Oscillation properties of solutions of a class of nonlinear parabolic equations

Jaroslav Jarošík, Takaši Kusano, Norio Yoshida

*Department of Mathematical Analysis, Faculty of Mathematics and Physics, Comenius University, 842 15 Bratislava, Slovak Republic

bDepartment of Applied Mathematics, Faculty of Science, Fukuoka University, Fukuoka 814-0180, Japan

cDepartment of Mathematics, Faculty of Science, Toyama University, Toyama 930-8555, Japan

Received 20 April 2001; received in revised form 14 October 2001

Abstract

Oscillations of solutions of a class of nonlinear parabolic equations are investigated, and the unboundedness of solutions is also studied as corollaries. Our approach is to employ the modifications of Picone-type identities for half-linear elliptic operators. © 2002 Elsevier Science B.V. All rights reserved.

MSC: 35B05

Keywords: Oscillation; Nonlinear parabolic equations; Picone-type identities

Since the work of McNabb [8] in 1962, the unboundedness of solutions of parabolic equations or systems has been studied by several authors. We refer the reader to Dunninger [4], Kusano and Narita [7] for parabolic equations or inequalities, and to Chan [1], Chan and Young [2,3], Kuk [5] for time-dependent matrix differential inequalities. As corollaries, all of them contain the results about the zeros of solutions or singularities of matrix solutions.

Recently Kusano et al. [6] have established Picone-type identities for the half-linear elliptic operators. The purpose of this paper is to obtain the oscillatory behavior or the unboundedness of solutions by extending the Picone identities to time-dependent nonlinear parabolic equations.

We are concerned with the oscillation properties of solutions of the nonlinear parabolic equation

$$\frac{\partial v}{\partial t} - P_2[v] = 0, \quad (x,t) \in \Omega \equiv G \times (0, \infty)$$

(1)

* Corresponding author.

0377-0427/02/$ - see front matter © 2002 Elsevier Science B.V. All rights reserved.

PII: S0377-0427(02)00360-6
where $G$ is a bounded domain in $\mathbb{R}^n$ with piecewise smooth boundary $\partial G$ and
\[ P_x[v] = \nabla \cdot (A(x,t)|\nabla v|^{p-2} \nabla v) + C(x,t)|v|^{p-1}v, \]
\(\nabla\) being the gradient \((\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})\).

\begin{align*}
\text{Theorem 1} \quad & \text{De/FFnition 4.} \\
\text{De/FFnition 3.} \quad & \text{De/FFnition 2.} \\
\text{De/FFnition 1.} \quad & \text{De/FFnition 1.} \\
\text{De/FFnition 3.} \quad & \text{any} \\
\n\text{Assume that:} \\
\text{A} \quad & \text{where} \\
\text{A} \quad & \text{where} \\
\n\text{(H}_2\text{)} \quad & A(x,t) \in C(\tilde{\Omega}; (0,\infty)) \text{ and } C(x,t) \in C(\tilde{\Omega}; \mathbb{R}). \\
\text{Definition 1.} \quad & \text{The domain } D_{P_x}(\Omega) \text{ of } P_x \text{ is defined to be the set of all functions } v \text{ of class } C^1(\tilde{\Omega}; \mathbb{R}) \\
& \text{with the property that } A(x,t)|\nabla v|^{p-1} \nabla v \in C^1(\Omega; \mathbb{R}) \cap C(\tilde{\Omega}; \mathbb{R}). \\
\text{Definition 2.} \quad & \text{By a } solution \text{ of (1) we mean a function } v \in D_{P_x}(\Omega) \text{ which satisfies Eq. (1).} \\
\text{Definition 3.} \quad & \text{A solution } v \text{ of (1) is said to be } oscillatory \text{ on } \tilde{\Omega} \text{ if } v \text{ has a zero in } \tilde{G} \times [t, \infty) \text{ for any } t > 0. \\
\text{Associated with (2) we consider the half-linear differential operator } p_x \text{ defined by} \\
\text{Definition 4.} \quad & \text{The domain } D_{P_x}(G) \text{ of } p_x \text{ is defined to be the set of all functions } u \text{ of class } C^1(\tilde{G}; \mathbb{R}) \\
\text{with the property that } A(x,t)|\nabla u|^{p-1} \nabla u \in C^1(G; \mathbb{R}) \cap C(\tilde{G}; \mathbb{R}). \\
\text{The following theorem was established by Kusano et al. [6, Theorem 1.1].} \\
\text{Theorem 1 (Picone-type identity).} \quad & \text{Assume that } u \in D_{P_x}(G), v \in D_{P_x}(\Omega) \text{ and } v \neq 0 \text{ in } G \times I, \text{ where } I \text{ is any interval in } \mathbb{R}. \text{ Then we have the identity} \\
& \nabla \cdot \left(\left(\frac{u}{\varphi(v)}\right)[\varphi(v)a(x)|\nabla u|^{p-1} \nabla u - \varphi(u)A(x,t)|\nabla v|^{p-1} \nabla v]\right) \\
& = (a(x) - A(x,t))|\nabla u|^{p+1} + (C(x,t) - c(x))|u|^{p+1} \\
& \quad + A(x,t) \left[|\nabla u|^{p+1} + \alpha \left|\frac{u}{v} \nabla v\right|^{p+1} - (\alpha + 1) \left|\frac{u}{v} \nabla v\right|^{p-1} (\nabla u) \cdot \left(\frac{u}{v} \nabla v\right)\right] \\
& \quad + \frac{u}{\varphi(v)}(\varphi(v)p_x[u] - \varphi(u)p_x[v]), \quad (x,t) \in G \times I, \\
& \text{where } \varphi(s) = |s|^{p-1}s.
Theorem 2. Assume that $(H_1)$–$(H_3)$ hold, and that there exists a nontrivial function $u \in \mathcal{D}_{P_x}(G)$ such that

