} - \phi_{f_1}^{\prime}\phi_{f_1}^{*})\right] \\
&\quad + CM\left\{\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right\}(\phi_{f_{20}} - \phi_{p_{20}}),
\end{aligned} \tag{3.27}$$

$$\begin{aligned}
0 &= \left[\frac{1}{r}\left\{\phi_{p_1}\phi_{p_1}^{*\prime\prime} - \phi_{p_1}^{\prime\prime}\phi_{p_1}^{*}) - \frac{1}{r}(\phi_{p_1}\phi_{p_1}^{*\prime} - \phi_{p_1}^{\prime}\phi_{p_1}^{*})\right\}\right]' \\
&\quad - i\alpha \operatorname{Re}\frac{C}{r^2}\left[(\phi_{p_1}\phi_{p_1}^{*\prime\prime} - \phi_{p_1}^{\prime\prime}\phi_{p_1}^{*}) - \frac{1}{r}(\phi_{p_1}\phi_{p_1}^{*\prime} - \phi_{p_1}^{\prime}\phi_{p_1}^{*})\right] \\
&\quad + CN\left\{\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right\}(\phi_{p_{20}} - \phi_{f_{20}}),
\end{aligned} \tag{3.28}$$

$$\begin{aligned}
&\left\{\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} - 4\alpha^2\right\}\left\{\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} - 4\alpha^2\right\}\phi_{f_{22}} \\
&\quad = -2i\alpha \operatorname{Re}(1-C)\{1 + 2K(1-r^2)\}\left\{\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} - 4\alpha^2\right\}\phi_{f_{22}} \\
&\quad + i\alpha \operatorname{Re}\frac{1-C}{r}\left[\phi_{f_1}\phi_{f_1}^{\prime\prime\prime} - \phi_{f_1}^{\prime}\phi_{f_1}^{\prime\prime} - \frac{3}{r}\phi_{f_1}\phi_{f_1}^{\prime\prime} + \frac{1}{r^2}(\phi_{f_1}^{\prime})^2 + \frac{3}{r^2}\phi_{f_1}\phi_{f_1}^{\prime} + \frac{2}{r}\alpha^2(\phi_{f_1}^{\prime})^2\right] \\
&\quad + CM\left[\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} - 4\alpha^2\right](\phi_{f_{22}} - \phi_{p_{22}}),
\end{aligned} \tag{3.29}$$

$$\begin{aligned}
&2i\alpha C \operatorname{Re}\left[\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} - 4\alpha^2\right]\phi_{p_{22}} \\
&\quad = -2i\alpha CK \operatorname{Re}\left[2K\left\{1 + \frac{4}{M} - r^2\right\}\right]\left\{\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} - 4\alpha^2\right\}\phi_{p_{22}} \\
&\quad + \frac{i\alpha \operatorname{Re} C}{r}\left[\phi_{p_1}\phi_{p_1}^{\prime\prime\prime} - \phi_{p_1}^{\prime}\phi_{p_1}^{\prime\prime} - \frac{3}{r}\phi_{p_1}\phi_{p_1}^{\prime\prime} + \frac{1}{r^2}(\phi_{p_1}^{\prime})^2 + \frac{3}{r^2}\phi_{p_1}\phi_{p_1}^{\prime} + \frac{2}{r}\alpha^2(\phi_{p_1}^{\prime})^2\right] \\
&\quad + CN\left\{\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} - 4\alpha^2\right\}(\phi_{p_{22}} - \phi_{f_{22}}),
\end{aligned} \tag{3.30}$$

$$\begin{aligned}
&\phi_{f_{20}}^{\prime}(1) + \frac{1}{2}\{\phi_{f_1}^{\prime\prime}(1) + \phi_{f_1}^{*\prime\prime}(1)\} - 3K = 0, \quad \phi_{f_{22}}^{\prime}(1) + \frac{1}{2}\phi_{f_1}^{\prime\prime}(1) - \frac{3}{2}K = 0, \\
&\phi_{f_{22}}(1) + \frac{1}{2}\phi_{f_1}^{\prime}(1) = 0, \quad \phi_{p_{22}}(1) + \frac{1}{4}\phi_{p_1}^{\prime}(1) = 0.
\end{aligned} \tag{3.31}$$

These differential equations along with the corresponding boundary conditions are sufficient to solve the problem up to the second order in ε. After finding the solutions of the above system of differential equations, we can determine ψ_{f_1} and ψ_{p_1}. The higher order terms may also be determined in the same way. It may be noted that the present problem, being one of moving boundaries, have nonhomogeneous boundary conditions and hence it is not an eigenvalue problem. This makes the solution of the problem much more difficult. The pumping of an originally stationary fluid, with a zero pressure gradient [*viz.* $\frac{\partial p}{\partial z} = 0$], however, removes K and all the coefficients in the differential equations (3.24), (3.25), and (3.27)–(3.30) become constant. For this particular case, it is possible to obtain the solution in a simpler way. Over this one, the simple solution for a Poiseuille flow can be superposed if $K = O(\varepsilon^2)$.

