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Deformation theory of objects in homotopy and derived categories II: Pro-representability of the deformation functor [☆]

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Abstract

This is the second paper in a series. In part I we developed deformation theory of objects in homotopy and derived categories of DG categories. Here we extend these (derived) deformation functors to an appropriate bicategory of artinian DG algebras and prove that these extended functors are pro-representable in a strong sense.

Keywords: Koszul duality; Deformation theory; Derived categories; Moduli spaces

Contents

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1. Introduction

In our paper [2] we developed a general deformation theory of objects in homotopy and derived categories of DG categories. The corresponding deformation pseudo-functors are defined on the category of artinian DG algebras dgart and take values in the 2-category **Gpd** of groupoids. More precisely if $\mathcal{A}$ is a DG category and E is a right DG module over $\mathcal{A}$ we defined four pseudo-functors

$$\mathrm{Def}^{\mathrm{h}}(E), \mathrm{coDef}^{\mathrm{h}}(E), \mathrm{Def}(E), \mathrm{coDef}(E) : \mathrm{dgart} \to \mathbf{Gpd}.$$

The first two are the *homotopy* deformation and co-deformation pseudo-functors, i.e. they describe deformations (and co-deformations) of E in the homotopy category of DG $\mathcal{A}^{op}$-modules; and the last two are their *derived* analogues. The pseudo-functors $\mathrm{Def}^{\mathrm{h}}(E)$, $\mathrm{coDef}^{\mathrm{h}}(E)$ are equivalent and depend only on the quasi-isomorphism class of the DG algebra $\mathrm{End}(E)$. The derived pseudo-functors $\mathrm{Def}(E)$, $\mathrm{coDef}(E)$ need some boundedness conditions to give the "right" answer and in that case they are equivalent to $\mathrm{Def}^{\mathrm{h}}(F)$ and $\mathrm{coDef}^{\mathrm{h}}(F)$ respectively for an appropriately chosen h-projective or h-injective DG module F which is quasi-isomorphic to E (one also needs to restrict the pseudo-functors to the category dgart_{-} of negative artinian DG algebras).

In this second paper we would like to discuss the pro-representability of these pseudo-functors. Recall that "classically" one defines representability only for functors with values in the category of sets (since the collection of morphisms between two objects in a category is a set). For example, given a moduli problem in the form of a pseudo-functor with values in the 2-category of groupoids one then composes it with the functor π_0 to get a set valued functor, which one then tries to (pro-)represent. This is certainly a loss of information. But in order to represent the original pseudo-functor one needs the source category to be a bicategory.

It turns out that there is a natural bicategory 2-adgalg of augmented DG algebras. (Actually we consider two versions of this bicategory, 2-adgalg and $2'$-adgalg, but then show that they are equivalent.) We consider its full subcategory 2-dgart_{-} whose objects are negative artinian DG algebras, and show that the derived deformation functors can be naturally extended to pseudo-functors

$$\mathrm{coDEF}_{-}(E): 2\text{-}\mathrm{dgart}_{-} \to \mathbf{Gpd}, \qquad \mathrm{DEF}_{-}(E): 2'\text{-}\mathrm{dgart}_{-} \to \mathbf{Gpd}.$$

Then (under some finiteness conditions on the graded algebra $\mathrm{Ext}(E,E) = H(\mathcal{C})$, where $\mathcal{C} = \mathbf{R}\mathrm{Hom}(E,E)$), we prove pro-representability of these pseudo-functors by the DG algebra $\hat{S} = (B\bar{A})^*$ which is the linear dual of the bar construction $B\bar{A}$ of the minimal A_∞-model of $\mathcal{C}$ (Theorems 14.1, 14.2, 15.1, 15.2).

This pro-representability appears to be more "natural" for the pseudo-functor coDEF_{-}, because the bar complex $B\bar{A} \otimes_{\tau_A} A$ is the "universal co-deformation" of A considered as an A_∞-module over A^{op}. The pro-representability of the pseudo-functor DEF_{-} may then be formally deduced from that of coDEF_{-}, but we can find the corresponding "universal deformation" (of A) only under an additional assumption on A (Theorem 15.12). We also make the equivalence $\mathrm{DEF}_{-}(E) \cong 1\text{-}\mathrm{Hom}(\hat{S}, -)$ explicit in this case (Corollary 15.15).

These theorems describe formal deformation theory of objects in derived categories. Our formal moduli spaces are in general "non-commutative DG schemes". In contrast, in the paper [10] global *commutative* moduli D^--stacks of objects in DG categories are studied. In [3] we treat in detail an example where we can construct a global moduli space of objects.

Namely, take some vector space V of dimension n, and consider the object $\mathcal{O}_{\mathbb{P}(W)} \in D^b_{coh}(\mathbb{P}(V))$, where $W \in \mathrm{Gr}(m,V)(k)$, $1 \leqslant m \leqslant n-1$. The corresponding DG algebra $\hat{S}$ satisfies the following property: $H^i(\hat{S}) = 0$ for $i \neq 0$, and for $m \neq 1$ the algebra $H^0(\hat{S})$ is non-commutative. This suggests the existence of a non-commutative space $\mathrm{NGr}(m,V)$ such that there is a k-point x associated with each subspace $W \subset V$ of dimension m. In [3] we construct these non-commutative spaces and call them "non-commutative Grassmanians". These non-commutative Grassmanians should be treated as true moduli spaces of objects $\mathcal{O}_{\mathbb{P}(W)} \subset D^b_{coh}(\mathbb{P}(V))$. One of their properties is the following: if $x \in \mathrm{NGr}(m,V)(k)$ is the point corresponding to $W \subset V$, then we have $\widehat{\mathcal{O}_x} \cong H^0(\hat{S})$.

We also note that the space $\mathrm{NGr}(\dim V - 1, V)$, which can be considered as a (dual) non-commutative projective space, is closely related to the non-commutative projective space of Kontsevich and Rosenberg [7]. The example of non-commutative Grassmanians should admit a generalization to a large class of families of objects in derived categories, for instance, "non-commutative Jacobians".

The first part of the paper is devoted to preliminaries on A_∞-algebras, A_∞-modules and A_∞-categories. The only non-standard point here is the DG category of A_∞ $A_{\mathcal{C}}$-modules for an A_∞-algebra A and a DG algebra $\mathcal{C}$, and the corresponding derived category $D_\infty(A_{\mathcal{C}})$. We also discuss certain functors defined by the bar complex of an augmented A_∞-algebra.

In the second part we introduce the Maurer–Cartan pseudo-functor $\mathcal{MC}(A)$: dgart $\to$ **Gpd** for a strictly unital A_∞-algebra A. The Maurer–Cartan groupoid $\mathcal{MC}_{\mathcal{R}}(A)$ can be described by means of some A_∞-category with the same objects, which are solutions of the generalized Maurer–Cartan equation (Section 5). We develop the obstruction theory for the Maurer–Cartan pseudo-functor (Proposition 6.1). Finally, we show the invariance of (quasi-)equivalence classes of the constructed A_∞-categories and Maurer–Cartan pseudo-functors under the quasi-isomorphisms of A_∞-algebras (Theorems 7.1, 7.2).

In the third part we define the bicategories 2-adgalg and 2′-adgalg and the pseudo-functors coDEF_- and DEF_- and discuss their relations. We also obtain here some results on the equivalences between the homotopy and derived (co-)deformation functors (Lemma 9.9, Theorem 11.8).

In the fourth part we prove the pro-representability theorems.

We freely use the notation and results of [2]. The reference to [2] appears in the form I, Theorem .... As in [2] our basic reference for bicategories is [1].

Part 1. A_∞-structures and the bar complex

2. Coalgebras

2.1. Coalgebras and comodules

We will consider DG coalgebras. For a DG coalgebra $\mathcal{G}$ we denote by $\mathcal{G}^{gr}$ the corresponding graded coalgebra obtained from $\mathcal{G}$ by forgetting the differential. Recall that if $\mathcal{G}$ is a DG coalgebra, then its graded dual $\mathcal{G}^*$ is naturally a DG algebra. Also given a finite-dimensional DG algebra $\mathcal{B}$ its dual $\mathcal{B}^*$ is a DG coalgebra.

A morphism of DG coalgebras $k \to \mathcal{G}$ (resp. $\mathcal{G} \to k$) is called a co-augmentation (resp. a co-unit) of $\mathcal{G}$ if it satisfies some obvious compatibility condition. We denote by $\bar{\mathcal{G}}$ the cokernel of the co-augmentation map.

Denote by $\bar{\mathcal{G}}_{[n]}$ the kernel of the n-th iterate of the co-multiplication map $\Delta^n : \bar{\mathcal{G}} \to \bar{\mathcal{G}}^{\otimes n}$. The DG coalgebra $\mathcal{G}$ is called *co-complete* if

$$\bar{\mathcal{G}} = \bigcup_{n \geqslant 2} \bar{\mathcal{G}}_{[n]}.$$

A $\mathcal{G}$-*comodule* means a left DG comodule over $\mathcal{G}$.

A $\mathcal{G}^{gr}$-comodule is *cofree* if it is isomorphic to $\mathcal{G} \otimes V$ with the obvious comodule structure for some graded vector space V.

Denote by $\mathcal{G}^{op}$ the DG coalgebra with the opposite co-multiplication.

Let $g : \mathcal{H} \to \mathcal{G}$ be a homomorphism of DG coalgebras. Then $\mathcal{H}$ is a DG $\mathcal{G}$-comodule with the co-action $g \otimes 1 \cdot \Delta_{\mathcal{H}} : \mathcal{H} \to \mathcal{G} \otimes \mathcal{H}$ and a DG $\mathcal{G}^{op}$-comodule with the co-action $1 \otimes g \cdot \Delta_{\mathcal{H}} : \mathcal{H} \to \mathcal{H} \otimes \mathcal{G}$.

Let M and N be a right and left DG $\mathcal{G}$-comodules respectively. Their cotensor product $M \square_{\mathcal{G}} N$ is defined as the kernel of the map

$$\Delta_M \otimes 1 - 1 \otimes \Delta_N : M \otimes N \to M \otimes \mathcal{G} \otimes N,$$

where $\Delta_M : M \to M \otimes \mathcal{G}$ and $\Delta_N : N \to \mathcal{G} \otimes N$ are the co-action maps.

A DG coalgebra $\mathcal{G}$ is a left and right DG comodule over itself. Given a DG $\mathcal{G}$-comodule M the co-action morphism $M \to \mathcal{G} \otimes M$ induces an isomorphism $M = \mathcal{G} \square_{\mathcal{G}} M$. Similarly for DG $\mathcal{G}^{op}$-modules.

Definition 2.1. The dual $\mathcal{R}^*$ of an artinian DG algebra $\mathcal{R}$ is called an *artinian* DG coalgebra.

Given an artinian DG algebra $\mathcal{R}$, its augmentation $\mathcal{R} \to k$ induces the co-augmentation $k \to \mathcal{R}^*$ and its unit $k \to \mathcal{R}$ induces the co-unit $\mathcal{R}^* \to k$.

2.2. *From comodules to modules*

If P is a DG comodule over a DG coalgebra $\mathcal{G}$, then P is naturally a DG module over the DG algebra $(\mathcal{G}^*)^{op}$. Namely, the $(\mathcal{G}^*)^{op}$-module structure is defined as the composition

$$P \otimes \mathcal{G}^* \xrightarrow{\Delta_P \otimes 1} \mathcal{G} \otimes P \otimes \mathcal{G}^* \xrightarrow{T \otimes 1} P \otimes \mathcal{G} \otimes \mathcal{G}^* \xrightarrow{1 \otimes \mathrm{ev}} P,$$

where $T : \mathcal{G} \otimes P \to P \otimes \mathcal{G}$ is the transposition map.

Similarly, if Q is a DG $\mathcal{G}^{op}$-comodule, then Q is a DG module over $\mathcal{G}^*$.

Let P and Q be a left and right DG $\mathcal{G}$-comodules respectively. Then $P \otimes Q$ is a DG $\mathcal{G}^*$-bi-module, i.e. a DG $\mathcal{G}^* \otimes \mathcal{G}^{*0}$-module by the above construction. Note that its center

$$Z(P \otimes Q) := \{x \in P \otimes Q \mid ax = (-1)^{\bar{a}\bar{x}} xa \text{ for all } a \in \mathcal{G}^*\}$$

is isomorphic to the cotensor product $Q \square_{\mathcal{G}} P$.

3. Preliminaries on A_∞-algebras, A_∞-categories and A_∞-modules

3.1. *A_∞-algebras and A_∞-modules*

The basic reference for A_∞-structures is [8].

Let $A = \bigoplus_{n \in \mathbb{Z}} A^n$ be a $\mathbb{Z}$-graded k-vector space. Put $BA = T(A[1]) = \bigoplus_{n \geqslant 0} A[1]^{\otimes n}$. Then the graded vector space BA has natural structure of a graded coalgebra with counit:

$$\Delta(a_1 \otimes \cdots \otimes a_n) = \sum_{m=0}^{n} (a_1 \otimes \cdots \otimes a_m) \otimes (a_{m+1} \otimes \cdots \otimes a_n),$$

$$\varepsilon(a_1 \otimes \cdots \otimes a_n) = \begin{cases} 0 & \text{for } n \geqslant 1; \\ 1 & \text{for } n = 0. \end{cases}$$

Here we put $a_1 \otimes \cdots \otimes a_n = 1$ for $n = 0$. Put also $\overline{BA} = BA/k$. Then $\overline{BA}$ is also a graded coalgebra, but it is non-counital. The most effective way to define the notion of a $\mathbb{Z}$-graded (non-unital) A_∞-algebra is the following:

Definition 3.1. A structure of a (non-unital) A_∞-algebra on $\mathbb{Z}$-graded vector space A is a coderivation $b : \overline{BA} \to \overline{BA}$ of degree 1 such that $b^2 = 0$, i.e. a structure of a DG coalgebra on the graded coalgebra $\overline{BA}$.

Such a coderivation is equivalent to a sequence of maps $b_n = b_n^A : A[1]^{\otimes n} \to A[1]$, $n \geqslant 1$, of degree 1 satisfying for each $n \geqslant 1$ the following identity:

$$\sum_{r+s+t=n} b_{r+1+t}\big(\mathbf{1}^{\otimes r} \otimes b_s \otimes \mathbf{1}^{\otimes t}\big) = 0. \tag{3.1}$$

Note that the coderivation $b : \overline{BA} \to \overline{BA}$ naturally extends to a coderivation $b : BA \to BA$ (which we denote by the same letter), thus BA also becomes a DG coalgebra, and $\varepsilon \cdot b = 0$. Thus, its dual $\hat{S} = (BA)^*$ is naturally a DG algebra.

Let $s : A \to A[1]$ be the translation map. Identify $A^{\otimes n}$ with $A[1]^{\otimes n}$ via the map $s^{\otimes n}$, and A with $A[1]$ via the map s. Let $m_n = m_n^A : A^{\otimes n} \to A$ be the maps corresponding to b_n. Then m_n has degree $(2-n)$ and this sequence of maps satisfies for each $n \geqslant 1$ the following identity:

$$\sum_{r+s+t=n} (-1)^{r+st} m_{r+1+t}\big(\mathbf{1}^{\otimes r} \otimes m_s \otimes \mathbf{1}^{\otimes t}\big) = 0. \tag{3.2}$$

In particular, m_1 is a differential on A, hence A is a complex. Further, m_2 is a morphisms of complexes and it is associative up to homotopy given by m_3. Thus, the cohomology $H(A)$ is naturally a (possibly non-unital) graded algebra. Further, if $m_n = 0$ for $n \geqslant 3$, then A is a (possibly non-unital) DG algebra.

Let A_1, A_2 be (non-unital) A_∞-algebras. The most effective way to define the notion of an A_∞-morphism between them is the following:

Definition 3.2. An A_∞-morphism $f : A_1 \to A_2$ is a (counital) homomorphism of DG coalgebras $f : BA_1 \to BA_2$ (which we denote by the same letter).

Thus, the assignment $A \mapsto BA$ is the full embedding of the category of (non-unital) A_∞-algebras and A_∞-morphisms to the category of counital DG coalgebras.

An A_∞-morphism $f : A_1 \to A_2$ is equivalent to a sequence of maps $\widetilde{f_n} : A[1]^{\otimes n} \to A[1]$, $n \geqslant 1$, of degree zero satisfying for each $n \geqslant 1$ the following identity:

$$\sum_{i_1+\cdots+i_s=n} b_s^{A_2}(\widetilde{f_{i_1}} \otimes \cdots \otimes \widetilde{f_{i_s}}) = \sum_{r+s+t} \widetilde{f_{r+1+t}}\big(\mathbf{1}^{\otimes r} \otimes b_s^{A_1} \otimes \mathbf{1}^{\otimes t}\big). \tag{3.3}$$

Let $f_n : A_1^{\otimes n} \to A_2$ be the maps corresponding to $\widetilde{f_n}$ with respect to our identifications. Then f_n has degree $(1-n)$ and this sequence of maps satisfies for each $n \geqslant 1$ the following identity:

$$\sum_{i_1+\cdots+i_s=n} (-1)^{\epsilon(i_1,\ldots,i_{s-1},s)} m_s^{A_2}(f_{i_1} \otimes \cdots \otimes f_{i_s}) = \sum_{r+s+t} (-1)^{r+st+s} f_{r+1+t}\big(\mathbf{1}^{\otimes r} \otimes m_s^{A_1} \otimes \mathbf{1}^{\otimes t}\big), \tag{3.4}$$

where $\epsilon(i_1,\ldots,i_{s-1},s) = (s-1)i_1 + \cdots + i_{s-1} + \frac{s(s+1)}{2}$.

In particular, f_1 is a morphism of complexes, and $H(f): H(A_1) \to H(A_2)$ is a morphism of (non-unital) graded associative algebras. An A_∞-morphism f is called quasi-isomorphism if f_1 is a quasi-isomorphism of complexes.

Further, we are going to define the DG category $A\text{-mod}_\infty$ of A_∞ A-modules for an A_∞-algebra A.

Definition 3.3. A structure of an A_∞-module over A on the graded vector space M is a differential $b^M : BA \otimes M[1] \to BA \otimes M[1]$ of degree 1, which defines a structure of a DG BA-comodule on the graded cofree $(BA)^{gr}$-comodule $BA \otimes M[1]$.

Such a structure is equivalent to a sequence of maps $b_n = b_n^M : A[1]^{\otimes(n-1)} \otimes M[1] \to M[1]$, $n \geqslant 1$, of degree 1, satisfying for each $n \geqslant 1$ the identity (3.1), where b_i is interpreted as b_i^A or b_i^M, according to the type of its arguments. It is also equivalent to the sequence of maps $m_n = m_n^M : A^{\otimes(n-1)} \otimes M \to M$, $n \geqslant 1$, of degree $(2-n)$ satisfying for each $n \geqslant 1$ the identity (3.2), where m_i is interpreted as m_i^A or m_i^M, according to the type of its arguments. In particular, $(m_1^M)^2 = 0$, hence M is a complex. Again, m_2^M is a morphism of complexes and m_2 is associative up to a homotopy given by m_3^M. Thus, $H(M)$ is naturally a graded $H(A)$-module.

If M and N are A_∞ A-modules, then we put

$$\operatorname{Hom}_{A\text{-mod}_\infty}(M, N) := \operatorname{Hom}_{BA\text{-comod}}\big(BA \otimes M[1], BA \otimes N[1]\big).$$

More explicitly,

$$\operatorname{Hom}^n_{A\text{-mod}_\infty}(M, N) = \prod_{m \geqslant 1} \operatorname{Hom}^n_k\big(A[1]^{\otimes(m-1)} \otimes M[1], N[1]\big),$$

and for $\phi = (\phi_m) \in \operatorname{Hom}^n_{A\text{-mod}_\infty}(M, N)$ one has

$$(d\phi)_m = \sum_{1 \leqslant i \leqslant m} b^N_{m-i+1}\big(\mathbf{1}^{\otimes(l-i)} \otimes \phi_i\big) - (-1)^n \sum_{r+s+t=m} \phi_{r+1+t}\big(\mathbf{1}^{\otimes r} \otimes b_s \otimes \mathbf{1}^{\otimes t}\big), \quad (3.5)$$

where b_s in the RHS is interpreted as b_s^A or b_s^M, according to the type of its arguments. If $\phi = (\phi_m) \in \operatorname{Hom}_{A\text{-mod}_\infty}(M, N)$ and $\psi = (\psi_m) \in \operatorname{Hom}_{A\text{-mod}_\infty}(N, L)$, then

$$(\psi \cdot \phi)_m = \sum_{1 \leqslant i \leqslant m} \psi_{m-i+1}\big(\mathbf{1}^{\otimes(m-i)} \otimes \phi_i\big). \quad (3.6)$$

We will write $\operatorname{Hom}_A(M, N)$ instead of $\operatorname{Hom}_{A\text{-mod}_\infty}(M, N)$.

The closed morphism $\phi \in \operatorname{Hom}^0(A\text{-mod}_\infty)$ is called quasi-isomorphism if ϕ_1 is a quasi-isomorphism of complexes.

The homotopy category $K_\infty(A)$ is defined as $\operatorname{Ho}(A\text{-mod}_\infty)$. It is always triangulated. It turns out that all acyclic A_∞ A-modules in $K_\infty(A)$ are already null-homotopic. Hence the corresponding derived category $D_\infty(A)$ is the same as $K_\infty(A)$. However, we will write $D_\infty(A)$ instead of $K_\infty(A)$.

Let $f : A_1 \to A_2$ be an A_∞-morphism. Then we have the DG functor $f_* : A_2\text{-mod}_\infty \to A_1\text{-mod}_\infty$, which we call the "restriction of scalars". Namely, if $M \in A_2\text{-mod}_\infty$, then $f_*(M)$

coincides with M as a graded vector space, and the differential on $BA_1 \otimes f_*(M)[1]$ coincides with the differential on $BA_1 \square_{BA_2} (BA_2 \otimes M[1])$ after the natural identification

$$BA_1 \otimes f_*(M)[1] \cong BA_1 \square_{BA_2} \big(BA_2 \otimes M[1]\big).$$

We also have the resulting exact functor $f_* : D_\infty(A_2) \to D_\infty(A_1)$. If f is a quasi-isomorphism, then the DG functor $f_* : A_2\text{-mod}_\infty \to A_1\text{-mod}_\infty$ is quasi-equivalence, and hence the functor $f_* : D_\infty(A_2) \to D_\infty(A_1)$ is an equivalence.

We would like also to define the A_∞-bimodules.

Definition 3.4. Let A_1 and A_2 be A_∞-algebras. A structure of an A_∞ A_1–A_2-bimodule on the graded vector space M is a differential $b^M : BA_1 \otimes M[1] \otimes BA_2 \to BA_1 \otimes M[1] \otimes BA_2$ which defines the structure of a DG comodule over $BA_1 \otimes (BA_2)^{op}$ on the $(BA_1 \otimes (BA_2)^{op})^{gr}$-bicomodule $BA_1 \otimes M[1] \otimes BA_2$.

Such a differential is given by a sequence of maps

$$b_{i,j} : A_1[1]^{\otimes i} \otimes M[1] \otimes A_2[1]^{\otimes j} \to M[1]$$

satisfying analogous equations. In particular, we have a regular A_1–A_2-bimodule $A_1 \otimes A_2$. In the case when $A_1 = A_2$, we have a diagonal bimodule A. The DG category A_1-mod-A_2 of A_∞ A_1–A_2-bimodules is defined analogously (see also [6]). Again, we define $K_\infty(A_1–A_2)$ as the homotopy category $\mathrm{Ho}(A_1\text{-mod-}A_2)$. All acyclic A_∞-bimodules in $K_\infty(A_1–A_2)$ are null-homotopic and hence the corresponding derived category $D_\infty(A_1–A_2)$ coincides with $K_\infty(A_1–A_2)$.

3.2. Strictly unital A_∞-algebras

Definition 3.5. An A_∞-algebra is called strictly unital if there exists an element $1_A \in A$ of degree zero satisfying the following properties:

(U1) $m_1(1_A) = 0$;
(U2) $m_2(a, 1_A) = m_2(1_A, a) = a$ for each $a \in A$;
(U3) for $n \geqslant 3$, $m_n(a_1, \dots, a_n)$ vanishes if at least one of a_i equals to 1_A.

Such an element 1_A is called a strict unit.

Clearly, if a strict unit exists then it is unique. An A_∞-morphism $f : A_1 \to A_2$ of strictly unital A_∞-algebras is called strictly unital if $f_1(1_{A_1}) = 1_{A_2}$, and for $n \geqslant 2$ $f_n(a_1, \dots, a_n)$ vanishes if at least one of a_i equals to 1_{A_1}. Further, an A_∞-module $M \in A\text{-mod}_\infty$ is called strictly unital if $m_2^M(1_A, m) = m$ for each $m \in M$ and for $n \geqslant 3$ $m_n^M(a_1, \dots, a_{n-1}, m) = 0$ if at least one of a_i equals to 1_A. If A is strictly unital then we denote by $D_\infty^{su}(A) \subset D_\infty(A)$ the full subcategory which consists of strictly unital A_∞ A-modules.

Analogously, if A_1 and A_2 are strictly unital A_∞-algebras, then we have a notion of strictly unital A_∞ A_1–A_2-bimodules, and we define $D_\infty^{su}(A_1–A_2) \subset D_\infty(A_1–A_2)$ as the full subcategory which consists of strictly unital A_∞-bimodules.

If $\mathcal{C}$ is a DG algebra then it is also a strictly unital A_∞-algebra with $m_n = 0$ for $n \geqslant 3$. We have an obvious DG functor $\mathcal{C}\text{-mod} \to \mathcal{C}\text{-mod}_\infty$. It induces an equivalence

$$D(\mathcal{C}) \xrightarrow{\sim} D_\infty^{su}(\mathcal{C}).$$

Let A be an arbitrary A_∞-algebra. Then its unitization $A_+ := k \cdot 1_+ \oplus A$, which is a strictly unital A_∞-algebra, is defined as follows:

$$m_n^{A_+}(a_1, \dots, a_n) = m_n^A(a_1, \dots, a_n) \quad \text{for any } a_1, \dots, a_n \in A,$$
$$m_1(1_+) = 0,$$
$$m_2^{A_+}(1_+, a) = m_2^{A_+}(a, 1_+) = a \quad \text{for each } a \in A_+,$$
$$m_n^{A_+}(a_1, \dots, a_n) = 0 \quad \text{if at least one of } a_i \text{ equals to } 1_+.$$

Clearly, the assignment $A \mapsto A_+$ defines faithful functor from the category of A_∞-algebras and A_∞-morphisms to the category of strictly unital A_∞-algebras and strictly unital A_∞-morphisms. Further, we have an obvious faithful DG functor $A\text{-mod}_\infty \to A_+\text{-mod}_\infty$. Its image consists of strictly unital A_∞-modules. The induced functor $D_\infty(A) \to D_\infty^{su}(A_+)$ is an equivalence.

We call A_∞-algebras of the form A_+ augmented A_∞-algebras. We also use the notation $A = \overline{A_+}$.

Definition 3.6. Let A be an augmented A_∞-algebra. Its bar–cobar construction $U(A)$, which is a DG algebra, together with a strictly unital A_∞ quasi-isomorphism $f_A : A \to U(A)$ are defined by the following universal property. If $\mathcal{B}$ is a DG algebra, and $f : A \to \mathcal{B}$ is a strictly unital A_∞-morphism then there exists a unique morphism of DG algebras $\varphi : U(A) \to \mathcal{B}$ such that $f = \varphi \cdot f_A$.

More explicitly, $U(A)$ equals to $T(\overline{B\bar{A}[-1]})$ as a graded algebra, and the differential comes from the differential and comultiplication on $\overline{B\bar{A}}$. The A_∞-morphism f_A is the obvious one.

3.3. Minimal models of A_∞-algebras

An A_∞-algebra A is called minimal if $m_1^A = 0$. Each (strictly unital) A_∞-algebra is quasi-isomorphic to the minimal (strictly unital) A_∞-algebra.

Proposition 3.7. *(See [8], Corollaire 1.4.1.4, Proposition 3.2.4.1.) Let A be an A_∞-algebra. There exists an A_∞-algebra structure on $H(A)$ such that*

a) *$m_1 = 0$ and m_2 is induced by m_2^A;*
b) *there exists an A_∞-quasi-isomorphism of A_∞-algebras $f : H(A) \to A$ such that f_1 induces the identity in cohomology.*

Moreover, if A is strictly unital then this A_∞-structure on $H(A)$ and the quasi-isomorphism can be chosen to be strictly unital.