\begin{align*}
p_x[u] &= 0 \quad \text{in } G, \\
u &= 0 \quad \text{on } \partial G, \\
\lim_{t \to \infty} \int_t^T V[u](s) \, ds &= \infty \quad \text{for any } T > 0,
\end{align*}

where

\[ V[u](t) \equiv \int_G [(a(x) - A(x,t))|\nabla u|^{\alpha+1} + (C(x,t) - c(x))|u|^{\alpha+1}] \, dx. \]

If $0 < \alpha \leq 1$, every solution $v \in \mathcal{D}_{P_x}(\Omega)$ of (1) which is nonoscillatory on $\tilde{\Omega}$ satisfies

\[ \lim_{t \to \infty} \int_G |u|^{\alpha+1} \Psi(|v|) \, dx = \infty, \]

where

\[ \Psi(s) = \begin{cases} 
\log s & \text{if } \alpha = 1, \\
s^{-\alpha+1} & \text{if } 0 < \alpha < 1. 
\end{cases} \]

If $\alpha > 1$, every solution $v \in \mathcal{D}_{P_x}(\Omega)$ of (1) is oscillatory on $\tilde{\Omega}$.

Proof. First we consider the case where $0 < \alpha \leq 1$. Let $v \in \mathcal{D}_{P_x}(\Omega)$ be a solution of (1) which is nonoscillatory on $\tilde{\Omega}$. Then there exists a number $t_0 > 0$ such that $v \neq 0$ on $\tilde{G} \times [t_0, \infty)$. Integrating the Picone identity (4) over $G$ and taking account of the following inequality [6, Lemma 2.1]:

\[ A(x,t) \left[ |\nabla u|^{\alpha+1} + \alpha \left| \frac{u}{v} \nabla v \right|^{\alpha+1} - (\alpha + 1) \left| \frac{u}{v} \nabla v \right|^{\alpha-1} \left( \nabla u \cdot \left( \frac{u}{v} \nabla v \right) \right) \right] \geq 0 \quad \text{in } \tilde{G} \times [t_0, \infty), \]

we observe that

\[ 0 \geq V[u](t) - \int_G \frac{u \phi(u)}{\phi(v)} P_x[v] \, dx \]
\[ = V[u](t) - \int_G |u|^{\alpha+1} \frac{1}{|v|^{\alpha-1} v} \frac{\partial v}{\partial t} \, dx, \quad t \geq t_0. \]

Since

\[ \frac{1}{|v|^{\alpha-1} v} \frac{\partial v}{\partial t} = \begin{cases} 
\frac{\partial}{\partial t} \log|v| & (\alpha = 1), \\
\frac{\partial}{\partial t} \left( \frac{1}{-\alpha + 1} |v|^{-\alpha+1} \right) & (\alpha \neq 1),
\end{cases} \]

then

\[ \frac{\partial}{\partial t} \left| \frac{u}{v} \nabla v \right|^{\alpha+1} = \frac{\partial}{\partial t} \left( \frac{|u|^{\alpha+1}}{|v|^{\alpha-1} v} \frac{\partial v}{\partial t} \right) \geq 0, \]

