HEEGAARD DIAGRAMS AND HOMOTOPY 3-SPHERES

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ALL SURFACES and three-manifolds are assumed to be orientable throughout the paper. A complete system (CS) $x = \{x_1, x_2, \ldots, x_n\}$ on a closed surface $S$ means a set of simple closed curves on $S$ which are:

1. mutually disjoint; and
2. do not separate $S$

and such that the set is maximal with respect to properties (1) and (2). [It follows that $n$ is the genus of $S$ and that the result of surgering $S$ along $x$ (i.e. along each $x_i$) is a 2-sphere.]

Given $(S, x)$, where $x$ is a CS on $S$, we can construct a solid handle body $T(x)$ as follows: glue a (thickened) 2-disc to $S$ along each $x_i$, thereby “realizing” the surgery of $S$ along $x$, and then glue in a 3-ball to the 2-sphere which results from this surgery:

A Heegaard diagram (H-diagram) is a closed surface $S$ with two complete systems $x, y$. The H-diagram defines a 3-manifold

$$M(x, y) = T(x) \cup_3 T(y).$$

$M(x, y)$ has a handle presentation in which the thickened discs attached to the $x_i$ (resp. $y_i$) are the 2-handles (resp. 1-handles), and the 3-ball which completes $T(x)$ (resp. $T(y)$) is the single 3-handle (resp. 0-handle). It follows from standard results (on existence of nice handle presentations, etc.) that any closed (orientable) 3-manifold can be obtained in this way.

This paper is concerned with the possibility of using H-diagrams to find counterexamples to the Poincaré conjecture.

There is a very elegant characterization of a homotopy 3-sphere in terms of any corresponding H-diagram:

**Theorem 1.** $S(x, y)$ is an H-diagram of a homotopy 3-sphere if and only if there is an embedding of $T(y)$ in $S^3$ such that $x_1, x_2, \ldots, x_n$ bound disjoint orientable surfaces $S_1, S_2, \ldots, S_n$ in $S^3 - T(y)$. 

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Theorem 1 is stated in Haken's paper [1] and attributed to Moise and others. For completeness, we give a proof of Theorem 1 in §1.

Elegant though Theorem 1 is, it is useless for detecting a possible counterexample to the Poincaré conjecture because it does not provide a computable way to recognize an H-diagram which represents a homotopy 3-sphere. Given an H-diagram there is no effective way to search through all possible knotted embeddings of $T(y)$ in $S^3$ or to check for a given knotted embedding whether $x_1, x_2, \ldots, x_n$ bound disjoint surfaces.

However, as we will show in this paper, Theorem 1 can be used to derive another characterization (Theorem 2) of an H-diagram of a homotopy 3-sphere, which leads at once to a computer program to list all such diagrams.

Before stating Theorem 2, we need to prove some, more-or-less well-known, facts about systems of curves on a surface.

We say complete systems $x$, $y$ are equivalent (written $x \sim y$) if $T(x)$ is homeomorphic to $T(y)$ by a homeomorphism fixed on $S$. (Clearly $M(x, y)$ depends only on the $\sim$ classes of $x$ and $y$.)

By a super-complete system (SCS) on $S$ we mean any set of disjoint simple closed curves on $S$ which contains a CS. Given an SCS $x$ on $S$, then $T(x)$ is constructed exactly as for a CS: the only difference is that there may be several 3-balls to be glued in at the end, and there is therefore a similar notion of equivalence ($\sim$) for SCSs.

Remark. If $x_+ \supset x$ are SCSs on $S$, then $x_+ \sim x$.

Proof. The new discs and 3-balls in $T(x_+)$ can be found inside the 3-ball(s) of $T(x)$. Q.E.D.

The remark implies that insertions and deletions of curves in an SCS does not change the $\sim$ class. We now prove the converse.

**Lemma 1.** Suppose $x$ is an SCS on $S$ and $y$ is any set of disjoint simple closed curves on $S$ such that each $y_i$ bounds a disc in $T(x)$. Then $x \sim y_+$ by insertions and deletions where $y_+ \supset y$.