4. PUMPING IN THE ABSENCE OF ZEROTH-ORDER PRESSURE GRADIENT

Let us consider that the fluid-particle mixture is stationary initially, i.e., the pressure gradient and hence K are zero. Under this assumption, the solution of equations (3.24) and (3.25), subject to the boundary conditions (3.26) together with the condition that the velocity, and hence $(1/r)\,(d\phi_{f_1}/dr)$, must remain finite at $r = 0$ yields

$$\phi_{f_1}(r) = C_{f_1}\, r\, I(\beta r) + C_{f_2}\, r\, I(\alpha r), \tag{4.1}$$

$$\phi_{p_1}(r) = C_{p_1}\, r\, I(\beta r) + C_{p_2}\, r\, I(\alpha r), \tag{4.2}$$

where

$$\beta^2 = \alpha^2 - i\alpha\,\mathrm{Re}\left[(1-C) + \frac{C\,M}{N - i\alpha\,\mathrm{Re}}\right], \tag{4.3}$$

$$C_{f_1} = -\frac{1}{2}\left[\frac{\alpha I_0(\alpha)}{\alpha\, I_1(\beta)\, I_0(\alpha) - \beta\, I_1(\alpha)\, I_0(\beta)}\right] \tag{4.4}$$

$$C_{f_2} = \frac{1}{2}\left[\frac{\beta\, I_0(\beta)}{\alpha\, I_1(\beta)\, I_0(\alpha) - \beta\, I_1(\alpha\, I_0(\beta)}\right], \tag{4.5}$$

$$C_{p_1} = \frac{C_{f_1} N}{N - i\alpha\,\mathrm{Re}}, \tag{4.6}$$

$$C_{p_2} = \frac{1 - C_{p_1}\, I_1(\beta)}{I_1(\alpha)}, \tag{4.7}$$

where I_0 and I_1 are the modified Bessel functions of the first kind.

Furthermore, in the expansion of ψ_{f_2}, ψ_{p_2}, we are interested in $\phi_{f_{20}}(r)$, $\phi_{p_{20}}(r)$ only, as our goal is to find out the mean flow only. So, under the assumption that $K = 0$, the differential equations (3.27) and (3.28), subject to the boundary conditions (3.31) are solved as follows.

The substitution of (4.1), (4.2) and their conjugates into (3.27), (3.28), and (3.31) yields

$$\begin{aligned}
&\left\{\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right\}\left\{\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right\}\phi_{f_{20}} \\
&= -\alpha^2\,\mathrm{Re}^2\,(1-C)^2\Bigg[\left\{\left(1 + \frac{C}{1-C}\frac{M}{N - i\alpha\,\mathrm{Re}}\right)\left(1 + \frac{C}{1-C}\frac{M}{N + i\alpha\,\mathrm{Re}}\right)\right\} \\
&\qquad \times \left(C_{f_1} C_{f_1}^* + \frac{C}{1-C}\frac{M}{N} C_{p_1} C_{p_1}^*\right) \\
&\qquad \{r\,\beta^*\, I_1(\beta r) I_0(\beta^* r) + r\,\beta\, I_0(\beta r) I_1(\beta^* r) - 2 I_1(\beta r)\; I_1(\beta^*(r)\} \\
&\quad + \left(1 + \frac{C}{1-C}\frac{M}{N + i\alpha\,\mathrm{Re}}\right)\{r\,\beta^*\, I_1(\alpha r)\, I_0(\beta^* r) + r\,\alpha\, I_0(\alpha r)\, I_1(\beta^* r) - 2 I_1(\beta^* r)\, I_1(\alpha r)\} \\
&\qquad \times \left(C_{f_1}^* C_{f_2} + \frac{C}{(1-C)}\frac{M}{N} C_{p_1}^* C_{p_2}\right) \\
&\quad + \left(1 + \frac{C}{1-C}\frac{M}{N - i\alpha\,\mathrm{Re}}\right)\left(C_{f_1} C_{f_2}^* + \frac{C}{1-C}\frac{M}{N} C_{p_1} C_{p_2}^*\right) \\
&\quad + \{r\,\beta\, I_0(\beta r)\, I_1(\alpha r) + r\,\alpha\, I_1(\beta r)\, I_0(\alpha r) - 2 I_1(\beta r)\, I_1(\alpha r)\}\Bigg],
\end{aligned} \tag{4.8}$$

