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On a Generalization of Krull Domains

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INTRODUCTION

Let R be an integral domain with quotient field L and let F be a family of valuations on L satisfying the following:

(1) Each $v \in F$ has rank one.

(2) $R = \cap \{R_v \mid v \in F\}$,

(3) $R_v = R_{P(v)}$ for each $v \in F$.

Here $R_v = \{x \in L \mid v(x) \geqslant 0\}$ and $P(v) = \{x \in R \mid v(x) > 0\}$. $P(v)$ is called the center of v on R. The family F is said to be of finite character if for any nonzero $x \in L$ there are only finitely many $v \in F$ such that $v(x) \neq 0$. We say R satisfies (#) with respect to F, if for distinct subsets F_1, F_2, of F, we have $\cap \{R_w \mid w \in F_1\} \neq \cap \{R_w \mid w \in F_2\}$ (This is a generalization of Gilmer's property (#) in [2].).

In [3], a semigroup $\mathcal{O}(R)$ of fractionary ideal classes was constructed using the family F. Necessary and sufficient conditions were determined for $\mathcal{O}(R)$ and $\mathcal{D}(R)$ to be isomorphic, where $\mathcal{D}(R)$ is the divisor group of R constructed in [1]. Following [1], a nonzero fractionary ideal A of R is said to be divisorial If it is the intersection of all principal fractionary ideals which contain it. if the elements of F are discrete and F is of finite character (i.e., if R is a Krull domain) then $P(v)$ is divisorial for each $v \in F$. In [3] it was shown that if F is a family of discrete valuations satisfying (1), (2), (3), then $\mathcal{O}(R) \cong \mathcal{D}(R)$ iff $P(v)$ is divisorial for each $v \in F$. An example was given to show that F need not be of finite character in order for each $P(v)$ to be divisorial. In this paper we investigate integral domains which have family F of valuations satisfying (1), (2), (3), above and (4) $P(v)$ is divisorial for each $v \in F$.

For the construction and properties of $\mathcal{D}(R)$, the reader is referred to [1]. All notation concerning $\mathcal{D}(R)$ will be that of [1]. Prime ideals are always nonzero and not all of R. Proper containment is indicated by $<$. Otherwise, the notation is that of [4] and [5].

1. Definition and Characterizations of K Domains

For completeness we briefly outline the construction of $\mathcal{Cl}(R)$. Thus, let $I(R)$ denote the collection of nonzero fractionary ideals of R. For $A, B \in I(R)$, define $A \sim B$ iff $v(A) = v(B)$ for all $v \in F$, where $v(A) = \inf\{v(a) \mid a \in A\}$. Then $\sim$ is an equivalence relation on R, $[A]$ denotes the equivalence class of of A and $\mathcal{Cl}(R)$ denotes the collection of all such equivalence classes. For $[A], [B] \in \mathcal{Cl}(R)$, put $[A] \leqslant [B]$ iff $v(A) \leqslant v(B)$ for all $v \in F$, and put $[A] + [B] = [AB]$. With these definitions $\mathcal{Cl}(R)$ is a partially ordered Abelian semigroup with $0 = [R]$ satisfying the cancellation law. In [3] we have shown that the map $g : \mathcal{Cl}(R) \to \mathscr{D}(R)$ defined by $g([A]) = \operatorname{div}_R(A)$ is an order preserving homomorphism from $\mathcal{Cl}(R)$ onto $\mathscr{D}(R)$.

Definition 1.1. R is called a K domain if there is a family F of valuations on L satisfying the following:

(i) Each $v \in F$ has rank one.

(ii) $R = \cap \{R_v \mid v \in F\}$.

(iii) $R_v = R_{P(v)}$ for each $v \in F$.

(iv) $P(v)$ is divisorial for each $v \in F$.

The family F is called a family of essential valuations for the K domain R.

