Cyclic Permutations and Nearly Symmetric Integer Vectors

A. Felzenbaum and A. Tamir*
Department of Statistics
Tel-Aviv University
Tel Aviv, Israel

Submitted by Hans Schneider

ABSTRACT

Given an integer vector $x^T=(x_1,\ldots,x_n)$ with the property $x_1>x_2>\cdots>x_n>0$, it is shown that the convex hull of the n cyclic permutations of x contains all the nearly symmetric integer vectors majorized by x. A generalization to noninteger vectors and an application to a class of integer symmetric optimization problems are also given.

Given a vector $x^T=(x_1,\ldots,x_n)$, let $\tilde{x}$ denote the n-dimensional vector obtained by arranging the coordinates of x in decreasing order. Hardy, Littlewood and Polya [3] introduced the following relation on R^n. A vector y is said to be majorized by a vector x if for $i=1,\ldots,n$, $\sum_{j=1}^{i}\tilde{y}_j\leqslant\sum_{j=1}^{i}\tilde{x}_j$, with equality holding for $i=n$. They proved that y is majorized by x if and only if y can be expressed as a convex combination of the $n!$ permuted vectors obtained from x. Equivalently, y is majorized by x if and only if $y=Sx$ for some doubly stochastic matrix S. In fact, by using known linear programming arguments one can easily show that y being majorized by x implies that y can be described as a convex combination of only n permuted vectors of x. For example, the symmetric vector denoted by $\bar{x}$, whose coordinates are all equal to $(1/n)\sum_{j=1}^{n}x_j$, can be described as $\bar{x}=(1/n)\sum_{j=1}^{n}P_jx$, where $P_1,P_2,\ldots,P_n$ are the n cyclic permutation matrices.

The principal purpose of this paper is to investigate the convex hull of the n cyclic permutations of a given integer vector x, i.e. the polytope generated by $P_1x,P_2x,\ldots,P_nx$. We use $A(x)$ to denote this polytope.

We show that if the integer vector $x^T=(x_1,x_2,\ldots,x_n)$ satisfies $x_1>x_2>\cdots>x_n$ or $x_1<x_2\cdots<x_n$, then $A(x)$ contains not only the symmetric point $\bar{x}$, but also $\binom{n}{t}$ integer vectors, where $t=\sum_{i=1}^{n}x_i \pmod n$. More

*Currently visiting at the Graduate School of Management, Northwestern University, Evanston, IL 60201.

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specifically, defining an integer vector y to be nearly symmetric if $|y_i - y_j| \leqslant 1$ for all $i,j=1,\dots,n$, it is shown that $A(x)$ contains all the nearly symmetric integer vectors y satisfying $\sum_{i=1}^n y_i = \sum_{i=1}^n x_i$. It is easily observed that each one of these nearly symmetric vectors y is majorized by the vector x. Hence, by results of [3], each such y can be expressed as a convex combination (which may depend on y) of some n permutations of x. Our result is stronger in the sense that it shows that the same set of n permutations, i.e. the cyclic ones, can be chosen for all (majorized) nearly symmetric integer vectors y.

Referring to a possible relaxation of the assumption $x_1 > x_2 \cdots > x_n$, we note that the strict inequalities cannot be weakened as illustrated by the 4-dimensional vector $x^T = (2,1,0,0)$, where $A(x)$ contains no nearly symmetric integer vectors. In fact, it can be verified that there exist no set of four permutations of the vector $(2,1,0,0)$ with the property that their convex hull contains all the nearly symmetric integer vectors which are majorized by $(2,1,0,0)$.

Given an integer vector $x^T = (x_1,\dots,x_n)$ and a nearly symmetric integer vector y which is majorized by x, our problem is to verify the existence of a solution $\lambda^T = (\lambda_1,\dots,\lambda_n)$ to the following linear program:

$$\sum_{i=1}^n \lambda_i P_i x = y, \qquad \lambda_i \geqslant 0,\ i = 1,\dots,n, \quad \sum_{i=1}^n \lambda_i = 1, \tag{1}$$

where $\{P_1, P_2, \dots, P_n\}$ is the group of cyclic permutations. We note that (1) has a solution for given x and y if and only if it has a solution for the vectors $x^T + (t,t,\dots,t)$ and $y^T + (t,t,\dots,t)$, where t is an arbitrary real number. Hence, we can assume that $x_i > 0$ for all the components of the integer vector x.