$$
\begin{aligned}
&CN\left[\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right]\psi_{p_{20}} \\
&\quad = CN\left[\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right]\psi_{f_{20}} \\
&\quad + \alpha^2\,\mathrm{Re}^2\,C\,(1-C)\left[\left\{\left(1+\frac{C}{1-C}\,\frac{M}{N-i\alpha\,\mathrm{Re}}\right)+\left(1+\frac{C}{1-C}\,\frac{M}{N+i\alpha\,\mathrm{Re}}\right)\right\}\right. \\
&\qquad \times C_{p_1}\,C^*_{p_1}\,\{r\,\beta^*\,I_1(\beta r)\,I_0(\beta^* r)+r\,\beta\,I_0(\beta r)\,I_1(\beta^* r)-2I_1(\beta r)\,I_1(\beta^* r)\} \\
&\quad + \left(1+\frac{C}{1-C}\,\frac{M}{N-i\alpha\,\mathrm{Re}}\right)C_{p_1}\,C^*_{p_2}\,\{r\,\beta\,I_0(\beta r)\,I_1(\alpha r)+r\,\alpha\,I_1(\beta r)\,I_0(\alpha r)-2I_1(\beta r)\,I_1(\alpha r)\} \\
&\quad + \left(1+\frac{C}{1-C}\,\frac{M}{N+i\alpha\,\mathrm{Re}}\right)C^*_{p_1}\,C_{p_2}\left\{r\,\beta^*\,I_1(\alpha r)\,I_0(\beta^* r)+r\,\alpha\,I_0(\alpha r)\,I_1(\beta^* r)\right. \\
&\quad \left.\left. -\,2I_1(\beta^* r)\,I_1(\alpha r)\right\}\right],
\end{aligned} \tag{4.9}
$$

$$
\begin{aligned}
\phi'_{f_{20}}(1) = -\frac{1}{2}\Big[&C_{f_1}\,\beta^2\,I_1(\beta)+(C_{f_2}+C^*_{f_2})\,\alpha^2\,I_1(\alpha)+C^*_{f_1}\,\beta^{*2}\,I_1(\beta^*) \\
&+C_{f_1}\,\beta\,I_0(\beta)+(C_{f_2}+C^*_{f_2})\,\alpha\,I_0(\alpha)+C^*_{f_1}\,\beta^*\,I_0(\beta^*)\Big] \equiv \zeta.
\end{aligned} \tag{4.10}
$$

The terms in brackets on the right hand side of (4.8) and (4.9) are complicated functions of r, and hence, the determination of the particular solutions of (4.8) and (4.9) corresponding to these groups of the terms is extremely complicated. To get rid of the tedious calculations and manipulations, we represent the results approximately by polynomials of the following forms:

$$
\left\{\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right\}\left\{\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right\}\phi_{f_{20}} = -(1-C)^2\,\alpha^2\,\mathrm{Re}^2\sum_{i=1}^{J}B_i\,r^{2i}, \tag{4.11}
$$

$$
CN\left\{\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right\}(\phi_{p_{20}}-\phi_{f_{20}}) = \alpha^2\,\mathrm{Re}^2\,C\,(1-C)\sum_{i=1}^{J'}B'_i\,r^{2i}; \tag{4.12}
$$

B_i and B'_i have been determined by a least squares procedure. They depend on C, α, and Re; thus, for the different ranges of C, α, and Re different B_i and B'_i have been used.

Solving (4.11) and (4.12) and differentiating once, we obtain

$$
\phi'_{f_{20}} = (L_1+L_2)\,r+2L_2\,r\,\ln(r)+L_3\,r^3-(1-C)^2\alpha^2\mathrm{Re}^2\sum_{i=1}^{J}\frac{B_i\,r^{(2i+3)}}{(2i+2)^2\,(2i)}, \tag{4.13}
$$

$$
\begin{aligned}
\phi'_{p_{20}} = {}&(L_4+L_2)\,r+2L_2\,r\,\ln(r)+L_3\,r^3-\alpha^2\,\mathrm{Re}^2\,(1-C)^2\sum_{i=1}^{J}\frac{B_i\,r^{(2i+3)}}{(2i+2)^2\,(2i)} \\
&+\frac{C\,(1-C)}{CN}\alpha^2\mathrm{Re}^2\sum_{i=1}^{J}\frac{B'_i\,r^{(2i+1)}}{2i},
\end{aligned} \tag{4.14}
$$

where L_1, L_2, L_3, and L_4 are constants.