3.4. Perfect A_∞-modules and A_∞-bimodules

Let A be a strictly unital A_∞-algebra. The category $\mathrm{Perf}(A)$ of perfect A_∞ A-modules is the minimal full thick triangulated subcategory of $D_\infty^{su}(A)$ which contains A.

Further, if A_1 and A_2 are strictly unital A_∞-algebras then the category $\mathrm{Perf}(A_1\text{–}A_2)$ of perfect A_∞ A_1–A_2-bimodules is the minimal full thick triangulated subcategory of $D_\infty^{su}(A_1\text{–}A_2)$ which contains $A_1 \otimes A_2$.

3.5. A_∞-categories

The notion of an A_∞-category is a straightforward generalization of the notion of an A_∞-algebra. Namely, a non-unital A_∞-category $\mathcal{A}$ is the following data:

- the class of objects of $\mathcal{A}$;
- for each two objects X_1, X_2 the graded vector space $\mathrm{Hom}(X_1, X_2)$;
- for each finite sequence of objects $X_0, X_1, \ldots, X_n \in \mathcal{A}$, $n \geqslant 1$, the map

$$m_n^{\mathcal{A}(X_0,\ldots,X_n)} : \mathrm{Hom}(X_{n-1}, X_n) \otimes \cdots \otimes \mathrm{Hom}(X_0, X_1) \to \mathrm{Hom}(X_0, X_n)$$

  of degree $(2-n)$, such that for any $Y_1, \ldots, Y_m \in \mathcal{A}$ the graded vector space $\bigoplus_{1 \leqslant i,j \leqslant m} \mathrm{Hom}(Y_i, Y_j)$ becomes an A_∞-algebra.

If $\mathcal{A}$ is an A_∞-category then $\mathrm{Ho}(\mathcal{A})$ is a pre-category, i.e. a "category" which may not have identity morphisms.

An element $\mathbf{1}_X \in \mathrm{Hom}(X, X)$ of degree zero is called a strict identity morphism if it satisfies the conditions (U1), (U2), (U3) from Definition 3.5, where a and a_i are arbitrary morphisms such that the equalities make sense. An A_∞-category is called strictly unital if each object has a strict identity morphism. If $\mathcal{A}$ is a strictly unital A_∞-category then $\mathrm{Ho}(\mathcal{A})$ is a true category.

A (strictly unital) A_∞-algebra can be thought of as a (strictly unital) A_∞-category with one object.

Let $\mathcal{A}_1, \mathcal{A}_2$ be A_∞-categories. An A_∞-functor $F : \mathcal{A}_1 \to \mathcal{A}_2$ is the following data:

- an object $F(X) \in \mathcal{A}_2$ for each object $X \in \mathcal{A}_1$;
- for each finite sequence of objects $X_0, X_1, \ldots, X_n \in \mathcal{A}_1$, $n \geqslant 1$, the map

$$F(X_0, \ldots, X_n) : \mathrm{Hom}(X_{n-1}, X_n) \otimes \cdots \otimes \mathrm{Hom}(X_0, X_1) \to \mathrm{Hom}\big(F(X_0), F(X_n)\big)$$

  of degree $(1-n)$, such that for any $Y_1, \ldots, Y_m \in \mathcal{A}_1$ we obtain an A_∞-morphism between A_∞-algebras

$$\bigoplus_{1 \leqslant i,j \leqslant m} \mathrm{Hom}(Y_i, Y_j) \to \bigoplus_{1 \leqslant i,j \leqslant m} \mathrm{Hom}\big(F(Y_i), F(Y_j)\big).$$

The definition of a strictly unital A_∞-functor between strictly unital A_∞-categories is analogous to the definition of a strictly unital A_∞-morphism between strictly unital A_∞-algebras.

A strictly unital A_∞-functor $F : \mathcal{A}_1 \to \mathcal{A}_2$ between strictly unital A_∞-categories is called quasi-equivalence if the following conditions hold:

– the map $F(X,Y):\mathrm{Hom}(X,Y)\to\mathrm{Hom}(F(X),F(Y))$ is a quasi-isomorphism of complexes for any $X,Y\in\mathcal{A}_1$;
– the induced functor $\mathrm{Ho}(F):\mathrm{Ho}(\mathcal{A}_1)\to\mathrm{Ho}(\mathcal{A}_2)$ is an equivalence.

3.6. *The tensor product of an A_∞-algebra and a DG algebra*

Let A be an A_∞-algebra and $\mathcal{C}$ be a DG algebra. Then their tensor $A\otimes\mathcal{C}$ is naturally an A_∞-algebra with the following multiplications:

$$m_1^{A\otimes\mathcal{C}}=m_1^A\otimes\mathbf{1}_{\mathcal{C}}+\mathbf{1}_{\mathcal{A}}\otimes d_{\mathcal{C}};$$
$$m_n^{A\otimes\mathcal{C}}(a_1\otimes c_1,\dots,a_n\otimes c_n)=(-1)^{\epsilon}m_n^A(a_1,\dots,a_n)\otimes(c_1\dots c_n)\quad\text{for } n\geqslant 2,$$

where $\epsilon=\sum_{i<j}\bar{a_j}\bar{c_i}$ (all a_i and c_i are homogeneous). If A is strictly unital, then $A\otimes\mathcal{C}$ is also strictly unital and $\mathbf{1}_{A\otimes\mathcal{C}}=\mathbf{1}_A\otimes\mathbf{1}_{\mathcal{C}}$.

Remark 3.8. The constructed tensor product is a specialization of the complicated construction of the tensor product of A_∞-algebras which was first proposed in [9]. We also remark that in the case when A_1 and A_2 are strictly unital A_∞-algebras, there is a canonical DG model for $A_1\otimes A_2$:

$$A_1\text{“}\otimes\text{”}A_2=\mathrm{End}_{A_1\text{-mod-}A_2^{op}}(A_1\otimes A_2),$$

see [6].

3.7. *The category of $A_{\mathcal{C}}$-modules for an A_∞-algebra A and a DG algebra $\mathcal{C}$*

Let A be an A_∞-algebra and let $\mathcal{C}$ be a DG algebra. We want to define the DG category of A_∞ $A_{\mathcal{C}}$-modules which is analogue of the category of $(A\otimes\mathcal{C})$-modules in the case when A is a DG algebra.

Definition 3.9. A structure of an A_∞ $A_{\mathcal{C}}$-module on the graded vector space M is the following data:

1) A structure of a $\mathcal{C}^{gr}$-module on M;
2) A differential $b^M: BA\otimes M[1]\to BA\otimes M[1]$ of degree 1 which makes $BA\otimes M[1]$ into a DG comodule over BA and into a DG module over $\mathcal{C}$.

If we are already given with the structure of a $\mathcal{C}^{gr}$-module on M then such a differential b^M is equivalent to the sequence of maps $b_n=b_n^M: A[1]^{\otimes(n-1)}\otimes M[1]\to M[1]$, $n\geqslant 1$, satisfying the following properties:

1) The maps b_n^M satisfy the identities (3.1) (in the same sense as for A_∞ A-modules);
2) The differential b_1^M makes $M[1]$ into a DG module over $\mathcal{C}$;
3) The maps b_n^M are $\mathcal{C}^{gr}$-linear for $n\geqslant 2$.

Further, the corresponding maps $m_n=m_n^M: A^{\otimes(n-1)}\otimes M\to M$ have to satisfy the following properties:

1) The maps m_n^M satisfy the identities (3.2) (in the same sense as for A_∞ A-modules);
2) The differential m_1^M makes M into a DG module over $\mathcal{C}$;
3) The maps m_n^M are $\mathcal{C}^{gr}$-linear for $n \geqslant 2$.

If M, N are A_∞ $A_{\mathcal{C}}$-modules then we put

$$\mathrm{Hom}_{A_{\mathcal{C}}\text{-mod}_\infty}(M, N) := \mathrm{Hom}_{BA\text{-comod}} \cap \mathrm{Hom}_{\mathcal{C}\text{-mod}}\big(BA \otimes M[1], BA \otimes N[1]\big).$$

More explicitly,

$$\mathrm{Hom}^n_{A_{\mathcal{C}}\text{-mod}_\infty}(M, N) = \prod_{m\geqslant 1} \mathrm{Hom}^n_{\mathcal{C}^{gr}}\big(A[1]^{\otimes(m-1)} \otimes M[1], N[1]\big),$$

the differential and the compositions are defined by the formulas (3.5) and (3.6) respectively.

We will write $\mathrm{Hom}_{A_{\mathcal{C}}}(M, N)$ instead of $\mathrm{Hom}_{A_{\mathcal{C}}\text{-mod}_\infty}(M, N)$.

Again, the homotopy category $K_\infty(A_{\mathcal{C}})$ is defined as $\mathrm{Ho}(A_{\mathcal{C}}\text{-mod}_\infty)$. The acyclic A_∞ $A_{\mathcal{C}}$-module in $K_\infty(A_{\mathcal{C}})$ are not null-homotopic in general, hence we define the derived category $D_\infty(A_{\mathcal{C}})$ as the Verdier quotient of $K_\infty(A_{\mathcal{C}})$ by the subcategory of acyclic A_∞ $A_{\mathcal{C}}$-modules.

Remark 3.10. Notice that the structure of an A_∞ $A_{\mathcal{C}}$-module is not equivalent to the structure of an A_∞ $A \otimes \mathcal{C}$-module. Moreover, there is a natural DG functor $A_{\mathcal{C}}\text{-mod}_\infty \to A_+ \otimes \mathcal{C}\text{-mod}_\infty$ which induces an equivalence $D_\infty(A_{\mathcal{C}}) \xrightarrow{\sim} D_\infty^{su}(A_+ \otimes \mathcal{C})$. Also, in the case when A is strictly unital, the DG functor $A_{\mathcal{C}}\text{-mod}_\infty \to A \otimes \mathcal{C}\text{-mod}_\infty$ induces an equivalence $D_\infty^{su}(A_{\mathcal{C}}) \xrightarrow{\sim} D_\infty^{su}(A \otimes \mathcal{C})$.

Definition 3.11. An A_∞ $A_{\mathcal{C}}$-module M is called h-projective (resp. h-injective) if for each acyclic $N \in A_{\mathcal{C}}\text{-mod}_\infty$ the complex $\mathrm{Hom}_{A_{\mathcal{C}}}(M, N)$ (resp. $\mathrm{Hom}_{A_{\mathcal{C}}}(N, M)$) is acyclic.

It turns out that an A_∞ $A_{\mathcal{C}}$-module is h-projective (resp. h-injective) iff it is such as a DG $\mathcal{C}$-module.

Proposition 3.12. *Let M be an A_∞ $A_{\mathcal{C}}$-module. Suppose that M is h-projective (resp. h-injective) as a DG $\mathcal{C}$-module. Then M is also h-projective (resp. h-injective) as an A_∞ $A_{\mathcal{C}}$-module.*

Proof. We will prove proposition for h-projectives. The proof for h-injectives is analogous.

So let $M \in A_{\mathcal{C}}\text{-mod}_\infty$ and suppose that M is h-projective as a DG $\mathcal{C}$-module. Let N be an acyclic A_∞ $A_{\mathcal{C}}$-module. The complex $K^\cdot = \mathrm{Hom}_{A_{\mathcal{C}}}(M, N)$ admits a decreasing filtration by subcomplexes

$$F^p K^\cdot = \prod_{n\geqslant p} \mathrm{Hom}_{\mathcal{C}^{gr}}\big(A^{\otimes n} \otimes M, N\big).$$

The subquotients

$$F^p K^\cdot / F^{p+1} K^\cdot = \mathrm{Hom}_{\mathcal{C}}\big(A^{\otimes p} \otimes M, N\big)$$

are acyclic since the DG modules $A^{\otimes p} \otimes M$ are h-projective. Since

$$K^{\cdot} = \varprojlim K^{\cdot}/F^p K^{\cdot},$$

the complex $K^{\cdot}$ is also acyclic. Therefore, M is h-projective as an A_∞ $A_{\mathcal{C}}$-module. □

We denote by $K_\infty^P(A_{\mathcal{C}}) \subset K_\infty(A_{\mathcal{C}})$ (resp. by $K_\infty^I(A_{\mathcal{C}}) \subset K_\infty(A_{\mathcal{C}})$) the full subcategory which consists of h-projective (resp. h-injective) A_∞ $A_{\mathcal{C}}$-modules.

Theorem 3.13. *For each $M \in A_{\mathcal{C}}\text{-mod}_\infty$, there exist quasi-isomorphisms $M \to I$, $P \to M$, where $I \in A_{\mathcal{C}}\text{-mod}_\infty$ is h-injective and $P \in A_{\mathcal{C}}\text{-mod}_\infty$ is h-projective. The natural functor $K_\infty^P(A_{\mathcal{C}}) \to D_\infty(A_{\mathcal{C}})$ (resp. $K_\infty^I(A_{\mathcal{C}}) \to D_\infty(A_{\mathcal{C}})$) is an equivalence.*

Proof. First we construct a quasi-isomorphism $pM \to M$ with h-projective P. Namely, let pM be the total complex of the bicomplex

$$\cdots \to \mathcal{C}^{\otimes n} \otimes M \xrightarrow{d^n} \mathcal{C}^{\otimes n-1} \otimes M \to \cdots \to \mathcal{C} \otimes M,$$

where d^n is the bar differential. Then pM is naturally an A_∞ $A_{\mathcal{C}}$-module. A quasi-isomorphism of complexes $pM \to M$ is a quasi-isomorphism in $A_{\mathcal{C}}\text{-mod}_\infty$ (with zero components $f_n : A^{\otimes n-1} \otimes M \to M$ for $n \geqslant 2$). Further, pM satisfies property (P) as a DG $\mathcal{C}$-module (I, Definition 3.2). Hence, pM is an h-projective $A_{\mathcal{C}}$-module.

The construction $M \to pM$ extends to the functor $p : K_\infty(A_{\mathcal{C}}) \to K_\infty^P(A_{\mathcal{C}})$ which is right adjoint to the inclusion $K_\infty^P(A_{\mathcal{C}}) \to K_\infty(A_{\mathcal{C}})$. The kernel of p consists of acyclic $A_{\mathcal{C}}$-modules. Thus, the functor $K_\infty^P(A_{\mathcal{C}}) \to D_\infty(A_{\mathcal{C}})$ is an equivalence.

Analogously, one can construct a functor $i : K_\infty(A_{\mathcal{C}}) \to K_\infty^I(A_{\mathcal{C}})$ which is left adjoint to the inclusion $K_\infty^I(A_{\mathcal{C}}) \to K_\infty(A_{\mathcal{C}})$. Thus, the functor $K_\infty^I(A_{\mathcal{C}}) \to D_\infty(A_{\mathcal{C}})$ is an equivalence. Theorem is proved. □

Notice that if $G : K_\infty(A_{\mathcal{C}}) \to \mathcal{T}$ is an exact functor between triangulated categories then we can define its left and right derived functors

$$\mathbf{L}G : D_\infty(A_{\mathcal{C}}) \to \mathcal{T}, \qquad \mathbf{R}G : D_\infty(A_{\mathcal{C}}) \to \mathcal{T}.$$

Namely, for each $M \in A_{\mathcal{C}}\text{-mod}_\infty$ choose quasi-isomorphisms $P \to M$, $M \to I$ with h-projective P and h-injective I, and put

$$\mathbf{L}G(M) = G(P), \qquad \mathbf{R}G(M) = G(I).$$

Proposition 3.14. *The derived categories $D_\infty(A_{\mathcal{C}})$ and $D(U(A_+) \otimes \mathcal{C})$ are naturally equivalent.*

Proof. Indeed, the "restriction of scalars" DG functor

$$f_{A*} : \big(U(A_+) \otimes \mathcal{C}\big)\text{-mod} \to A_{\mathcal{C}}\text{-mod}_\infty$$

admits a right adjoint DG functor

$$f_A^! : A_{\mathcal{C}}\text{-mod}_\infty \to \big(U(A_+) \otimes \mathcal{C}\big)\text{-mod},$$

given by the formula

$$f_A^!(M) = \mathrm{Hom}_A\big(U(A_+), M\big).$$

For any $M \in (U(A_+) \otimes \mathcal{C})$-mod, $N \in A_{\mathcal{C}}$-mod$_\infty$, the adjunction morphisms $M \to f_A^! f_{A*} M$, $f_{A*} f_A^! N \to N$ are quasi-isomorphisms. Moreover, both f_{A*} and $f_A^!$ preserve acyclic modules. Thus, the induced functors

$$f_{A*} : D\big(U(A_+) \otimes \mathcal{C}\big) \to D_\infty(A_{\mathcal{C}}), \qquad f_A^! : D_\infty(A_{\mathcal{C}}) \to D\big(U(A) \otimes \mathcal{C}\big)$$

are mutually inverse equivalences. □

3.8. The bar complex

Let A be an A_∞-algebra. The graded vector space $BA \otimes A[1] \otimes BA$ carries a natural differential which makes it into a DG bicomodule over BA. Namely, such a differential is determined by its components

$$b_{i,j} : A[1]^{\otimes i} \otimes A[1] \otimes A[1]^{\otimes j} \to A[1],$$

and we put $b_{i,j} = b^A_{i+j+1}$.

In particular, $BA \otimes A$ is an A_∞-module over A^{op}. It is called the bar complex and is denoted by $BA \otimes_{\tau_A} A$.

Now let A be an augmented A_∞-algebra, and put $\hat{S} = (B\bar{A})^*$. The graded vector space $B\bar{A} \otimes A[1] \otimes B\bar{A}$ also carries a natural differential which makes it into a DG bicomodule over $B\bar{A}$. In particular, $B\bar{A} \otimes A$ is an A_∞-module over $\bar{A}^{op}$. It is also called the bar complex and is denoted by $B\bar{A} \otimes_{\tau_A} A$. Note that $B\bar{A} \otimes_{\tau_A} A$ is a $B\bar{A}$-comodule, and hence is a $\hat{S}^{op}$-module. This makes it into an object of $\bar{A}^{op}_{\hat{S}^{op}}$-mod$_\infty$.

Analogously, we have an A_∞ $\bar{A}_{\hat{S}}$-module $A \otimes_{\tau_A} B\bar{A}$.

4. Some functors defined by the bar complex

Definition 4.1. An augmented A_∞-algebra $\mathcal{C}$ is called

a) *nonnegative* if $\mathcal{C}^i = 0$ for $i < 0$;
b) *connected* if $\mathcal{C}^0 = k$;
c) *locally finite* if $\dim_k \mathcal{C}^i < \infty$ for all i.

We say that $\mathcal{C}$ is admissible if it satisfies a), b), c).

4.1. The functor Δ

Fix an augmented A_∞-algebra $\mathcal{C}$. Consider the bar construction $B\bar{\mathcal{C}}$, the corresponding DG algebra $\hat{S} = (B\bar{\mathcal{C}})^*$ and the A_∞ $\bar{\mathcal{C}}^{op}_{\hat{S}^{op}}$-module $B\bar{\mathcal{C}} \otimes_{\tau_{\mathcal{C}}} \mathcal{C}$ (the bar complex). If $\mathcal{C}$ is connected and nonnegative, then $B\bar{\mathcal{C}}$ is concentrated in nonnegative degrees and consequently $\hat{S}$ is concentrated in nonpositive degrees.

Let $\mathcal{B}$ be a DG algebra. Denote by $D_f(\mathcal{B}^{op}) \subset D(\mathcal{B}^{op})$ the full triangulated subcategory consisting of DG modules with finite-dimensional cohomology.

Lemma 4.2. *Assume that DG algebra $\mathcal{B}$ is augmented and local and complete. Also assume that $\mathcal{B}^i = 0$ for $i > 0$. Then the category $D_f(\mathcal{B}^{op})$ is the triangulated envelope of the DG $\mathcal{B}^{op}$-module k.*

Proof. Denote by $\langle k\rangle \subset D(\mathcal{B}^{op})$ the triangulated envelope of k.

Let M be a DG $\mathcal{B}^{op}$-module with finite-dimensional cohomology. First assume that M is concentrated in one degree. Then $\dim M < \infty$. Since $\mathcal{B}^{\mathrm{gr}}$ is a complete local algebra the module M has a filtration with subquotients isomorphic to k. Thus $M \in \langle k\rangle$.

In the general case by I, Lemma 3.19 we may and will assume that $M^i = 0$ for $|i| \gg 0$. Let s be the least integer such that $M^s \neq 0$. The kernel K of the differential $d : M^s \to M^{s+1}$ is a DG $\mathcal{B}^{op}$-submodule. By the above argument $K \in \langle k\rangle$. If $K \neq 0$ then by induction on the dimension of the cohomology we obtain that $M/K \in \langle k\rangle$. Hence also $M \in \langle k\rangle$. If $K = 0$, then the DG $\mathcal{B}^{op}$-submodule $\tau_{<s+1}M$ (I, Lemma 3.19) is acyclic, and hence M is quasi-isomorphic to $\tau_{\geqslant s+1}M$. But we may assume that $\tau_{\geqslant s+1}M \in \langle k\rangle$ by descending induction on s. □

Choose a quasi-isomorphism of A_∞ $\bar{\mathcal{C}}^{op}_{\hat{S}^{op}}$-modules $B\bar{\mathcal{C}} \otimes_{\tau_\mathcal{C}} \mathcal{C} \to J$, where J satisfies the property (I) as $\hat{S}^{op}$-module (hence is h-injective).

Consider the contravariant DG functor $\Delta : \hat{S}^{op}\text{-mod} \to \bar{\mathcal{C}}^{op}\text{-mod}_\infty$ defined by

$$\Delta(M) := \mathrm{Hom}_{\hat{S}^{op}}(M, J).$$

This functor extends trivially to derived categories $\Delta : D(\hat{S}^{op}) \to D_\infty(\bar{\mathcal{C}}^{op})$.

Theorem 4.3. *Assume that the DG algebra $\mathcal{C}$ is admissible. Then*

a) *The contravariant functor Δ is full and faithful on the category $D_f(\hat{S}^{op})$.*
b) *$\Delta(k)$ is isomorphic to $\mathcal{C}$.*

Proof. By Lemma 4.2 the category $D_f(\hat{S}^{op})$ is the triangulated envelope of the DG $\hat{S}^{op}$-module k. So for the first statement of the theorem it suffices to prove that the map $\Delta : \mathrm{Ext}_{\hat{S}^{op}}(k,k) \to \mathrm{Ext}_{\mathcal{C}^{op}}(\Delta(k), \Delta(k))$ is an isomorphism. The following proposition implies the theorem.

Proposition 4.4. *Under the assumptions of the above theorem the following holds:*

a) *The complex $\mathbf{R}\mathrm{Hom}_{\hat{S}^{op}}(k,k)$ is quasi-isomorphic to $\mathcal{C}$.*
b) *The natural morphism of complexes $\mathrm{Hom}_{\hat{S}^{op}}(k, B\bar{\mathcal{C}} \otimes_{\tau_\mathcal{C}} \mathcal{C}) \to \mathrm{Hom}_{\hat{S}^{op}}(k, J)$ is a quasi-isomorphism.*
c) *$\Delta(k)$ is quasi-isomorphic to $\mathcal{C}$.*
d) *$\Delta : \mathrm{Ext}_{\hat{S}^{op}}(k,k) \to \mathrm{Ext}_{\mathcal{C}^{op}}(\Delta(k), \Delta(k))$ is an anti-isomorphism.*

Proof. a) Recall the A_∞ $\bar{\mathcal{C}}_{\hat{S}}$-module $\mathcal{C} \otimes_{\tau_\mathcal{C}} B\bar{\mathcal{C}}$ (Section 3.8). Consider the corresponding A_∞ $\bar{\mathcal{C}}^{op}_{\hat{S}^{op}}$-module $P := \mathrm{Hom}_k(\mathcal{C} \otimes_{\tau_\mathcal{C}} B\bar{\mathcal{C}}, k)$. Since $\mathcal{C}$ is locally finite and bounded below and $B\bar{\mathcal{C}}$ is bounded below the graded $\hat{S}^{op}$-module P^{gr} is isomorphic to $(\hat{S} \otimes \mathrm{Hom}_k(\mathcal{C}, k))^{gr}$. Since the complex $\mathrm{Hom}_k(\mathcal{C}, k)$ is bounded above and the DG algebra $\hat{S}$ is concentrated in nonpositive degrees the DG $\hat{S}^{op}$-module P has the property (P) (and hence is h-projective). Thus $\mathbf{R}\mathrm{Hom}_{\hat{S}^{op}}(k, k) = \mathrm{Hom}_{\hat{S}^{op}}(P, k) = \mathrm{Hom}_k(\mathrm{Hom}_k(\mathcal{C}, k), k) = \mathcal{C}$. This proves a).

b) Since $\mathrm{Hom}_{\hat{S}^{op}}(k, B\bar{\mathcal{C}} \otimes_{\tau_C} \mathcal{C}) = \mathcal{C}$ the assertion follows from a).

c) follows from b).

d) follows from a) and c). □

This proves the theorem. □

Remark 4.5. Notice that for any augmented A_∞-algebra $\mathcal{C}$ we have $\mathrm{Hom}_{\hat{S}^{op}}(k, B\bar{\mathcal{C}} \otimes_{\tau_C} \mathcal{C}) = \mathcal{C}$. Thus the A_∞ $\bar{\mathcal{C}}^{op}_{\hat{S}^{op}}$-module $B\bar{\mathcal{C}} \otimes_{\tau_C} \mathcal{C}$ is a "homotopy $\hat{S}$-co-deformation" of $\mathcal{C}$. Proposition 4.4 implies that for an admissible $\mathcal{C}$ this A_∞ $\bar{\mathcal{C}}^{op}_{\hat{S}^{op}}$-module is a "derived $\hat{S}$-co-deformation" of $\mathcal{C}$. (Of course we have only defined co-deformations along artinian DG algebras.)

4.2. The functor ∇

Now we define another functor $\nabla : D(\hat{S}^{op}) \to D_\infty(\bar{\mathcal{C}}^{op})$, which is closely related to Δ.

Denote by m the augmentation ideal of $\hat{S}$. For a DG $\hat{S}^{op}$-module M denote $M_n := M/m^n M$ and

$$\hat{M} = \varprojlim_n M_n.$$

Fix a DG $\hat{S}^{op}$-module N. Choose a quasi-isomorphism $P \to N$ with an h-projective P. Define

$$\nabla(N) := \varinjlim \Delta(P_n) = \varinjlim \mathrm{Hom}_{\hat{S}^{op}}(P_n, J).$$

Denote by $\mathrm{Perf}(\hat{S}^{op}) \subset D(\hat{S}^{op})$ the minimal full triangulated subcategory which contains the DG $\hat{S}^{op}$-module $\hat{S}$ and is closed with respect to taking of direct summands.

Theorem 4.6. *Assume that the A_∞-algebra $\mathcal{C}$ is admissible and finite-dimensional. Then*

a) *The contravariant functor $\nabla : D(\hat{S}^{op}) \to D_\infty(\bar{\mathcal{C}}^{op})$ is full and faithful on the subcategory* $\mathrm{Perf}(\hat{S}^{op})$.
b) $\nabla(\hat{S})$ *is isomorphic to k.*

Proof. Denote by $m \subset \hat{S}^{op}$ the maximal ideal and put $S_n := \hat{S}^{op}/m^n\hat{S}^{op}$. Since the A_∞-algebra $\mathcal{C}$ is finite-dimensional S_n is also finite-dimensional for all n. We need a few lemmas.

Lemma 4.7. *Let K be a DG $\hat{S}^{op}$-module such that* $\dim_k K < \infty$. *Then the natural morphism of complexes*

$$\mathrm{Hom}_{\hat{S}^{op}}(K, B\mathcal{C} \otimes_{\tau_C} \mathcal{C}) \to \mathrm{Hom}_{\hat{S}^{op}}(K, J)$$

is a quasi-isomorphism.

Proof. Notice that since the algebra $\hat{S}$ is local, every element $x \in m$ acts on K as a nilpotent operator. It follows that $m^n K = 0$ for $n \gg 0$. For the same reason the DG $\hat{S}^{op}$-module K has a filtration with subquotients isomorphic to k. Thus we may prove the assertion by induction on $\dim K$. If $K = k$, then this is part b) of Proposition 4.4. Otherwise we can find a short exact sequence of DG $\hat{S}^{op}$-modules

$$0 \to M \to K \to N \to 0,$$

such that $\dim M, \dim N < \dim K$.

