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Orthogonal sequences in non-archimedean locally convex spaces

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ABSTRACT

The problem of the existence of (orthogonal) bases and basic sequences in non-archimedean locally convex spaces is studied. To this end we derive a characterization of compactoidity in terms of orthogonal sequences (Theorem 2.2).

INTRODUCTION

Throughout K denotes a non-archimedean non-trivially valued field which is complete under the metric induced by the valuation $|\ | : K \to [0, \infty)$. For fundamentals of locally convex spaces over K we refer to [9], [6].

In this paper all locally convex spaces are over K and assumed to be Hausdorff. A sequence $x_1, x_2, \dots$ in a locally convex space E is called *a (topological) base of E* if each $x \in E$ can be written uniquely as $x = \sum_{n=1}^{\infty} \lambda_n x_n$ with $\lambda_n \in K$. If the coefficient functionals $x \mapsto \lambda_n$ $(n \in \mathbb{N})$ are continuous then $x_1, x_2, \dots$ is called a *Schauder base*. As in the real or complex case one proves that every base in a Fréchet space is Schauder. Clearly every locally convex space with a (Schauder) base is (strictly) of countable type i.e. there is a countable set whose K-linear hull is dense. Conversely, any infinite-dimensional Banach space of countable type is known to be linearly homeomorphic to c_0, hence it has a Schauder base ([5], 3.16(ii)).

It is still unknown whether a Fréchet space of countable type has a Schauder base. For a partial result, see Theorem 3.5.

A sequence in a locally convex space is called a *basic sequence* if it is a Schauder base of its closed linear span. This leads to the question as to whether a Fréchet space has – at least – a basic sequence; a partial answer will be given in Corollary 3.1.

In §1 we compare the notion of orthogonality of a sequence introduced by N. De Grande-De Kimpe in [2] with the concept of basic sequence. In §2 we characterize compactoidity in terms of orthogonal sequences and in §3 we apply this to obtain results on existence of basic sequences in certain locally convex spaces and the non-archimedean counterpart of the Bessaga-Pelczynski selection principle (Corollaries 3.2 and 3.3).

NOTATIONS AND TERMINOLOGY

For a set X in a K-vector space we denote by $[\![X]\!]$ its linear span, and by co X its absolutely convex hull i.e. the smallest module over the ring $\{\lambda \in K : |\lambda| \leq 1\}$ that contains X.

c_0 is the K-Banach space of all sequences in K converging to 0, where for $x = (\xi_1, \xi_2, \ldots) \in c_0$ we set $\|x\| := \max |\xi_n|$. c_{00} is the subspace of c_0 consisting of all sequences $(\xi_1, \xi_2 \ldots)$ such that $\xi_n = 0$ for large n. Let E be a Hausdorff locally convex space over K. By E^* we denote its algebraic dual, by E' its topological dual. E is called *dual-separating* if for each $x \in E$, $x \neq 0$ there exists an $f \in E'$ such that $f(x) \neq 0$. Then the *weak topology* $\sigma(E, E')$ is Hausdorff. The closure of a set $X \subset E$ is written $\overline{X}$. Instead of $\overline{\text{co } X}$ we write $\overline{\text{co}}\, X$.

The completion of E is denoted by $E^\wedge$. If τ is the topology of E we denote the topology on $E^\wedge$ again by τ.

1. ORTHOGONAL AND BASIC SEQUENCES

Let p be a (non-archimedean) seminorm on a K-vector space E, let $t \in (0, 1]$. Recall that a sequence $x_1, x_2, \ldots$ in E is called *t-orthogonal with respect to p* if for each $n \in \mathbb{N}$ and $\lambda_1, \lambda_2, \ldots, \lambda_n \in K$ we have

$$(*) \qquad p\left(\sum_{i=1}^{n} \lambda_i x_i\right) \geq t \max_{1 \leq i \leq n} p(\lambda_i x_i).$$

If $t = 1$ then $x_1, x_2, \ldots$ is called *orthogonal with respect to p* and $(*)$ can be written as

$$p\left(\sum_{i=1}^{n} \lambda_i x_i\right) = \max_{1 \leq i \leq n} p(\lambda_i x_i).$$

Definition 1.1. A sequence $x_1, x_2, \ldots$ in a locally convex space E is called 'orthogonal' in E if the collection $\mathcal{P}$ of all continuous seminorms p for which $x_1, x_2, \ldots$ is orthogonal with respect to p forms a base of continuous seminorms.