For any nonzero fractionary ideal A of R, $\tilde{A} = \cap \{Rx \mid A \subseteq Rx\}$. Thus A is divisorial iff $A = \tilde{A}$ (see [1]).

Proposition 1.2. *If F is a family of essential valuations for the K domain R then each $v \in F$ is discrete.*

Proof. We observe that R is the intersection of rank one valuation rings and, hence, is completely integrally closed. Thus $\mathscr{D}(R)$ is a group (see [1]). Now let $v \in F$. If v is not discrete then $P = P^2$. Then $\operatorname{div}_R(P) = \operatorname{div}_R(P^2)$, i.e., $2 \operatorname{div}_R(P) = \operatorname{div}_R(P)$ and $\operatorname{div}_R(P) = 0$. But then $\tilde{P} = R$ contradicting $\tilde{P} = P < R$.

It is well-known that if R is a Krull domain with family F of essential rings R_P where P ranges over the collection of all minimal primes of R. Thus when R is a Krull domain, the family F is uniquely determined by R.

The following proposition shows that if R is a K domain with family F of essential valuations then F is uniquely determined by R.

Proposition 1.3. *Let F be a family of essential valuations for the K domain R. Then the valuation rings R_v, $v \in F$ are identical with the quotient rings R_P where P runs over all divisorial primes of R.*

Proof. It is clear that the collection of valuation rings R_v, $v \in F$, is contained in the collection of quotient rings R_P, P a divisorial prime. On the other hand, let P be a divisorial prime in R. If $P \neq P(v)$ for any $v \in F$, then $[P] = 0$, i.e., $v(P) = 0$ for all $v \in F$. But then $g([P]) = \text{div}_R(P) = 0$ and $\tilde{P} = R$, contradicting $\tilde{P} = P < R$.

In 1.2 it was shown that if F is the family of essential valuations of a K domain R then each $v \in F$ is discrete. In practice it is often obvious whether or not a valuation is discrete. We now study necessary and sufficient conditions for each $P(v)$ to be divisorial when the elements of F are assumed to be discrete. Thus, let F be a family of valuations on L satisfying:

(1) each $v \in F$ has rank one and is discrete;

(2) $R = \cap \{R_v \mid v \in F\}$;

(3) $R_v = R_{P(v)}$ for each $v \in F$.

The following two propositions are of use and interest because they give necessary and sufficient conditions for a single $P(v)$, $v \in F$, to be divisorial.

PROPOSITION 1.4. *Let F be a family of valuations satisfying* (1), (2), (3). *For $v \in F$, $P(v)$ is divisorial iff $P(v) < P(v)\,[R : P(v)]$.*

Proof. Let $P(v) = P$. If $P < P(R : P)$, then $[P] > 0$ and $[P(R : P)] = 0$. Now $R : P = R : \tilde{P}$ and $P \subseteq \tilde{P}$, so $P(R : P) \subseteq \tilde{P}(R : \tilde{P}) \subseteq R$. Thus, $0 = [P] + [R : P] = [\tilde{P}] + [R : \tilde{P}]$. Since $\mathcal{A}(R)$ is a cancellative semigroup we have $P = \tilde{P}$. For if $P < \tilde{P}$ then $[\tilde{P}] = 0$. On the other hand, suppose $P = \tilde{P}$. If $P = P(R : P)$ then $\text{div}_R(P) = \text{div}_R[P(R : P)] = 0$ so that $\tilde{P} = R$. This contradicts $\tilde{P} = P < R$.

PROPOSITION 1.5. *Let F be a family of valuations satisfying* (1), (2), (3). *For $v \in F$, $P(v)$ is divisorial iff $R < R : P(v)$.*

Proof. Let $P = P(v)$, $v \in F$.