Sublemma. *The sequence of complexes*

$$0 \to \mathrm{Hom}_{\hat{S}^{op}}(N, B\mathcal{C} \otimes_{\tau_{\mathcal{C}}} \mathcal{C}) \to \mathrm{Hom}_{\hat{S}^{op}}(K, B\mathcal{C} \otimes_{\tau_{\mathcal{C}}} \mathcal{C}) \to \mathrm{Hom}_{\hat{S}^{op}}(M, B\mathcal{C} \otimes_{\tau_{\mathcal{C}}} \mathcal{C}) \to 0$$

is exact.

Proof. We only need to prove the surjectivity of the map

$$\mathrm{Hom}_{\hat{S}^{op}}(K, B\mathcal{C} \otimes_{\tau_{\mathcal{C}}} \mathcal{C}) \to \mathrm{Hom}_{\hat{S}^{op}}(M, B\mathcal{C} \otimes_{\tau_{\mathcal{C}}} \mathcal{C}).$$

Let $n \gg 0$ be such that $m^n K = m^n M = 0$. Let ${}_n(B\mathcal{C} \otimes_{\tau_{\mathcal{C}}} \mathcal{C}) \subset (B\mathcal{C} \otimes_{\tau_{\mathcal{C}}} \mathcal{C})$ denote the DG $\hat{S}^{op}$-submodule consisting of elements x such that $m^n x = 0$. Then ${}_n(B\mathcal{C} \otimes_{\tau_{\mathcal{C}}} \mathcal{C})$ is a DG S_n-module and $\mathrm{Hom}_{\hat{S}^{op}}(K, B\mathcal{C} \otimes_{\tau_{\mathcal{C}}} \mathcal{C}) = \mathrm{Hom}_{S_n}(K, {}_n(B\mathcal{C} \otimes_{\tau_{\mathcal{C}}} \mathcal{C}))$ and similarly for M.

Note that ${}_n(B\mathcal{C} \otimes_{\tau_{\mathcal{C}}} \mathcal{C})$ as a graded S_n-module is isomorphic to $S_n^* \otimes \mathcal{C}$, hence is a finite direct sum of shifted copies of the injective graded module S_n^*. Hence the above map of complexes is surjective. □

Now we can prove the lemma.

Consider the commutative diagram of complexes

$$\begin{array}{ccccccccc}
0 & \to & \mathrm{Hom}_{\hat{S}^{op}}(N, B\mathcal{C} \otimes_{\tau_{\mathcal{C}}} \mathcal{C}) & \to & \mathrm{Hom}_{\hat{S}^{op}}(K, B\mathcal{C} \otimes_{\tau_{\mathcal{C}}} \mathcal{C}) & \to & \mathrm{Hom}_{\hat{S}^{op}}(M, B\mathcal{C} \otimes_{\tau_{\mathcal{C}}} \mathcal{C}) & \to & 0 \\
 & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\
0 & \to & \mathrm{Hom}_{\hat{S}^{op}}(N, J) & \to & \mathrm{Hom}_{\hat{S}^{op}}(K, J) & \to & \mathrm{Hom}_{\hat{S}^{op}}(M, J) & \to & 0,
\end{array}$$

where the bottom row is exact since J^{gr} is an injective graded $\hat{S}^{op}$-module (because J satisfies property (I)). By the induction assumption α and γ are quasi-isomorphisms. Hence also β is such. □

We are ready to prove the theorem.

It follows from Lemma 4.7 that $\nabla(\hat{S})$ is quasi-isomorphic to

$$\varinjlim \mathrm{Hom}_{\hat{S}^{op}}(S_n, B\bar{\mathcal{C}} \otimes_{\tau_{\mathcal{C}}} \mathcal{C}) = \varinjlim \mathrm{Hom}_{S_n}\big(S_n, {}_n(B\bar{\mathcal{C}} \otimes_{\tau_{\mathcal{C}}} \mathcal{C})\big) = \varinjlim\big({}_n(B\bar{\mathcal{C}} \otimes_{\tau_{\mathcal{C}}} \mathcal{C})\big) = B\bar{\mathcal{C}} \otimes_{\tau_{\mathcal{C}}} \mathcal{C}.$$

This proves the second assertion. The first one follows from the next lemma.

Lemma 4.8. *For any augmented A_∞-algebra $\mathcal{C}$ the complex* $\mathrm{Hom}_{\bar{\mathcal{C}}^{op}}(k,k)$ *is quasi-isomorphic to* $\hat{S}^{op}$.

Proof. This follows straightforwardly from the definition of the DG category of A_∞ $\bar{\mathcal{C}}^{op}$-modules. □

This proves the theorem. □

4.3. The functor Ψ

Finally consider the covariant functor $\Psi : D(\hat{S}) \to D_\infty(\bar{\mathcal{C}}^{op})$ defined by

$$\Psi(M) := (B\bar{\mathcal{C}} \otimes_{\tau_\mathcal{C}} \mathcal{C}) \overset{\mathbf{L}}{\otimes}_{\hat{S}} M.$$

Theorem 4.9. *For any augmented A_∞-algebra $\mathcal{C}$ the following holds:*

a) *The functor Ψ is full and faithful on the subcategory* $\mathrm{Perf}(\hat{S})$.
b) $\Psi(\hat{S}) = k$.

Proof. b) is obvious and a) follows from Lemma 4.8 above. □

Part 2. Maurer–Cartan pseudo-functor for A_∞-algebras

5. The definition

Let A be a strictly unital A_∞-algebra, and $\mathcal{R}$ be an artinian DG algebra with the maximal ideal $\mathfrak{m}$. Recall that $A \otimes \mathcal{R}$ is naturally a strictly unital A_∞-algebra (see Section 3.6). We define the set $MC(A \otimes \mathfrak{m})$ as the set of $\alpha \in (A \otimes \mathfrak{m})^1$ such that the generalized Maurer–Cartan equation holds:

$$\sum_{n\geqslant 1}(-1)^{\frac{n(n+1)}{2}} m_n(\alpha,\dots,\alpha) = 0. \tag{5.1}$$

This equation is well defined since $\mathfrak{m} \subset \mathcal{R}$ is nilpotent ideal. Below for convenience we will write α^n instead of $\underbrace{\alpha,\dots,\alpha}_{n}$.

There is a natural A_∞-category $\mathcal{MC}_\infty^{\mathcal{R}}(A)$ with the set of objects $MC(A \otimes \mathfrak{m})$. Namely, for $\alpha_1, \alpha_2 \in MC(A \otimes \mathfrak{m})$ we define

$$\mathrm{Hom}_{\mathcal{MC}_\infty^{\mathcal{R}}(A)}(\alpha_1,\alpha_2) := (A \otimes \mathcal{R})^{gr}$$

as a graded vector space. Further, for $\alpha_0, \alpha_1, \dots, \alpha_m \in MC(A \otimes \mathfrak{m})$ and for homogeneous $x_1 \in \mathrm{Hom}(\alpha_0,\alpha_1), \dots, x_n \in \mathrm{Hom}(\alpha_{n-1},\alpha_n)$ we define

$$m_n^{\mathcal{MC}_\infty^{\mathcal{R}}(A)(\alpha_0,\dots,\alpha_n)}(x_n,\dots,x_1) = \sum_{i_0,\dots,i_n\geqslant 0}(-1)^{\epsilon} m_{n+i_0+\dots+i_n}^{A\otimes\mathcal{R}}\big(\alpha_n^{i_n}, x_n, \alpha_{n-1}^{i_{n-1}},\dots,\alpha_1^{i_1}, x_1, \alpha_0^{i_0}\big),$$

where

$$\epsilon = \sum_{n\geqslant k>j\geqslant 0} (\bar{x_k} + i_k)i_j + \sum_{k=0}^{n} \frac{i_k(i_k+1)}{2} + \sum_{k=1}^{n} k i_k.$$

One checks without difficulties that this indeed defines an A_∞-category and that $\mathbf{1} \in (A\otimes\mathcal{R})^{gr} = \mathrm{Hom}(\alpha,\alpha)$ is a strict identity for each $\alpha \in \mathcal{MC}_\infty^{\mathcal{R}}(A)$. Below we will write $m_n^{\alpha_0,\dots,\alpha_n}$ instead of $m_n^{\mathcal{MC}_\infty^{\mathcal{R}}(A)(\alpha_0,\dots,\alpha_n)}$.

Remark 5.1. The Maurer–Cartan equation and the formulas for higher multiplications are the same as in the definition of the A_∞-category of one-sided twisted complexes, see [5]. Note that in the case of one-sided twisted complexes all the solutions of Maurer–Cartan equation are automatically "nilpotent".

Now we define the Maurer–Cartan pseudo-functor $\mathcal{MC}(A)$: dgart $\to$ **Gpd** as follows. Let $\mathcal{R}$ and $\mathfrak{m}$ be as above. The objects of the groupoid $\mathcal{MC}_{\mathcal{R}}(A)$ are the same as the objects of $\mathcal{MC}_\infty^{\mathcal{R}}(A)$. For $\alpha,\beta \in \mathcal{MC}_\infty^{\mathcal{R}}(A)$, let $G(\alpha,\beta)$ be the set of elements $g \in \mathbf{1} + (A\otimes\mathfrak{m})^0$ such that

$$m_1^{\alpha,\beta}(g) = \sum_{i_0,i_1\geqslant 0} (-1)^{i_0i_1+\frac{i_0(i_0+1)}{2}+\frac{i_1(i_1-1)}{2}} m_{1+i_0+i_1}^{A\otimes\mathcal{R}}\big(\beta^{i_1}, g, \alpha^{i_0}\big) = 0.$$

Then we have an obvious action of the group $(A\otimes\mathfrak{m})^{-1}$ on the set $G(\alpha,\beta)$:

$$h: g \mapsto g + m_1^{\alpha,\beta}(h) = g + \sum_{i_0,i_1\geqslant 0} (-1)^{i_0i_1+\frac{i_0(i_0-1)}{2}+\frac{i_1(i_1-1)}{2}} m_{1+i_0+i_1}^{A\otimes\mathcal{R}}\big(\beta^{i_1}, g, \alpha^{i_0}\big).$$

We define $\mathrm{Hom}_{\mathcal{MC}_{\mathcal{R}}(A)}(\alpha,\beta)$ as the set of orbits $G(\alpha,\beta)/(A\otimes\mathfrak{m})^{-1}$. The composition of morphisms in $\mathcal{MC}_{\mathcal{R}}(A)$ is induced by $m_2^{\mathcal{MC}_\infty^{\mathcal{R}}(A)}$. It follows from the axioms of A_∞-structures that we obtain a well-defined category.

Proposition 5.2. *The category $\mathcal{MC}_{\mathcal{R}}(A)$ is a groupoid.*

Proof. Let $g \in \mathrm{Hom}_{\mathcal{MC}_{\mathcal{R}}(A)}(\alpha,\beta)$. Prove that it has a left inverse $g' \in \mathrm{Hom}_{\mathcal{MC}_{\mathcal{R}}(A)}(\beta,\alpha)$.

Let $\tilde{g} \in G(\alpha,\beta)$ be a lift of g. First prove that there exists $\tilde{g}' \in 1 + (A\otimes\mathfrak{m})^0$ such that

$$m_2^{\alpha,\beta,\alpha}\big(\tilde{g}', \tilde{g}\big) = 1. \tag{5.2}$$

Let n be the minimal positive integer such that $\mathfrak{m}^n = 0$. The proof is by induction over n.

For $n = 1$, there is nothing to prove.

Suppose that the induction hypothesis holds for $n = m \geqslant 1$. Prove it for $n = m + 1$. From the induction hypothesis it follows that there exists $\tilde{g}' \in \mathrm{Hom}_{\mathcal{MC}_{\mathcal{R}}(A)}(\beta,\alpha)$ such that $m_2^{\alpha,\beta,\alpha}(\tilde{g}', \tilde{g}) = 1 + x$, where $x \in (A\otimes\mathfrak{m}^{n-1})^0$. Then we obviously have

$$m_2^{\alpha,\beta,\alpha}\big(\tilde{g}' - x, \tilde{g}\big) = 1.$$

Thus, the induction hypothesis is proved for $n = m + 1$.The statement is proved.

Further, take $\tilde{g}' \in 1 + (A \otimes \mathfrak{m})^0$ such that (5.2) holds. To prove that g has a left inverse it suffices to prove that

$$m_1^{\beta,\alpha}(\tilde{g}') = 0.$$

From the equality (5.2), and since $m_1^{\alpha,\beta}(g) = 0$, we obtain that

$$m_2^{\alpha,\beta,\alpha}(m_1^{\beta,\alpha}(\tilde{g}'), \tilde{g}) = 0.$$

Suppose that $m_1^{\beta,\alpha}(\tilde{g}') \neq 0$. Take the maximal positive integer m such that $m_1^{\beta,\alpha}(\tilde{g}') \in (A \otimes \mathfrak{m}^m)^0$. Then we obviously obtain that $m_2^{\alpha,\beta,\alpha}(m_1^{\beta,\alpha}(\tilde{g}'), \tilde{g}) \in (A \otimes \mathfrak{m}^m)^0 \setminus (A \otimes \mathfrak{m}^{m+1})^0$, this leads to contradiction.

Thus, g has a left inverse. Analogously, it has a right inverse, hence g is invertible. Therefore, the category $\mathcal{MC}_{\mathcal{R}}(A)$ is a groupoid. □

Clearly, the assignment $\mathcal{R} \mapsto \mathcal{MC}_{\mathcal{R}}(A)$ defines a pseudo-functor from dgart to **Gpd**. We denote this pseudo-functor by $\mathcal{MC}(A)$ and call it Maurer–Cartan pseudo-functor.

Notice that if A is a DG algebra, i.e. $m_n^A = 0$ for $n \geqslant 3$, then $\mathcal{MC}_{\infty}^{\mathcal{R}}(A)$ is a DG category. Further, for $\phi \in \mathrm{Hom}(\alpha, \beta)$ we have

$$d^{\mathcal{MC}_{\mathcal{R}}^{\infty}(A)}(x) = d^{A\otimes\mathcal{R}}(x) + \beta x - (-1)^{\bar{x}} x\alpha,$$

and the composition in $\mathcal{MC}_{\mathcal{R}}^{\infty}(A)$ is just the product in $A \otimes \mathcal{R}$. It follows that the constructed Maurer–Cartan pseudo-functor coincides in this case with that constructed in [2, Section 5].

Remark 5.3. The Maurer–Cartan groupoid $\mathcal{MC}_{\mathcal{R}}(A)$ can be extended to a ∞-groupoid $\mathcal{MC}_{\mathcal{R}}^{\infty}(A)$ so that $\mathcal{MC}_{\mathcal{R}}(A) = \pi_0(\mathcal{MC}_{\mathcal{R}}^{\infty}(A))$. Further, the assignment $\mathcal{R} \to \mathcal{MC}_{\mathcal{R}}^{\infty}(A)$ defines a pseudo-functor $\mathcal{MC}^{\infty}(A) : \mathrm{dgart} \to \mathbf{Gpd}^{\infty}$, where $\mathbf{Gpd}^{\infty}$ is a ∞-category of ∞-groupoids.

6. Obstruction theory

Fix a strictly unital A_{∞}-algebra A.

Let $\mathcal{R}$ be an artinian DG algebra with the maximal ideal $\mathfrak{m}$. Further, let n be the minimal positive integer such that $\mathfrak{m}^{n+1} = 0$. Put $\mathcal{I} = \mathfrak{m}^n$, $\bar{\mathcal{R}} = \mathcal{R}/\mathcal{I}$, and $\pi : \mathcal{R} \to \bar{\mathcal{R}}$ — the projection morphism. The next proposition describes the obstruction theory for lifting of objects and morphisms along the functor

$$\pi^* : \mathcal{MC}_{\mathcal{R}}(A) \to \mathcal{MC}_{\bar{\mathcal{R}}}(A).$$

Proposition 6.1.

1) *There exists a map $o_2 : Ob(\mathcal{MC}_{\bar{\mathcal{R}}}(A)) \to H^2(A \otimes \mathcal{I})$ such that $\alpha \in \mathcal{MC}_{\bar{\mathcal{R}}}(A)$ is in the image of π^* if and only if $o_2(\alpha) = 0$. Furthermore, if $\alpha, \beta \in \mathcal{MC}_{\bar{\mathcal{R}}}(A)$ are isomorphic then $o_2(\alpha) = 0$ iff $o_2(\beta) = 0$.*

2) *Let $\xi \in Ob(\mathcal{MC}_{\bar{\mathcal{R}}}(A))$ be such that the fiber $(\pi^*)^{-1}(\xi)$ is non-empty. Then there exists a simply transitive action of the group $Z^1(A\otimes\mathcal{I})$ on the set $Ob((\pi^*)^{-1}(\xi))$. Let $\xi_1,\xi_2\in Ob(\mathcal{MC}_{\bar{\mathcal{R}}}(A))$ be isomorphic objects such that both fibers $(\pi^*)^{-1}(\xi_1)$, $(\pi^*)^{-1}(\xi_2)$ are non-empty, and let $f:\xi_1\to\xi_2$ be a morphism. Take the action of $Z^1(A\otimes\mathcal{I})$ on $Ob((\pi^*)^{-1}(\xi_2))$ as above and the action on $Ob((\pi^*)^{-1}(\xi_1))$ which is inverse to the above action. Then there is a (non-canonical) $Z^1(A\otimes\mathcal{I})$-equivariant map*

$$\widetilde{o_1}: Ob\bigl(\bigl(\pi^*\bigr)^{-1}(\xi_1)\bigr)\times Ob\bigl(\bigl(\pi^*\bigr)^{-1}(\xi_2)\bigr)\to Z^1(A\otimes\mathcal{I}),$$

such that the composition of it with the projection

$$Z^1(A\otimes\mathcal{I})\to H^1(A\otimes\mathcal{I}),$$

which we denote by o_1^f, is canonically defined and satisfies the following property: for $\alpha_1\in Ob((\pi^*)^{-1}(\xi_1))$, $\alpha_2\in Ob((\pi^*)^{-1}(\xi_2))$ there exists a morphism $\gamma:\alpha_1\to\alpha_2$ such that $\pi^*(\gamma)=f$ iff $o_1^f(\alpha_1,\alpha_2)=0$.

3) *Let $\tilde{\alpha},\tilde{\beta}\in\mathcal{MC}_{\mathcal{R}}(A)$ be objects and let $f:\alpha\to\beta$ be a morphism from $\alpha=\pi^*(\tilde{\alpha})$ to $\beta=\pi^*(\tilde{\beta})$. Suppose that the set $(\pi^*)^{-1}(f)$ of morphisms $\tilde{f}:\tilde{\alpha}\to\tilde{\beta}$ such that $\pi^*(\tilde{f})=f$ is non-empty. Then there is a simple transitive action of the group $\operatorname{Im}(H^0(A\otimes\mathcal{I})\to H^0(A\otimes\mathfrak{m},m_1^{\alpha,\beta}))$ on the set $(\pi^*)^{-1}(f)$. In particular, the difference map*

$$o_0:\bigl(\pi^*\bigr)^{-1}(f)\times\bigl(\pi^*\bigr)^{-1}(f)\to\operatorname{Im}\bigl(H^0(A\otimes\mathcal{I})\to H^0\bigl(A\otimes\mathfrak{m},m_1^{\alpha,\beta}\bigr)\bigr)$$

satisfies the following property: if $\tilde{f},\tilde{f}'\in(\pi^*)^{-1}(f)$ then $\tilde{f}=\tilde{f}'$ iff $o_0(\tilde{f},\tilde{f}')=0$.

Proof. 1) Let $\alpha\in\mathcal{MC}_{\bar{\mathcal{R}}}(A)$. Take some $\tilde{\alpha}\in(A\otimes\mathfrak{m})^1$ such that $\pi(\tilde{\alpha})=\alpha$. Then we have

$$\sum_{n\geqslant1}(-1)^{\frac{n(n+1)}{2}+1}m_n^{A\otimes\mathcal{R}}(\tilde{\alpha},\dots,\tilde{\alpha})\in(A\otimes\mathcal{I})^2.$$

A straightforward applying of (3.2) shows that

$$\sum_{n\geqslant1}(-1)^{\frac{n(n+1)}{2}+1}m_n^{A\otimes\mathcal{R}}(\tilde{\alpha},\dots,\tilde{\alpha})\in Z^2(A\otimes\mathcal{I}).$$

Further, if $\tilde{\alpha}'\in A\otimes\mathfrak{m}$ is another lift of α then

$$\sum_{n\geqslant1}(-1)^{\frac{n(n+1)}{2}+1}m_n^{A\otimes\mathcal{R}}\bigl(\tilde{\alpha}',\dots,\tilde{\alpha}'\bigr)-\sum_{n\geqslant1}(-1)^{\frac{n(n+1)}{2}+1}m_n^{A\otimes\mathcal{R}}(\tilde{\alpha},\dots,\tilde{\alpha})$$
$$=m_1^{A\otimes\mathcal{R}}\bigl(\tilde{\alpha}'-\tilde{\alpha}\bigr). \tag{6.1}$$

Hence, we obtain the well-defined element $o_2(\alpha)\in H^2(A\otimes\mathcal{I})$ and therefore the map $o_2: Ob(\mathcal{MC}_{\bar{\mathcal{R}}}(A))\to H^2(A\otimes\mathcal{I})$. The first property of o_2 is obviously satisfied.

Further, let $\alpha, \beta \in \mathcal{MC}_{\bar{\mathcal{R}}}(A)$, and $f : \alpha \to \beta$ be a morphism. Suppose that $o_2(\alpha) = 0$. Take some $\tilde{\alpha} \in (\pi^*)^{-1}(\alpha)$. Further, take some $\tilde{f} \in 1 + (A \otimes \mathfrak{m})^0$ such that $\pi(\tilde{f})$ represents f, and $\tilde{\beta} \in (A \otimes \mathfrak{m})^1$ such that $\pi(\tilde{\beta}) = \beta$. We have that

$$\sum_{i_0, i_1 \geqslant 0} (-1)^{i_0 i_1 + \frac{i_0(i_0+1)}{2} + \frac{i_1(i_1+3)}{2}} m^{A\otimes\mathcal{R}}_{1+i_0+i_1}\left(\tilde{\beta}^{i_1}, \tilde{f}, \tilde{\alpha}^{i_0}\right) \in (A \otimes \mathcal{I})^1.$$

A straightforward applying of (3.2) shows that

$$m_1^{A\otimes\mathcal{R}}\left(\sum_{i_0, i_1 \geqslant 0} (-1)^{i_0 i_1 + \frac{i_0(i_0+1)}{2} + \frac{i_1(i_1+3)}{2}} m^{A\otimes\mathcal{R}}_{1+i_0+i_1}\left(\tilde{\beta}^{i_1}, \tilde{f}, \tilde{\alpha}^{i_0}\right)\right)$$
$$= m_2^{A\otimes\mathcal{R}}\left(\sum_{n\geqslant 1} (-1)^{\frac{n(n+1)}{2}+1} m_n^{A\otimes\mathcal{R}}(\tilde{\beta}, \dots, \tilde{\beta}), \tilde{f}\right) = \sum_{n\geqslant 1} (-1)^{\frac{n(n+1)}{2}+1} m_n^{A\otimes\mathcal{R}}(\tilde{\beta}, \dots, \tilde{\beta}).$$

Therefore, $o_2(\beta) = 0$. This proves 1).

2) Let $\eta \in Z^1(A \otimes \mathcal{I})$. It follows from (6.1) that the formula

$$\eta : \alpha \mapsto \alpha + \eta$$

defines a simply transitive action of the group $Z^1(A \otimes \mathcal{I})$ on the set $Ob((\pi^*)^{-1}(\xi))$. Let ξ_1, ξ_2, f be as in proposition. Take some $\tilde{f} \in 1 + (A \otimes \mathfrak{m})^0$ such that $\overline{\pi(\tilde{f})} = f$. Define $\widetilde{o_1}$ by the formula

$$\widetilde{o_1}(\alpha, \beta) = o_1^{\tilde{f}}(\alpha, \beta) = m_1^{\alpha,\beta}(\tilde{f}).$$

It is easy to see that the image of $o_1^{\tilde{f}}$ lies in $Z^1(A \otimes \mathcal{I})$ and that $o_1^{\tilde{f}}$ is $Z^1(A \otimes \mathcal{I})$-equivariant. If $\tilde{f}'$ is another lift of f, then there exists $h \in (A \otimes \mathfrak{m})^{-1}$ such that

$$v = \tilde{f}' - \tilde{f} - m_1^{\alpha,\beta}(h) \in (A \otimes \mathcal{I})^0.$$

Further,

$$o_1^{\tilde{f}'}(\alpha, \beta) - o_1^{\tilde{f}}(\alpha, \beta) = m_1^{A\otimes\mathcal{R}}(v),$$

hence the map o_1^f is canonically defined.

Suppose that $o_1^f(\alpha, \beta) = 0$ for some $\alpha \in (\pi^*)^{-1}(\xi_1)$, $\beta \in (\pi^*)^{-1}(\xi_2)$. Let $\tilde{f}$ be as above. Then there exists $x \in (A \otimes \mathcal{I})^0$ such that

$$m_1^{\alpha,\beta}(\tilde{f}) = m_1^{A\otimes\mathcal{R}}(x).$$

We have $\tilde{f} - x \in G(\alpha, \beta)$, and $\pi^*(\overline{\tilde{f} - x}) = f$.

Conversely, suppose that there exists a morphism $\gamma \in \mathrm{Hom}_{\mathcal{MC}_{\mathcal{R}}(A)}(\alpha, \beta)$ for some $\alpha \in (\pi^*)^{-1}(\xi_1)$, $\beta \in (\pi^*)^{-1}(\xi_2)$, such that $\pi^*(\gamma) = f$. Let $\tilde{\gamma} \in G(\alpha, \beta)$ be a representative of γ. Then we have $o_1^{\tilde{\gamma}}(\alpha, \beta) = 0$, hence $o_1^f(\alpha, \beta) = 0$. This proves 2).

3) First we define the action of the group $Z^0(A \otimes \mathcal{I})$ on the set $(\pi^*)^{-1}(f)$ by the formula

$$\eta : \bar{\bar{\tilde{f}}} \to \overline{\tilde{f} + \eta},$$

where $\eta \in Z^0(A \otimes \mathcal{I})$, and $\tilde{f} \in G(\alpha, \beta)$ is such that $\pi^*(\bar{\bar{\tilde{f}}}) = f$. Clearly, this is correct. Further, if $\eta = m_1^{A\otimes\mathcal{R}}(\zeta)$ for some $\zeta \in (A \otimes \mathfrak{m})^{-1}$, then

$$\eta(\bar{\bar{\tilde{f}}}) = \overline{\tilde{f} + m_1^{\alpha,\beta}(\zeta)} = \bar{\bar{\tilde{f}}}.$$

Hence, we have an action of $\mathrm{Im}(H^0(A \otimes \mathcal{I}) \to H^0(A \otimes \mathfrak{m}, m_1^{\alpha,\beta}))$ on the set $(\pi^*)^{-1}(f)$.

Tautologically, this action is simple.

Prove that it is transitive. Let $\tilde{f}, \tilde{f}' \in G(\alpha, \beta)$ be such that $\pi^*(\bar{\bar{\tilde{f}}}) = \pi^*(\bar{\bar{\tilde{f}}}') = f$. Then, by definition, there exists $h \in (A \otimes \mathfrak{m})^{-1}$ such that

$$\tilde{f}' - \tilde{f} - m_1^{\alpha,\beta}(h) \in (A \otimes \mathcal{I})^0.$$

Replacing $\tilde{f}$ by $\tilde{f} + m_1^{\alpha,\beta}(h)$, we obtain $\tilde{f}' = \tilde{f} + \eta$, where $\eta \in (A \otimes \mathcal{I})^0$. Since $\tilde{f}, \tilde{f}' \in G(\alpha, \beta)$, we have that $\eta \in Z^0(A \otimes \mathcal{I})$. This shows transitivity and proves 3).

Proposition is proved. □

Remark 6.2. One can also construct the obstruction theory for lifting of objects and all k-morphisms along the ∞-functor

$$\pi^* : \mathcal{MC}_{\mathcal{R}}^{\infty}(A) \to \mathcal{MC}_{\bar{\mathcal{R}}}^{\infty}(A).$$

7. Invariance theorems

Let A_1, A_2 be strictly unital A_∞-algebras and $f : A_1 \to A_2$ be a strictly unital A_∞-morphism between them given by a sequence of maps

$$f_n : A_1^{\otimes n} \to A_2.$$

Further, let $\mathcal{R}$ be an artinian DG algebra with the maximal ideal $\mathfrak{m}$.