Remark 1.2.

(i) A sequence is 'orthogonal' in E if and only if it is 'orthogonal' in its (algebraic) linear span. This can be shown by straightforward arguments. (Let $x_1, x_2, \ldots$ be 'orthogonal' in $D := [\![x_1, x_2, \ldots]\!]$, let $\mathcal{P}$ be a base of continuous seminorms on D for which $x_1, x_2, \ldots$ are orthogonal. Let $\mathcal{P}^* := \{t : t$ continuous seminorm on E, $t|D \in \mathcal{P}\}$. To prove that $\mathcal{P}^*$ is a base of continuous seminorms on E, let q be a continuous seminorm on E. There is a $p \in \mathcal{P}$ such that $q \leq p$ on D. Then p extends to a continuous seminorm $\widetilde{p}$ on E, and $\max(\widetilde{p}, q) \in \mathcal{P}^*$, $q \leq \max(\widetilde{p}, q)$.) It enables us to speak about 'orthogonal' sequences without specifying a subspace.

(ii) A sequence $x_1, x_2, \ldots$ in E is 'orthogonal' if and only if there exists a collection $\mathcal{P}$ of continuous seminorms generating the topology such that $x_1, x_2, \ldots$ is orthogonal with respect to each $p \in \mathcal{P}$.

(iii) Let $x_1, x_2, \ldots \in E$ be an 'orthogonal' sequence, let $x_n \neq 0$ for each n, let $\mathcal{P}$ be as in Definition 1.1. If $x \in E$ can be expressed as $\sum_{n=1}^{\infty} \lambda_n x_n$ with $\lambda_n \in K$ then $p(x) = \max_n p(\lambda_n x_n)$ for each $p \in \mathcal{P}$. It follows that the λ_n are unique; in particular the $x_1, x_2, \ldots$ are linearly independent. Also, if $x_1, x_2, \ldots$ is a topological base it is automatically a Schauder base.

Lemma 1.3. *Let $x_1, x_2, \ldots$ be a linearly independent sequence in a locally convex space E, let $D := [\![x_1, x_2, \ldots]\!]$. For each $n \in \mathbb{N}$, let $f_n \in D^*$ be given by $f_n(x_m) = \delta_{nm}$ $(m \in \mathbb{N})$. Then $x = \sum_n f_n(x) x_n$ for each $x \in D$, and $x_1, x_2, \ldots$ is 'orthogonal' if and only if the maps $x \mapsto f_n(x) x_n$ $(x \in D)$ are equicontinuous.*

Proof. Let $x_1, x_2, \ldots$ be 'orthogonal', let $\mathcal{P}$ be as in Definition 1.1. For each $n \in \mathbb{N}$, put $\delta_n(x) = f_n(x) x_n$ $(x \in D)$. For each $p \in \mathcal{P}$ we have $p = \max_n p \circ \delta_n$ and the equicontinuity of $\{\delta_n : n \in \mathbb{N}\}$ follows. Conversely, assume $\{\delta_n : n \in \mathbb{N}\}$ is equicontinuous. For each $p \in \mathcal{P}$ put

$$p^*(x) = \max_n p(f_n(x) x_n) \qquad (x \in D).$$

Then $p \leq p^*$ on D and $x_1, x_2, \ldots$ is orthogonal with respect to p^*. By equicontinuity p^* is continuous and the set $\{p^* : p \in \mathcal{P}\}$ is a base of continuous seminorms on D. □

Proposition 1.4. *Each 'orthogonal' sequence of non-zero vectors is a basic sequence.*