($\Rightarrow$) $R : P = \{x \in L \mid xP \subseteq R\}$, so $R \subseteq R : P$. Now R and $R : P$ are divisorial, so if $R < R : P$, then $\text{div}_R(R) = 0 \neq \text{div}_R(R : P)$. Thus, if P is divisorial and $R = R : P$, then

$$0 = \text{div}_R[P(R : P)] = \text{div}_R(P) + \text{div}_R(R : P) = \text{div}_R(P) + 0 = \text{div}_R(P).$$

But then $\tilde{P} = R$ contradicts $\tilde{P} = P < R$.

($\Rightarrow$) If $R < R : P$ then $\text{div}_R(R : P) \neq 0$. Then

$$0 = \text{div}_R[P(R : P)] = \text{div}_R(P) + \text{div}_R(R : P) \quad \text{and} \quad \text{div}_R(P) \neq 0.$$

Hence, $\tilde{P} \neq R$. It follows that $\tilde{P} = P$ as in the proof of 1.4.

For $v \in F$, we let $F_v = F - \{v\}$. The proof of the following lemma is substantially the same as Lemma 1 of [2] and is omitted.

LEMMA 1.6. *R satisfies (#) with respect to F iff $\cap \{R_w \mid w \in F_v\} \not\subseteq R_v$ for any $v \in F$.*

PROPOSITION 1.7. *Let R be an integral domain with family F satisfying* (1), (2), (3) *above. R is a K domain iff R satisfies (#) with respect to F.*

Proof. ($\Rightarrow$) This follows from Proposition 1.3 and Lemma 1.6.

($\Leftarrow$) Suppose R satisfies (#) with respect to F and let $v \in F$. By Lemma 1.6, there is $x \in \cap \{R_w \mid w \in F_v\}$ such that $x \notin R_v$. Then $w(x) \geqslant 0$ for all $w \in F_v$. Since $x \notin R_v$, $v(x) = -n$ for some integer $n > 0$. We may assume that $v[P(v)] = 1$ for each $v \in F$ since the elements of F are discrete. Let $y \in P(v)$ be such that $v(y) = 1$. Then $v(xy^{n-1}) = -1$, so $z = xy^{n-1} \notin R$. But $z \in R : P(v)$ since for each $u \in F$ we have $u[zP(v)] \geqslant 0$. Thus, $R < R : P(v)$ and $P(v)$ is divisorial by Proposition 1.5.

We can now state the following theorem.

THEOREM 1.8. *Let R be an integral domain with family F of valuations satisfying* (1), (2), (3), *above. The following are equivalent.*

(i) *R is a K domain with family F of essential valuations.*

(ii) *$\mathcal{C}\ell(R)$ is a group.*

(iii) *The map $g : \mathcal{C}\ell(R) \to \mathcal{D}(R)$ is an isomorphism.*

(iv) *$v(A) = v(\tilde{A})$ for all $v \in F$, $A \in I(R)$.*

(v) *$P(v) < P(v)\,[R : P(v)]$ for each $v \in F$.*

(vi) *$R < R : P(v)$ for each $v \in F$.*

(vii) *R satisfies (#) with respect to F.*

Proof. The equivalence of (i) through (iv) is found in [3]. The equivalence of (i), (v), (vi), and (vii) follows from the above propositions.

2. EXTENSIONS OF K DOMAINS

Throughout this section we shall assume that R is an integral domain with quotient field L and family F of valuations satisfying conditions (1), (2), (3), of Section 1.

Let L' be a finite, algebraic extension of L with R' denoting the integral closure of R in L'. Let F' denote the collection of extensions of elements of F to valuations on L'. By Theorem 30, p. 87 of [5], F' is a family of valuations on L', satisfying (1), (2), (3), with R replaced by R'. For $w \in F'$, w is the extension of some $v \in F$. Letting $P' = P'(w)$, we have $P' \cap R = P$, where $P = P(v)$. The above notation and hypothesis are assumed in the following proposition.