The conditions that the velocity must remain finite at $r=0$, and therefore $\phi'_{f_{20}}$ must remain finite at $r=0$, yields $L_2=0$. Defining the following functions

$$
G(r) = (1-C)^2\sum_{i=2}^{J}\frac{B_i\,r^{(2i+3)}}{(2i)\,(2i+2)^2} \tag{4.15}
$$

$$
F(r) = (1-C)^2\sum_{i=2}^{J}\frac{B_i\,r^{(2i+3)}}{(2i)\,(2i+2)}-\frac{C\,(1-C)}{CN}\sum_{i=2}^{J'}\frac{B_i\,r^{(2i+1)}}{2i}, \tag{4.16}
$$

the boundary condition (4.10) is written as

$$\phi'_{f_{20}}(1) = L_1 + L_3 - \alpha^2 \mathrm{Re}^2 G(1) \equiv \zeta. \tag{4.17}$$

Now, using (4.15)–(4.17) and $L_2 = 0$ into (4.13) and (4.14) we obtain

$$\phi'_{f_{20}} = [\zeta + \alpha^2 \mathrm{Re}^2 G(1)]\, r + L_3\,(r^3 - r) - \alpha^2 \mathrm{Re}^2 G(r), \tag{4.18}$$

$$\phi'_{p_{20}} = [\zeta + \alpha^2 \mathrm{Re}^2 G(1)]\, r + L_3 \left\{ r^3 - r - \frac{4}{M} r \right\} - \alpha^2 \mathrm{Re}^2 F(r). \tag{4.19}$$

One of the constants, L_3, still remains arbitrary and will depend upon the mean pressure gradient. To this end, if we consider a time-average of (2.1) for the solution given by (2.1), (2.2), (2.4), (2.5), (3.1)–(3.3), (3.7), (3.8), (3.17), (3.24), and (3.25), we find that

$$\left(\frac{\partial p}{\partial z}\right)_2 = \Delta - \frac{4L_3}{\mathrm{Re}}, \tag{4.20}$$

where

$$\begin{aligned}
\Delta = -\alpha^2 \mathrm{Re} \Bigg[& -\frac{G''(r)}{r} + \frac{G'(r)}{r^2} - \frac{G(r)}{r^3} \\
& - \left\{ \left((1-C) + \frac{C\,M}{N + i\alpha \mathrm{Re}} \right) + \left((1-C) + \frac{C\,M}{N - i\alpha \mathrm{Re}} \right) \right\} \\
& \quad \left\{ (1-C)\, C_{f_1} C^*_{f_1} + C \frac{M}{N} C_{p_1} C^*_{p_1} \right\} I_1(\beta^* r)\, I_1(\beta r) \\
& - \left((1-C) + \frac{C\,M}{N - i\alpha \mathrm{Re}} \right) \left\{ (1-C)\, C_{f_1} C^*_{f_2} + C \frac{M}{N} C_{p_1} C^*_{p_2} \right\} I_1(\beta r)\, I_1(\alpha r) \\
& \left((1-C) + \frac{C\,M}{N + i\alpha \mathrm{Re}} \right) \left\{ (1-C)\, C^*_{f_1} C_{f_2} + C \frac{M}{N} C^*_{p_1} C_{p_2} \right\} I_1(\beta^* r)\, I_1(\alpha r) \Bigg] - \frac{4L_3}{\mathrm{Re}}.
\end{aligned}$$

It may be noted that unlike the two-dimensional case, the time-averaged pressure gradient is not constant over time, but has a perturbation which depends upon the radius. Since $\Delta = 0$ on $r = 0$, one may determine L_3 by solving the above equation. L_3 is found to be proportional to the axial time-averaged pressure gradient accompanying the peristaltic motion. Having specified L_3 in this way, the solution, averaged over time, for the mean axial velocity may be given by

$$\bar{u}_f(r) = -\frac{\varepsilon^2}{r} \phi'_{f_{20}} = \varepsilon^2 \left[-\alpha^2 \mathrm{Re}^2 \left\{ G(1) - \frac{G(r)}{r} \right\} - \zeta + L_3\,(1 - r^2) \right], \tag{4.21}$$

$$\bar{u}_p(r) = -\frac{\varepsilon^2}{r} \phi'_{p_{20}} = \varepsilon^2 \left[-\alpha^2 \mathrm{Re}^2 \left\{ G(1) - \frac{F(r)}{r} \right\} - \zeta + L_3 \left(1 + \frac{4}{M} - r^2 \right) \right]. \tag{4.22}$$