Proof. Suppose $y_1$ bounds the disc $D$ in $T(x)$ and let $D_1, \ldots, D_4$ denote the discs bounded by $x = \{x_1, \ldots, x_4\}$. We simplify the transverse intersection

$$Q = D \cap (D_1 \cup \ldots \cup D_4).$$

If $Q$ contains a closed curve, then the usual "push across a 3-ball" argument isotopes $D$ so as to simplify $Q$. If not, then $Q$ consists of arcs. Choose an outermost arc $z$ with endpoints $a, b$ in $x_i$, say. Let $\beta$ be the arc in $y_1$ from $a$ to $b$ on the outside. Define two new curves $x'_i, x''_i$ by cutting $x_i$ at $a, b$, inserting $\beta$ and then pushing away from $D$ and $x_i$ a little:

![Fig. 2]
Then if we replace $x_i$ by $x'_i$ and $x''_i$ in $x$ (two insertions and one deletion), we have $x'_i$, $x''_i$ bounding discs $D'_i$, $D''_i$ such that $Q' = D_i \cap (D_1 \cup \ldots \cup D'_i \cup D''_i \cup \ldots ; D_j)$ is simpler. By a finite number of such moves we have $Q = \emptyset$ and $y_i$ can be inserted. We continue in the same way to insert $y_2, y_3, \ldots$ and since the only deleted curves are ones which meet a $y_i$ and since the $y_i$ are mutually disjoint, the process ends with the required system $y_\alpha$. Q.E.D.

**Corollary 1.** $x \sim y \iff x$ is obtained from $y$ by insertions and deletions.

**Proof.** $\Leftarrow$ See Remark. $\Rightarrow$ Identify $T(x)$ and $T(y)$ by the homeomorphism, then $y$ satisfies the hypotheses of the lemma and hence $y_\alpha \supset y$ is obtained from $x$ by insertion and deletion. But $y_\alpha$ then deletes down to $y$. Q.E.D.

Suppose now that we have any set of simple closed curves $x = \{x_1, \ldots, x_n\}$ on a surface $S$ and that $z$ is a further curve. Then, by orienting $S$ and all the curves, we can read off a (cyclic) word $\omega(z, x)$ in the symbols $x_1, \ldots, x_n$ by traversing $z$ once (in the given direction) and reading $x_i$ or $x_i^{-1}$ for each transverse crossing with $x_i$ by the rules:

![Fig. 3.](image)

If $\omega(z, x)$ reduces by cancellation to the empty word, then we write $\omega(z, x) = e$ and clearly this statement about $\omega$ is independent of all chosen orientations.

**Lemma 2.** Suppose $x$ is an SCS on $S$, then $x$ bounds a disc in $T(x) \iff \omega(z, x) = e$.

**Proof.** $\Rightarrow$ Suppose $z$ bounds $D$ and $x_1, \ldots, x_n$ bound $D_1, \ldots, D_n$. The triviality of $\omega(z, x)$ follows from inspecting the arcs in the transverse intersection $Q = D_i \cap (D_1 \cup \ldots \cup D_j)$.

![Fig. 4.](image)

An outermost arc $a$ corresponds to a subword $x_i x_i^{-1}$ or $x_i^{-1} x_i$ in $\omega = \omega(z, x)$ which cancels to yield $\omega'$, say. Then by induction on the number of arcs in $Q$, $\omega' = e$ and hence $\omega = e$.

$\Leftarrow$ $\omega(z, x) = e$ implies that $a$ represents 1 in $\pi_1(T(x))$ and hence $x$ bounds a disc by Dehn's lemma. (Actually, this is a very special case of Dehn's lemma which has an elementary proof using handle slides (see [5])) Q.E.D.

Combining Lemmas 1 and 2 we have the combinatorial criterion for equivalence of SCSs:

**Corollary 2.** $x \sim y \iff \omega(y_i, x) = e$ for each $y_i \in y$. 
Proof. Follows at once from Lemma 2. By Lemma 2, each $y_i$ bounds a disc in $T(x)$ and hence $x \sim y_i \Rightarrow y$ by Lemma 1. Q.E.D.

Now let $T$ denote the standard solid handlebody of genus $n$ embedded in $\mathbb{R}^3$. The boundary of $T$ is denoted $S$, and there are two standard CSs on $S$, $a = \{a_1, \ldots, a_n\}$ bounding discs outside $T$ and $b = \{b_1, \ldots, b_n\}$ bounding discs inside $T$:

![Diagram of a handlebody](image)

Fig. 5.