Summing the elements of $\sum_{i=1}^n \lambda_i P_i x = y$ yields

$$\left(\sum_{i=1}^n x_i\right)\left(\sum_{i=1}^n \lambda_i\right) = \sum_{i=1}^n y_i.$$

Since y is majorized by x, the latter equality implies that $\sum_{i=1}^n \lambda_i = 1$. Therefore we may focus on solving the system

$$\sum_{i=1}^n \lambda_i P_i x = y, \qquad \lambda_i \geqslant 0, \quad i = 1,2,\dots,n \tag{2}$$

To prove the existence of a solution to (2) for integer vectors x and y satisfying the above assumptions, we shall first study the solvability of (2) for a more general setting. Thus, suppose now that x and y are any two vectors in R^n, which are not necessarily integral.

The equations in (2) can be written as $C\lambda = y$, where

$$C = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ x_n & x_1 & \cdots & x_{n-1} \\ \vdots & & & \vdots \\ x_3 & & & x_2 \\ x_2 & x_3 & \cdots & x_n x_1 \end{bmatrix}. \tag{3}$$

The matrix C is recognized in the literature as a cyclic matrix [1,4,6]. It is known that

$$\det C = \prod_{i=1}^{n} \sum_{j=1}^{n} \left(\alpha_i^{j-1} x_j\right), \tag{3a}$$

where $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ are the distinct nth roots of 1 [4,6].

LEMMA 1. *Let C be a cyclic matrix, and assume that either one of the following is satisfied*:

(i) $0 < x_1 < x_2 \cdots < x_n$,
(ii) $x_1 > x_2 \cdots > x_n > 0$

Then $\det C \neq 0$.

Proof. Suppose that $\det C = 0$. Then from (3a) $f(Z) = \sum_{j=1}^{n} x_j Z^{j-1} = 0$, where $Z^n = 1$. From [5, p. 105] it follows that $|Z| \leqslant \max_{1 \leqslant j \leqslant n-1}(x_j / x_{j+1}) = k$. Now, if the first condition holds, then $|Z| \leqslant k < 1$, contradicting $Z^n = 1$.

To obtain the contradiction with the second condition being met, we observe that $f(Z) = 0$ with $Z^n = 1$ imply that $g(V) = \sum_{j=1}^{n} x_j V^{n-j} = 0$ has a solution with $V^n = 1$. Again, it follows that $|V| \leqslant \max_{1 < j \leqslant n}(x_j / x_{j-1}) < 1$, which contradicts $V^n = 1$. ∎

As a corollary of the above lemma, we have that the linear system $C\lambda = y$ has a unique solution λ for any vectors x and y, provided either one of the conditions of Lemma 1 is satisfied. Next, we provide conditions under which this unique solution λ is nonnegative and satisfies $\sum_{i=1}^{n} \lambda_i = 1$.

THEOREM 2. *Let* $x^T=(x_1,\dots,x_n)\in R^n$ *satisfy* $0<x_1<x_2<\cdots<x_n$, *and let* $y^T=(y_1,\dots,y_n)\in R^n$ *satisfy* $\sum_{i=1}^n y_i=\sum_{i=1}^n x_i$. *Define*

$$m=\min_{i=1,\dots,n-1}(x_{i+1}-x_i)\quad\textit{and}\quad M=\max_{i=1,2,\dots,n}(y_i-y_{i\oplus 1}),$$

where $i\oplus 1=i+1 \pmod n$. *If* $m\geqslant M$, *then the linear system*

$$\begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ x_n & x_1 & \cdots & x_{n-1} \\ \vdots & & & \vdots \\ x_2 & x_3 & \cdots & x_1 \end{bmatrix}\lambda=y \tag{4}$$

has a unique solution $\lambda=(\lambda_1,\dots,\lambda_n)^T$, *which also satisfies* $\sum_{i=1}^n\lambda_i=1$ *and* $\lambda\geqslant 0$.