From (4.21), the critical reversal flow condition, that the velocity along the centre-line of the tube is zero may be introduced. Using (4.20), this condition may be expressed by

$$\frac{\partial \bar{p}}{\partial z}_{2\mathrm{crit}} = \frac{4}{\mathrm{Re}} [-\alpha^2 \mathrm{Re}^2 G(1) - \zeta]. \tag{4.23}$$

For particle-free flow our results tally exactly with those of Yin and Fung [6].

Table 1. Variation of ζ with C, α, and Re.

Re	α	C		
		0.0	0.1	0.2
600	0.12	−22.70283	−29.80276	−36.47531
	0.15	−35.89977	−23.21400	−8.484365
	0.18	13.13472	21.316	26.34322
	0.21	30.33152	31.931	32.56042
700	0.12	−41.07535	−38.48836	−28.82742
	0.15	−6.854490	16.87302	23.29600
	0.18	29.79772	31.38101	32.00893
	0.21	32.56108	32.11160	31.42037
800	0.12	−18.60755	−3.477511	8.680878
	0.15	26.51192	29.41717	30.90509
	0.18	32.08697	31.80008	31.22109
	0.21	30.79297	29.78825	28.74751

An Example: Peristaltic Flow in the Ureter

In order to illustrate the applicability of the analytical work presented above, let us consider as a specific example, the flow through the ureter.

It is known that for an adult human being, the average value of the length of the ureter is about 300 mm and when fully distended, it has a diameter of a few mm, but it can easily collapse to a slit-shaped cross-section with essentially zero lumen. Under normal circumstances, the ureter undergoes peristalsis, i.e., successive waves of active muscular contraction pass along its wall from the kidney (renal pelvis) to the bladder at intervals of about 10 to 60 seconds (in men). The amplitude of the wave is of the order of 5 mm and the speed is typically about 20 to 60 mm/s, so that the passage of one wave through the whole ureter takes about 10 to 15 seconds. The length of each wave usually ranges between 60 and 100 mm. The frequency of contractions varies from one individual to another, and is about 1 to 8 per minute. Each contraction lasts about 1.5 to 9 sec, the dilating phase is about twice as long as the contracting phase. Pressure during the contraction varies from 2 to 8 mm Hg at the pelvis, 2 to 10 mm Hg in the upper segments of the ureter, and 2 to 14 mmHg in the lower segment of the ureter. The computational results presented below have been obtained by using the above data, which have been reported by Orkins [7], Bergman [8], Boyarsky [9], Griffiths [10], and Wienberg [11].

For the flow of urine through the ureter, numerical values of the mean axial velocity, $\bar{u}_f(r)$, given by (4.21), have been computed for various values of C, α, and Re. They reveal that the main contribution for the mean axial velocity comes from ζ and the mean parabolic term $L_3\,(1-r^2)$. It is also observed that the value of ζ (which owes its origin to the nonslip condition imposed on the wall of the tube) increases, in general, with the increase of C, α, and Re, although no exact trend is evident for the range considered here. The computed values of ζ have been displayed in Table 1.

From the time-averaged second order pressure gradient evaluated on the axis, we get the parabolic term $L_3\,(1-r^2)$, consisting of the mean pressure gradient. The mean pressure gradient has a perturbation function $\Delta(r)$. So a pressure perturbation function may be defined as

$$H(r) = \frac{\Delta(r)}{\alpha^2\,\mathrm{Re}}.$$

We observe a slight increase in $H(r)$ with the increase of C and Re. Its value, however, decreases with the increase of α(cf. Figure 1).