PROPOSITION 2.1. *If P is divisorial then P' is divisorial.*

Proof. We may assume without loss of generality that $w[P'(w)] = 1$ for each $w \in F'$. Since P is divisorial, by Proposition 1.5 we have $R < R : P$. Thus, there is $x \in L - R$ such that $xP \subseteq R$. Then $w(x) = -n$ for some integer $n \geqslant 1$. Let $P_1',..., P_t'$ be the distinct (from P' and each other) primes of R' which correspond to the other extensions of $v \in F$. Let $x_1 \in P_1' - P',..., x_t \in P_t' - P'$ and let $a = x_1 x_2 ... x_t$ ($a = 1$ if P' is the unique prime lying above P.) Let $y \in P'$ be such that $w(y) = 1$, and consider $z = a^n x y^{n-1}$. We have $w(z) = -1$ and $u(z) \geqslant 0$ for all $u \in F'$, $u \neq w$. It follows that $zP' \subseteq R'$ but $z \notin R'$ so that $R' < R' : P'$. Hence, P' is divisorial by Proposition 1.5.

We can now state the following theorem.

THEOREM 2.2. *Let R be a K domain with quotient field L and family F of essential valuations. Let L' be a finite, algebraic extension of L, let F' denote the family of extensions of elements of F to L', and let R' denote the integral closure of R in L'. Then R' is a K domain with F' as family of essential valuations.*

Now let X be an indeterminate. Let F' denote the collection of canonical extensions of elements of F to $L(X)$ and let G denote the family of $a(x)$-adic valuations on $L(X)$, where $a(x)$ is a nonconstant irreducible polynomial in $L[X]$. It follows that $F' \cup G$ is a family of valuations on $L(X)$ satisfying (1), (2), (3), of Section 1 with R replaced by $R[X]$ (see [5]).

The following lemma is found in [3]. We repeat the proof here for completeness. We also observe that only conditions (2), (3) are used in the proof.

LEMMA 2.3. *If R satisfies (#) with respect to F then $R[X]$ satisfies (#) with respect to $F' \cup G$.*

Proof. Let $w \in F' \cup G$. If $w \in G$ then w is an $a(x)$-adic valuation for some nonconstant irreducible polynomial $a(x) \in L[X]$. Without loss of generality we may assume that $a(x) \in R[X]$. Suppose $a(x) = a_n X^n + \cdots + a_1 x + a_0$, $a_i \in R$. Let $b = \prod_{a_k \neq 0} a_k$. Then $b \neq 0$ since $a_n \neq 0$ and

$$v(b) = \sum_{a_k \neq 0} v(a_k) \geqslant \min_{0 \leqslant j \leqslant n} v(a_j) \geqslant 0$$

for all $v \in F$ since $a_k \in R$ for all k and every $v \in F$ is nonnegative on R. Then for $v' \in F'$ we have $v'[b/a(x)] = v'(b) - v'[a(x)] = v(b) - \min_{0 \leqslant j \leqslant n} v(a_j) \geqslant 0$. If $u \in G$, $u \neq w$, then u is a $q(x)$-adic valuation for some nonconstant irreducible polynomial $q(x)$ which does not divide $a(x)$, and, hence, $u[b/a(x)] = 0$. So $b/a(x) \in \cap \{R[x]_u \mid u \in (F' \cup G)_w\}$. However, $b/a(x) \notin R[x]_w$ since $w[b/a(x)] = -1 < 0$. This if $w \in G$ then $\cup \{R[x]_u \mid u \in (F' \cup G)_w\} \not\subseteq R[x]_w$. On the other hand, if $w \in F'$, then $w = v'$ for some $v \in F$. Since $\cap \{R_u \mid u \in F_v\} \not\subseteq R_v$, there is $x \in \cap \{R_u \mid u \in F_v\}$ with $x \notin R_v$. It follows that $x \in \cap \{R[x]_w \mid w \in (F' \cup G)_{v'}\}$, $x \notin R[x]_{v'}$. Thus, R satisfies (#) with respect to $F' \cup G$ by Lemma 1.6.