Proof. Let $x_1, x_2, \ldots$ be 'orthogonal' in a locally convex space E. To prove the statement we may assume that E is complete. Let D, f_n, δ_n be as in Lemma 1.3. Each f_n extends uniquely to an $\overline{f}_n \in (\overline{D})'$; put $\overline{\delta}_n(x) = \overline{f}_n(x) x_n$ $(n \in \mathbb{N}, x \in \overline{D})$. By Remark 1.2 (iii) it suffices to prove that $x = \sum_{n=1}^{\infty} \overline{\delta}_n(x)$ for each $x \in \overline{D}$. By Lemma 1.3 the set $\{\delta_n : n \in \mathbb{N}\}$ is equicontinuous, hence so is $\{\overline{\delta}_n : n \in \mathbb{N}\}$. Since $\lim_{n \to \infty} \delta_n(x) = 0$ for all $x \in D$ we have $\lim_{n \to \infty} \overline{\delta}_n(x) = 0$ for all $x \in \overline{D}$. By completeness and equicontinuity the formula $Tx = \sum_{n=1}^{\infty} \overline{\delta}_n(x)$ defines a continuous linear map $T : \overline{D} \to \overline{D}$. But T is the identity on D, hence on $\overline{D}$. □

Corollary 1.5. *Let $x_1, x_2, \dots$ be an 'orthogonal' sequence in a locally convex space E, let $x_n \neq 0$ for each n, let $E = [\![x_1, x_2, \dots]\!]$. Then $x_1, x_2, \dots$ is a Schauder base of E.*

Remark 1.6. The converse of Proposition 1.4 does not hold in general. In fact, let $\alpha \in K$, $0 < |\alpha| < 1$ and let $E := c_{00}$. Set $x_n := (1, \alpha, \alpha^2, \dots, \alpha^{n-1}, 0, 0, \dots)$ $(n \in \mathbb{N})$.

It is easily seen that $x_1, x_2, \dots$ is a Schauder base of c_{00} but t-orthogonal (with respect to the norm) for no $t \in (0, 1]$ and therefore not 'orthogonal' (see 2.3). However:

Proposition 1.7. *In a Fréchet space every basic sequence is 'orthogonal'; every Schauder base is an 'orthogonal' base.*

Proof. We need only to prove the first statement. Let $x_1, x_2, \dots$ be a basic sequence, let $D := \overline{[\![x_1, x_2, \dots]\!]}$, let $f_n \in D'$ be such that $f_n(x_m) = \delta_{mn}$ $(m, n \in \mathbb{N})$. Then $x = \sum_{n=1}^{\infty} f_n(x)x_n$ for each $x \in D$, so the maps $x \mapsto f_n(x)x_n$ $(x \in D)$ are pointwise bounded. But D is Fréchet, hence barrelled and so the above maps are equicontinuous. By Lemma 1.3 the sequence $x_1, x_2, \dots$ is 'orthogonal'. □

Remark 1.8. The second conclusion of the proposition holds for ℓ^∞-barrelled spaces, see [4].

2. A CHARACTERIZATION OF COMPACTOIDS

Recall that a subset X of a locally convex space E is called *a compactoid* if for each zero neighbourhood U in E there exists a finite set $F \subset E$ such that $X \subset U + \text{co}\, F$.

From [7] we quote the following result.

Theorem 2.1. *Let X be a bounded set in a normed space $E = (E, \|\ \|)$ over K. Then X is a compactoid if and only if for each $t \in (0, 1]$, each t-orthogonal sequence in X (with respect to $\|\ \|$) tends to* 0.

Actually, in [7] it was supposed that E is a Banach space, but trivially the result holds for general normed spaces.

In this note we prove the following generalization.

Theorem 2.2. *Let X be a bounded set in a locally convex space E over K. Then X is a compactoid if and only if each 'orthogonal' sequence in X tends to* 0.

First, we prove that this is, indeed, a generalization of Theorem 2.1.