which implies that

\[ |u|^{\alpha+1} \geq \frac{1}{|v|^{\alpha-1} v} \frac{\partial v}{\partial t} \quad \text{in } \tilde{G} \times (t_0, \infty). \]
(6) implies that
\[
\frac{d}{dr} \left( \int_G |u|^{x+1} \log|v| \, dx \right) \geq V[u](t), \quad (x = 1),
\]
\[
\frac{d}{dr} \left( \frac{1}{-x+1} \int_G |u|^{x+1}|v|^{-x+1} \, dx \right) \geq V[u](t), \quad (x \neq 1)
\]
for \( t \geq t_0 \). Integrating (7), (8) over \([t_0, T]\), we obtain
\[
 z(T) - z(t_0) \geq \int_{t_0}^{T} V[u](s) \, ds, \quad (x = 1),
\]
\[
 \frac{1}{-x+1}(z(T) - z(t_0)) \geq \int_{t_0}^{T} V[u](s) \, ds, \quad (x \neq 1),
\]
where
\[
z(t) \equiv \int_G |u|^{x+1} \Psi(|v|) \, dx.
\]
In case \( 0 < x \leq 1 \), inequalities (9), (10) imply that
\[
 \lim_{T \to \infty} z(T) = \infty,
\]
which is equivalent to (5).

Next we treat the case where \( x > 1 \). Suppose that there exists a nonoscillatory solution \( v \in \mathcal{D}_{p_s}(\Omega) \) on \( \tilde{\Omega} \) of (1). Proceeding as in the proof of the first statement, we see that (10) holds. Since \( -x + 1 < 0 \), (10) implies that
\[
 \frac{1}{x-1} z(t_0) \geq \int_{t_0}^{T} V[u](s) \, ds.
\]
The right side of the above inequality tends to \( \infty \) as \( T \to \infty \), and hence we are led to a contradiction. The proof is complete. \( \square \)

**Corollary 1.** Let \( 0 < x \leq 1 \). Assume that \( (H_1)-(H_3) \) hold, and that there exists a nontrivial function \( u \in \mathcal{D}_{p_s}(G) \) such that
\[
p_s[u] = 0 \quad \text{in } G,
\]
\[
u = 0 \quad \text{on } \partial G,
\]
\[
\lim_{t \to \infty} \int_{T}^{t} V[u](s) \, ds = \infty \quad \text{for any } T > 0.
\]
Then every bounded solution \( v \in \mathcal{D}_{p_s}(\Omega) \) of (1) is oscillatory on \( \tilde{\Omega} \).
Proof. Let \( v \in \mathcal{D}_P(\Omega) \) be any bounded solution of (1). Then, we find that \( \Psi(|v|) \) is bounded from above, and therefore \( \int_G |u|^{x+1} \Psi(|v|) \, dx \) is bounded from above. Since (5) does not hold, it follows from Theorem 2 that the bounded solution \( v \) is oscillatory on \( \tilde{\Omega} \). □

Corollary 2. Let \( 0 < x \leq 1 \). Assume that the same hypotheses as those of Theorem 2 hold. If \( v \in \mathcal{D}_P(\Omega) \) is a solution of (1) which is nonoscillatory on \( \tilde{\Omega} \), then \( v \) is unbounded in \( \Omega \).

Proof. Since \( v \) is nonoscillatory on \( \tilde{\Omega} \), \( v \) satisfies condition (5). Hence, \( |v| \) cannot be bounded from above in \( \Omega \), that is, \( v \) is unbounded in \( \Omega \). □

The following theorem is a special case of the Picone-type identity given by Kusano et al. [6, Theorem 1.1].

Theorem 3. Let \( v \in \mathcal{D}_P(\Omega) \) and let \( v \neq 0 \) in \( G \times I \), where \( I \) is any interval in \( \mathbb{R} \). Then the following identity holds for any \( u \in C^1(G; \mathbb{R}) \):

\[
A(x,t) \left[ |\nabla u|^{x+1} + \alpha \left| \frac{u}{v} \nabla v \right|^{x+1} - (x + 1) \left| \frac{u}{v} \nabla v \right|^{x-1} (\nabla u) \cdot \left( \frac{u}{v} \nabla v \right) \right] \\
+ \nabla \cdot \left( \frac{|u|^{x+1}}{\varphi(v)} A(x,t)|\nabla v|^{x-1} \nabla v \right) \\
= A(x,t)|\nabla u|^{x+1} - C(x,t)|u|^{x+1} + \frac{|u|^{x+1}}{\varphi(v)} P_2[v], \quad (x,t) \in G \times I. \tag{12}
\]