Recalling Theorem 1.8, and retaining the above notation, we have the following immediate corollary.

THEOREM 2.4. *Let R be a K domain with family F of essential valuations and let x be an indeterminate. Then $R[x]$ is a K domain with $F' \cup G$ as family of essential valuations.*

The proof of the following corollary is immediate by induction.

COROLLARY 2.5. *Let R be a K domain and let $x_1, \ldots, x_n$ be indeterminates. Then $R[x_1, \ldots, x_n]$ is also a K domain.*

It is well-known that when R is a Krull domain with family F of essential valuations, then a domain T such that $R \subseteq T \subseteq L$ is a Krull domain if there is a subfamily G of F such that $T = \cap \{R_v \mid v \in G\}$. For K domains, the author has been able to prove the following.

PROPOSITION 2.6. *Let R be a K domain with family F of essential valuations and let $G \subseteq F$. The domain $T = \cap \{R_v \mid v \in G\}$ is a K domain with G as family of essential valuations.*

Proof. It is easy to show that G satisfies (1), (2), (3), of Section 1 and that T satisfies (#) with respect to G.

It is also a well-known fact that if R is a Krull domain and if S is a multiplicative system in R then R_S is a Krull domain. In the next section we give an example to show that this is not true for K domains in general.

3. An Example

In this section we give an example of a K domain R which is not a Krull domain. We show that there is a multiplicative system S in R such that R_S is not a K domain. We also indicate that $C(R) \cong C(R[x_1, \ldots, x_n])$, where $C(R)$ denotes the class group of R and $x_1, \ldots, x_n$ are indeterminates.

Let R denote the ring of entire functions, let C denote the set of complex numbers and let Z denote the additive group of integers. For $a \in C$, define $v_a : R - \{0\} \to Z$ by $v_a(f) = n$ if a is a zero of order $n \geqslant 0$ of f. Define $v_a(0) = +\infty$ for all $a \in C$. Each v_a can be extended to a valuation on the quotient field of R. Let F denote the collection of all such valuations. The following properties are easy to verify:

(i) Each $v \in F$ is rank one, discrete.

(ii) $R = \cap \{R_v \mid v \in F\}$.

(iii) $R_v = R_{P(v)}$ for each $v \in F$.

(iv) $P(v_a) = (z - a) R$ and, hence, $P(v)$ is divisorial for each $v \in F$.

(v) F is not of finite character.

Thus, R is a K domain which is not a Krull domain.

In the above example it can also be shown that $P(v)$ is maximal for each $v \in F$ and that R has the following property:

(*) Every rank one, discrete valuation on the quotient field of R which is nonnegative on R is equivalent to some $v \in F$.

Returning now to the general situation, let D be a K domain with quotient field Q and family F of essential valuations. Consider the following statement about D.

(**) D_S is a K domain for every multiplicative system S in R.

(**) is true when D is a Krull domain.

The above example does show that (**) is false for K domains in general. For let $\{z_m\}_{m=1}^{\infty}$ be a sequence of complex numbers such that $\lim z_m = \infty$. For $m = 1, 2, \ldots$, let f_m be an entire function whose only zeroes are z_m, $z_{m+1}, \ldots$. The ideal A generated by $\{f_m \mid m = 1, 2, \ldots\}$ is proper and not contained in $P(v)$ for any $v \in F$. Thus, A is contained in a maximal ideal M which is not equal to $P(v)$ for any $v \in F$. It follows the R_M is not a K domain since R satisfies (*).

Let $B(D)$ denote the subgroup of $\mathfrak{A}(D)$ generated by principal fractionary ideals of D. $C(D) = \mathfrak{A}(D)/B(D)$ is called the class group of D. When D is a Krull domain and $x_1, \ldots, x_n$ are indeterminates it is known that $C(D) \simeq C(D[x_1, \ldots, x_n])$. This statement is true for the example R above but it is unknown to the autuor if this is true for K domains in general.

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