The third perturbation term contributing to the velocity is $G(1) - G(r)/r = E(r)$ (say). At the boundary, $E(r)$ is minimum and increases inwards along the radius, and finally becomes $G(1)$

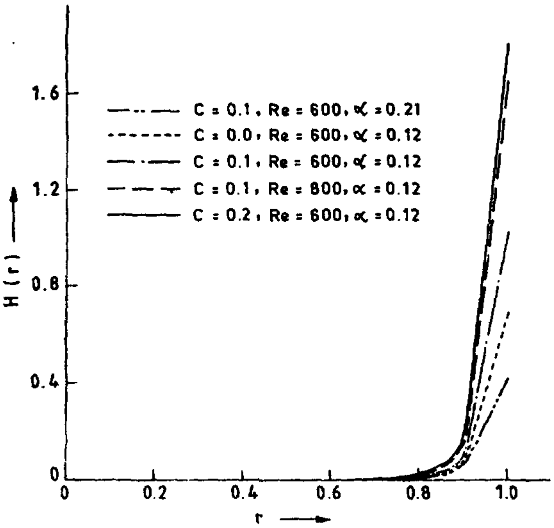


Figure 1. Variation of the pressure perturbation function in the radial direction for some selected cases.

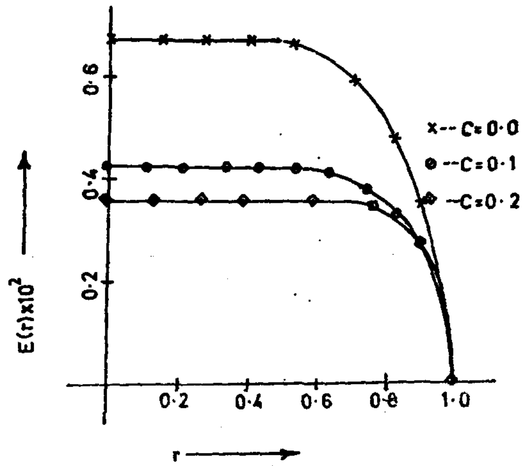


Figure 2. Mean velocity perturbation function $E(r)$ for various values of particle-concentration (Re = 600 and $\alpha = 0.12$).

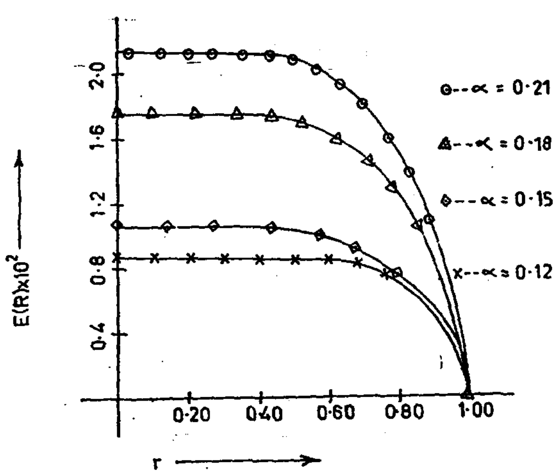


Figure 3. Mean velocity perturbation function for various values of the wave number ($C = 0.1$ and Re = 700).

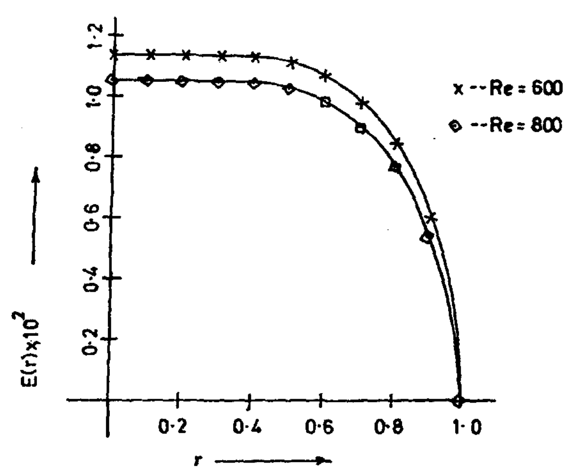


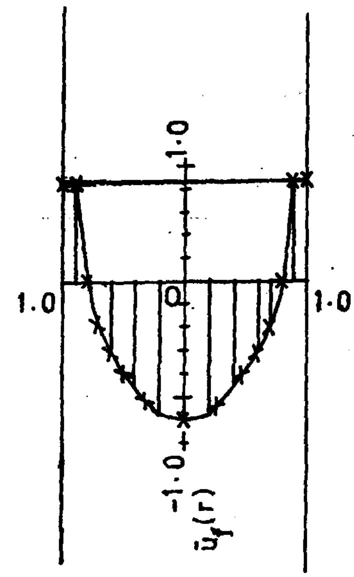
Figure 4. Mean velocity perturbation function $E(r)$ for various values of Reynolds number ($C = 0.1$ and $\alpha = 0.18$).

on the centre line. Its variation with C, α, and Re is not very prominent. The variation of $E(r)$ with r has been displayed in Figures 2–4 for different values of C, α, and Re.