An H-system is a CS $\{x_1, \ldots, x_t, y_{t+1}, \ldots, y_n\}$ on $S$ such that

1. $\omega(a_i, x_i) = e$, $i = 1, 2, \ldots, n$ where $x = \{x_1, \ldots, x_t\}$, and
2. $\omega(y_i, b) = e$, $i = t + 1, t + 2, \ldots, n$.

By Lemma 2, condition (2) is equivalent to saying that each $y_i$ bounds a disc in $T$, and hence by Lemma 1 we can extend $y_{t+1}, \ldots, y_n$ to a CS $y_+ = \{y_1, \ldots, y_n\}$ equivalent to $b$ (and clearly $y_+$ is determined up to equivalence). Now let $S'$ be the result of surgering $S$ along $y_{t+1}, \ldots, y_n$, then there are two CSs on $S'$, namely $x$ and $y = \{y_1, \ldots, y_t\}$ (and $y$ is again clearly determined up to equivalence). Thus the H-system gives rise to the associated H-diagram $S'(x, y)$.

**Theorem 2.** Let $S'(x, y)$ be an H-diagram associated to an H-system. Then $M(x, y)$ is a homotopy 3-sphere and every H-diagram for a homotopy 3-sphere arises in this way.

Since the data for an H-system are clearly effectively computable and since the process of constructing $S'(x, y)$ from the data is algorithmic, Theorem 2 leads at once to a computer program to list all H-diagrams of homotopy 3-spheres (at least up to equivalence of one of the systems) and hence to list all candidates for a counterexample to Poincaré conjecture.

There is also an algorithm to compute the Rohlin invariant of a homotopy 3-sphere from its H-diagram. (Lickorish's proof [3] that $\Omega_1 = 0$ and the new proof [4] are both algorithmic: they provide algorithms to convert an H-diagram into a surgery description. From the surgery description the Rohlin invariant can be computed, see [2].) Therefore there is a computer program which would effectively search for a strong counterexample to the Poincaré conjecture, i.e. a homotopy 3-sphere of Rohlin invariant 1†.

§1. PROOF OF THEOREM 1

Theorem 1 is proved using transversality and the following well-known lemma:

**Lemma 3.** $M^3$ is a homotopy 3-sphere if and only if there is a degree 1 map $f : S^3 \to M^3$.

**Proof of Lemma 3.** Suppose $M^3$ is a homotopy 3-sphere. Choose standard embeddings of $D^3$ in $S^3$ and $M^3$. $M^3 - D^3$ is contractible (by Whitehead's theorem, homology version) and hence there is a homotopy to zero of $S^2 = \partial D^3$ in $M^3 - D^3$. Define the degree 1 map

†This paper was written before Casson proved that the Rohlin invariant of a homotopy 3-sphere must be zero!
$f: S^3 \rightarrow M^3$ by mapping $D^3$ to $D^3$ by the identity and $S^3 - D^3$ to $N^3 - D^3$ by the homotopy of $S^2$ to zero.

Conversely, suppose $f: S^3 \rightarrow M^3$ is a degree 1 map. If $M^3$ is non-simply-connected, then there is a lift $\tilde{f}: S^3 \rightarrow \tilde{M}^3$. Now either $\tilde{M}^3$ is compact (in which case $\tilde{f}: \tilde{M}^3 \rightarrow M^3$ has finite degree $> 1$) or $\tilde{M}^3$ is non-compact (in which case $H_3(\tilde{M}^3) = 0$). In either case, $f_*: H_3(S^3) \rightarrow H_3(M^3)$ fails to be an isomorphism, which is a contradiction. So, we can assume $M^3$ is simply-connected and then, since $f$ is a homology equivalence (using duality), $f$ is a homotopy equivalence by Whitehead. Q.E.D.

Proof of Theorem 1. Suppose $M^3$ is a homotopy 3-sphere and

$$M^3 = D^3 \cup h_1 \cup \ldots \cup h_n \cup j_1 \cup \ldots \cup j_k \cup B^3$$

is a given nice handle decomposition of $M$, where $h_j$ are 1-handles and $j_k$ are 2-handles. That is, $S(x, y)$ is an H-diagram for $M^3$ where $S = \partial(D^3 \cup h_1 \cup \ldots \cup h_n)$, $y_i$ is the b-sphere of $h_i$ and $x_k$ is the a-sphere of $j_k$, for each $i, k$.