Then we have a (strictly unital) A_∞-functor

$$f_{\mathcal{R}}^* : \mathcal{MC}_{\infty}^{\mathcal{R}}(A_1) \to \mathcal{MC}_{\infty}^{\mathcal{R}}(A_2)$$

defined by the formulas

$$f_{\mathcal{R}}^*(\alpha) = \sum_{n\geqslant 1}(-1)^{\frac{n(n-1)}{2}} f_n(\alpha, \dots, \alpha);$$

$$f_{\mathcal{R}}^*(\alpha_0, \dots, \alpha_n)(x_1, \dots, x_n) = \sum_{i_0,\dots,i_n\geqslant 0} (-1)^{\epsilon} f_{n+i_0+\dots+i_n}\big(\alpha_n^{i_n}, x_n, \alpha_{n-1}^{i_{n-1}}, \dots, \alpha_1^{i_1}, x_1, \alpha_0^{i_0}\big),$$

where

$$\epsilon = \sum_{n\geqslant k>j\geqslant 0}(\bar{x_k}+i_k)i_j + \sum_{k=0}^{n}\frac{i_k(i_k-1)}{2} + \sum_{k=1}^{n} k i_k.$$

One checks without difficulties that these formulas indeed define a strictly unital A_∞-functor.

It induces a functor $f^*_{\mathcal{R}} : \mathcal{MC}_{\mathcal{R}}(A_1) \to \mathcal{MC}_{\mathcal{R}}(A_2)$ and we obtain a morphism of pseudo-functors

$$f^* : \mathcal{MC}(A_1) \to \mathcal{MC}(A_2).$$

The following theorems show that for quasi-isomorphic strictly unital A_∞-algebras the corresponding Maurer–Cartan A_∞-categories (resp. Maurer–Cartan pseudo-functors) are quasi-equivalent (resp. equivalent).

Theorem 7.1. *Let $f : A_1 \to A_2$ be a strictly unital quasi-isomorphism of strictly unital A_∞-algebras and let $\mathcal{R}$ be an artinian DG algebra with the maximal ideal $\mathfrak{m}$. Then the A_∞-functor*

$$f^*_{\mathcal{R}} : \mathcal{MC}^{\mathcal{R}}_\infty(A_1) \to \mathcal{MC}^{\mathcal{R}}_\infty(A_2)$$

is a quasi-equivalence.

Proof. 1) Prove that for any $\alpha, \beta \in \mathcal{MC}^{\mathcal{R}}_\infty(A_1)$ the morphism of complexes

$$f^*_{\mathcal{R}}(\alpha,\beta) : \mathrm{Hom}_{\mathcal{MC}^{\mathcal{R}}_\infty(A_1)}(\alpha,\beta) \to \mathrm{Hom}_{\mathcal{MC}^{\mathcal{R}}_\infty(A_2)}\big(f^*(\alpha), f^*(\beta)\big)$$

is quasi-isomorphism. Note that both complexes have finite filtrations by subcomplexes $A_1 \otimes \mathfrak{m}^i$ and $A_2 \otimes \mathfrak{m}^i$. The morphism $f^*_{\mathcal{R}}(\alpha,\beta)$ is compatible with these filtrations and induces quasi-isomorphisms on the subquotients. Hence, it is quasi-isomorphism.

2) Now we prove that the functor

$$\mathrm{Ho}\big(f^*\big) : \mathrm{Ho}\big(\mathcal{MC}^{\mathcal{R}}_\infty(A_1)\big) \to \mathrm{Ho}\big(\mathcal{MC}^{\mathcal{R}}_\infty(A_2)\big)$$

is an equivalence. We have already proved that it is full faithful, hence it remains to prove that it is essentially surjective. We will prove the stronger statement: the functor

$$f^*_{\mathcal{R}} : \mathcal{MC}_{\mathcal{R}}(A_1) \to \mathcal{MC}_{\mathcal{R}}(A_2)$$

is essentially surjective.

Let n be the minimal positive integer such that $\mathfrak{m}^n = 0$. The proof is by induction over n.

For $n = 1$, there is nothing to prove.

Suppose that the induction hypothesis holds for $n = m$. Prove it for $n = m + 1$. Let $\mathcal{I}$, $\bar{\mathcal{R}}$, $\bar{\mathfrak{m}}$, π be as above. A straightforward checking shows that the following diagram commutes:

$$\begin{array}{ccc} Ob(\mathcal{MC}_{\bar{\mathcal{R}}}(A_1)) & \xrightarrow{f^*_{\bar{\mathcal{R}}}} & Ob(\mathcal{MC}_{\bar{\mathcal{R}}}(A_2)) \\ \downarrow{\scriptstyle o_2} & & \downarrow{\scriptstyle o_2} \\ H^2(A_1 \otimes \mathcal{I}) & \xrightarrow{\sim} & H^2(A_2 \otimes \mathcal{I}), \end{array} \tag{7.1}$$

where the map o_2 is defined in Proposition 6.1.

Let $\alpha \in \mathcal{MC}_{\mathcal{R}}(A_2)$. By the induction hypothesis, there exists $\beta \in \mathcal{MC}_{\bar{\mathcal{R}}}(A_1)$ such that $f^*_{\bar{\mathcal{R}}}(\beta)$ is isomorphic to $\pi^*(\alpha)$ in $\mathcal{MC}_{\bar{\mathcal{R}}}(A_2)$. Since the diagram (7.1) commutes, we have that $o_2(\beta) = 0$. Thus, by Proposition 6.1, the fiber $(\pi^*)^{-1}(\beta)$ is nonempty. Fix some $\tilde{\beta} \in (\pi^*)^{-1}(\beta)$. Let $\gamma : f^*_{\bar{\mathcal{R}}}(\beta) \to \pi^*(\alpha)$ be a morphism. A straightforward checking shows that the following diagram commutes:

$$\begin{array}{ccc} Ob((\pi^*)^{-1}(\beta)) & \xrightarrow{f^*_{\mathcal{R}}} & Ob((\pi^*)^{-1}(f^*_{\bar{\mathcal{R}}}(\beta))) \\ \downarrow{\scriptstyle o_1^{\mathrm{id}_\beta}(*,\tilde{\beta})} & & \downarrow{\scriptstyle o_1^{\gamma}(f^*_{\mathcal{R}}(*),\alpha) - o_1^{\gamma}(f^*_{\mathcal{R}}(\tilde{\beta}),\alpha)} \\ H^1(A_1 \otimes \mathcal{I}) & \xrightarrow{\sim} & H^1(A_2 \otimes \mathcal{I}), \end{array} \tag{7.2}$$

where the vertical arrows are defined in Proposition 6.1. Since the map $o_1^{\mathrm{id}_\beta}(*, \tilde{\beta})$ is surjective and the diagram (7.2) commutes, there exists an object $\tilde{\beta}' \in Ob((\pi^*)^{-1}(\beta))$ such that $o_1^{\gamma}(f^*_{\mathcal{R}}(\tilde{\beta}'), \alpha) = 0$. Then, by Proposition 6.1, there exists a morphism $\tilde{\gamma} : f^*_{\mathcal{R}}(\tilde{\beta}') \to \alpha$ (such that $\pi^*(\tilde{\gamma}) = \gamma$). Therefore, the functor $f^*_{\mathcal{R}}$ is essentially surjective, and the induction hypothesis is proved for $n = m + 1$. The statement is proved.

Theorem is proved. □

Theorem 7.2. *Let $f : A_1 \to A_2$ be a strictly unital quasi-isomorphism of strictly unital A_∞-algebras. Then the morphism of pseudo-functors*

$$f^* : \mathcal{MC}(A_1) \to \mathcal{MC}(A_2)$$

is an equivalence.

Proof. Fix an artinian DG algebra $\mathcal{R}$ with the maximal ideal $\mathfrak{m}$. We must prove that the functor

$$f^*_{\mathcal{R}} : \mathcal{MC}_{\mathcal{R}}(A_1) \to \mathcal{MC}_{\mathcal{R}}(A_2)$$

is an equivalence.

In the proof of the previous theorem we have already shown that it is essentially surjective. So it remains to prove that it is full and faithful.

Let n be the minimal positive integer such that $\mathfrak{m}^n = 0$. The proof is by induction over n.

For $n = 1$, there is nothing to prove.

Suppose that the induction hypothesis holds for $n = m \geqslant 1$. Prove it for $n = m + 1$.

Full. Let $\alpha, \beta \in \mathcal{MC}_{\mathcal{R}}(A_1)$ and let $\gamma : f^*_{\mathcal{R}}(\alpha) \to f^*_{\mathcal{R}}(\beta)$ be a morphism. By induction hypothesis, there exists a morphism $g : \pi^*(\alpha) \to \pi^*(\beta)$ such that

$$f^*_{\mathcal{R}}(g) = \pi^*(\gamma).$$

A straightforward checking shows that the following diagram commutes:

$$\begin{array}{ccc} (\pi^*)^{-1}(\pi^*(\alpha)) \times (\pi^*)^{-1}(\pi^*(\beta)) & \longrightarrow & (\pi^*)^{-1}(\pi^*(f^*_{\mathcal{R}}(\alpha))) \times (\pi^*)^{-1}(\pi^*(f^*_{\mathcal{R}}(\beta))) \\ \Big\downarrow o_1^g & & \Big\downarrow o_1^{\pi^*(\gamma)} \\ H^1(A_1 \otimes \mathcal{I}) & \xrightarrow{\sim} & H^1(A_2 \otimes \mathcal{I}). \end{array} \tag{7.3}$$

By Proposition 6.1 and since the diagram (7.3) commutes there exists a morphism $\tilde{g} : \alpha \to \beta$ such that $\pi^*(\tilde{g}) = g$. Further, a straightforward checking shows that the following diagram commutes:

$$\begin{array}{ccc} (\pi^*)^{-1}(g) & \xrightarrow{o_0(*,\tilde{g})} & \mathrm{Im}(H^0(A_1 \otimes \mathcal{I}) \to H^0(A_1 \otimes \mathfrak{m}, m_1^{\alpha,\beta})) \\ \Big\downarrow f^*_{\mathcal{R}} & & \sim \Big\downarrow \\ (\pi^*)^{-1}(\pi^*(\gamma)) & \xrightarrow{o_0(f^*_{\mathcal{R}}(*),\gamma) - o_0(f^*_{\mathcal{R}}(\tilde{g}),\gamma)} & \mathrm{Im}(H^0(A_2 \otimes \mathcal{I}) \to H^0(A_2 \otimes \mathfrak{m}, m_1^{f^*_{\mathcal{R}}(\alpha), f^*_{\mathcal{R}}(\beta)})). \end{array} \tag{7.4}$$

Since the upper arrow is surjective, there exists a morphism $\tilde{g}' \in (\pi^*)^{-1}(g)$ such that

$$o_0\big(f^*_{\mathcal{R}}(\tilde{g}'), \gamma\big) = 0,$$

i.e. $f^*_{\mathcal{R}}(\tilde{g}') = \gamma$. Hence, the functor $f^*_{\mathcal{R}}$ is full.

Faithful. Let $\gamma_1, \gamma_2 : \alpha \to \beta$ be two morphisms in $\mathcal{MC}_{\mathcal{R}}(A_1)$. Suppose that $f^*_{\mathcal{R}}(\gamma_1) = f^*_{\mathcal{R}}(\gamma_2)$. Then we have also $f^*_{\mathcal{R}}(\pi^*(\gamma_1)) = f^*_{\mathcal{R}}(\pi^*(\gamma_2))$, hence by induction hypothesis $\pi^*(\gamma_1) = \pi^*(\gamma_2)$. A straightforward checking shows that the following diagram commutes:

$$\begin{array}{ccc} (\pi^*)^{-1}(\pi^*(\gamma_1)) \times (\pi^*)^{-1}(\pi^*(\gamma_1)) & \xrightarrow{o_0} & \mathrm{Im}(H^0(A_1 \otimes \mathcal{I}) \to H^0(A_1 \otimes \mathfrak{m}, m_1^{\alpha,\beta})) \\ \Big\downarrow f^*_{\mathcal{R}} & & \sim \Big\downarrow \\ (\pi^*)^{-1}(\pi^*(f^*_{\mathcal{R}}(\gamma_1))) \times (\pi^*)^{-1}(\pi^*(f^*_{\mathcal{R}}(\gamma_1))) & \xrightarrow{o_0} & \mathrm{Im}(H^0(A_2 \otimes \mathcal{I}) \to H^0(A_2 \otimes \mathfrak{m}, m_1^{f^*_{\mathcal{R}}(\alpha), f^*_{\mathcal{R}}(\beta)})). \end{array} \tag{7.5}$$

By Proposition 6.1 and since the diagram (7.5) commutes we have that

$$o_0(\gamma_1, \gamma_2) = 0,$$

hence $\gamma_1 = \gamma_2$. Thus, the functor $f^*_{\mathcal{R}}$ is full.

The induction hypothesis is proved for $n = m + 1$. The statement is proved.
Theorem is proved. $\square$

Remark 7.3. It can be proved that an A_∞-quasi-isomorphism $f : A_1 \to A_2$ induces an equivalence of ∞-groupoids $f^*_{\mathcal{R}} : \mathcal{MC}^\infty_{\mathcal{R}}(A_1) \to \mathcal{MC}^\infty_{\mathcal{R}}(A_2)$.

8. Twisting cochains

Let $\mathcal{G}$ be a co-augmented DG coalgebra. Let A be an arbitrary A_∞-algebra. Then the graded vector space $\mathrm{Hom}_k(\mathcal{G}, A)$ has natural structure of an A_∞-algebra. If $\dim \mathcal{G} < \infty$ or $\dim A < \infty$, then $\mathrm{Hom}_k(\mathcal{G}, A)$ is canonically identified with $A \otimes \mathcal{G}^*$ as an A_∞-algebra.

Suppose that the DG coalgebra $\mathcal{G}$ is co-complete. The map $\tau : \mathcal{G} \to A$ of degree 1 is called a twisting cochain if it passes through $\bar{\mathcal{G}}$ and satisfies the generalized Maurer–Cartan equation (5.1) as an element of A_∞-algebra $\mathrm{Hom}_k(\mathcal{G}, A)$. This is well defined since $\mathcal{G}$ is co-complete. If $\mathcal{R}$ is an artinian DG algebra and A is strictly unital, then we have natural bijection between the set of twisting cochains $\tau : \mathcal{R}^* \to A$ and the set $MC(A \otimes \mathfrak{m})$. In the case when A is augmented, the twisting cochain is called admissible if it passes through $\bar{A}$. Tautologically, admissible twisting cochains $\mathcal{G} \to A$ are in one-to-one correspondence with twisting cochains $\mathcal{G} \to \bar{A}$.

Proposition 8.1. *Let A be an A_∞-algebra. The composition $\tau_A : BA \to A$ of the natural projection $BA \to A[1]$ with the shift map $A[1] \to A$ is the universal twisting cochain. That is, if $\mathcal{G}$ is a co-augmented co-complete DG coalgebra and $\tau : \mathcal{G} \to A$ is a twisting cochain then there exists a unique homomorphism $g_\tau : \mathcal{G} \to BA$ of co-augmented DG coalgebras, such that $\tau_A \cdot g_\tau = \tau$.*

It follows that if A is augmented then the composition of $\tau_{\bar{A}}$ with the embedding $\bar{A} \hookrightarrow A$, which we also denote by τ_A, is the universal admissible twisting cochain in the same sense.

Proof. A straightforward checking. $\square$

Further, if $\mathcal{G}$ is a co-augmented co-complete DG coalgebra, and $\tau : \mathcal{G} \to A$ is a twisting cochain then

$$\mathcal{G} \otimes_\tau A := \mathcal{G} \,\square_{BA}\, (BA \otimes_{\tau_A} A)$$

is an object of $A^{op}_{(\mathcal{G}^*)^{op}}$-$\mathrm{mod}_\infty$.

Proposition 8.2. *Let $f : A_1 \to A_2$ be an A_∞-quasi-isomorphism of A_∞-algebras, $\mathcal{G}$ be a co-augmented co-complete DG coalgebra, and $\tau : \mathcal{G} \to A_1$ be a twisting cochain. Then there is a natural homotopy equivalence in $A^{op}_{1(\mathcal{G}^*)^{op}}$-$\mathrm{mod}_\infty$:*

$$\mathcal{G} \otimes_\tau A_1 \to f_*(\mathcal{G} \otimes_{f \cdot \tau} A_2).$$

Proof. We have a natural homotopy equivalence of DG bicomodules over BA_1:

$$BA_1 \otimes A_1[1] \otimes BA_1 \to BA_1 \,\square_{BA_2}\, \big(BA_2 \otimes A_2[1] \otimes BA_2\big) \,\square_{BA_2}\, BA_1.$$

Co-tensoring it on the left by $\mathcal{G}$, we obtain the required homotopy equivalence. $\square$

Now let A be an augmented A_∞-algebra. If $\mathcal{G}$ is a co-augmented co-complete DG coalgebra and $\tau:\mathcal{G}\to A$ is an admissible twisting cochain then

$$\mathcal{G}\otimes_\tau A := \mathcal{G}\,\square_{B\bar{A}}\,(B\bar{A}\otimes_{\tau_A} A)$$

is an object of $\bar{A}^{op}_{(\mathcal{G}^*)^{op}}$-mod$_\infty$.

Proposition 8.3. *Let $f: A_1\to A_2$ be an A_∞-quasi-isomorphism of augmented A_∞-algebras, $\mathcal{G}$ be a co-augmented co-complete DG coalgebra and $\tau:\mathcal{G}\to A_1$ be an admissible twisting cochain. Then there is a natural homotopy equivalence in $\bar{A}_1{}^{op}_{(\mathcal{G}^*)^{op}}$-mod$_\infty$:*

$$\mathcal{G}\otimes_\tau A_1\to f_*(\mathcal{G}\otimes_{f\cdot\tau} A_2).$$

Proof. We have a natural homotopy equivalence of DG bicomodules over $B\bar{A}_1$:

$$B\bar{A}_1\otimes A_1[1]\otimes B\bar{A}_1\to B\bar{A}_1\,\square_{B\bar{A}_2}\,\big(B\bar{A}_2\otimes A_2[1]\otimes B\bar{A}_2\big)\,\square_{B\bar{A}_2}\,B\bar{A}_1.$$

Co-tensoring it on the left by $\mathcal{G}$, we obtain the required homotopy equivalence. □

Let $\mathcal{R}$ be an artinian DG algebra, and $\tau:\mathcal{R}^*\to A$ be an admissible twisting cochain. Then by Proposition 8.1 we have a natural morphism of DG coalgebras $g_\tau:\mathcal{R}^*\to B\bar{A}$. Further, we have the dual morphism of DG algebras $g_\tau^*:\hat{S}\to\mathcal{R}$. In particular, $\mathcal{R}$ becomes a DG $\hat{S}^{op}$-module.

Lemma 8.4. *In the above notation A_∞ $\bar{A}^{op}_{\hat{S}^{op}}$-modules $\mathrm{Hom}_{\hat{S}^{op}}(\mathcal{R}, B\bar{A}\otimes_{\tau_A} A)$ and $\mathcal{R}^*\otimes_\tau A$ are isomorphic.*

Proof. Evident. □

If $\tau:\mathcal{R}^*\to A$ is an admissible twisting cochain and $\alpha\in\mathcal{MC}_{\mathcal{R}}(A)$ is the corresponding object, then we will write also $A\otimes_\alpha\mathcal{R}^*$ instead of $\mathcal{R}^*\otimes_\tau A$.

Further, for $\alpha\in\mathcal{MC}_{\mathcal{R}}(A)$ corresponding to an admissible twisting cochain we put

$$A\otimes_\alpha\mathcal{R} := \mathrm{Hom}_{\mathcal{R}}\big(\mathcal{R}^*, A\otimes_\alpha\mathcal{R}^*\big).$$

This is an object of $\bar{A}^{op}_{\mathcal{R}^{op}}$-mod$_\infty$. Its $(\mathcal{R}^{op})^{gr}$-module structure is obvious and A_∞-module structure can also be given by the explicit formulas:

$$m_n^{A\otimes_\alpha\mathcal{R}}(m,a_1,\dots,a_{n-1}) = m_n^{0,\dots,0,\alpha}(m,a_1\otimes\mathbf{1}_{\mathcal{R}},\dots,a_{n-1}\otimes\mathbf{1}_{\mathcal{R}}). \tag{8.1}$$

Proposition 8.5. *Let $f: A_1\to A_2$ be an A_∞-quasi-isomorphism of augmented A_∞-algebras, $\mathcal{R}$ be an artinian DG algebra and let $\alpha\in\mathcal{MC}_{\mathcal{R}}(A_1)$. Then there is a natural homotopy equivalence in $\bar{A}_1{}^{op}_{\mathcal{R}^{op}}$-mod$_\infty$:*

$$A_1\otimes_\alpha\mathcal{R}\to f_*(A_2\otimes_{f^*_{\mathcal{R}}(\alpha)}\mathcal{R}).$$

Proof. The required homotopy equivalence is obtained by applying the functor $\operatorname{Hom}_{\mathcal{R}}(\mathcal{R}^*, -)$ to the homotopy equivalence

$$A_1 \otimes_\alpha \mathcal{R}^* \to f_*\big(A_2 \otimes_{f^*_{\mathcal{R}}(\alpha)} \mathcal{R}^*\big)$$

from Proposition 8.3. □

Note that if A is a DG algebra, then $A \otimes_\alpha \mathcal{R}^*$ and $A \otimes_\alpha \mathcal{R}$ are the DG modules from $\operatorname{coDef}^{\mathrm{h}}_{\mathcal{R}}(A)$ and $\operatorname{Def}^{\mathrm{h}}_{\mathcal{R}}(A)$ respectively, which correspond to α.

Finally, if A is a strictly unital but not necessarily augmented A_∞-algebra, $\mathcal{R}$ is an artinian DG algebra, α is an object of $\mathcal{MC}_{\mathcal{R}}(A)$ and $\tau : \mathcal{R}^* \to A$ is the corresponding twisting cochain then we also write $A \otimes_\alpha \mathcal{R}^*$ instead of $\mathcal{R}^* \otimes_\tau A$. Further, we put

$$A \otimes_\alpha \mathcal{R} = \operatorname{Hom}_{\mathcal{R}}\big(\mathcal{R}^*, A \otimes_\alpha \mathcal{R}^*\big).$$

This is the object of $A^{op}_{\mathcal{R}^{op}}\text{-mod}_\infty$. Again, its $(\mathcal{R}^{op})^{gr}$-module structure is obvious and the A_∞-module structure is given by the formulas (8.1). The following proposition is absolutely analogous to the previous one and we omit the proof.

Proposition 8.6. *Let $f : A_1 \to A_2$ be a strictly unital A_∞-morphism of strictly unital A_∞-algebras, $\mathcal{R}$ be an artinian DG algebra and let $\alpha \in \mathcal{MC}_{\mathcal{R}}(A_1)$. Then there is a natural homotopy equivalence in $A^{op}_{1\mathcal{R}^{op}}\text{-mod}_\infty$:*

$$A_1 \otimes_\alpha \mathcal{R} \to f_*(A_2 \otimes_{f^*_{\mathcal{R}}(\alpha)} \mathcal{R}).$$

Part 3. The pseudo-functors DEF and coDEF

9. The bicategory 2-adgalg and deformation pseudo-functor coDEF

Let $\mathcal{E}$ be a bicategory and $F, G : \mathcal{E} \to \mathbf{Gpd}$ two pseudo-functors. A morphism $\epsilon : F \to G$ is called an equivalence if for each $X \in Ob\,\mathcal{E}$ the functor $\epsilon_X : F(X) \to G(X)$ is an equivalence of categories.

Definition 9.1. We define the bicategory 2-adgalg of augmented DG algebras as follows. The objects are augmented DG algebras. For DG algebras $\mathcal{B}, \mathcal{C}$ the collection of 1-morphisms 1-$\operatorname{Hom}(\mathcal{B}, \mathcal{C})$ consists of pairs (M, θ), where

- $M \in D(\mathcal{B}^{op} \otimes \mathcal{C})$ is such that there exists an isomorphism (in $D(\mathcal{C})$) $\mathcal{C} \to \nu_* M$ (where $\nu_* : D(\mathcal{B}^{op} \otimes \mathcal{C}) \to D(\mathcal{C})$ is the functor of restriction of scalars corresponding to the natural homomorphism $\nu : \mathcal{C} \to \mathcal{B}^{op} \otimes \mathcal{C}$); and
- $\theta : k \overset{\mathbf{L}}{\otimes}_{\mathcal{C}} M \to k$ is an isomorphism in $D(\mathcal{B}^{op})$.

The composition of 1-morphisms

$$1\text{-}\operatorname{Hom}(\mathcal{B}, \mathcal{C}) \times 1\text{-}\operatorname{Hom}(\mathcal{C}, \mathcal{D}) \to 1\text{-}\operatorname{Hom}(\mathcal{B}, \mathcal{D})$$

is defined by the tensor product $\cdot \overset{\mathbf{L}}{\otimes}_{\mathcal{C}} \cdot$. Given 1-morphisms $(M_1, \theta_1), (M_2, \theta_2) \in 1\text{-}\operatorname{Hom}(\mathcal{B}, \mathcal{C})$ a 2-morphism $f : (M_1, \theta_1) \to (M_2, \theta_2)$ is an isomorphism (in $D(\mathcal{B}^{op} \otimes \mathcal{C})$) $f : M_2 \to M_1$ (not from M_1 to M_2!) such that $\theta_1 \cdot k \overset{\mathbf{L}}{\otimes}_{\mathcal{C}} (f) = \theta_2$. So in particular the category 1-Hom$(\mathcal{B}, \mathcal{C})$ is a groupoid. Denote by 2-dgart the full subbicategory of 2-adgalg consisting of artinian DG algebras. Similarly we define the full subbicategories 2-dgart$_+$, 2-dgart$_-$, 2-art, 2-cart (I, Definition 2.3).

Remark 9.2. Assume that augmented DG algebras $\mathcal{B}$ and $\mathcal{C}$ are such that $\mathcal{B}^i = \mathcal{C}^i = 0$ for $i > 0$, $\dim \mathcal{B}^i, \dim \mathcal{C}^i < \infty$ for all i and $\dim H(\mathcal{C}) < \infty$. Denote by $\langle k \rangle \subset D(\mathcal{B}^{op} \otimes \mathcal{C})$ the triangulated envelope of the DG $\mathcal{B}^{op} \otimes \mathcal{C}$-module k. Let $(M, \theta) \in$ 1-Hom$(\mathcal{B}, \mathcal{C})$. Then by I, Corollary 3.22 $M \in \langle k \rangle$.

For any augmented DG algebra $\mathcal{B}$ we obtain a pseudo-functor $h_{\mathcal{B}}$ between the bicategories 2-adgalg and **Gpd** defined by $h_{\mathcal{B}}(\mathcal{C}) =$ 1-Hom$(\mathcal{B}, \mathcal{C})$.

Note that a usual homomorphism of augmented DG algebras $\gamma : \mathcal{B} \to \mathcal{C}$ defines the structure of a DG $\mathcal{B}^{op}$-module on $\mathcal{C}$ with the canonical isomorphism of DG $\mathcal{B}^{op}$-modules $\mathrm{id} : k \overset{\mathbf{L}}{\otimes}_{\mathcal{C}} \mathcal{C} \to k$. Thus it defines a 1-morphism $(\mathcal{C}, \mathrm{id}) \in$ 1-Hom$(\mathcal{B}, \mathcal{C})$. This way we get a pseudo-functor $\mathcal{F}$: adgalg $\to$ 2-adgalg, which is the identity on objects.