Proposition 2.3. *Let $x_1, x_2, \dots$ be a sequence in a normed space $(E, \|\ \|)$ over K. Then $x_1, x_2, \dots$ is 'orthogonal' in the sense of Definition* 1.1 *if and only if, for some $t \in (0, 1]$, it is t-orthogonal with respect to $\|\ \|$.*

Proof. Let $x_1, x_2, \ldots$ be 'orthogonal'. Let $\mathcal{P}$ be as in Definition 1.1. Then there is a $p \in \mathcal{P}$ for which $\|\ \| \leq p$. By continuity of p we have $p \leq c\|\ \|$ for some constant $c \geq 1$. Then, for each $n \in \mathbb{N}$ and $\lambda_1, \ldots, \lambda_n \in K$ we have

$$\left\| \sum_{i=1}^{n} \lambda_i x_i \right\| \geq c^{-1} p\left(\sum_{i=1}^{n} \lambda_i x_i \right) = c^{-1} \max_{1 \leq i \leq n} p(\lambda_i x_i) \geq c^{-1} \max_{1 \leq i \leq n} \|\lambda_i x_i\|,$$

showing that $x_1, x_2, \ldots$ is c^{-1}-orthogonal with respect to $\|\ \|$.

Conversely, suppose that $x_1, x_2, \ldots$ is t-orthogonal with respect to $\|\ \|$ for some $t \in (0, 1]$. To show 'orthogonality' we may suppose $x_n \neq 0$ for each n. Then the $x_1, x_2, \ldots$ are linearly independent. For all $n \in \mathbb{N}$ and $\lambda_1, \ldots, \lambda_n \in K$ we have

$$\left\| \sum_{i=1}^{n} \lambda_i x_i \right\| \leq \max_{1 \leq i \leq n} \|\lambda_i x_i\| \leq t^{-1} \left\| \sum_{i=1}^{n} \lambda_i x_i \right\|.$$

So, for each $x \in D := [\![x_1, x_2, \ldots]\!]$, $x = \sum_{i=1}^{n} \lambda_i x_i$, we have

$$\|x\| \leq \widetilde{p}(x) \leq t^{-1}\|x\|,$$

where $\widetilde{p}(x) := \max_{1 \leq i \leq n} \|\lambda_i x_i\|$. We see that $\widetilde{p}$ is a norm on D defining the topology and that $x_1, x_2, \ldots$ is orthogonal with respect to $\widetilde{p}$. Now use Remark 1.2 (i). □

For the proof of Theorem 2.2 we need the following easy observations. Let p be a seminorm on a K-vector space E. Let $\pi_p : E \to E_p := E/\mathrm{Ker}\, p$ be the quotient map. The formula $\overline{p}(\pi_p(x)) = p(x)$ defines a norm $\overline{p}$ on E_p. Propositions 2.4 and 2.5 are well-known.

Proposition 2.4. *Let $t \in (0, 1]$, let $x_1, x_2, \ldots$ be a sequence in E. Then $x_1, x_2, \ldots$ is t-orthogonal with respect to p if and only if $\pi_p(x_1), \pi_p(x_2), \ldots$ is t-orthogonal with respect to $\overline{p}$.*

Proposition 2.5. *Let E be a locally convex space over K, let $\mathcal{P}$ be a base of continuous seminorms, let $X \subset E$. Then X is a compactoid if and only if $\pi_p(X)$ is a compactoid in E_p for each $p \in \mathcal{P}$.*

Proposition 2.6. *Let $x_1, x_2, \ldots$ be a sequence in a locally convex space E over K and suppose there is a base $\mathcal{P}$ of continuous seminorms and a map $p \mapsto t_p$ of $\mathcal{P}$ into $(0, 1]$ such that, for each $p \in \mathcal{P}$, $x_1, x_2, \ldots$ is t_p-orthogonal with respect to p. Then $x_1, x_2, \ldots$ is an 'orthogonal' sequence in E.*

Proof. Let $D := [\![x_1, x_2, \ldots]\!]$. For each $p \in \mathcal{P}$, $n \in \mathbb{N}$ and $\lambda_1, \ldots, \lambda_n \in K$ we have

$$p\left(\sum_{i=1}^{n} \lambda_i x_i \right) \geq t_p \max_{1 \leq i \leq n} p(\lambda_i x_i).$$