Theorem 4. Assume that (H1), (H2) hold, and that there is a nontrivial function \( u \in C^1(\tilde{G}; \mathbb{R}) \) such that \( u = 0 \) on \( \partial G \) and

\[
\lim_{t \to \infty} \int_T^t M[u](s) \, ds = -\infty \quad \text{for any } T > 0,
\]

where

\[
M[u](t) = \int_G \left[ A(x,t)|\nabla u|^{x+1} - C(x,t)|u|^{x+1} \right] \, dx.
\]

If \( 0 < x \leq 1 \), then every solution \( v \in \mathcal{D}_P(\Omega) \) of (1) which is nonoscillatory on \( \tilde{\Omega} \) satisfies (5). If \( x > 1 \), then every solution \( v \in \mathcal{D}_P(\Omega) \) of (1) is oscillatory on \( \tilde{\Omega} \).

Proof. First we consider the case where \( 0 < x \leq 1 \). Let \( v \in \mathcal{D}_P(\Omega) \) be a solution of (1) which is nonoscillatory on \( \tilde{\Omega} \). There exists a number \( t_0 > 0 \) such that \( v \neq 0 \) on \( \tilde{G} \times [t_0, \infty) \). Since the first term of the left side of (12) is nonnegative, integrating (12) over \( G \) yields

\[
0 \leq M[u](t) + \int_G |u|^{x+1} \frac{1}{|v|^{x-1}P_2[v]} \, dx \\
= M[u](t) + \int_G |u|^{x+1} \frac{1}{|v|^{x-1}P_2[v]} \, dx, \quad t \geq t_0.
\]
Arguing as in the proof of Theorem 2, we obtain

\[ z(T) - z(t_0) \geq -\int_{t_0}^{T} M[u](s) \, ds \quad (\alpha = 1), \]

\[ \frac{1}{-\alpha + 1} (z(T) - z(t_0)) \geq -\int_{t_0}^{T} M[u](s) \, ds \quad (\alpha \neq 1), \]

where \( z(t) \) is given by (11). By the same arguments as in the proof of Theorem 2 we conclude that \( v \) satisfies (5). The case where \( \alpha > 1 \) can be treated similarly, and we are also led to a contradiction. The proof is complete.

**Corollary 3.** Let \( 0 < \alpha \leq 1 \). Assume that \( (H_1), (H_2) \) hold, and that there is a nontrivial function \( u \in C^1(\tilde{G}; \mathbb{R}) \) such that \( u = 0 \) on \( \partial G \) and

\[ \lim_{t \to \infty} \int_{t}^{T} M[u](s) \, ds = -\infty \quad \text{for any } T > 0. \]

Then every bounded solution \( v \in D_{P\alpha}(\Omega) \) of (1) is oscillatory on \( \tilde{\Omega} \).

**Corollary 4.** Let \( 0 < \alpha \leq 1 \). Assume that the same hypotheses as those of Theorem 4 hold. If \( v \in D_{P\alpha}(\Omega) \) is a solution of (1) which is nonoscillatory on \( \tilde{\Omega} \), then \( v \) is unbounded in \( \tilde{\Omega} \).

The proofs of Corollaries 3 and 4 are quite similar to those of Corollaries 1 and 2, respectively, and will be omitted.

**Example 1.** We consider the parabolic equation

\[ \frac{\partial v}{\partial t} - \left( \frac{\partial}{\partial x} \left( A_0 \left| \frac{\partial v}{\partial x} \right|^{\alpha-1} \frac{\partial v}{\partial x} \right) + C_0 |v|^{\alpha-1}v \right) = 0, \quad (x,t) \in (-1,1) \times (0,\infty), \]

where \( \alpha = 2 \) or \( \frac{1}{2} \), and \( A_0 \) and \( C_0 \) are positive constants. Here \( n = 1 \), \( A(x,t) = A_0 > 0 \), \( C(x,t) = C_0 > 0 \), \( G = (-1,1) \) and \( \Omega = (-1,1) \times (0,\infty) \). First we deal with the case where \( \alpha = 2 \). Letting \( u = 1 - x^2 \), we find that \( u(-1) = u(1) = 0 \). A simple calculation shows that

\[ M[u](t) = \int_{-1}^{1} [A_0 |u'(x)|^3 - C_0 |u(x)|^3] \, dx \]

\[ = 4A_0 - \frac{32}{35} C_0. \]