The notation core $(h_i)$ denotes the (1-dimensional) core of $h_i$ and, similarly, core $(j_k)$, the (2-dimensional) core of $j_k$.

Using the proof of Lemma 3 there is a degree 1 map $f: S^3 \rightarrow M^3$ such that $f: f^{-1}(D^3) \rightarrow D^3$ is a homeomorphism. Make $f$ transverse to the cores of the 1-handles $h_1, h_2, \ldots, h_n$, then $f^{-1}(\text{core } (h_i))$, for each $i$, consists of an arc starting and ending on $\partial D^3$ and a number of "spare" circles. By thinking of $h_i$ as a disc bundle over the core, it may be assumed that $f^{-1}(h_i)$ is a thickened version of $f^{-1}(\text{core } (h_i))$, i.e. it consists of a "tube" (the arc thickened) and a number of "spare tubes" which are the thickened "spare" circles: the "spare tubes" are, in fact, solid tori.

Next make $f$ transverse to the cores of the 2-handles and then $f^{-1}(\text{core } (j_k))$, for each $k$, is a surface with boundary lying on the "tubes" and "spare tubes" $f^{-1}(h_i)$ and on $\partial D^3$.

Call the collection $D^3, f^{-1}(h_i), f^{-1}(\text{core } (j_k))$ a transversality diagram for $f$. Then the diagram determines $f$ up to homotopy since each element of the diagram is mapped to a contractible subset of $M^3$. It follows that we can make "abstract" changes to the diagram and then use the changed diagram to redefine the map $f$. In particular, any free components of the diagram may be deleted and hence it may be assumed that the diagram is connected.
We now explain how to eliminate the "spare tubes". Since the diagram is connected there must be a connected surface \( S \) in the diagram (part of \( f^{-1} (\text{core (} j_k \text{)}) \), say) with one boundary component \( \partial \) meeting a tube and another lying on a "spare tube". At this point we need to observe that \( D^3 \) together with the tubes (not the spare tubes) is in fact a copy of \( T(y) = D^3 \cup h_1 \cup \ldots \cup h_n \) and hence \( \partial \) is a copy of \( x_k \) (the \( a \)-sphere of \( j_k \)). So, we can choose a point of \( S \) on the "spare tube" and join it by an arc \( \beta \) in \( S \) to a corresponding point on \( \partial \) (that is, a point which has the same image under \( f \) in \( x_k \)). Now perform a bridge move on the diagram using the arc \( \beta \) as pictured in Fig. 7. The figure explains how the surface \( S \) and any other surfaces (typified by \( S' \)) incident to the tubes are modified.

![Fig. 7.](image)

(This move can in fact be realized by a homotopy of \( f \).) The move reduces the number of "spare tubes" by one and hence, by induction, we can assume that there are no spare tubes. After eliminating the spare tubes, \( f^{-1}(T(y)) = D^3 \cup \text{tubes} \), is a copy of \( T(y) \) and \( f^{-1} (\text{core (} j_k \)) \), \( k = 1, 2, \ldots, n \), are disjoint surfaces spanning the copies of \( x_k \); in other words, we have found the required embedding of \( T(y) \) in \( S^3 \) so that \( x_1, x_2, \ldots, x_n \) bound disjoint surfaces in \( \overline{S^3 - T(y)} \).

For the converse, if such an embedding is given, then it may be regarded as a transversality diagram and hence it defines a degree 1 map \( S^3 \to M^3 \), and therefore \( M^3 \) is a homotopy sphere by Lemma 3. Q.E.D.

### §2. PROOF OF THEOREM 2

Throughout this section, \( T, S, a, b \) are the standard objects and systems as in the definition of an H-system. We need a geometrical interpretation of condition (1) in that definition.

**Lemma 4.** Let \( x = \{x_1, \ldots, x_t\} \) be a set of disjoint curves on \( S \), then

\[
\omega(a_i, x) = e, \quad i = 1, 2, \ldots, n,
\]

\( \iff \) \( x_1, \ldots, x_t \) bound disjoint surfaces

\( S_1, S_2, \ldots, S_t \) in \( \mathbb{R}^3 - T \).