Proof. The existence of a unique solution to (4) is ensured by Lemma 1. Furthermore, summing the n equations of (4) and using the relation $\sum_{i=1}^n y_i=\sum_{i=1}^n x_i>0$, we have $\sum_{i=1}^n\lambda_i=1$. Thus it suffices to show that $\lambda\geqslant 0$. Generate a new linear system as follows.

For $k=1,\dots,n$, the kth row of the new system is obtained by subtracting the $(k\oplus 1)$st row of (4) from the kth row of (4). [$k\oplus j=k+j \pmod n$.] If $A=(a_{ij})$ is the matrix associated with this new system, then

$$a_{ij}=x_{(n-i)\oplus j\oplus 1}-x_{(n-i)\oplus j},\qquad i,j=1,\dots,n, \tag{5}$$

where $x_0=x_n$. From (5) we note that $a_{ij}\geqslant m>0$ for $j\neq i$, while $a_{ij}=x_1-x_n$ for $j=i$. We also have

$$\sum_{i=1}^n a_{ij}=0,\qquad j=1,\dots,n. \tag{6}$$

The right-hand-side vector, $\bar{\bar{y}}$ of the new system $A\lambda=\bar{\bar{y}}$ is defined by $\bar{\bar{y}}_k=y_k-y_{k\oplus 1}$.

Suppose that $\lambda\ngeqslant 0$, and let $J=\{j|\lambda_j>0\}$. J is not empty, and $\sum_{j\in J}\lambda_j>1$. Moreover, since $a_{ij}\geqslant m$ for $j\neq i$, and since the only negative coefficient in any row i $(i=1,\dots,n)$ is x_1-x_n, which is associated with λ_i, we have

$$\sum_{j\in J}a_{ij}\lambda_j>m\qquad\text{for all } i,\ i\notin J. \tag{7}$$

Let i be such that $i \notin J$. Applying (7) to the ith equation of the system $A\lambda = \bar{\bar{y}}$ and using the relation $\bar{\bar{y}}_i \leqslant M \leqslant m$ yield

$$-\sum_{j \notin J} a_{ij}\lambda_j = -\bar{\bar{y}}_i + \sum_{j \in J} a_{ij}\lambda_j > -\bar{\bar{y}}_i + m \geqslant 0, \qquad i \notin J. \tag{8}$$

Since $\lambda_j \leqslant 0$ for $j \notin J$,

$$\sum_{j \notin J} a_{ij}|\lambda_j| > 0 \qquad \text{for} \quad i \text{ satisfying } i \notin J. \tag{9}$$

Summing (9) over all i such that $i \notin J$, we have

$$\sum_{j \notin J} |\lambda_j| \Big(\sum_{i \notin J} a_{ij} \Big) > 0. \tag{10}$$

We complete the proof by showing that

$$\sum_{i \notin J} a_{ij} \leqslant 0 \qquad \text{for} \quad j \notin J.$$

Using (5)–(6) and the fact that $a_{ij} \geqslant m > 0$ for $i \neq j$, it is sufficient to observe that one of the elements in the sum $\sum_{i \notin J} a_{ij}$ is the unique negative element which exists in each column, i.e. the element $x_1 - x_n$. But the latter is trivially implied by $j \notin J$, since by choosing $i = j$ we see that $a_{ij} = x_1 - x_n$ is an element in that sum. ■

As a simple corollary of Theorem 2, we obtain the conditions referring to the case $x_1 > x_2 > \cdots > x_n > 0$.

COROLLARY 3. *Let $x^T = (x_1, \dots, x_n) \in R^n$ satisfy $x_1 > x_2 > \cdots > x_n > 0$, and let $y^T = (y_1, \dots, y_n) \in R^n$ satisfy $\sum_{i=1}^n y_i = \sum_{i=1}^n x_i$. Define*

$$m = \min_{i=1,\dots,n-1} (x_i - x_{i+1}) \quad \text{and} \quad M = \max_{i=1,\dots,n} (y_{i \oplus 1} - y_i),$$

where $i \oplus 1 = i + 1 \pmod n$. If $m \geqslant M$, then the linear system (4) has a unique solution $\lambda = (\lambda_1, \dots, \lambda_n)^T$, which also satisfies $\sum_{i=1}^n \lambda_i = 1$ and $\lambda \geqslant 0$.

The specialization of the above results to the case of majorized nearly symmetric integer vectors is now straightforward.