Hence, it is clear that

\[ \lim_{t \to \infty} \int_{t}^{T} M[u](s) \, ds = -\infty \]

for any \( T > 0 \) if \( A_0 < \frac{8}{35} C_0 \). Theorem 4 implies that every solution \( v \) of (13) with \( \alpha = 2 \) is oscillatory on \( \tilde{\Omega} \). Next we treat the case where \( \alpha = \frac{1}{2} \). Choosing \( u = 1 - x^2 \), we see that \( u(-1) = u(1) = 0 \).
and
\[
M[u](t) = \int_{-1}^{1} [A_0|u'(x)|^{3/2} - C_0|u(x)|^{3/2}] \, dx
\]
\[
= \frac{8}{5} \sqrt{2} A_0 - \frac{3}{8} \pi C_0.
\]
If \( A_0 < \frac{15}{128} \sqrt{2} \pi C_0 \), we observe that
\[
\lim_{t \to \infty} \int_{t}^{t'} M[u](s) \, ds = -\infty
\]
for any \( T > 0 \). It follows from Theorem 4 that every solution \( v \) of (13) with \( \alpha = \frac{1}{2} \) which is nonoscillatory on \( \bar{\Omega} \) satisfies
\[
\lim_{t \to \infty} \int_{-1}^{1} (1 - x^2)^{3/2} |v|^{1/2} \, dx = \infty.
\]

**Example 2.** We consider the parabolic equation
\[
\frac{\partial v}{\partial t} - \left( \frac{\partial}{\partial x} \left( \frac{\partial v}{\partial x} \right) \right) C_0|v| = 0, \quad (x,t) \in (0,\pi) \times (0,\infty),
\]
where \( A_0 \) and \( C_0 \) are positive constants satisfying \( A_0 < C_0 \). Here \( n=1, \alpha=2, A(x,t)=A_0 > 0, C(x,t)=C_0 > 0, G = (0,\pi) \) and \( \Omega = (0,\pi) \times (0,\infty) \). Letting \( u = \sin x \), we observe that \( u(0) = u(\pi) = 0 \) and
\[
M[u](t) = \int_{0}^{\pi} [A_0|u'(x)|^{3} - C_0|u(x)|^{3}] \, dx
\]
\[
= 2A_0 \int_{0}^{\pi/2} \cos^3 x \, dx - 2C_0 \int_{0}^{\pi/2} \sin^3 x \, dx
\]
\[
= \frac{4}{3} (A_0 - C_0) < 0.
\]
Hence, it is obvious that
\[
\lim_{t \to \infty} \int_{t}^{t'} M[u](s) \, ds = -\infty
\]
for any \( T > 0 \). It follows from Theorem 4 that every solution \( v \) of (14) is oscillatory on \( \bar{\Omega} \).

**Example 3.** We consider the parabolic equation
\[
\frac{\partial v}{\partial t} - \left( \frac{\partial}{\partial x} \left( \frac{\partial v}{\partial x} \right) \right) + e^t = 0, \quad (x,t) \in (0,\pi/2) \times (0,\infty).
\]
Here \( n=1, \alpha=\frac{1}{2}, A(x,t)=1, C(x,t)=e^{t/2}, G = (0,\pi/2) \) and \( \Omega = (0,\pi/2) \times (0,\infty) \). Choosing \( u = x \cos x \), we see that \( u(0) = u(\pi/2) = 0 \) and
\[
M[u](t) = \int_{0}^{\pi/2} [u'(x)]^{3/2} - e^{t/2} |u(x)|^{3/2} \, dx
\]
\[
= K_1 - K_2 e^{t/2}.
\]
where

\[ K_1 = \int_0^{\pi/2} |\cos x - x \sin x|^{3/2} \, dx > 0, \]

\[ K_2 = \int_0^{\pi/2} |x \cos x|^{3/2} \, dx > 0. \]

Therefore, we obtain

\[ \int_T^t M[u](s) \, ds = \int_T^t [K_1 - K_2 e^{s/2}] \, ds \]

\[ = K_1 (t - T) - 2K_2 (e^{t/2} - e^{T/2}) \]

\[ = e^{t/2} (-2K_2 + K_1 t e^{-t/2}) - K_1 T + 2K_2 e^{T/2}, \]

and hence

\[ \lim_{t \to \infty} \int_T^t M[u](s) \, ds = -\infty \]

for any \( T > 0 \). It follows from Theorem 4 that every solution \( v \) of (15) which is nonoscillatory on \([0, \pi/2] \times [0, \infty)\) satisfies

\[ \lim_{t \to \infty} \int_0^{\pi/2} |x \cos x|^{3/2} |v|^{1/2} \, dx = \infty. \]

For example, \( v = e^t \) is such a solution.

References