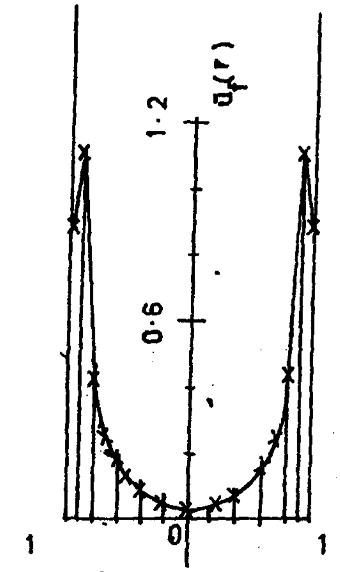
In case the mean pressure gradient $\left(\frac{\partial p}{\partial z}\right)_2$ exceeds a certain critical value, a reversal of flow takes place (Figure 5a). For the critical value of the mean pressure gradient, the mean axial velocity $\bar{u}_f(r)$ is zero at the centre of the tube (cf. Figure 5b). For the sake of comparison, velocity profiles have been presented for different variations of the parameters in Figures 5a–5d.

At higher Reynolds number, the mean critical pressure gradient turns out to be negative in some cases, and the presence of the solid particles favours the backward flow (cf. Figure 6).

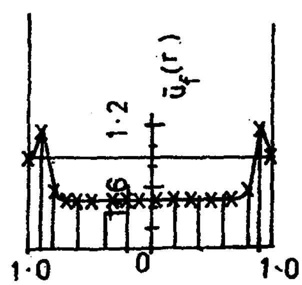
It was observed by Srivastava and Srivastava [2] that for the two-dimensional flow, the flow reversal does not take place at the boundary. However, present study indicates (cf. Figure 7) that for the axisymmetric case, there is a possibility of flow reversal for the mean flow even at the boundaries.



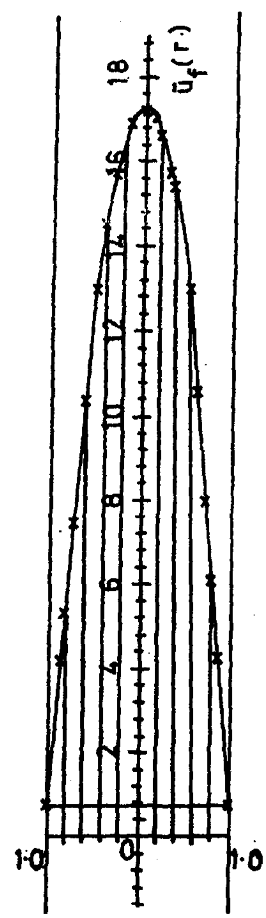
(a) Time-averaged mean axial velocity profile for ($C = 0.2$, $\alpha = 0.12$, Re $= 600$, $\varepsilon = 0.15$, $\left(\frac{\partial p}{\partial z}\right)_2 = 0.5 > \left(\frac{\partial p}{\partial z}\right)_{2crit}$.



(b) Time-averaged mean axial velocity profile for ($C = 0.2$, $\alpha = 0.12$, Re $= 600$, $\varepsilon = 0.15$, $\left(\frac{\partial p}{\partial z}\right)_2 > \left(\frac{\partial p}{\partial z}\right)_{2crit}$.



(c) Time-averaged mean axial velocity profile for ($C = 0.2$, $\alpha = 0.12$, Re $= 600$, $\varepsilon = 0.15$, $\left(\frac{\partial p}{\partial z}\right)_2 = 0$).



(d) Time-averaged mean axial velocity profile ($C = 0.2$, $\alpha = 0.12$, Re $= 600$, $\varepsilon = 0.15$, $\left(\frac{\partial p}{\partial z}\right)_2 = -5 < \left(\frac{\partial p}{\partial z}\right)_{2crit}$.

Figure 5.