Lemma 9.3. *Assume that augmented DG algebras $\mathcal{B}$ and $\mathcal{C}$ are concentrated in degree zero (hence have zero differential). Also assume that these algebras are local (with maximal ideals being the augmentation ideals). Then*

a) *the map $\mathcal{F} : \mathrm{Hom}(\mathcal{B}, \mathcal{C}) \to \pi_0(1\text{-}\mathrm{Hom}(\mathcal{B}, \mathcal{C}))$ is surjective, i.e. every 1-morphism from $\mathcal{B}$ to $\mathcal{C}$ is isomorphic to $\mathcal{F}(\gamma)$ for a homomorphism of algebras γ;*
b) *the 1-morphisms $\mathcal{F}(\gamma_1)$ and $\mathcal{F}(\gamma_2)$ are isomorphic if and only if γ_2 is the composition of γ_1 with the conjugation by an invertible element in $\mathcal{C}$;*
c) *in particular, if $\mathcal{C}$ is commutative then the map of sets $\mathcal{F} : \mathrm{Hom}(\mathcal{B}, \mathcal{C}) \to \pi_0(1\text{-}\mathrm{Hom}(\mathcal{B}, \mathcal{C}))$ is a bijection.*

Proof. a) For any $(M, \theta) \in$ 1-Hom$(\mathcal{B}, \mathcal{C})$ the DG $\mathcal{B}^{op} \otimes \mathcal{C}$-module M is isomorphic (in $D(\mathcal{B}^{op} \otimes \mathcal{C})$ to $H^0(M)$. Thus we may assume that M is concentrated in degree 0. By assumption there exists an isomorphism of $\mathcal{C}$-modules $\mathcal{C} \to M$. Multiplying this isomorphism by a scalar we may assume that it is compatible with the isomorphisms $\mathrm{id} : k \overset{\mathbf{L}}{\otimes}_{\mathcal{C}} \mathcal{C} \to k$ and $\theta : k \overset{\mathbf{L}}{\otimes}_{\mathcal{C}} M \to k$. A choice of such an isomorphism defines a homomorphism of algebras $\mathcal{B}^{op} \to \mathrm{End}_{\mathcal{C}}(\mathcal{C}) = \mathcal{C}^{op}$. Since $\mathcal{B}$ and $\mathcal{C}$ are local this is a homomorphism of augmented algebras. Thus (M, θ) is isomorphic to $\mathcal{F}(\gamma)$.

b) Let $\gamma_1, \gamma_2 : \mathcal{B} \to \mathcal{C}$ be homomorphisms of algebras. A 2-morphism $f : \mathcal{F}(\gamma_1) \to \mathcal{F}(\gamma_2)$ is simply an isomorphism of the corresponding $\mathcal{B}^{op} \otimes \mathcal{C}$-modules $f : \mathcal{C} \to \mathcal{C}$, which commutes with the augmentation. Being an isomorphism of $\mathcal{C}$-modules it is the right multiplication by an invertible element $c \in \mathcal{C}$. Hence for every $b \in \mathcal{B}$ we have $c^{-1}\gamma_1(b)c = \gamma_2(b)$.

c) This follows from a) and b). □

Remark 9.4. If in the definition of 1-morphisms 1-Hom$(\mathcal{B}, \mathcal{C})$ we do not fix an isomorphism θ, then we obtain a special case of a "quasi-functor" between the DG categories $\mathcal{B}$-mod and $\mathcal{C}$-mod. This notion was first introduced by Keller in [4] for DG modules over general DG categories.

The next proposition asserts that the deformation functor coDef has a natural “lift” to the bicategory 2-dgart.

Proposition 9.5. *There exist a pseudo-functor* $\mathrm{coDEF}(E)$ *from* 2-dgart *to* **Gpd** *which is an extension to* 2-dgart *of the pseudo-functor* coDef, *i.e. there is an equivalence of pseudo-functors* $\mathrm{coDef}(E) \simeq \mathrm{coDEF}(E) \cdot \mathcal{F}$.

Proof. Given artinian DG algebras $\mathcal{R}$, $\mathcal{Q}$ and $M = (M, \theta) \in 1\text{-}\mathrm{Hom}(\mathcal{R}, \mathcal{Q})$ we need to define the corresponding functor

$$M^! : \mathrm{coDef}_{\mathcal{R}}(E) \to \mathrm{coDef}_{\mathcal{Q}}(E).$$

Let $S = (S, \sigma) \in \mathrm{coDef}_{\mathcal{R}}(E)$. Put

$$M^!(S) := \mathbf{R}\mathrm{Hom}_{\mathcal{R}^{op}}(M, S) \in D\big(\mathcal{A}^{op}_{\mathcal{Q}}\big).$$

We claim that $M^!(S)$ defines an object in $\mathrm{coDef}_{\mathcal{Q}}(E)$, i.e. $\mathcal{R}\mathrm{Hom}_{\mathcal{Q}^{op}}(k, M^!(S))$ is naturally isomorphic to E (by the isomorphisms θ and σ).

Indeed, choose quasi-isomorphisms $P \to k$ and $S \to I$ for $P \in \mathcal{P}(\mathcal{A}^{op}_{\mathcal{Q}})$ and $I \in \mathcal{I}(\mathcal{A}^{op}_{\mathcal{R}})$. Then

$$\mathbf{R}\mathrm{Hom}_{\mathcal{Q}^{op}}\big(k, M^!(S)\big) = \mathrm{Hom}_{\mathcal{Q}^{op}}\big(P, \mathrm{Hom}_{\mathcal{R}^{op}}(M, I)\big).$$

By I, Lemma 3.17 the last term is equal to $\mathrm{Hom}_{\mathcal{R}^{op}}(P \otimes_{\mathcal{Q}} M, I)$. Now the isomorphism θ defines an isomorphism between $P \otimes_{\mathcal{Q}} M = k \overset{\mathbf{L}}{\otimes}_{\mathcal{Q}} M$ and k, and we compose it with the isomorphism $\sigma : E \to \mathbf{R}\mathrm{Hom}_{\mathcal{R}^{op}}(k, I) = i^! S$.

So $M^!$ is a functor from $\mathrm{coDef}_{\mathcal{R}}(E)$ to $\mathrm{coDef}_{\mathcal{Q}}(E)$.

Given another artinian DG algebra $\mathcal{Q}'$ and $M' \in 1\text{-}\mathrm{Hom}(\mathcal{Q}, \mathcal{Q}')$ there is a natural isomorphism of functors

$$\Big(M' \overset{\mathbf{L}}{\otimes}_{\mathcal{Q}} M\Big)^!(-) \simeq M'^! \cdot M^!(-).$$

(This follows again from I, Lemma 3.17.)

Also a 2-morphism $f \in 2\text{-}\mathrm{Hom}(M, M_1)$ between objects $M, M_1 \in 1\text{-}\mathrm{Hom}(\mathcal{R}, \mathcal{Q})$ induces an isomorphism of the corresponding functors $M^! \xrightarrow{\sim} M_1^!$.

Thus we obtain a pseudo-functor $\mathrm{coDEF}(E) : 2\text{-dgart} \to \mathbf{Gpd}$, such that $\mathrm{coDEF}(E) \cdot \mathcal{F} = \mathrm{coDef}(E)$. $\square$

We denote by $\mathrm{coDEF}_+(E)$, $\mathrm{coDEF}_-(E)$, $\mathrm{coDEF}_0(E)$, $\mathrm{coDEF}_{\mathrm{cl}}(E)$ the restriction of the pseudo-functor $\mathrm{coDEF}(E)$ to subbicategories 2-dgart_+, 2-dgart_-, 2-art and 2-cart respectively.

Proposition 9.6. *A quasi-isomorphism* $\delta : E_1 \to E_2$ *of DG* $\mathcal{A}^{op}$*-modules induces an equivalence of pseudo-functors*

$$\delta^* : \mathrm{coDEF}(E_2) \to \mathrm{coDEF}(E_1)$$

defined by $\delta^*(S, \sigma) = (S, \sigma \cdot \delta)$.

Proof. This is clear. □

Proposition 9.7. *Let $F : \mathcal{A} \to \mathcal{A}'$ be a DG functor which induces a quasi-equivalence $F^{\text{pre-tr}} : \mathcal{A}^{\text{pre-tr}} \to \mathcal{A}'^{\text{pre-tr}}$ (this happens for example if F is a quasi-equivalence). Then for any $E \in D(\mathcal{A}^{op})$ the pseudo-functors* $\operatorname{coDEF}_-(E)$ *and* $\operatorname{coDEF}_-(\mathbf{R}F^!(E))$ *are equivalent (hence also* $\operatorname{coDEF}(F_*(E'))$ *and* $\operatorname{coDEF}_-(E')$ *are equivalent for any $E' \in D(\mathcal{A}'^0)$).*

Proof. The proof is similar to the proof of I, Proposition 10.11. Namely let $R, Q \in 2\text{-}\operatorname{dgart}_-$ and $M \in 1\text{-}\operatorname{Hom}(R, Q)$. The DG functor $F^!$ induces a commutative functorial diagram

$$\begin{array}{ccc} D(\mathcal{A}^{op}_{\mathcal{R}}) & \xrightarrow{\mathbf{R}(F\otimes \mathrm{id})^!} & D(\mathcal{A}'^0_{\mathcal{R}}) \\ \Big\downarrow{\scriptstyle \mathbf{R}i^!} & & \Big\downarrow{\scriptstyle \mathbf{R}i^!} \\ D(\mathcal{A}^{op}) & \xrightarrow{\mathbf{R}F^!} & D(\mathcal{A}'^0) \end{array}$$

(and a similar diagram for $\mathcal{Q}$ instead of $\mathcal{R}$) which is compatible with the functors

$$M^! : D\big(\mathcal{A}^{op}_{\mathcal{R}}\big) \to D\big(\mathcal{A}^{op}_{\mathcal{Q}}\big) \quad \text{and} \quad M^! : D\big(\mathcal{A}'^0_{\mathcal{R}}\big) \to D\big(\mathcal{A}'^0_{\mathcal{Q}}\big).$$

Thus we obtain a morphism of pseudo-functors

$$F^! : \operatorname{coDEF}_-(E) \to \operatorname{coDEF}_-\big(\mathbf{R}F^!(E)\big).$$

By I, Corollary 3.15 the functors $\mathbf{R}F^!$ and $\mathbf{R}(F \otimes \mathrm{id})^!$ are equivalences. □

Corollary 9.8. *Assume that DG algebras $\mathcal{B}$ and $\mathcal{C}$ are quasi-isomorphic. Then the pseudo-functors* $\operatorname{coDEF}_-(\mathcal{B})$ *and* $\operatorname{coDEF}_-(\mathcal{C})$ *are equivalent.*

Proof. We may assume that there exists a homomorphism of DG algebras $\mathcal{B} \to \mathcal{C}$ which is a quasi-isomorphism. Then put $\mathcal{A} = \mathcal{B}$ and $\mathcal{A}' = \mathcal{C}$ in the last proposition. □

The following lemma is stronger then I, Corollary 11.15 for the pseudo-functors coDef_- and $\operatorname{coDef}^{\mathrm{h}}_-$.

Lemma 9.9. *Let $\mathcal{B}$ be a DG algebra. Suppose that the following conditions hold:*

a) $H^{-1}(\mathcal{B}) = 0$;
b) *the graded algebra $H(\mathcal{B})$ is bounded below.*

Then the pseudo-functors $\operatorname{coDef}_-(\mathcal{B})$ *and* $\operatorname{coDef}^{\mathrm{h}}_-(\mathcal{B})$ *are equivalent.*

Proof. Fix some negative artinian DG algebra $\mathcal{R} \in \operatorname{dgart}_-$. Take some $(T, \mathrm{id}) \in \operatorname{coDef}^{\mathrm{h}}_{\mathcal{R}}(\mathcal{B})$. Due to I, Corollary 11.4b) it suffices to prove that $i^! T = \mathbf{R}i^! T$. Let A be a strictly unital minimal model of $\mathcal{B}$, and let $f : A \to \mathcal{B}$ be a strictly unital A_∞ quasi-isomorphism. By our assumption on $H(\mathcal{B})$, A is bounded below.

By Theorem 7.2 there exists an object $\alpha \in \mathcal{MC}_{\mathcal{R}}(A)$ such that $S \cong \mathcal{B} \otimes_{f^*_{\mathcal{R}}(\alpha)} \mathcal{R}^*$. The DG $\mathcal{R}^{op}$-modules $\mathcal{B} \otimes_{f^*_{\mathcal{R}}(\alpha)} \mathcal{R}^*$ and $f_*(\mathcal{B} \otimes_{f^*_{\mathcal{R}}(\alpha)} \mathcal{R}^*)$ are naturally identified. Further, by Proposition 8.6 we have natural homotopy equivalence (in $A^{op}_{\mathcal{R}^{op}}\text{-mod}_\infty$)

$$\gamma : A \otimes_\alpha \mathcal{R}^* \to f_*\big(\mathcal{B} \otimes_{f^*_{\mathcal{R}}(\alpha)} \mathcal{R}^*\big).$$

Thus, it remains to prove that

$$i^!\big(A \otimes_\alpha \mathcal{R}^*\big) = \mathbf{R}i^!\big(A \otimes_\alpha \mathcal{R}^*\big).$$

We claim that $A \otimes_\alpha \mathcal{R}^*$ is h-injective. Indeed, since A is bounded below and $\mathcal{R} \in \text{dgart}_-$, this DG $\mathcal{R}^{op}$-module has a decreasing filtration by DG $\mathcal{R}^{op}$-submodules $A^{\geqslant i} \otimes \mathcal{R}^*$ with subquotients being cofree DG $\mathcal{R}^{op}$-modules $A^i \otimes \mathcal{R}^*$. Thus $A \otimes_\alpha \mathcal{R}^*$ satisfies property (I) as DG $\mathcal{R}^{op}$-module and hence is h-injective. Lemma is proved. □

The next result implies stronger statement for pseudo-functor coDef_- then I, Proposition 11.16.

Proposition 9.10. *Let* $E \in \mathcal{A}^{op}\text{-mod}$. *Assume that*

a) $\text{Ext}^{-1}(E, E) = 0$;
b) *the graded algebra* $\text{Ext}(E, E)$ *is bounded below;*
c) *there exists a bounded below h-projective or h-injective DG* $\mathcal{A}^{op}$*-module* F *which is quasi-isomorphic to* E.

Put $\mathcal{B} = \text{End}(F)$. *Then the pseudo-functors* $\text{coDEF}_-(\mathcal{B})$ *and* $\text{coDEF}_-(E)$ *are equivalent.*

Proof. Consider the DG functor

$$\mathcal{L} := \Sigma^F \cdot \psi^* : \mathcal{B}^{op}\text{-mod} \to \mathcal{A}^{op}\text{-mod}, \qquad \mathcal{L}(N) = N \otimes_{\mathcal{C}} F$$

as in I, Remark 11.17. It induces the equivalence of pseudo-functors

$$\text{coDef}^{\text{h}}(\mathcal{L}) : \text{coDef}^{\text{h}}_-(\mathcal{B}) \xrightarrow{\sim} \text{coDef}^{\text{h}}_-(F),$$

i.e. for every artinian DG algebra $\mathcal{R} \in \text{dgart}_-$ the corresponding DG functor

$$\mathcal{L}_{\mathcal{R}} : (\mathcal{B} \otimes \mathcal{R})^{op}\text{-mod} \to \mathcal{A}^{op}_{\mathcal{R}}\text{-mod}$$

induces the equivalence of groupoids $\text{coDef}^{\text{h}}_{\mathcal{R}}(\mathcal{B}) \xrightarrow{\sim} \text{coDef}^{\text{h}}_{\mathcal{R}}(F)$ (I, Propositions 9.2, 9.4). By I, Theorem 11.6b) there is a natural equivalences of pseudo-functors

$$\text{coDef}^{\text{h}}_-(F) \simeq \text{coDef}_-(E).$$

By Lemma 9.9 there is an equivalence of pseudo-functors

$$\text{coDef}^{\text{h}}_-(\mathcal{B}) \simeq \text{coDef}_-(\mathcal{B}).$$

Hence the functor $\mathbf{L}\mathcal{L}$ induces the equivalence

$$\mathbf{L}\mathcal{L}:\mathrm{coDef}_-(\mathcal{B})\xrightarrow{\sim}\mathrm{coDef}_-(E).$$

Fix $\mathcal{R},\mathcal{Q}\in 2\text{-}\mathrm{dgart}\_$ and $M\in 1\text{-}\mathrm{Hom}(\mathcal{R},\mathcal{Q})$. We need to show that there exists a natural isomorphism

$$\mathbf{L}\mathcal{L}_{\mathcal{Q}}\cdot M^!\simeq M^!\cdot\mathbf{L}\mathcal{L}_{\mathcal{R}}$$

between functors from $\mathrm{coDef}_{\mathcal{R}}(\mathcal{B})$ to $\mathrm{coDef}_{\mathcal{Q}}(E)$.

Since the cohomology of M is finite-dimensional, and the DG algebra $\mathcal{R}\otimes\mathcal{Q}$ has no components in positive degrees, by I, Corollary 3.21 we may assume that M is finite-dimensional.

Lemma 9.11. *Let* (S,id) *be an object in* $\mathrm{coDef}^{\mathrm{h}}_{\mathcal{R}}(\mathcal{B})$ *or in* $\mathrm{coDef}^{\mathrm{h}}_{\mathcal{R}}(F)$. *Then S is acyclic for the functor* $\mathrm{Hom}_{\mathcal{R}^{op}}(M,S)$, *i.e.* $M^!(S)=\mathrm{Hom}_{\mathcal{R}^{op}}(M,S)$.

Proof. In the proof of Lemma 9.9 (resp. in I, Lemma 11.8) we showed that S is h-injective when considered as a DG $\mathcal{R}^{op}$-module. □

Choose $(S,\mathrm{id})\in\mathrm{coDef}^{\mathrm{h}}(\mathcal{B})$. By the above lemma $M^!(S)=\mathrm{Hom}_{\mathcal{R}^{op}}(M,S)$.

We claim that the DG $\mathcal{B}^{op}$-module $\mathrm{Hom}_{\mathcal{R}^{op}}(M,S)$ is h-projective. Indeed, first notice that the graded $\mathcal{R}^{op}$-module S is injective being isomorphic to a direct sum of copies of shifted graded $\mathcal{R}^{op}$-module $\mathcal{R}^*$ (the abelian category of graded $\mathcal{R}^{op}$-modules is locally notherian, hence a direct sum of injectives is injective). Second, the DG $\mathcal{R}^{op}$-module M has a (finite) filtration with subquotients isomorphic to k. Thus the DG $\mathcal{B}^{op}$-module $\mathrm{Hom}_{\mathcal{R}^{op}}(M,S)$ has a filtration with subquotients isomorphic to $\mathrm{Hom}_{\mathcal{R}^{op}}(k,S)=i^!S\simeq\mathcal{B}$. So it has property (P).

Hence $\mathbf{L}\mathcal{L}\cdot M^!(S)=\mathrm{Hom}_{\mathcal{R}^{op}}(M,S)\otimes_{\mathcal{B}}F$. For the same reasons $M^!\cdot\mathbf{L}\mathcal{L}_{\mathcal{R}}(S)=\mathrm{Hom}_{\mathcal{R}^{op}}(M,S\otimes_{\mathcal{B}}F)$. The isomorphism

$$\mathrm{Hom}_{\mathcal{R}^{op}}(M,S)\otimes_{\mathcal{B}}F=\mathrm{Hom}_{\mathcal{R}^{op}}(M,S\otimes_{\mathcal{B}}F)$$

follows from the fact that S as a graded module is a tensor product of graded $\mathcal{C}^{op}$ and $\mathcal{R}^{op}$ modules and also because $\dim_k M<\infty$. □

10. Deformation pseudo-functor coDEF for an augmented A_∞-algebra

Let A be an augmented A_∞-algebra. We are going to define the pseudo-functor $\mathrm{coDEF}(A):2\text{-}\mathrm{dgart}\to\mathbf{Gpd}$.

Let $\mathcal{R}$ be an artinian DG algebra. An object of the groupoid $\mathrm{coDEF}_{\mathcal{R}}(A)$ is a pair (S,σ), where $S\in D_\infty(\bar{A}^{op}_{\mathcal{R}^{op}})$, and σ is an isomorphism (in $D_\infty(\bar{A}^{op})$)

$$\sigma:A\to\mathbf{R}i^!(S).$$

A morphism $f:(S,\sigma)\to(T,\tau)$ in $\mathrm{coDEF}_{\mathcal{R}}(A)$ is an isomorphism (in $D(\bar{A}^{op}_{\mathcal{R}^{op}})$) $f:S\to T$ such that

$$\mathbf{R}i^!(f)\circ\sigma=\tau.$$

This defines the pseudo-functor coDEF(A) on objects. Further, let $(M,\theta) \in$ 1-Hom$(\mathcal{R},\mathcal{Q})$. Define the corresponding functor

$$M^! : \mathrm{coDEF}_{\mathcal{R}}(A) \to \mathrm{coDEF}_{\mathcal{Q}}(A)$$

as follows. For an object $(S,\sigma) \in \mathrm{coDEF}_{\mathcal{R}}(A)$ put

$$M^!(S) = \mathbf{R}\mathrm{Hom}_{\mathcal{R}^{op}}(M,S) \in D_\infty\big(\bar{A}^{op}_{\mathcal{Q}^{op}}\big).$$

Then we have natural isomorphisms in $D_\infty(\bar{A})$:

$$\mathbf{R}\mathrm{Hom}_{\mathcal{Q}^{op}}\big(k, M^!(S)\big) \cong \mathbf{R}\mathrm{Hom}_{\mathcal{R}^{op}}\Big(k \overset{\mathbf{L}}{\otimes}_{\mathcal{R}^{op}} M, S\Big) \cong \mathbf{R}\mathrm{Hom}_{\mathcal{R}^{op}}(k,S) = \mathbf{R}i^!(S)$$

(the second isomorphism is induced by θ). Thus, $M^!$ is a functor form $\mathrm{coDEF}_{\mathcal{R}}(A)$ to $\mathrm{coDEF}_{\mathcal{Q}}(A)$.

If $\mathcal{Q}'$ is another artinian DG algebra and $(M',\theta') \in$ 1-Hom$(\mathcal{Q},\mathcal{Q}')$ then there is a natural isomorphism of functors

$$\Big(M' \overset{\mathbf{L}}{\otimes}_{\mathcal{Q}} M\Big)^! \cong M'^! \cdot M^!.$$

Further, if $f \in$ 2-Hom$((M,\theta),(M,\theta_1))$ is a 2-morphism between objects $(M,\theta),(M,\theta_1) \in$ 1-Hom$(\mathcal{R},\mathcal{Q})$ then it induces an isomorphism between the corresponding functors $M^! \simeq M_1^!$.

Thus we obtain a pseudo-functor coDEF(A) : 2-dgart $\to$ **Gpd**. We denote by coDEF$_-(A)$ its restriction to the sub-2-category 2-dgart$_-$.

Proposition 10.1. *Let A be an augmented A_∞-algebra and $U(A)$ its bar–cobar construction. Then there is a natural equivalence of pseudo-functors* coDEF$(U(A)) \cong$ coDEF(A).

Proof. Let $f_A : A \to U(A)$ be the universal strictly unital A_∞-morphism. Let $\mathcal{R}$ be an artinian DG algebra. Recall that by Proposition 3.14 we have an equivalence

$$f_{A*} : D\big((U(A)\otimes\mathcal{R})^{op}\big) \to D_\infty\big(\bar{A}^{op}_{\mathcal{R}^{op}}\big).$$

Moreover, the following diagram of functors commutes up to an isomorphism:

$$\begin{array}{ccc} D((U(A)\otimes\mathcal{R})^{op})D_\infty & \xrightarrow{f_{A*}} & (\bar{A}^{op}_{\mathcal{R}^{op}}) \\ \downarrow{\scriptstyle \mathbf{R}i^!} & & \downarrow{\scriptstyle \mathbf{R}i^!} \\ D(U(A)^{op}) & \xrightarrow{f_{A*}} & D_\infty(\bar{A}^{op}). \end{array}$$

Hence, the functor f_{A*} induces an equivalence of groupoids $\mathrm{coDEF}_{\mathcal{R}}(U(A)) \to \mathrm{coDEF}_{\mathcal{R}}(A)$ and we obtain the required equivalence of pseudo-functors. □

Corollary 10.2. *Let A be an augmented A_∞-algebra and let $\mathcal{B}$ be a DG algebra quasi-isomorphic to A. Then the pseudo-functor* coDEF(A) *and* coDEF$(\mathcal{B})$ *are equivalent.*

Proof. Indeed, by Proposition 10.1 the pseudo-functors coDEF(A) and DEF($U(A)$) are equivalent, and by Corollary 9.8 the pseudo-functors coDEF($U(A)$) and coDEF($\mathcal{B}$) are equivalent. □

Corollary 10.3. *Let A be an admissible A_∞-algebra, and $\mathcal{R}$ be an artinian negative DG algebra. Then for any $(S,\sigma) \in \mathrm{coDEF}_{\mathcal{R}}(A)$ there exists a morphism of DG algebras $\hat{S} \to \mathcal{R}$ such that the pair (T, id), where $T = \mathrm{Hom}_{\hat{S}^{op}}(\mathcal{R}, B\bar{A} \otimes_{\tau_A} A)$, defines an object of $\mathrm{coDEF}_{\mathcal{R}}(A)$ which is isomorphic to (S,σ).*

Proof. This follows easily from Proposition 10.1, the proof of Lemma 9.9 in the case $\mathcal{B} = U(A)$, and Lemma 8.4. □

11. The bicategory 2′-adgalg and deformation pseudo-functor DEF

It turns out that the deformation pseudo-functor Def lifts naturally to a different version of a bicategory of augmented DG algebras. We denote this bicategory 2′-adgalg. It differs from 2-adgalg in two respects: the 1-morphisms are objects in $D(\mathcal{B} \otimes \mathcal{C}^{op})$ (instead of $D(\mathcal{B}^{op} \otimes \mathcal{C})$) and 2-morphisms go in the opposite direction. We will relate the bicategories 2-adgalg and 2′-adgalg (and the pseudo-functors coDEF and DEF) in Section 13 below.

Definition 11.1. We define the bicategory 2′-adgalg of augmented DG algebras as follows. The objects are augmented DG algebras. For DG algebras $\mathcal{B}, \mathcal{C}$ the collection of 1-morphisms 1-Hom($\mathcal{B}, \mathcal{C}$) consists of pairs (M, θ), where

- $M \in D(\mathcal{B} \otimes \mathcal{C}^{op})$ and there exists an isomorphism (in $D(\mathcal{C}^{op})$) $\mathcal{C} \to \nu_* M$ (where $\nu_* : D(\mathcal{B} \otimes \mathcal{C}^{op}) \to D(\mathcal{C}^{op})$ is the functor of restriction of scalars corresponding to the natural homomorphism $\nu : \mathcal{C}^{op} \to \mathcal{B} \otimes \mathcal{C}^{op}$); and
- $\theta : M \overset{\mathbf{L}}{\otimes}_{\mathcal{C}} k \to k$ is an isomorphism in $D(\mathcal{B})$.

The composition of 1-morphisms

$$1\text{-}\mathrm{Hom}(\mathcal{B}, \mathcal{C}) \times 1\text{-}\mathrm{Hom}(\mathcal{C}, \mathcal{D}) \to 1\text{-}\mathrm{Hom}(\mathcal{B}, \mathcal{D})$$

is defined by the tensor product $\cdot \overset{\mathbf{L}}{\otimes}_{\mathcal{C}} \cdot$. Given 1-morphisms $(M_1, \theta_1), (M_2, \theta_2) \in 1\text{-}\mathrm{Hom}(\mathcal{B}, \mathcal{C})$ a 2-morphism $f : (M_1, \theta_1) \to (M_2, \theta_2)$ is an isomorphism (in $D(\mathcal{B} \otimes \mathcal{C}^{op})$) $f : M_1 \to M_2$ such that $\theta_1 = \theta_2 \cdot ((f) \overset{\mathbf{L}}{\otimes}_{\mathcal{C}} k)$. So in particular the category 1-Hom($\mathcal{B}, \mathcal{C}$) is a groupoid. Denote by 2′-dgart the full subbicategory of 2′-adgalg consisting of artinian DG algebras. Similarly we define the full subbicategories 2′-dgart$_+$, 2′-dgart$_-$, 2′-art, 2′-cart (I, Definition 2.3).

Remark 11.2. The exact analogue of Remark 9.2 holds for the bicategory 2′-adgalg.

For any augmented DG algebra $\mathcal{B}$ we obtain a pseudo-functor $h'_{\mathcal{B}}$ between the bicategories 2′-adgalg and **Gpd** defined by $h'_{\mathcal{B}}(\mathcal{C}) = 1\text{-}\mathrm{Hom}(\mathcal{B}, \mathcal{C})$.

Note that a usual homomorphism of DG algebras $\gamma : \mathcal{B} \to \mathcal{C}$ defines the structure of a $\mathcal{B}$-module on $\mathcal{C}$ with the canonical isomorphism of DG $\mathcal{B}$-modules $\mathcal{C} \overset{\mathbf{L}}{\otimes}_{\mathcal{C}} k$. Thus it defines a 1-morphism $(\mathcal{C}, \mathrm{id}) \in 1\text{-}\mathrm{Hom}(\mathcal{B}, \mathcal{C})$. This way we get a pseudo-functor $\mathcal{F}' : \mathrm{adgalg} \to$ 2′-adgalg, which is the identity on objects.