THEOREM 4. *Let $x^T=(x_1,\dots,x_n)\in R^n$ be an integer vector satisfying either one of the following*:

(i) $0<x_1<x_2<\cdots<x_n$,
(ii) $x_1>x_2>\cdots>x_n>0$.

If y is a nearly symmetric integer vector which is majorized by x, then the linear system (2) *has a unique solution λ, which also satisfies $\sum_{i=1}^n\lambda_i=1$ and $\lambda\geqslant 0$, i.e., y is in the convex hull of the n cyclic permutations of x.*

If an n-dimensional integer vector x has at least one nearly symmetric, but not symmetric, integer vector, majorized by x, then there exist at least n linearly independent such vectors y. This implies that at least n of the $n!$ permutations of x are needed to span the entire set of nearly symmetric integer vectors which are majorized by x. The set of n cyclic permutations is, therefore, minimal in this respect.

We also state that at least $n/2$ cyclic permutations are required to span a nearly symmetric integer vector, provided the conditions of Theorem 4 are met. To see this, consider the system (4), and observe that $\lambda_i+\lambda_{i\oplus 1}=0$ implies that $y_i>y_{i\oplus 1}>y_{i\oplus 2}$ (or $y_i<y_{i\oplus 1}<y_{i\oplus 2}$), thus contradicting the property that the components of y may only take on one of two different values. In fact, $n/2$ is a tight bound, since for an even n and the vectors $x^T=(n,n-1,\dots,1), y^T=(n/2+1,n/2,n/2+1,\dots,n/2)$, we obtain $\lambda^T=(2/n,0,2/n,0,\dots,2/n,0)$.

As a corollary of Theorem 4, we have the following result.

THEOREM 5. *Let $x^T=(x_1,x_2,\dots,x_n)$ be an integer vector with $x_i\neq x_j$ for $i\neq j$, and let P be the permutation arranging the coordinates of x in decreasing order. Then the convex hull of the vectors $\{P_1Px,P_2Px,\dots,P_nPx\}$ contains the $\binom{n}{t}$ nearly symmetric integer vectors which are majorized by x. (t is given by $t=\sum_{i=1}^n x_i$ (mod n), and $\{P_1,\dots,P_n\}$ is the group of cyclic permutations.)*

Theorem 5 can be applied to provide additional insight into the class of integer symmetric optimization problems considered by Greenberg and Pierskala [2]. They introduced several definitions.

DEFINITION 1. A set X is S-convex if $x\in X$ implies $Sx\in X$ for all doubly stochastic matrices S.

DEFINITION 2. A function f is S-concave on an S-convex set X if for all doubly stochastic matrices S

$$f(Sx)\geqslant f(x)\qquad \text{for all}\quad x\in X.$$

Using the results in [3], the following is proved in [2].

THEOREM 6. *If X is S-convex and f is S-concave on X, then for any integer point $x \in X$, there exists a nearly symmetric integer point y such that $f(y) \geqslant f(x)$.*

We shall now show that Theorem 5 enables us to relax the S-concavity property of f in Theorem 6. Given a permutation matrix P, define

$$X(P) = \{x \in X,\ x_i \neq x_j, i \neq j, \text{ and the coordinates of } Px \text{ are in decreasing order}\}.$$

Replace the S-concavity property by

(i) For any permuation matrix P

$$f(Sx) \geqslant f(x)$$

for all $x \in X(P)$ and for all matrices S which are convex combinations of $\{P_1P, P_2P, \dots, P_nP\}$, and

(ii) For any doubly stochastic matrix S

$$f(Sx) \geqslant f(x) \qquad \text{for all} \quad x \notin \bigcup_P X(P).$$

Now, Theorem 5 ensures that for any integer point x in $\bigcup X(P)$, there exists a nearly symmetric integer point y such that $f(y) \geqslant f(x)$, while from (ii), combined with the results of [3], we obtain the same property for $x \notin \bigcup X(P)$.

Finally, we note that $X - \bigcup X(P)$ is contained in the union of $n(n-1)/2$ $(n-1)$-dimensional hyperplanes. Hence, if X is a convex, n-dimensional set, then the S-concavity property is relaxed on $\bigcup X(P)$, which is dense in X.

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