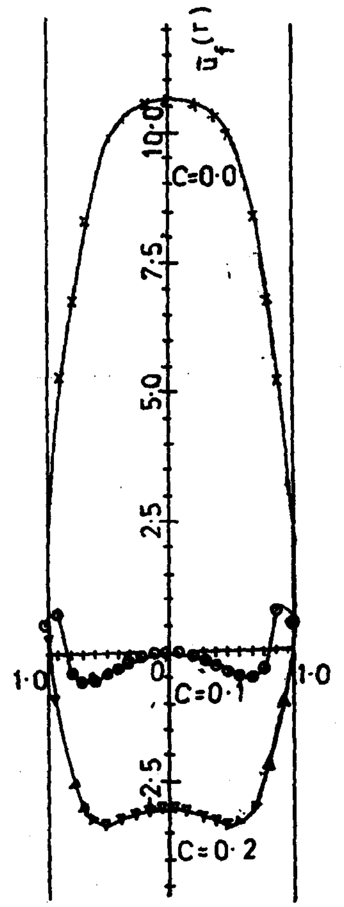


Figure 6. Variation of time-averaged mean velocity profile with particle-concentration ($\alpha = 0.15$, Re $=$ 600, $\varepsilon = 0.15$, $\left(\frac{\partial p}{\partial z}\right)_2 = 0.15$).

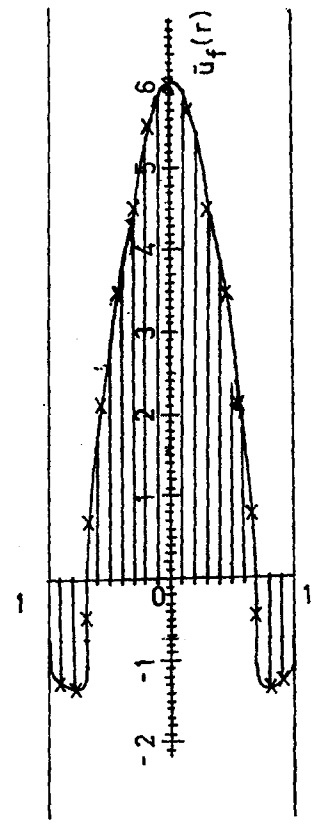


Figure 7. Time-averaged mean axial velocity profile for ($C = 0.2$, $\alpha = 0.21$, Re $= 600$, $\varepsilon =$ 0.15, $\left(\frac{\partial p}{\partial z}\right)_2 = -5$).

5. CONCLUSION

We observe a qualitative similarity between the axisymmetric flow and the two-dimensional channel flow for the mixture of fluid and solid particles. Unlike the case of two-dimensional channel flow, the second order pressure gradient (averaged over a period), however, varies along the radius in the case of the cylindrical flow. From the analysis, we are in a position to conclude that the mean flow induced by the peristaltic motion is proportional to the square of the amplitude ratio and depends on the mean pressure gradient also (induced by the peristaltic motion). Moreover, at a certain critical value of the pressure gradient, the reversal of flow takes place, which is favoured by the presence of particles. Further, contrary to the two-dimensional flow, the mean flow in axisysmmetric case may exhibit the reversal of flow at the boundaries also.

REFERENCES

1. T.K. Hung and T.D. Brown, Solid-particle motion in two-dimensional peristaltic flows, *J. Fluid Mech.* **73**, 77–96 (1976).
2. L.M. Srivastava and V.P. Srivastava, Peristaltic transport of a particle-fluid suspension, *J. Biomech. Engg.* **111**, 157–165 (1989).
3. D.A. Drew, Stability of a Stokes layer of a dusty gas, *Phys. Fluids* **22**, 2081–2084 (1979).
4. C.K.W. Tam, The drag on a cloud of spherical particles in low Reynolds number flow, *J. Fluid Mech.* **38**, 537–546 (1969).
5. S.E. Charm and G.S. Kurland, *Blood Flow and Micro-Circulation*, John Wiley, New York, (1974).
6. F. Yin and Y.C. Fung, Peristaltic waves in a circular cylindrical tube, *J. Applied Mech.* **36**, 93–112 (1969).

7. L.A. Orkins, *Trauma in the Ureter: Pathogenesis and Management*, F.A. Davis Co., Philadelphia, PA, (1964).
8. H. Bergeman, *The Ureter*, Harper & Row, New York, (1967).
9. S. Boyarsky, *Neurogenic Bladder*, Williams and Wilkins Co., Baltimore, (1967).
10. D.J. Griffiths, Flow of urine through the ureter: A collapsible, muscular tube undergoing peristalsis, *J. Biomech. Engg.* **111**, 206–211 (1989).
11. S.R. Weinberg, Physiology of the Ureter, In *The Ureter*, (Edited by H. Bergeman), pp. 48–66, Harper & Row, (1967).