Remark 11.3. The precise analogue of Lemma 9.3 holds for the bicategory $2'$-adgalg and the pseudo-functor $\mathcal{F}'$.

Proposition 11.4. *There exist a pseudo-functor* $\mathrm{DEF}(E)$ *from* $2'$-dgart *to* **Gpd** *and which is an extension to* $2'$-dgart *of the pseudo-functor* $\mathrm{Def}(E)$*, i.e. there is an equivalence of pseudo-functors* $\mathrm{Def}(E) \simeq \mathrm{DEF}(E) \cdot \mathcal{F}'$.

Proof. Let $\mathcal{R}$, $\mathcal{Q}$ be artinian DG algebras. Given $(M, \theta) \in 1\text{-}\mathrm{Hom}(\mathcal{R}, \mathcal{Q})$ we define the corresponding functor

$$M^* : \mathrm{Def}_{\mathcal{R}}(E) \to \mathrm{Def}_{\mathcal{Q}}(E)$$

as follows

$$M^*(S) := S \overset{\mathbf{L}}{\otimes}_{\mathcal{R}} M$$

for $(S, \sigma) \in \mathrm{Def}_{\mathcal{R}}(E)$. Then we have the canonical isomorphism

$$M^*(S) \overset{\mathbf{L}}{\otimes}_{\mathcal{Q}} k = S \overset{\mathbf{L}}{\otimes}_{\mathcal{R}} \left(M \overset{\mathbf{L}}{\otimes}_{\mathcal{Q}} k \right) \xrightarrow{\theta} S \overset{\mathbf{L}}{\otimes}_{\mathcal{R}} k \xrightarrow{\sigma} E.$$

So that $M^*(S) \in \mathrm{Def}_{\mathcal{Q}}(E)$ indeed.

Given another artinian DG algebra $\mathcal{Q}'$ and $M' \in 1\text{-}\mathrm{Hom}(Q, Q')$ there is a natural isomorphism of functors

$$M'^* \cdot M^* = \left(M \overset{\mathbf{L}}{\otimes}_{\mathcal{Q}} M' \right)^*.$$

Also a 2-morphism $f \in 2\text{-}\mathrm{Hom}(M, M_1)$ between $M, M_1 \in 1\text{-}\mathrm{Hom}(\mathcal{R}, \mathcal{Q})$ induces an isomorphism of corresponding functors $M^* \xrightarrow{\sim} M_1^*$.

Thus we obtain a pseudo-functor $\mathrm{DEF}(E) : 2'\text{-dgart} \to \mathbf{Gpd}$, such that $\mathrm{DEF}(E) \cdot \mathcal{F}' = \mathrm{Def}(E)$. □

We denote by $\mathrm{DEF}_+(E)$, $\mathrm{DEF}_-(E)$, $\mathrm{DEF}_0(E)$, $\mathrm{DEF}_{\mathrm{cl}}(E)$ the restriction of the pseudo-functor $\mathrm{DEF}(E)$ to subbicategories $2'$-dgart_+, $2'$-dgart_-, $2'$-art and $2'$-cart respectively.

Proposition 11.5. *A quasi-isomorphism* $\delta : E_1 \to E_2$ *of DG* $\mathcal{A}^{op}$*-modules induces an equivalence of pseudo-functors*

$$\delta_* : \mathrm{DEF}(E_1) \to \mathrm{DEF}(E_2)$$

defined by $\delta_*(S, \sigma) = (S, \delta \cdot \sigma)$.

Proof. This is clear. □

Proposition 11.6. *Let* $F : \mathcal{A} \to \mathcal{A}'$ *be a DG functor which induces a quasi-equivalence* $F^{\text{pre-tr}} : \mathcal{A}^{\text{pre-tr}} \to \mathcal{A}'^{\text{pre-tr}}$ *(this happens for example if* F *is a quasi-equivalence). Then for any* $E \in D(\mathcal{A}^{op})$ *the pseudo-functors* $\mathrm{DEF}_-(E)$ *and* $\mathrm{DEF}_-(\mathbf{L}F^*(E))$ *are equivalent (hence also* $\mathrm{DEF}_-(F_*(E'))$ *and* $\mathrm{DEF}_-(E')$ *are equivalent for any* $E' \in D(\mathcal{A}'^0)$*).*

Proof. The proof is similar to the proof of I, Proposition 10.4. Let $\mathcal{R}, \mathcal{Q} \in \mathrm{dgart}_-$ and $M \in$ 1-$\mathrm{Hom}(\mathcal{R}, \mathcal{Q})$. The DG functor F induces a commutative functorial diagram

$$\begin{array}{ccc} D(\mathcal{A}^{op}_{\mathcal{R}}) & \xrightarrow{\mathbf{L}(F\otimes \mathrm{id})^*} & D(\mathcal{A}'^{0}_{\mathcal{R}}) \\ \Big\downarrow{\scriptstyle \mathbf{L}i^*} & & \Big\downarrow{\scriptstyle \mathbf{L}i^*} \\ D(\mathcal{A}^{op}) & \xrightarrow{\mathbf{L}F^*} & D(\mathcal{A}'^{0}) \end{array}$$

(and a similar diagram for $\mathcal{Q}$ instead of $\mathcal{R}$) which is compatible with the functors

$$M^* : D\big(\mathcal{A}^{op}_{\mathcal{R}}\big) \to D\big(\mathcal{A}^{op}_{\mathcal{Q}}\big) \quad \text{and} \quad M^* : D\big(\mathcal{A}'^{0}_{\mathcal{R}}\big) \to D\big(\mathcal{A}'^{0}_{\mathcal{Q}}\big).$$

Thus we obtain a morphism of pseudo-functors

$$F^* : \mathrm{DEF}_-(E) \to \mathrm{DEF}_-\big(\mathbf{L}F^*(E)\big).$$

By I, Corollary 3.15 the functors $\mathbf{L}F^*$ and $\mathbf{L}(F \otimes \mathrm{id})^*$ are equivalences, hence this morphism F^* is an equivalence. □

Corollary 11.7. *Assume that DG algebras $\mathcal{B}$ and $\mathcal{C}$ are quasi-isomorphic. Then the pseudo-functors* $\mathrm{DEF}_-(\mathcal{B})$ *and* $\mathrm{DEF}_-(\mathcal{C})$ *are equivalent.*

Proof. We may assume that there exists a morphism of DG algebras $\mathcal{B} \to \mathcal{C}$ which is a quasi-isomorphism. Then put $\mathcal{A} = \mathcal{B}$ and $\mathcal{A}' = \mathcal{C}$ in the last proposition. □

The following theorem is stronger then I, Corollary 11.15 for the pseudo-functors Def_- and $\mathrm{Def}^{\mathrm{h}}_-$.

Theorem 11.8. *Let $E \in \mathcal{A}^{op}$-*mod *be a DG module. Suppose that the following conditions hold:*

a) $\mathrm{Ext}^{-1}(E, E) = 0$*;*
b) *the graded algebra* $\mathrm{Ext}(E, E)$ *is bounded above.*

Let $F \to E$ be a quasi-isomorphism with h-projective F. Then the pseudo-functors $\mathrm{Def}_-(E)$ *and* $\mathrm{Def}^{\mathrm{h}}_-(F)$ *are equivalent.*

Proof. Replace the pseudo-functor $\mathrm{Def}(E)$ by the equivalent pseudo-functor $\mathrm{Def}(F)$. Fix some negative artinian DG algebra $\mathcal{R} \in \mathrm{dgart}_-$.

Due to I, Corollary 11.4a) it suffices to prove that for each $(S, \mathrm{id}) \in \mathrm{Def}^{\mathrm{h}}_{\mathcal{R}}(F)$ one has $i^*(S) = \mathbf{L}i^*(S)$. Consider the DG algebra $\mathcal{B} = \mathrm{End}(F)$. First we will prove the following special case:

Lemma 11.9. *The pseudo-functors* $\mathrm{Def}_-(\mathcal{B})$ *and* $\mathrm{Def}^{\mathrm{h}}_-(\mathcal{B})$ *are equivalent.*

Proof. Take some $(S, \sigma) \in \mathrm{Def}^{\mathrm{h}}(\mathcal{B})$. Let A be a strictly unital minimal model of $\mathcal{B}$, and let $f : A \to \mathcal{B}$ be a strictly unital A_∞ quasi-isomorphism. By our assumption on $Ext(E, E) \cong H(\mathcal{B})$, A is bounded above.

By Theorem 7.2 there exists an object $\alpha \in \mathcal{MC}_{\mathcal{R}}(A)$ such that $S \cong \mathcal{B} \otimes_{f^*_{\mathcal{R}}(\alpha)} \mathcal{R}$. The DG $\mathcal{R}^{op}$-modules $\mathcal{B} \otimes_{f^*_{\mathcal{R}}(\alpha)} \mathcal{R}$ and $f_*(\mathcal{B} \otimes_{f^*_{\mathcal{R}}(\alpha)} \mathcal{R})$ are naturally identified. Further, by Proposition 8.6 we have natural homotopy equivalence (in $A^{op}_{\mathcal{R}^{op}}\text{-mod}_\infty$)

$$\gamma : \mathcal{R} \otimes_\alpha A \to f_*(\mathcal{B} \otimes_{f^*(\alpha)} \mathcal{R}).$$

Thus, it remains to prove that

$$i^*(A \otimes_\alpha \mathcal{R}) = \mathbf{L}i^*(A \otimes_\alpha \mathcal{R}).$$

We claim that $\mathcal{R} \otimes_\alpha A$ is h-projective. Indeed, since A is bounded above and $\mathcal{R} \in \text{dgart}_-$, this DG $\mathcal{R}^{op}$-module has an increasing filtration by DG $\mathcal{R}^{op}$-submodules $A^{\geqslant i} \otimes \mathcal{R}$ with subquotients being free DG $\mathcal{R}^{op}$-modules $A^i \otimes \mathcal{R}$. Thus $A \otimes_\alpha \mathcal{R}$ satisfies property (P) as DG $\mathcal{R}^{op}$-module and hence is h-projective. Lemma is proved. □

Now take some $(S, \text{id}) \in \text{Def}^{\text{h}}(F)$. We claim that S is h-projective. Recall the DG functor

$$\Sigma_{\mathcal{R}} : (\mathcal{B} \otimes \mathcal{R})^{op}\text{-mod} \to \mathcal{A}^{op}_{\mathcal{R}}\text{-mod}, \quad \Sigma(M) = M \otimes_{\mathcal{B}} F.$$

From I, Proposition 9.2e) we know that $S \cong \Sigma_{\mathcal{R}}(S')$ for some $(S', \text{id}) \in \text{Def}^{\text{h}}(\mathcal{B})$. By the above lemma and I, Proposition 11.2, DG $(\mathcal{B} \otimes \mathcal{R})^{op}$-module S' is h-projective. Since the DG functor $\Sigma_{\mathcal{R}}$ preserves h-projectives, it follows that S is also h-projective. Theorem is proved. □

The next proposition is the analogue of Proposition 9.10 for the pseudo-functor DEF_-. Note that here we do not need boundedness assumptions on the h-projective DG module.

Proposition 11.10. *Let $E \in \mathcal{A}^{op}\text{-mod}$ be a DG module. Suppose that the following conditions hold:*

a) $\text{Ext}^{-1}(E, E) = 0$*;*
b) *the graded algebra* $\text{Ext}(E, E)$ *is bounded above.*

Put $\mathcal{B} = \mathbf{R}\text{Hom}(E, E)$. Then pseudo-functors $\text{DEF}_-(\mathcal{B})$ and $\text{DEF}_-(E)$ are equivalent.

Proof. Take some h-projective F quasi-isomorphic to E and replace $\text{DEF}_-(E)$ by the equivalent pseudo-functor $\text{DEF}_-(F)$. We may assume that $\mathcal{B} = \text{End}(F)$.

By I, Proposition 9.2e) the DG functor $\Sigma = \Sigma^F : \mathcal{B}^{op}\text{-mod} \to \mathcal{A}^{op}\text{-mod}$, $\Sigma(N) = N \otimes_{\mathcal{B}} F$ induces an equivalence of pseudo-functors

$$\text{Def}^{\text{h}}(\Sigma) : \text{Def}^{\text{h}}(\mathcal{B}) \to \text{Def}^{\text{h}}(F).$$

By Lemma 11.8 we have that the pseudo-functors $\text{Def}_-(F)$ and $\text{Def}^{\text{h}}_-(F)$ (resp. $\text{Def}_-(\mathcal{B})$ and $\text{Def}^{\text{h}}_-(\mathcal{B})$) are equivalent. We conclude that Σ also induces an equivalence of pseudo-functors

$$\text{Def}_-(\Sigma) : \text{Def}_-(\mathcal{B}) \to \text{Def}_-(F).$$

Let us prove that it extends to an equivalence

$$\mathrm{DEF}_-(\Sigma):\mathrm{DEF}_-(\mathcal{B})\to\mathrm{DEF}_-(F).$$

Let $\mathcal{R},\mathcal{Q}\in\mathrm{dgart}_-$, $M\in 1\text{-}\mathrm{Hom}(\mathcal{R},\mathcal{Q})$. We need to show that the functorial diagram

$$\begin{array}{ccc}\mathrm{DEF}_{\mathcal{R}}(\mathcal{B}) & \xrightarrow{\mathrm{DEF}_{\mathcal{R}}(\Sigma)} & \mathrm{DEF}_{\mathcal{R}}(F)\\ {\scriptstyle M^*}\downarrow & & \downarrow{\scriptstyle M^*}\\ \mathrm{DEF}_{\mathcal{Q}}(\mathcal{B}) & \xrightarrow{\mathrm{DEF}_{\mathcal{Q}}(\Sigma)} & \mathrm{DEF}_{\mathcal{Q}}(F)\end{array}$$

commutes. This follows from the natural isomorphism

$$N\otimes_{\mathcal{B}}F\otimes_{\mathcal{R}}M\cong N\otimes_{\mathcal{R}}M\otimes_{\mathcal{B}}F. \qquad \square$$

12. Deformation pseudo-functor DEF for an augmented A_∞-algebra

Let A be an augmented A_∞-algebra. We are going to define the pseudo-functor $\mathrm{DEF}(A):2'\text{-}\mathrm{dgart}\to\mathbf{Gpd}$.

Let $\mathcal{R}$ be an artinian DG algebra. An object of the groupoid $\mathrm{DEF}_{\mathcal{R}}(A)$ is a pair (S,σ), where $S\in D_\infty(\bar{A}^{op}_{\mathcal{R}^{op}})$, and σ is an isomorphism (in $D_\infty(\bar{A}^{op})$)

$$\sigma:\mathbf{L}i^*(S)\to A.$$

A morphism $f:(S,\sigma)\to(T,\tau)$ in $\mathrm{DEF}_{\mathcal{R}}(A)$ is an isomorphism (in $D(\bar{A}^{op}_{\mathcal{R}^{op}})$) $f:S\to T$ such that

$$\tau\circ\mathbf{L}i^*(f)=\sigma.$$

This defines the pseudo-functor $\mathrm{DEF}(A)$ on objects. Further, let $(M,\theta)\in 1\text{-}\mathrm{Hom}(\mathcal{R},\mathcal{Q})$. Define the corresponding functor

$$M^*:\mathrm{DEF}_{\mathcal{R}}(A)\to\mathrm{DEF}_{\mathcal{Q}}(A)$$

as follows. For an object $(S,\sigma)\in\mathrm{DEF}_{\mathcal{R}}(A)$ put

$$M^*(S)=S\overset{\mathbf{L}}{\otimes}_{\mathcal{R}}M\in D_\infty\big(\bar{A}^{op}_{\mathcal{Q}^{op}}\big).$$

Then we have natural isomorphisms in $D_\infty(\bar{A})$:

$$M^*(S)\overset{\mathbf{L}}{\otimes}_{\mathcal{Q}}k=S\overset{\mathbf{L}}{\otimes}_{\mathcal{R}}\Big(M\overset{\mathbf{L}}{\otimes}_{\mathcal{Q}}k\Big)\cong S\overset{\mathbf{L}}{\otimes}_{\mathcal{R}}k\cong A$$

(the second isomorphism is induced by θ). Thus, M^* is a functor form $\mathrm{DEF}_{\mathcal{R}}(A)$ to $\mathrm{DEF}_{\mathcal{Q}}(A)$.

If $\mathcal{Q}'$ is another artinian DG algebra and $(M',\theta') \in 1\text{-}\mathrm{Hom}(\mathcal{Q},\mathcal{Q}')$ then there is a natural isomorphism of functors

$$\left(M' \overset{\mathbf{L}}{\otimes}_{\mathcal{Q}} M\right)^* \cong M'^* \cdot M^*.$$

Further, if $f \in 2\text{-}\mathrm{Hom}((M,\theta),(M,\theta_1))$ is a 2-morphism between objects $(M,\theta),(M,\theta_1) \in 1\text{-}\mathrm{Hom}(\mathcal{R},\mathcal{Q})$ then it induces an isomorphism between corresponding functors $M^* \to M_1^*$.

Thus we obtain a pseudo-functor $\mathrm{DEF}(A): 2\text{-}\mathrm{dgart} \to \mathbf{GPd}$. We denote by $\mathrm{DEF}_-(A)$ its restriction to the sub-2-category $2\text{-}\mathrm{dgart}_-$.

Proposition 12.1. *Let A be an augmented A_∞-algebra and $U(A)$ its universal DG algebra. Then there is a natural equivalence of pseudo-functors* $\mathrm{DEF}(U(A)) \cong \mathrm{DEF}(A)$.

Proof. The proof is the same as of Proposition 10.1 and we omit it. □

Corollary 12.2. *Let A be an augmented A_∞-algebra and let $\mathcal{B}$ be a DG algebra quasi-isomorphic to A. Then the pseudo-functor* $\mathrm{DEF}(A)$ *and* $\mathrm{DEF}(\mathcal{B})$ *are equivalent.*

Proof. Indeed, by Proposition 10.1 the pseudo-functors $\mathrm{DEF}(A)$ and $\mathrm{DEF}(U(A))$ are equivalent, and by Corollary 11.7 the pseudo-functors $\mathrm{DEF}(U(A))$ and $\mathrm{DEF}(\mathcal{B})$ are equivalent. □

Corollary 12.3. *Let A be an admissible A_∞-algebra, and $\mathcal{R}$ be an artinian negative DG algebra. Then for any $(S,\sigma) \in \mathrm{DEF}_{\mathcal{R}}(A)$ there exists an $\alpha \in \mathcal{MC}_{\mathcal{R}}(A)$ such that the pair (T, id), where $T = A \otimes_\alpha \mathcal{R}$, defines an object of* $\mathrm{DEF}_{\mathcal{R}}(A)$ *which is isomorphic to (S,σ).*

Proof. This follows easily from Proposition 12.1 and the proof of Lemma 11.8 in the case $\mathcal{B} = U(A)$. □

13. Comparison of pseudo-functors $\mathbf{coDEF}_-(E)$ and $\mathbf{DEF}_-(E)$

We have proved in I, Corollary 11.9 that under some conditions on E the pseudo-functors $\mathrm{coDef}_-(E)$ and $\mathrm{Def}_-(E)$ from dgart_- to **Gpd** are equivalent. Note that we cannot speak about an equivalence of pseudo-functors $\mathrm{coDEF}_-(E)$ and $\mathrm{DEF}_-(E)$ since they are defined on different bicategories. So our first goal is to establish an equivalence of the bicategories $2\text{-}\mathrm{adgalg}$ and $2'\text{-}\mathrm{adgalg}$ in the following sense: we will construct pseudo-functors

$$\mathcal{D}: 2\text{-}\mathrm{adgalg} \to 2'\text{-}\mathrm{adgalg},$$

$$\mathcal{D}': 2'\text{-}\mathrm{adgalg} \to 2\text{-}\mathrm{adgalg},$$

which have the following properties:

1) $\mathcal{D}$ (resp. $\mathcal{D}'$) is the identity on objects;
2) for each $\mathcal{B},\mathcal{C} \in Ob(2\text{-}\mathrm{adgalg})$ they define mutually inverse equivalences of groupoids

$$\mathcal{D}: \mathrm{Hom}_{2\text{-}\mathrm{adgalg}}(\mathcal{B},\mathcal{C}) \to \mathrm{Hom}_{2'\text{-}\mathrm{adgalg}}(\mathcal{B},\mathcal{C}),$$
$$\mathcal{D}': \mathrm{Hom}_{2'\text{-}\mathrm{adgalg}}(\mathcal{B},\mathcal{C}) \to \mathrm{Hom}_{2\text{-}\mathrm{adgalg}}(\mathcal{B},\mathcal{C}).$$

Fix augmented DG algebras $\mathcal{B}$, $\mathcal{C}$ and let M be a DG $\mathcal{C}\otimes\mathcal{B}^{op}$-module. Define the DG $\mathcal{B}\otimes\mathcal{C}^{op}$-module $\mathcal{D}(M)$ as

$$\mathcal{D}(M) := \mathbf{R}\operatorname{Hom}_{\mathcal{C}}(M, \mathcal{C}).$$

Further, let N be a DG $\mathcal{B}\otimes\mathcal{C}^{op}$-module. Define the DG $\mathcal{B}^{op}\otimes\mathcal{C}$-module $\mathcal{D}'(N)$ as

$$\mathcal{D}'(N) = \mathbf{R}\operatorname{Hom}_{\mathcal{C}^{op}}(N, \mathcal{C}).$$

Proposition 13.1. *The operations $\mathcal{D}$, $\mathcal{D}'$ as above induces the pseudo-functors:*

$$\mathcal{D} : 2\text{-}\mathrm{adgalg} \to 2'\text{-}\mathrm{adgalg},$$
$$\mathcal{D}' : 2'\text{-}\mathrm{adgalg} \to 2\text{-}\mathrm{adgalg},$$

so that the properties 1) *and* 2) *hold.*

Proof. To simplify the notation denote by $\operatorname{Hom}(-,-)$ and $\operatorname{Hom}'(-,-)$ the morphisms in the bicategories 2-adgalg and $2'$-adgalg respectively.

We will prove that for augmented DG algebras $\mathcal{B}$ and $\mathcal{C}$ we have a (covariant) functor

$$\mathcal{D} : \operatorname{Hom}(\mathcal{B}, \mathcal{C}) \to \operatorname{Hom}'(\mathcal{B}, \mathcal{C}),$$

and the functorial diagram

$$\begin{array}{ccccc} \operatorname{Hom}(\mathcal{B}_1,\mathcal{B}_2) & \times & \operatorname{Hom}(\mathcal{B}_2,\mathcal{B}_3) & \longrightarrow & \operatorname{Hom}(\mathcal{B}_1,\mathcal{B}_3) \\ \downarrow{\scriptstyle \mathcal{D}} & & \downarrow{\scriptstyle \mathcal{D}} & & \downarrow{\scriptstyle \mathcal{D}} \\ \operatorname{Hom}'(\mathcal{B}_1,\mathcal{B}_2) & \times & \operatorname{Hom}'(\mathcal{B}_2,\mathcal{B}_3) & \longrightarrow & \operatorname{Hom}'(\mathcal{B}_1,\mathcal{B}_3) \end{array}$$

commutes for every triple of augmented algebras $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$.

Let $(M,\theta) \in 1\text{-}\operatorname{Hom}(\mathcal{B},\mathcal{C})$. Choose a quasi-isomorphism $f : \mathcal{C} \to \nu_* M$ of DG $\mathcal{C}$-modules. It induces the quasi-isomorphism

$$\mathcal{D}(f) : \nu_*\mathcal{D}(M) \to \mathbf{R}\operatorname{Hom}_{\mathcal{C}}(\mathcal{C},\mathcal{C}) = \mathcal{C}$$

of DG $\mathcal{C}^{op}$-modules. Moreover, we claim that the quasi-isomorphism $\theta : k \overset{\mathbf{L}}{\otimes}_{\mathcal{C}} M \to k$ induces a quasi-isomorphism

$$\mathcal{D}(\theta) : \mathcal{D}(M) \overset{\mathbf{L}}{\otimes}_{\mathcal{C}} k \to k^* = k.$$

Indeed, we may and will assume that the DG $\mathcal{C}\otimes\mathcal{B}^{op}$-module M is h-projective. Then by I, Lemma 3.23 it is also h-projective as a DG $\mathcal{C}$-module. Therefore by Lemma 13.3 below

$$\mathcal{D}(M) \overset{\mathbf{L}}{\otimes}_{\mathcal{C}} k = \operatorname{Hom}_{\mathcal{C}}(M,\mathcal{C}) \otimes_{\mathcal{C}} k.$$

Note that the obvious morphism of DG $\mathcal{B}$-modules

$$\delta : \mathrm{Hom}_{\mathcal{C}}(M, \mathcal{C}) \otimes_{\mathcal{C}} k \to \mathrm{Hom}_{\mathcal{C}}(M, k)$$

is a quasi-isomorphism. Indeed, the DG $\mathcal{C}$-module M is homotopy equivalent to $\mathcal{C}$. Hence it suffices to check that δ is an isomorphism when $M = \mathcal{C}$, which is obvious, since both sides are equal to k. Now notice the obvious canonical isomorphisms

$$\mathrm{Hom}_{\mathcal{C}}(M, k) = \mathrm{Hom}_k(k \otimes_{\mathcal{C}} M, k) = (k \otimes_{\mathcal{C}} M)^* \xrightarrow{\theta^*} k^* = k.$$

Thus indeed, $(\mathcal{D}(M), \mathcal{D}(\theta))$ is an object in $\mathrm{Hom}'(\mathcal{B}, \mathcal{C})$ and therefore we have a (covariant) functor

$$\mathcal{D} : \mathrm{Hom}(\mathcal{B}, \mathcal{C}) \to \mathrm{Hom}'(\mathcal{B}, \mathcal{C}).$$

Let now $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3 \in Ob(2\text{-}\mathrm{adgalg})$ and $M_1 \in 1\text{-}\mathrm{Hom}(\mathcal{B}_1, \mathcal{B}_2)$, $M_2 \in 1\text{-}\mathrm{Hom}(\mathcal{B}_2, \mathcal{B}_3)$. Then

$$M_2 \overset{\mathbf{L}}{\otimes}_{\mathcal{B}_2} M_1 \in 1\text{-}\mathrm{Hom}(\mathcal{B}_1, \mathcal{B}_3) \quad \text{and} \quad \mathcal{D}(M_1) \overset{\mathbf{L}}{\otimes}_{\mathcal{B}_2} \mathcal{D}(M_2) \in 1\text{-}\mathrm{Hom}'(\mathcal{B}_1, \mathcal{B}_3).$$

We claim that the DG $\mathcal{B}_1 \otimes \mathcal{B}_3^{op}$-modules

$$\mathcal{D}\Big(M_2 \overset{\mathbf{L}}{\otimes}_{\mathcal{B}_2} M_1\Big) \quad \text{and} \quad \mathcal{D}(M_1) \overset{\mathbf{L}}{\otimes}_{\mathcal{B}_2} \mathcal{D}(M_2)$$

are canonically quasi-isomorphic.

Indeed, we may and will assume that M_1 and M_2 are h-projective as DG $\mathcal{B}_2 \otimes \mathcal{B}_1^{op}$- and $\mathcal{B}_3 \otimes \mathcal{B}_2^{op}$-modules respectively. Then by Lemma 13.3 below it suffices to prove that the morphism of DG $\mathcal{B}_1 \otimes \mathcal{B}_3^{op}$-modules

$$\epsilon : \mathrm{Hom}_{\mathcal{B}_2}(M_1, \mathcal{B}_2) \otimes_{\mathcal{B}_2} \mathrm{Hom}_{\mathcal{B}_3}(M_2, \mathcal{B}_3) \to \mathrm{Hom}_{\mathcal{B}_3}(M_2 \otimes_{\mathcal{B}_2} M_1, \mathcal{B}_3)$$

defined by

$$\epsilon(f \otimes g)(m_2 \otimes m_1) := (-1)^{\bar{f}(\bar{g} + \bar{m}_2)} g\big(m_2 f(m_1)\big)$$

is a quasi-isomorphism. To prove that ϵ is a quasi-isomorphism we may replace the DG $\mathcal{B}_2$-module M_1 by $\mathcal{B}_2$. Then ϵ is an isomorphism.