**Proof.** \( \iff \) Denote by \( D_i \) the disc bounded by \( a_i \) outside \( T \). The triviality of \( \omega(a_i, x) \) follows by inspecting the transverse intersection \( D_i \cap (S_1 \cup \ldots \cup S_t) \) exactly as in the first half of the proof of Lemma 2.

\( \iff \) The triviality of \( \omega(a_i, x) \) means that we can find disjoint arcs in \( D_i \) for each \( i \) which joins pairs \( (x_j, x_j^{-1}) \) in \( a_i \cap x \) and such that all points of \( a_i \cap x \) are paired off as in Fig. 4. (That is, we
have transverse intersection sets, as if the required surfaces existed.) Now thicken each $D_i$ and thicken the arcs as well, then in the complement we have a 3-ball with a number of disjoint closed curves in its boundary which can be capped by disjoint 2-balls to complete the required surfaces. Q.E.D.

As an aside here, we remark that if we combine Lemmas 2 and 4 we see that a complete system $x$ on $S$ which bounds disjoint surfaces outside $T$ is equivalent to the standard system $a$.

Proceeding now with the proof of Theorem 2, suppose that $x_1, \ldots, x_r, y_1, y_2, \ldots, y_n$ is an $H$-diagram and that $y_i = y_1 \ldots y_n$ is a complete system equivalent to $b$. Therefore the $y_i$ bound disjoint discs $D_1, \ldots, D_n$ inside $T$. Now let $T'$ be the result of cutting $T$ along $D_1, \ldots, D_n$, then $T'$ is a copy of $T(y)$, where $y = y_1 \ldots y_n$ and $T'$ is embedded in $S^3$. But by Lemma 4, $x_1, \ldots, x_r$ bound disjoint surfaces in $S^3 - T$ and therefore in $S^3 - T'$, and it follows from Theorem 1 that $M(x, y)$ is a homotopy 3-sphere.

Conversely, suppose that $M(x, y)$ is a homotopy sphere where $x, y$ are complete systems on $S'$, say (of genus $t$), and let $T$ denote $T(y)$. By Theorem 1 there is an embedding of $T'$ in $S^3$ such that $x_1, \ldots, x_r$ bound disjoint surfaces $S_1, S_2, \ldots, S_r$ in $S^3 - T$. Now $T'$ can be regarded as the regular neighbourhood of a 1-dimensional complex $K'$, say, and by choosing a triangulation of $S^3$, whose 1-skeleton $K'$ contains $K'$ as a subset, we can extend $T'$ to an unknotted handlebody $T''$, say, and we can think of $T''$ as obtained from $T'$ by attaching a sequence of 1-handles $h_{i+1}, h_{i+2}, \ldots, h_n$ say. We need to choose $T''$ so that the $h_i$ miss the surfaces $S_i$. To do this, we start by assuming that the chosen triangulation of $S^3$ includes each $S_i$ as a subcomplex, then $K' \subset K''$ where $K_1 - T$ lies in $\cup S_i$. Thus we can think of $T''$ as obtained from $T'$ by first attaching handles $h_{i+1}, \ldots, h_n$ whose cores lie in $\cup S_i$ and then attaching handles $h_{i+1}, \ldots, h_n$ disjoint from $\cup S_i$. Now slide the attaching tubes of the $h_{i+1}, \ldots, h_n$ off the $h_{i+1}, \ldots, h_n$ (by the usual "reordering" argument for handles) and then push the $h_{i+1}, \ldots, h_n$ to one side of the $S_i$. Thus, we now have $T''$ (unknotted) obtained from $T'$ by attaching 1-handles $h_{i+1}, \ldots, h_n$ disjoint from $\cup S_i$. Since $T''$ is unknotted, we can ambiently isotope it to standard position, i.e. to $T$. Write $y_i$ for the belt sphere of $h_i$, $i = t+1, \ldots, n$, then $y_i$ bounds the co-core $D_i$ of $h_i$, and if we cut $T$ along each of the $D_i$ we regain $T''$ (up to ambient isotopy). Now $x_1, \ldots, x_r$ bound disjoint surfaces in $S^3 - T$ and $y_{t+1}, \ldots, y_n$ bound disjoint discs in $T$. Therefore by Lemmas 2 and 4, $x_1, \ldots, x_r, y_1, \ldots, y_n$ is an $H$-system and the theorem is proved. Q.E.D.

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