Thus, the operation $\mathcal{D}$ induces a pseudo-functor

$$\mathcal{D} : 2\text{-}\mathrm{adgalg} \to 2'\text{-}\mathrm{adgalg}.$$

Analogously, the operation $\mathcal{D}'$ induces a pseudo-functor

$$\mathcal{D}' : 2'\text{-}\mathrm{adgalg} \to 2\text{-}\mathrm{adgalg}.$$

It is clear that for $M \in 1\text{-}\mathrm{Hom}(\mathcal{B}, \mathcal{C})$ (resp. $N \in 1\text{-}\mathrm{Hom}'(\mathcal{B}, \mathcal{C})$) the canonical morphism $M \to \mathcal{D}'\mathcal{D}(M)$ (resp. $N \to \mathcal{D}\mathcal{D}'(M)$) is an isomorphism. Thus, the compositions $\mathcal{D}'\mathcal{D}$ and $\mathcal{D}\mathcal{D}'$ are equivalent to the identity.

Proposition is proved. □

Corollary 13.2. *For any augmented DG algebra $\mathcal{B}$ the pseudo-functor $\mathcal{D}: 2\text{-}\operatorname{adgalg} \to 2'\text{-}\operatorname{adgalg}$ induces a morphism of pseudo-functors*

$$h_{\mathcal{B}} \to h'_{\mathcal{B}} \cdot \mathcal{D},$$

which is an equivalence.

Similarly, the pseudo-functor $\mathcal{D}': 2'\text{-}\operatorname{adgalg} \to 2\text{-}\operatorname{adgalg}$ induces an equivalence of pseudo-functors

$$h'_{\mathcal{B}} \to h_{\mathcal{B}} \cdot \mathcal{D}'.$$

Proof. This is clear. □

Lemma 13.3. *Let $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3 \in Ob(2\text{-}\operatorname{adgalg})$, $M_1 \in 1\text{-}\operatorname{Hom}(\mathcal{B}_1, \mathcal{B}_2)$, $M_2 \in 1\text{-}\operatorname{Hom}(\mathcal{B}_2, \mathcal{B}_3)$. Assume that M_1 and M_2 are h-projective as DG $\mathcal{B}_2 \otimes \mathcal{B}_1^{op}$- and $\mathcal{B}_3 \otimes \mathcal{B}_2^{op}$-modules respectively. Then*

a) *The DG $\mathcal{B}_2^{op}$-module $\operatorname{Hom}_{\mathcal{B}_2}(M_1, \mathcal{B}_2)$ is h-projective.*
b) *The DG $\mathcal{B}_3$-module $M_2 \otimes_{\mathcal{B}_2} M_1$ is h-projective.*

Proof. a) Since M_1 is h-projective as a DG $\mathcal{B}_2 \otimes \mathcal{B}_1^{op}$-module, it is also such as a DG $\mathcal{B}_2$-module (I, Lemma 3.23). We denote this DG $\mathcal{B}_2$-module again by M_1.

Choose a quasi-isomorphism of DG $\mathcal{B}_2$-modules $f: \mathcal{B}_2 \to M_1$. This is a homotopy equivalence since both $\mathcal{B}_2$ and M_1 are h-projective. Thus it induces a homotopy equivalence of DG $\mathcal{B}_2^{op}$-modules

$$f^*: \operatorname{Hom}_{\mathcal{B}_2}(\mathcal{B}_2, \mathcal{B}_2) \to \operatorname{Hom}_{\mathcal{B}_2}(M_1, \mathcal{B}_2).$$

But the DG $\mathcal{B}_2^{op}$-module $\operatorname{Hom}_{\mathcal{B}_2}(\mathcal{B}_2, \mathcal{B}_2) = \mathcal{B}_2$ is h-projective. Hence so is $\operatorname{Hom}_{\mathcal{B}_2}(M_1, \mathcal{B}_2)$.

b) The proof is similar. Namely, the DG $\mathcal{B}_3$-module $M_2 \otimes_{\mathcal{B}_2} M_1$ is homotopy equivalent to $M_2 \otimes_{\mathcal{B}_2} \mathcal{B}_2 = M_2$, which is homotopy equivalent to $\mathcal{B}_3$. □

Theorem 13.4. *Assume that the DG $\mathcal{A}^{op}$-module E has the following properties:*

i) $\operatorname{Ext}^{-1}(E, E) = 0$.
ii) *There exists a bounded above h-projective or h-injective DG $\mathcal{A}^{op}$-module P quasi-isomorphic to E.*
iii) *There exists a bounded below h-projective or h-injective DG $\mathcal{A}^{op}$-module I which is quasi-isomorphic to E.*

Then the pseudo-functors $\operatorname{coDEF}_{-}(E)$ and $\operatorname{DEF}_(E) - \cdot\mathcal{D}$ from $2\text{-}\operatorname{dgart}_{-}$ to **Gpd** *are equivalent.*

Hence also the pseudo-functors $\operatorname{DEF}_{-}(E)$ and $\operatorname{coDEF}_(E) - \cdot\mathcal{D}'$ from $2'\text{-}\operatorname{dgart}_{-}$ to **Gpd** *are equivalent.*

Proof. Let $\mathcal{R} \in \operatorname{dgart}_{-}$. Recall (I, Theorem 11.13) the DG functor

$$\epsilon_{\mathcal{R}}: \mathcal{A}_{\mathcal{R}}^{op}\text{-mod} \to \mathcal{A}_{\mathcal{R}}^{op}\text{-mod}$$

defined by

$$\epsilon_{\mathcal{R}}(M) = M \otimes_{\mathcal{R}} \mathcal{R}^*;$$

and the corresponding derived functor

$$\mathbf{L}\epsilon_{\mathcal{R}} : D\big(\mathcal{A}_{\mathcal{R}}^{op}\big) \to D\big(\mathcal{A}_{\mathcal{R}}^{op}\big).$$

We know (I, Theorem 11.13) that under the assumptions i), ii), iii) this functor induces an equivalence of groupoids

$$\mathbf{L}\epsilon_{\mathcal{R}} : \mathrm{Def}_{\mathcal{R}}(E) \to \mathrm{coDef}_{\mathcal{R}}(E).$$

Let now $\mathcal{Q} \in$ dgart_ and $M \in 1\text{-}\mathrm{Hom}(\mathcal{R}, \mathcal{Q})$. It suffices to prove that the functorial diagram

$$\begin{array}{ccc} \mathrm{Def}_{\mathcal{R}}(E) & \xrightarrow{\mathbf{L}\epsilon_{\mathcal{R}}} & \mathrm{coDef}_{\mathcal{R}}(E) \\ {\scriptstyle \mathcal{D}(M)^*}\downarrow & & \downarrow{\scriptstyle M^!} \\ \mathrm{Def}_{\mathcal{Q}}(E) & \xrightarrow{\mathbf{L}\epsilon_{\mathcal{Q}}} & \mathrm{coDef}_{\mathcal{Q}}(E) \end{array}$$

naturally commutes.

Choose a bounded above h-projective or h-injective P quasi-isomorphic to E. By I, Theorem 11.6a) the groupoids $\mathrm{Def}_{\mathcal{R}}(E)$ and $\mathrm{Def}^{\mathrm{h}}_{\mathcal{R}}(P)$ are equivalent. Hence given $(S, \mathrm{id}) \in \mathrm{Def}^{\mathrm{h}}_{\mathcal{R}}(P)$ it suffices to prove that there exists a natural isomorphism of objects in $D(\mathcal{A}_{\mathcal{Q}}^{op})$

$$M^! \cdot \mathbf{L}\epsilon_{\mathcal{R}}(S) \simeq \mathbf{L}\epsilon_{\mathcal{Q}} \cdot \mathcal{D}(M)^*(S),$$

i.e.

$$\mathbf{R}\,\mathrm{Hom}_{\mathcal{R}^{op}}\Big(M, S \overset{\mathbf{L}}{\otimes}_{\mathcal{R}} \mathcal{R}^*\Big) \simeq S \overset{\mathbf{L}}{\otimes}_{\mathcal{R}} \mathbf{R}\,\mathrm{Hom}_{\mathcal{Q}}(M, \mathcal{Q}) \overset{\mathbf{L}}{\otimes}_{\mathcal{Q}} \mathcal{Q}^*.$$

We may and will assume that the DG $\mathcal{Q} \otimes \mathcal{R}^{op}$-module M is h-projective. In the proof of I, Lemma 11.7 we showed that the DG $\mathcal{A}_{\mathcal{R}}^{op}$-module S is h-projective as a DG $\mathcal{R}^{op}$-module. Therefore by Lemma 13.3a) it suffices to prove that the morphism of DG $\mathcal{A}_{\mathcal{Q}}^{op}$-modules

$$\eta : S \otimes_{\mathcal{R}} \mathrm{Hom}_{\mathcal{Q}}(M, \mathcal{Q}) \otimes_{\mathcal{Q}} \mathcal{Q}^* \to \mathrm{Hom}_{\mathcal{R}^{op}}\big(M, S \otimes_{\mathcal{R}} \mathcal{R}^*\big)$$

defined by

$$\eta(s \otimes f \otimes g)(m)(r) = sg\big(f(mr)\big)$$

is a quasi-isomorphism.

It suffices to prove that η is a quasi-isomorphism of DG $\mathcal{Q}^{op}$-modules. Notice that just the $\mathcal{R}^{op}$-module structure on S is important for us. Furthermore we may assume that S satisfies property (P) as DG $\mathcal{R}^{op}$-module (I, Definition 3.2). Thus it suffices to prove that η is a quasi-isomorphism if $S = \mathcal{R}$. Then

$$\eta : \mathrm{Hom}_{\mathcal{Q}}(M, \mathcal{Q}) \otimes_{\mathcal{Q}} \mathcal{Q}^* \to \mathrm{Hom}_{\mathcal{R}^{op}}\big(M, \mathcal{R}^*\big).$$

We have the canonical isomorphisms

$$\mathrm{Hom}_{\mathcal{R}^{op}}\big(M, \mathrm{Hom}_k(\mathcal{R}, k)\big) = \mathrm{Hom}_k(M \otimes_{\mathcal{R}} \mathcal{R}, k) = M^*.$$

Also, since the DG $\mathcal{Q}^{op}$-module M is homotopy equivalent to $\mathcal{Q}$, we have the homotopy equivalences

$$\mathrm{Hom}_{\mathcal{Q}}(M, \mathcal{Q}) \otimes_{\mathcal{Q}} \mathcal{Q}^* \simeq \mathrm{Hom}_{\mathcal{Q}}(\mathcal{Q}, \mathcal{Q}) \otimes_{\mathcal{Q}} \mathcal{Q}^* \simeq \mathcal{Q}^* \simeq M^*. \qquad \square$$

The next theorem is closely related to the previous one. It asserts the stronger statement in the case when E is a DG algebra considered as a DG module over itself.

Theorem 13.5. *Let $\mathcal{B}$ be a DG algebra. Suppose that the following conditions hold:*

a) $H^{-1}(\mathcal{B}) = 0$;
b) *the cohomology algebra $H(\mathcal{B})$ is bounded above and bounded below. Then the pseudo-functors* $\mathrm{coDEF}_-(\mathcal{B})$ *and* $\mathrm{DEF}_-(\mathcal{B}) \cdot \mathcal{D}$ *from* 2-dgart$_-$ *to* **Gpd** *are equivalent.*

Proof. Let $\mathcal{R}$ be a negative artinian DG algebra. Recall the DG functors

$$\epsilon_{\mathcal{R}} : (\mathcal{B} \otimes \mathcal{R})^{op}\text{-mod} \to (\mathcal{B} \otimes \mathcal{R})^{op}\text{-mod}, \qquad \epsilon_{\mathcal{R}}(M) = M \otimes_{\mathcal{R}} \mathcal{R}^*,$$
$$\eta_{\mathcal{R}} : (\mathcal{B} \otimes \mathcal{R})^{op}\text{-mod} \to (\mathcal{B} \otimes \mathcal{R})^{op}\text{-mod}, \qquad \eta_{\mathcal{R}}(M) = \mathrm{Hom}_{\mathcal{R}}\big(\mathcal{R}^*, M\big).$$

By I, Proposition 4.7 they induce quasi-inverse equivalences

$$\epsilon_{\mathcal{R}} : \mathrm{Def}^{\mathrm{h}}_{\mathcal{R}}(\mathcal{B}) \to \mathrm{coDef}^{\mathrm{h}}_{\mathcal{R}}(\mathcal{B}),$$
$$\eta_{\mathcal{R}} : \mathrm{coDef}^{\mathrm{h}}_{\mathcal{R}}(\mathcal{B}) \to \mathrm{Def}^{\mathrm{h}}_{\mathcal{R}}(\mathcal{B}).$$

By Theorem 11.8 the pseudo-functors $\mathrm{Def}_-(\mathcal{B})$ and $\mathrm{Def}^{\mathrm{h}}_-(\mathcal{B})$ are equivalent. By Lemma 9.9 the pseudo-functors $\mathrm{coDef}_-(\mathcal{B})$ and $\mathrm{coDef}^{\mathrm{h}}_-(\mathcal{B})$. It follows that the derived functors $\mathbf{L}\epsilon_{\mathcal{R}}$, $\mathbf{R}\eta_{\mathcal{R}}$ induce mutually inverse equivalences

$$\mathbf{L}\epsilon_{\mathcal{R}} : \mathrm{Def}_{\mathcal{R}}(\mathcal{B}) \to \mathrm{coDef}_{\mathcal{R}}(\mathcal{B}),$$
$$\mathbf{R}\eta_{\mathcal{R}} : \mathrm{coDef}_{\mathcal{R}}(\mathcal{B}) \to \mathrm{Def}_{\mathcal{R}}(\mathcal{B}).$$

Let now $\mathcal{Q} \in \mathrm{dgart}_-$ and $M \in 1\text{-}\mathrm{Hom}(\mathcal{R}, \mathcal{Q})$. It suffices to prove that the functorial diagram

$$\begin{array}{ccc} \mathrm{Def}_{\mathcal{R}}(\mathcal{B}) & \xrightarrow{\mathbf{L}\epsilon_{\mathcal{R}}} & \mathrm{coDef}_{\mathcal{R}}(\mathcal{B}) \\ {\scriptstyle \mathcal{D}(M)^*}\downarrow & & \downarrow{\scriptstyle M^!} \\ \mathrm{coDef}_{\mathcal{Q}}(\mathcal{B}) & \xrightarrow{\mathbf{L}\epsilon_{\mathcal{Q}}} & \mathrm{coDef}_{\mathcal{Q}}(\mathcal{B}) \end{array}$$

naturally commutes. This fact is absolutely analogous to the analogous fact from the proof of the previous theorem. □

Part 4. Pro-representability theorems

14. Pro-representability of the pseudo-functor coDEF$_-$

The next theorem claims that under some conditions on the DG algebra $\mathcal{C}$ that the functor $\mathrm{coDEF}_-(\mathcal{C})$ is pro-representable.

Theorem 14.1. *Let $\mathcal{C}$ be a DG algebra such that the cohomology algebra $H(\mathcal{C})$ is admissible finite-dimensional. Let A be a strictly unital minimal model of $\mathcal{C}$. Then the pseudo-functor* $\mathrm{coDEF}_-(\mathcal{C})$ *is pro-representable by the DG algebra $\hat{S} = (B\bar{A})^*$. That is, there exists an equivalence of pseudo-functors* $\mathrm{coDEF}_-(\mathcal{C}) \simeq h_{\hat{S}}$ *from* 2-dgart$_-$ *to* **Gpd**.

As a corollary, we obtain the following

Theorem 14.2. *Let $E \in \mathcal{A}^{op}$*-mod. *Assume that the following conditions hold:*

a) *the graded algebra* $\mathrm{Ext}(E, E)$ *is admissible finite-dimensional;*
b) *E is quasi-isomorphic to a bounded below F which is h-projective or h-injective.*

Then the pseudo-functor $\mathrm{coDEF}_-(E)$ *is pro-representable by the DG algebra $\hat{S} = (B\bar{A})^*$, where A is a strictly unital minimal model of* $\mathbf{R}\mathrm{Hom}(E, E)$.

Proof. By Proposition 9.10 the pseudo-functors $\mathrm{coDEF}_-(E)$ and $\mathrm{coDEF}_-(\mathbf{R}\mathrm{Hom}(E, E))$ are equivalent. So it remains to apply Theorem 14.1. □

Proof. Note that we have natural quasi-isomorphism of DG algebras $U(A) \to \mathcal{C}$, hence $\mathrm{coDEF}_-(\mathcal{C}) \simeq \mathrm{coDEF}_-(U(A))$. Further, by Proposition 10.1 we have $\mathrm{coDEF}_-(U(A)) \simeq \mathrm{coDEF}_-(A)$. We will construct an equivalence of pseudo-functors $\Theta : h_{\hat{S}} \to \mathrm{coDEF}_-(A)$.

Consider the A_∞ $\bar{A}^{op}_{\hat{S}^{op}}$-module $B\bar{A} \otimes A$. Choose a quasi-isomorphism $B\bar{A} \otimes A \to J$, where J is an h-injective A_∞ $\bar{A}^{op}_{\hat{S}^{op}}$-module. Note that J is also h-injective as a DG $\hat{S}^{op}$-module.

Given an artinian DG algebra $\mathcal{R}$ and a 1-morphism $(M, \theta) \in 1\text{-}\mathrm{Hom}(\hat{S}, \mathcal{R})$ we define

$$\Theta(M) := \mathrm{Hom}_{\hat{S}^{op}}(M, J).$$

We have $\mathbf{R}\mathrm{Hom}_{\mathcal{R}^{op}}(k, \mathrm{Hom}_{\hat{\mathcal{S}}^{op}}(M, J)) = \mathbf{R}\mathrm{Hom}_{\hat{\mathcal{S}}^{op}}(k \overset{\mathbf{L}}{\otimes}_{\mathcal{R}} M, J)$. Hence the quasi-isomorphism $\theta : k \overset{\mathbf{L}}{\otimes}_{\mathcal{R}} M \to k$ induces a quasi-isomorphism

$$\mathbf{R}\mathrm{Hom}_{\mathcal{R}^{op}}\big(k, \Theta(M)\big) \simeq \mathbf{R}\mathrm{Hom}_{\hat{\mathcal{S}}^{op}}(k, J) = \mathrm{Hom}_{\hat{\mathcal{S}}^{op}}(k, J),$$

and by Proposition 4.4 the last term is canonically quasi-isomorphic to A as an A_∞ A^{op}-module.

If we are given with another artinian DG algebra $\mathcal{Q}$ and a 1-morphism $(N, \delta) \in 1\text{-}\mathrm{Hom}(\mathcal{R}, \mathcal{Q})$, then the object $\Theta(N \overset{\mathbf{L}}{\otimes}_{\mathcal{R}} M)$ is canonically quasi-isomorphic to the object $\mathbf{R}\mathrm{Hom}(N, \Theta(M))$. Thus, Θ is a morphism of pseudo-functors.

It remains to prove that for each $\mathcal{R} \in 2\text{-}\mathrm{dgart}\_$ the induced functor $\Theta_{\mathcal{R}} : 1\text{-}\mathrm{Hom}(\hat{\mathcal{S}}, \mathcal{R}) \to \mathrm{coDEF}_{\mathcal{R}}(A)$ is an equivalence of groupoids. So fix a DG algebra $\mathcal{R} \in 2\text{-}\mathrm{dgart}\_$.

Surjective on isomorphism classes. Let (S, σ) be an object of $\mathrm{coDEF}_{\mathcal{R}}(A)$. By Corollary 10.3, there exists a morphism of DG algebras $\phi : \hat{\mathcal{S}} \to \mathcal{R}$ such that the pair (T, id), where $T = \mathrm{Hom}_{\hat{\mathcal{S}}^{op}}(\mathcal{R}, B\bar{A} \otimes A)$, defines an object of $\mathrm{coDEF}_{\mathcal{R}}(A)$ which is isomorphic to (S, σ). Further, by Proposition 4.7 the morphism $\mathrm{Hom}_{\hat{\mathcal{S}}^{op}}(\mathcal{R}, B\bar{A} \otimes A) \to \mathrm{Hom}_{\hat{\mathcal{S}}^{op}}(\mathcal{R}, J)$ is quasi-isomorphism. Therefore, the object (T, id) is isomorphic to $\Theta(M)$, where $M = \mathcal{R}$ is DG $\hat{\mathcal{S}}^{op} \otimes \mathcal{R}$-module via the homomorphism ϕ.

Full and faithful. Consider the above Θ as a contravariant DG functor from $\mathcal{R} \otimes \hat{\mathcal{S}}^{op}$-mod to $\bar{A}^{op}_{\mathcal{R}^{op}}$-mod$_\infty$. Define the contravariant DG functor $\Phi : \bar{A}^{op}_{\mathcal{R}^{op}}$-mod$_\infty \to \mathcal{R} \otimes \hat{\mathcal{S}}^{op}$-mod defined by the similar formula:

$$\Phi(N) = \mathrm{Hom}_{\bar{A}^{op}}(N, J).$$

These DG functors induce the corresponding DG functors between derived categories

$$\Theta : D\big(\mathcal{R} \otimes \hat{\mathcal{S}}^{op}\big) \to D_\infty\big(\bar{A}^{op}_{\hat{\mathcal{S}}^{op}}\big), \qquad \Phi : D_\infty\big(\bar{A}^{op}_{\hat{\mathcal{S}}^{op}}\big) \to D\big(\mathcal{R} \otimes \hat{\mathcal{S}}^{op}\big).$$

Denote by $\langle k \rangle \subset D(\mathcal{R} \otimes \hat{\mathcal{S}}^{op})$ and $\langle A \rangle \subset D_\infty(\bar{A}^{op}_{\hat{\mathcal{S}}^{op}})$ the triangulated envelopes of the DG $\mathcal{R} \otimes \hat{\mathcal{S}}^{op}$-module k and A_∞ $\bar{A}^{op}_{\mathcal{R}^{op}}$-module A respectively.

Lemma 14.3. *The functors Θ and Φ induce mutually inverse anti-equivalences of the triangulated categories $\langle k \rangle$ and $\langle A \rangle$.*

Proof. For $M \in \mathcal{R} \otimes \hat{\mathcal{S}}^{op}$-mod, and $N \in \bar{A}^{op}_{\mathcal{R}^{op}}$-mod$_\infty$ we have the functorial closed morphisms

$$\beta_M : M \to \Phi\big(\Theta(M)\big), \qquad \beta_M(x)_1(f) = (-1)^{|f||x|} f(x), \qquad \beta_M(x)_n = 0 \quad \text{for } n \geqslant 2;$$
$$\gamma_N : N \to \Theta\big(\Phi(N)\big),$$
$$(\gamma_N)_n(a_1, \ldots, a_{n-1}, y)(f) = (-1)^{n(|a_1|+\cdots+|a_{n-1}|+|y|)} f_n(a_1, \ldots, a_{n-1}, y).$$

By Proposition 4.4 the A_∞ $\bar{A}^{op}_{\hat{\mathcal{S}}^{op}}$-module $\Theta(k)$ is quasi-isomorphic to A. Further, $\Phi(A)$ is quasi-isomorphic to J and hence to k. Therefore, β_k and γ_A are quasi-isomorphisms, and lemma is proved. $\square$

Note that for $(M,\theta) \in 1\text{-}\mathrm{Hom}(\hat{S},\mathcal{R})$ (resp. for $(S,\sigma) \in \mathrm{coDEF}_{\mathcal{R}}(A)$) $M \in \langle k \rangle$ (resp. $S \in \langle A \rangle$). Hence the functor $\Theta_{\mathcal{R}} : 1\text{-}\mathrm{Hom}(\hat{S},\mathcal{R}) \to \mathrm{coDEF}_{\mathcal{R}}(A)$ is fully faithful. This proves the theorem. □

15. Pro-representability of the pseudo-functor DEF_-

Pro-representability Theorems 14.1 and 14.2 imply analogous results for the pseudo-functor DEF_-. Namely, we have the following theorems.

Theorem 15.1. *Let $\mathcal{C}$ be a DG algebra such that the cohomology algebra $H(\mathcal{C})$ is admissible finite-dimensional. Let A be a strictly unital minimal model of $\mathcal{C}$. Then the pseudo-functor* $\mathrm{DEF}_-(\mathcal{C})$ *is pro-representable by the DG algebra $\hat{S} = (B\bar{A})^*$. That is, there exists an equivalence of pseudo-functors* $\mathrm{DEF}_-(\mathcal{C}) \simeq h'_{\hat{S}}$ *from* $2'\text{-}\mathrm{dgart}_-$ *to* **Gpd**.

Proof. By Theorem 14.1 we have the equivalence

$$\mathrm{coDEF}_-(\mathcal{C}) \simeq h_{\hat{S}}$$

of pseudo-functors from $2\text{-}\mathrm{dgart}$ to **Gpd**.

By Theorem 13.5 we have the equivalence

$$\mathrm{coDEF}_-(\mathcal{C}) \simeq \mathrm{DEF}_-(\mathcal{C}) \cdot \mathcal{D}$$

of pseudo-functors from $2\text{-}\mathrm{dgart}$ to **Gpd**.

Further, by Corollary 13.2

$$h_{\hat{S}} \simeq h'_{\hat{S}} \cdot \mathcal{D}.$$

Hence $\mathrm{DEF}_-(\mathcal{C}) \cdot \mathcal{D} \simeq h'_{\hat{S}} \cdot \mathcal{D}$ and therefore

$$\mathrm{DEF}_-(\mathcal{C}) \simeq h'_{\hat{S}}. \qquad \square$$

We get the following corollary.

Theorem 15.2. *Let $E \in \mathcal{A}^{op}$-mod. Assume that the graded algebra* $\mathrm{Ext}(E,E)$ *is admissible finite-dimensional. Then the pseudo-functor* $\mathrm{DEF}_-(E)$ *is pro-representable by the DG algebra $\hat{S} = (B\bar{A})^*$, where A is a strictly unital minimal model of the DG algebra* $\mathbf{R}\mathrm{Hom}(E,E)$.

Proof. Indeed, by Proposition 11.10 the pseudo-functors $\mathrm{DEF}_-(E)$ and $\mathrm{DEF}_-(\mathbf{R}\mathrm{Hom}(E,E))$ are equivalent. And by Theorem 15.1 the pseudo-functors $\mathrm{DEF}_-(\mathbf{R}\mathrm{Hom}(E,E))$ and $h'_{\hat{S}}$ are equivalent. □

We would like to mention here several examples.

Example 15.3. Let X be a commutative scheme over k of finite type, and let $x \in X(k)$ be a regular k-point. Take the skyscraper sheaf $\mathcal{O}_x \in D^b_{coh}(X)$. Then one can show that $\mathrm{Ext}^{\cdot}(\mathcal{O}_x,\mathcal{O}_x) \cong \Lambda(T_x X)$, and the DG algebra $\mathbf{R}\mathrm{Hom}(\mathcal{O}_x,\mathcal{O}_x)$ is formal. It follows that $H^i(\hat{S}) = 0$ for $i \neq 0$, and $H^0(\hat{S}) \cong k[[t_1,\dots,t_n]]$, where $n = \dim_x X$.

Example 15.4. Let X be a proper curve of genus g over k and $\mathcal{L} \in D^b_{coh}(X)$ a line bundle over X. Then $\mathrm{Ext}^0(\mathcal{L}, \mathcal{L}) = k$, $\mathrm{Ext}^1(\mathcal{L}, \mathcal{L}) = k^g$, and $\mathrm{Ext}^i(\mathcal{L}, \mathcal{L}) = 0$ for $i \neq 0, 1$. It follows that the DG algebra $\hat{S}$ is concentrated in degree zero and is isomorphic to the algebra of non-commutative power series in g variables.

Example 15.5. Let V be a vector space of dimension n, and let $W \subset V$ be a subspace of dimension m, $1 \leqslant m \leqslant n-1$. Put $E = \mathcal{O}_{\mathbb{P}(W)} \in D^b_{coh}(\mathbb{P}(V))$. One can show that the graded algebra $A = \mathrm{Ext}^{\cdot}(E, E)$ is isomorphic to $\sum_{0 \leqslant i \leqslant n-m} \mathrm{Sym}^i(W^{\vee}) \otimes \Lambda^i(V/W)$. The later algebra can be shown to be quadratic Koszul. Again, one can show that the DG algebra $\mathbf{R}\,\mathrm{Hom}(E, E)$ is formal. It follows that $H^i(\hat{S}) = 0$ for $i \neq 0$, and $H^0(\hat{S})$ is a (completion of) Koszul dual to A. For $m \neq 1$, we have that the algebra $H^0(\hat{S})$ is non-commutative.

In the proof of Theorem 14.1 we showed that the bar complex $B\bar{A} \otimes_{\tau_A} A$ is the "universal co-deformation" of the A_∞ $\bar{A}^{op}$-module A. However, Theorem 15.1 is deduced from Theorem 14.1 without finding the analogous "universal deformation" of the A_∞ $\bar{A}^{op}$-module A. We do not know if this "universal deformation" exists in general (under the assumptions of Theorem 15.1). But we can find it and hence give a direct proof of Theorem 15.1 if the minimal model A of $\mathcal{C}$ satisfies an extra assumption $(*)$ below.

For the rest of this section we assume that the DG algebra $\mathcal{C}$ has an augmented minimal model A.

Definition 15.6. Let A be an augmented A_∞-algebra. Consider k as a left A_∞ A-module. We say that A satisfies the condition $(*)$ if the canonical morphism

$$k \to \mathrm{Hom}_{\bar{A}^{op}}\big(\mathrm{Hom}_{\bar{A}}(k, A), A\big)$$

of left A_∞ $\bar{A}$-modules is a quasi-isomorphism.

Example 15.7. Let A be an augmented A_∞-algebra. If k lies in $\mathrm{Perf}(A)$ then A satisfies the condition $(*)$.

In particular, suppose that A is homologically smooth and compact. That is, the diagonal A_∞ A–A-bimodule A lies in $\mathrm{Perf}(A\text{–}A)$ (smoothness), and $\dim H(A) < \infty$ (compactness). Then the A_∞ A-module is perfect iff it has finite-dimensional total cohomology. Thus, $k \in \mathrm{Perf}(A)$ and A satisfies the condition $(*)$.

Example 15.8. Let A be an augmented A_∞-algebra which is left and right Gorenstein of dimension d. This means that

$$\mathrm{Ext}^p_{\bar{A}}(k, A) = \begin{cases} k, & \text{if } p = d \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\mathrm{Ext}^p_{\bar{A}^{op}}(k, A) = \begin{cases} k, & \text{if } p = d \\ 0, & \text{otherwise.} \end{cases}$$

Then A satisfies the condition $(*)$.

For the rest of this section assume that A is admissible, finite-dimensional and satisfies the condition $(*)$.

Denote by $\mathcal{E}$ the A_∞ $\bar{A}^{op}_{\hat{S}^{op}}$-module

$$\mathcal{E} := \mathrm{Hom}_A(k, A).$$

This A_∞-module is isomorphic to $A \otimes \hat{S}$ as a graded $(\hat{S}^{op})^{gr}$-module and can be given explicitly by the formula

$$m_n^{\mathcal{E}}(m, a_1, \ldots, a_{n-1}) = m_n^{\mathcal{MC}_\infty^{\hat{S}}(A)(0,\ldots,0,\tau_A)}(m, a_1 \otimes \mathbf{1}_{\hat{S}}, \ldots, a_{n-1} \otimes \mathbf{1}_{\hat{S}}). \tag{15.1}$$

Remark 15.9. The definition of the A_∞-category $\mathcal{MC}_\infty^{\hat{S}}(A)$ is the same as if $\hat{S}$ would be artinian. It is correct because $\hat{S}$ is complete in $\mathfrak{m}$-adic topology and A is finite-dimensional. In the above formula τ_A is considered as an element of $A \otimes \hat{S} = \mathrm{Hom}_k(B\bar{A}, A)$. We denote the A_∞ $\bar{A}^{op}_{\hat{S}^{op}}$-module $\mathcal{E}$ by $A \otimes_{\tau_A} \hat{S}$.

We claim that $\mathcal{E}$ is the "universal deformation" of A. This is justified by Theorem 15.12 below. Let us start with a few lemmas.

Lemma 15.10. *The object $\mathcal{E}$ considered as a DG $\hat{S}^{op}$-module is h-projective.*

Proof. Notice that the stupid filtration $A^{\geqslant i}$ of the complex A is finite. Since A is admissible it follows that the differential $m_1^{\mathcal{E}}$ preserves the $(\hat{S}^{op})^{gr}$-submodule $A^{\geqslant i} \otimes \hat{S}$. Hence the DG $\hat{S}^{op}$-module $\mathcal{E} = A \otimes_{\tau_A} \hat{S}$ has a finite filtration by DG $\hat{S}^{op}$-submodules $A^{\geqslant i} \otimes \hat{S}$ with subquotients being free $\hat{S}^{op}$-modules $A^i \otimes \hat{S}$. Thus the DG $\hat{S}^{op}$-module $\mathcal{E}$ is h-projective. □

Lemma 15.11. *The A_∞ $\bar{A}^{op}$-module $\mathcal{E} \overset{\mathbf{L}}{\otimes}_{\hat{S}} k$ is canonically quasi-isomorphic to A.*

Proof. By Lemma 15.10 and Remark 15.9 we have

$$\mathcal{E} \overset{\mathbf{L}}{\otimes}_{\hat{S}} k = \mathcal{E} \otimes_{\hat{S}} k = A \otimes_{\tau_A} \hat{S} \otimes_{\hat{S}} k,$$

and the last A_∞ A^{op}-module is isomorphic to A since $\tau_A \in A \otimes \mathfrak{m}$, where $\mathfrak{m} \subset \hat{S}$ is the augmentation ideal. □

Now we are ready to define a morphism of pseudo-functors

$$\Psi : h'_{\hat{S}} \to \mathrm{DEF}_-(A).$$

Let $\mathcal{R} \in \mathrm{dgart}\_$ and $M = (M, \theta) \in 1\text{-}\mathrm{Hom}(\hat{S}, \mathcal{R})$. We put

$$\Psi(M) := \mathcal{E} \overset{\mathbf{L}}{\otimes}_{\hat{S}} M \in D_\infty\big(\bar{A}^{op}_{\mathcal{R}^{op}}\big).$$

Notice that the structure isomorphism $\theta : M \overset{\mathbf{L}}{\otimes}_{\mathcal{R}} k \to k$ defines an isomorphism

$$\mathbf{L}i^*\Psi(M) = \Psi(M) \overset{\mathbf{L}}{\otimes}_{\mathcal{R}} k = \mathcal{E} \overset{\mathbf{L}}{\otimes}_{\hat{S}} M \overset{\mathbf{L}}{\otimes}_{\mathcal{R}} k \xrightarrow{\sim} \mathcal{E} \overset{\mathbf{L}}{\otimes}_{\hat{S}} k,$$

and the last term is canonically quasi-isomorphic to A as an A_∞ $\bar{A}^{op}$-module by Lemma 15.11. Hence $\Psi(M)$ is indeed an object in the groupoid $\mathrm{DEF}_{\mathcal{R}}(A)$.

If $\delta : M \to N$ is a 2-morphism, where $M, N \in 1\text{-}\mathrm{Hom}(\hat{S}, \mathcal{R})$, then $\Psi(\delta) : \Psi(M) \to \Psi(N)$ is a morphism of objects in the groupoid $\mathrm{DEF}_{\mathcal{R}}(A)$. Thus Ψ is indeed a morphism of pseudo-functors.

Theorem 15.12. *The morphism $\Psi : h'_{\hat{S}} \to \mathrm{DEF}_{-}(A)$ is an equivalence.*

Proof. It remains to show that for each $\mathcal{R} \in \mathrm{dgart}_{-}$ the induced functor

$$\Psi : 1\text{-}\mathrm{Hom}(\hat{S}, \mathcal{R}) \to \mathrm{DEF}_{\mathcal{R}}(A)$$

is an equivalence of groupoids.

We fix $\mathcal{R}$.

Surjective on isomorphism classes. Let $(S, \sigma) \in \mathrm{DEF}_{\mathcal{R}}(A)$. By Corollary 12.3 there exists an element $\alpha \in \mathcal{MC}_{\mathcal{R}}(A)$ such that the pair (T, id), where $T = A \otimes_\alpha \mathcal{R}$, defines an object of $\mathrm{DEF}_{\mathcal{R}}(A)$ and (T, id) is isomorphic to (S, σ) in $\mathrm{DEF}_{\mathcal{R}}(A)$. The element α corresponds to a (unique) admissible twisting cochain $\tau : \mathcal{R}^* \to A$, which in turn corresponds to a homomorphism of DG coalgebras $g_\tau : \mathcal{R}^* \to B\bar{A}$ (Proposition 8.1). By dualizing we obtain a homomorphism of DG algebras $g_\tau^* : \hat{S} \to \mathcal{R}$ and hence the corresponding object $M_\alpha = ({}_{\hat{S}}\mathcal{R}_{\mathcal{R}}, \mathrm{id}) \in 1\text{-}\mathrm{Hom}(\hat{S}, \mathcal{R})$.

Lemma 15.13. *The object $\Psi(M_\alpha) \in \mathrm{Def}_{\mathcal{R}}(A)$ is isomorphic to (T, id).*

Proof. By Remark 15.9

$$\mathcal{E} = A \otimes_{\tau_A} \hat{S},$$

and hence by Lemma 15.10

$$\Psi(M_\alpha) = (A \otimes_{\tau_A} \hat{S}) \otimes_{g_\tau^*} \mathcal{R}.$$

Notice that the image of τ_A under the map

$$\mathbf{1}_A \otimes g_\tau^* : A \otimes \hat{S} \to A \otimes \mathcal{R}$$

coincides with τ. Thus $\Psi(M_\alpha) = T$. $\square$

Full and faithful. Let us define a functor $\Pi : \mathrm{DEF}_{\mathcal{R}}(A) \to 1\text{-}\mathrm{Hom}(\hat{S}, \mathcal{R})$ as follows: for $S = (S, \sigma) \in \mathrm{DEF}_{\mathcal{R}}(A)$ we put

$$\Pi(S) := \mathrm{Hom}_{\bar{A}^{op}}(\mathcal{E}, S) \in D(\hat{S}^{op} \otimes \mathcal{R}).$$

We claim that $\Pi(S)$ is an object in 1-Hom$(\hat{S}, \mathcal{R})$, i.e. it is quasi-isomorphic to $\mathcal{R}$ as a DG $\mathcal{R}^{op}$-module and the isomorphism σ defines an isomorphism $\Pi(S) \overset{\mathbf{L}}{\otimes}_{\mathcal{R}} k \xrightarrow{\sim} k$.

Indeed, again by Corollary 12.3 we may and will assume that $(S, \sigma) = (T, \mathrm{id})$, where $T = A \otimes_\alpha \mathcal{R}$, $\alpha \in \mathcal{MC}_{\mathcal{R}}(A)$. We have

$$\Pi(T) = \mathrm{Hom}_{A^{op}}(\mathcal{E}, A \otimes_\alpha \mathcal{R}) = \mathrm{Hom}_{A^{op}}\big(\mathrm{Hom}_A(k, A), A\big) \otimes_\alpha \mathcal{R}.$$

Since the A_∞-algebra A satisfies the condition $(*)$ the last term as a DG $\mathcal{R}^{op}$-module is canonically quasi-isomorphic to $k \otimes \mathcal{R} = \mathcal{R}$. Thus we have a canonical isomorphism $\Pi(S) \overset{\mathbf{L}}{\otimes}_{\mathcal{R}} k \xrightarrow{\sim} k$.

Note that the functors Ψ and Π are adjoint:

$$\mathrm{Hom}_{\mathrm{DEF}_{\mathcal{R}}(A)}\big(\Psi(M), S\big) = \mathrm{Hom}_{1\text{-}\mathrm{Hom}(\hat{S}, \mathcal{R})}\big(M, \Pi(S)\big).$$

Now let us consider Ψ and Π as functors simply between the derived categories $D(\hat{S} \otimes \mathcal{R}^{op})$ and $D_\infty(\bar{A}^{op}_{\mathcal{R}^{op}})$. (They remain adjoint.) Denote by $\langle k\rangle \subset D(\hat{S} \otimes \mathcal{R}^{op})$ and $\langle A\rangle \subset D_\infty(\bar{A}^{op}_{\mathcal{R}^{op}})$ the triangulated envelopes of the DG module k and the A_∞ $\bar{A}^{op}_{\mathcal{R}^{op}}$-module A respectively. Let $(S, \sigma) \in \mathrm{DEF}_{\mathcal{R}}(A)$. By Corollary 12.3 we may and will assume that $(S, \sigma) = (T, \mathrm{id})$, where $T = A \otimes_\alpha \mathcal{R}$, $\alpha \in \mathcal{MC}_{\mathcal{R}}(A)$. Hence $S \in \langle A\rangle$. Choose $(M, \theta) \in$ 1-Hom$(\hat{S}, \mathcal{R})$. Since the DG algebra $\hat{S} \otimes \mathcal{R}^{op}$ is local and complete by Lemma 4.2 we have $M \in \langle k\rangle$. Therefore it suffices to prove the following lemma.

Lemma 15.14. *The functors Ψ and Π induce mutually inverse equivalences of triangulated categories $\langle k\rangle$ and $\langle A\rangle$.*

Proof. It suffices to prove that the adjunction maps $k \to \Pi\Psi(k)$ and $\Psi\Pi(A) \to A$ are isomorphisms.

We have $\Pi\Psi(k) = \mathrm{Hom}_{\bar{A}^{op}}(\mathcal{E}, \mathcal{E} \overset{\mathbf{L}}{\otimes}_{\hat{S}} k) = \mathrm{Hom}_{A^{op}}(\mathcal{E}, A)$ (Lemma 15.11). Hence $k \to \Pi\Psi(k)$ is a quasi-isomorphism because A satisfies property $(*)$.

Vice versa, $\Psi\Pi(A) = \mathcal{E} \overset{\mathbf{L}}{\otimes}_{\hat{S}} (\mathrm{Hom}_{\bar{A}^{op}}(\mathcal{E}, A)) = \mathcal{E} \overset{\mathbf{L}}{\otimes}_{\hat{S}} k$, since A satisfies property $(*)$. But $\mathcal{E} \overset{\mathbf{L}}{\otimes}_{\hat{S}} k = A$ by Lemma 15.11.

This proves the lemma. □

Theorem is proved. □

15.1. Explicit equivalence $\mathrm{DEF}_-(E) \cong h'_{\hat{S}}$

Let $E \in \mathcal{A}^{op}$-mod. Suppose that the graded algebra $\mathrm{Ext}(E, E)$ is admissible and finite-dimensional. Let A be a strictly unital minimal model of the DG algebra $\mathbf{R}\mathrm{Hom}(E, E)$. Suppose that A satisfies the condition $(*)$ above. Further, let $F \to E$ be a quasi-isomorphism with h-projective F, $\mathcal{C} = \mathrm{End}(F)$ and let $f : A \to \mathcal{C}$ be a strictly unital A_∞-quasi-isomorphism. By Theorem 15.12, the A_∞ $\bar{A}^{op}_{\hat{S}^{op}}$-module $\mathrm{Hom}_{\bar{A}}(k, A)$ is the "universal deformation" of the A_∞ $\bar{A}^{op}$-module A. It follows from the equivalence $\mathrm{DEF}_-(A) \cong \mathrm{DEF}_-(\mathcal{C})$ (Corollary 12.2) that the $(\mathcal{C} \otimes \hat{S})^{op}$-module

$$\mathrm{Hom}_{\bar{A}}(k, \mathcal{C}) = \mathcal{C} \otimes_{f^*(\tau_A)} \hat{S}$$

is the "universal deformation" of the DG $\mathcal{C}^{op}$-module $\mathcal{C}$.

Put

$$\mathcal{F} = \mathrm{Hom}_{\bar{A}}(k, \mathcal{C}) \otimes_{\mathcal{C}} F = (\mathcal{C} \otimes_{f^*(\tau_A)} \hat{S}) \otimes_{\mathcal{C}} F.$$

Then $\mathcal{F}$ is a DG $\mathcal{A}^{op}_{\hat{S}}$-module. We claim that it is a “universal deformation” of the DG module E. More precisely, we get the following

Corollary 15.15. *Let E and $\mathcal{F}$ be as above. Then the functors $\Phi_{\mathcal{R}} : D(\hat{S} \otimes \mathcal{R}^{op}) \to D(\mathcal{A}^{op}_{\mathcal{R}})$,*

$$\Phi_{\mathcal{R}}(M) = \mathcal{F} \overset{\mathbf{L}}{\otimes}_{\hat{S}} M,$$

induce the equivalence of pseudo-functors

$$\Phi : h'_{\hat{S}} \to \mathrm{DEF}_{-}(E)$$

from dgart_ *to* **Gpd**.

Proof. Indeed, the morphism $\Phi : h'_{\hat{S}} \to \mathrm{DEF}_{-}(E)$ is isomorphic to the composition of the equivalence

$$\Psi : h'_{\hat{S}} \to \mathrm{DEF}_{-}(A)$$

from Theorem 15.12, the equivalence

$$\mathrm{DEF}_{-}(A) \cong \mathrm{DEF}_{-}(\mathcal{C})$$

from Corollary 12.2, and the equivalence

$$\mathrm{DEF}_{-}(\Sigma) : \mathrm{DEF}_{-}(\mathcal{C}) \to \mathrm{DEF}_{-}(E)$$

from the proof of Proposition 11.10. □

16. Classical pro-representability

Recall that for a small groupoid $\mathcal{M}$ one denotes by $\pi_0(\mathcal{M})$ the set of isomorphism classes of objects in $\mathcal{M}$.

All our deformation functors have values in the 2-category of groupoids **Gpd**. We may compose those pseudo-functors with π_0 to obtain functors with values in the category Set of sets. Classically pro-representability theorems are statements about these compositions. Out pro-representability Theorems 14.1, 14.2, 15.1, 15.2 have some “classical” implications which we discuss next.

Definition 16.1. Denote by alg and calg the full subcategories of the category adgalg (I, Section 2) consisting of local (!) augmented algebras (resp. local commutative augmented algebras) concentrated in degree zero. That is we consider the categories of usual local augmented (resp. commutative local augmented) algebras. Then we have the full subcategories art ⊂ alg and cart $\subset$ calg of (local augmented) artinian (resp. commutative artinian) algebras (I, Definitions 2.1–2.3). Note that for $\mathcal{B}, \mathcal{C} \in$ alg the group of units of $\mathcal{C}$ acts by conjugation on the set $\mathrm{Hom}(\mathcal{B}, \mathcal{C})$. We call this the adjoint action. The orbits of this action define an equivalence relation on $\mathrm{Hom}(\mathcal{B}, \mathcal{C})$ and we denote by alg / ad the corresponding quotient category, where $\mathrm{Hom}_{\mathrm{alg}/\mathrm{ad}}(\mathcal{B}, \mathcal{C})$ is the set of equivalence classes. Let

$$q : \mathrm{alg} \to \mathrm{alg}/\mathrm{ad}$$

be the quotient functor. We obtain the corresponding full subcategory art / ad $\subset$ alg / ad.

Remark 16.2. Note that if $\mathcal{B}, \mathcal{C} \in$ alg and $\mathcal{C}$ is commutative then the adjoint action on $\mathrm{Hom}(\mathcal{B}, \mathcal{C})$ is trivial.

Recall the pseudo-functor $\mathcal{F} : \mathrm{adgalg} \to 2\text{-}\mathrm{adgalg}$ from Section 9. We denote also by $\mathcal{F}$ its restriction to the full subcategory alg. Since the functor q and the pseudo-functor $\mathcal{F}$ are the identity on objects we will write $\mathcal{B}$ instead of $q(\mathcal{B})$ or $\mathcal{F}(\mathcal{B})$ for $\mathcal{B} \in$ alg.

Fix $\mathcal{B} \in$ alg. We consider two functors from alg to Set which are defined by $\mathcal{B}$: $h_{\mathcal{B}} \cdot q$ and $\pi_0 \cdot h_{\mathcal{B}} \cdot \mathcal{F}$. Namely, for $\mathcal{C} \in$ alg:

$$h_{\mathcal{B}} \cdot q(\mathcal{C}) = \mathrm{Hom}_{\mathrm{alg}/\mathrm{ad}}(\mathcal{B}, \mathcal{C}),$$
$$\pi_0 \cdot h_{\mathcal{B}} \cdot \mathcal{F}(\mathcal{C}) = \pi_0\big(1\text{-}\mathrm{Hom}_{2\text{-}\mathrm{adgalg}}(\mathcal{B}, \mathcal{C})\big).$$

Lemma 16.3. *For any $\mathcal{B} \in$ alg the above functors $h_{\mathcal{B}} \cdot q$ and $\pi_0 \cdot h_{\mathcal{B}} \cdot \mathcal{F}$ from* alg *to* Set *are isomorphic.*

Proof. This is proved in Lemma 9.3a), b). □

Corollary 16.4. *For any $\mathcal{B} \in$ alg the functors $h_{\mathcal{B}}$ and $\pi_0 \cdot h_{\mathcal{B}} \cdot \mathcal{F}$ from* calg *to* Set *are isomorphic.*

Proof. This follows from Lemma 16.3 and Remark 16.2. □

Definition 16.5. Let A be an augmented A_∞-algebra. We call A Koszul if the DG algebra $\hat{S} := (B\bar{A})^*$ is quasi-isomorphic to $H^0(\hat{S})$.

Note that the augmented A_∞-algebras coming from Examples 15.3, 15.4, 15.5 are formal and quadratic Koszul, hence Koszul in our sense.

Lemma 16.6. *Let $\phi : \mathcal{B} \to \mathcal{C}$ be a quasi-isomorphism of augmented DG algebras. Then it induces a morphism $\phi_* : h_{\mathcal{C}} \to h_{\mathcal{B}}$ of pseudo-functors from* 2-adgalg *to* **Gpd**. *This morphism is an equivalence.*

Proof. Indeed, for $\mathcal{E} \in 2\text{-}\mathrm{adgalg}$ and $M \in 1\text{-}\mathrm{Hom}(\mathcal{C}, \mathcal{E})$ denote by $\phi_* M \in 1\text{-}\mathrm{Hom}(\mathcal{B}, \mathcal{E})$ the DG $\mathcal{B} \otimes \mathcal{E}^{op}$-module obtained from M by restriction of scalars. This functor ϕ_* defines an equivalence of derived categories

$$\phi_* : D\big(\mathcal{C} \otimes \mathcal{E}^{op}\big) \to D\big(\mathcal{B} \otimes \mathcal{E}^{op}\big)$$

since ϕ is a quasi-isomorphism. Hence it defines an equivalence of groupoids

$$\phi_* : 1\text{-}\mathrm{Hom}(\mathcal{C}, \mathcal{E}) \to 1\text{-}\mathrm{Hom}(\mathcal{B}, \mathcal{E}). \qquad \square$$

Theorem 16.7. *Let $\mathcal{C}$ be a DG algebra such that the strictly unital minimal model A of $\mathcal{C}$ (Definition* 4.1*) is a Koszul A_∞-algebra. Put $\hat{S} = (B\bar{A})^*$. Then*

a) *there exists an isomorphism of functors from* art *to* Set

$$h_{H^0(\hat{S})} \cdot q \simeq \pi_0 \cdot \mathrm{coDef}_0(\mathcal{C});$$

b) *there exists an isomorphism of functors from* cart *to* Set

$$h_{H^0(\hat{S})} \simeq \pi_0 \cdot \mathrm{coDef}_{\mathrm{cl}}(\mathcal{C}).$$

Proof. a) Note that the DG algebra $\hat{S}$ is concentrated in nonpositive degrees. Hence we have a natural homomorphism of augmented DG algebras $\hat{S} \to H^0(\hat{S})$ which is a quasi-isomorphism. Hence by Lemma 16.6 the pseudo-functors

$$h_{\hat{S}}, h_{H^0(\hat{S})} : 2\text{-}\mathrm{adgalg} \to \mathbf{Gpd}$$

are equivalent. Notice that $\hat{S}$ is a local algebra and the homomorphism $\hat{S} \to H^0(\hat{S})$ is surjective. Hence the algebra $H^0(\hat{S})$ is also local.

By Theorem 14.1 we have an equivalence of pseudo-functors

$$\mathrm{coDEF}_0(\mathcal{C}) \simeq h_{\hat{S}} : 2\text{-}\mathrm{art} \to \mathbf{Gpd}.$$

Thus $\mathrm{coDEF}_0(\mathcal{C}) \simeq h_{H^0(\hat{S})}$. By Proposition 9.5

$$\mathrm{coDEF}_-(\mathcal{C}) \cdot \mathcal{F} \simeq \mathrm{coDef}_-(\mathcal{C}).$$

Therefore

$$\mathrm{coDef}_0(\mathcal{C}) \simeq h_{H^0(\hat{S})} \cdot \mathcal{F} : \mathrm{art} \to \mathbf{Gpd}.$$

Finally, by Lemma 16.3

$$\pi_0 \cdot \mathrm{coDef}_0(\mathcal{C}) \simeq h_{H^0(\hat{S})} \cdot q : \mathrm{art} \to \mathrm{Set}.$$

This proves a).

b) This follows from a) and Remark 16.2 $\square$

Remark 16.8. Under the assumptions of Theorem 16.7 the same conclusion holds for pseudo-functors $\mathrm{coDef}^{\mathrm{h}}(\mathcal{C})$, $\mathrm{Def}(\mathcal{C})$, $\mathrm{Def}^{\mathrm{h}}(\mathcal{C})$ instead of $\mathrm{coDef}(\mathcal{C})$. Indeed, by Lemmas 11.8, 9.9 and Theorem 13.5 there are equivalences of pseudo-functors

$$\mathrm{coDef}_-(\mathcal{C}) \simeq \mathrm{coDef}^{\mathrm{h}}_-(\mathcal{C}) \simeq \mathrm{DEF}_-(\mathcal{C}) \simeq \mathrm{Def}^{\mathrm{h}}_-(\mathcal{C}).$$

Theorem 16.9. *Let* $E \in \mathcal{A}^{op}$-mod. *Assume that* E *is quasi-isomorphic to a bounded below* $F \in \mathcal{A}^{op}$-mod *which is h-projective or h-injective. Also assume that the graded algebra* $\mathrm{Ext}(E)$ *is admissible and finite-dimensional, and the strictly unital minimal model* A *of the DG algebra* $\mathrm{End}(F)$ *is Koszul. Put* $\hat{S} = (B\bar{A})^*$. *Then*

a) *there exists an isomorphism of functors from* art *to* Set

$$h_{H^0(\hat{S})} \cdot q \simeq \pi_0 \cdot \mathrm{coDef}_0(E);$$

b) *there exists an isomorphism of functors from* cart *to* Set

$$h_{H^0(\hat{S})} \simeq \pi_0 \cdot \mathrm{coDef}_{\mathrm{cl}}(E).$$

Proof. By I, Proposition 11.16 the pseudo-functors $\mathrm{coDef}_-(E)$ and $\mathrm{coDef}_-(\mathrm{End}(F))$ are equivalent. So the theorem follows from Theorem 16.7. □

Theorem 16.10. *Let* $E \in \mathcal{A}^{op}$-mod. *Assume that the graded algebra* $\mathrm{Ext}(E)$ *is admissible and finite-dimensional, and the strictly unital minimal model* A *of the DG algebra* $\mathbf{R}\mathrm{Hom}(E, E)$ *is Koszul. Put* $\hat{S} = (B\bar{A})^*$. *Then*

a) *there exists an isomorphism of functors from* art *to* Set

$$h_{H^0(\hat{S})} \cdot q \simeq \pi_0 \cdot \mathrm{Def}_0(E);$$

b) *there exists an isomorphism of functors from* cart *to* Set

$$h_{H^0(\hat{S})} \simeq \pi_0 \cdot \mathrm{Def}_{\mathrm{cl}}(E).$$

Proof. By I, Proposition 11.16 the pseudo-functors $\mathrm{Def}_-(E)$ and $\mathrm{Def}_-(\mathbf{R}\mathrm{Hom}(E, E))$ are equivalent. So the theorem follows from Theorem 16.7 and Remark 16.8. □

If, in addition, the A_∞-algebra A in the above theorem satisfies condition $(*)$ (Definition 15.6), then the equivalences a), b) can be made explicit. Namely, we get the following

Corollary 16.11. *Let* E, $\hat{S}$ *be as in Theorem* 16.10 *and* $\mathcal{F}$ *be as in Corollary* 15.15. *Then the equivalence* $h_{H^0(\hat{S})} \cdot q \to \mathrm{Def}_0(E)$ *of functors from* art *to* Set, *and the equivalence* $h_{H^0(\hat{S})} \to \mathrm{Def}_0(E)$ *of functors from* cart *to* Set *are induced by the functors* $\Phi_{\mathcal{R}} : D(H^0(\hat{S}) \otimes \mathcal{R}^{op}) \to D(\mathcal{A}^{op}_{\mathcal{R}})$,

$$\Phi_{\mathcal{R}}(M) = \mathcal{F} \overset{\mathbf{L}}{\otimes}_{\hat{S}} M.$$

Proof. This follows from Corollary 15.15 □

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