If $\sum_{i=1}^{n} \lambda_i x_i = 0$ then $p(\lambda_i x_i) = 0$ for all i, so the formula

$$\widetilde{p}\left(\sum_{i=1}^{n} \lambda_i x_i\right) = \max_{1 \leq i \leq n} p(\lambda_i x_i)$$

defines a seminorm $\widetilde{p}$ on D for which $p \leq \widetilde{p} \leq t_p^{-1} p$. Then $x_1, x_2, \ldots$ is orthogonal with respect to $\widetilde{p}$, and since $\{\widetilde{p} : p \in \mathcal{P}\}$ induces the topology of D the 'orthogonality' of $x_1, x_2, \ldots$ follows after applying Remark 1.2 (i). □

Proof of Theorem 2.2. Suppose X is a compactoid, and let $x_1, x_2, \ldots$ be an 'orthogonal' sequence in X. Let p be a continuous seminorm on E for which $x_1, x_2, \ldots$ is orthogonal with respect to p. It suffices to prove that $p(x_n) \to 0$. By Proposition 2.5 the set $\pi_p(X)$ is a compactoid in E_p and by Proposition 2.4, $\pi_p(x_1), \pi_p(x_2), \ldots$ is orthogonal with respect to $\bar{p}$. By Theorem 2.1 we have $\bar{p}(\pi_p(x_n)) \to 0$ i.e. $p(x_n) \to 0$.

Conversely, let X be bounded and let each 'orthogonal' sequence in X tend to 0. Suppose X is not a compactoid; we derive a contradiction.

By Proposition 2.5 there is a continuous seminorm p on E such that $\pi_p(X)$ is not a compactoid in E_p. By Theorem 2.1 there exists a sequence $x_1, x_2, \ldots$ in X such that, for some $t_p \in (0, 1]$, $\pi_p(x_1), \pi_p(x_2), \ldots$ is t_p-orthogonal in E_p but $\bar{p}(\pi_p(x_n)) \nrightarrow 0$, i.e. $x_1, x_2, \ldots$ is t_p-orthogonal with respect to p (Proposition 2.4) and $p(x_n) \nrightarrow 0$. Without loss, assume $p(x_n) \geq \alpha > 0$ for all n.

Now let $\mathcal{P}$ be the collection of all continuous seminorms on E that are $\geq p$. Then $\mathcal{P}$ is a base of continuous seminorms. Let $q \in \mathcal{P}$. By boundedness of X we have $M := \sup_n q(x_n) < \infty$.

For $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n \in K$ we have

$$\begin{aligned} q(\textstyle\sum_{i=1}^{n} \lambda_i x_i) &\geq p(\textstyle\sum_{i=1}^{n} \lambda_i x_i) \geq t_p \max_i |\lambda_i| p(x_i) \geq t_p \alpha \max_i |\lambda_i| \\ &\geq t_p \alpha M^{-1} \max_i |\lambda_i| q(x_i) \geq t_p \alpha M^{-1} q(\textstyle\sum_{i=1}^{n} \lambda_i x_i). \end{aligned}$$

We see that $x_1, x_2, \ldots$ is $t_p \alpha M^{-1}$-orthogonal with respect to q.

By Proposition 2.6 the sequence $x_1, x_2, \ldots$ is 'orthogonal' so by assumption, $x_n \to 0$ conflicting $p(x_n) \geq \alpha$. □

Combining Theorem 2.2 and Propositions 1.4 and 1.7 we obtain the following.

Corollary 2.7. *A bounded subset X of a Fréchet space is a compactoid if and only if each basic sequence in X tends to* 0.

Remark 2.8. The above corollary cannot be extended to non-complete spaces. In fact, let $x_1, x_2, \ldots$ be as in Remark 1.6. Clearly $x_1, x_2, \ldots$ is Cauchy in c_{00} so $X := \{x_1, x_2, \ldots\}$ is a compactoid. But the basic sequence $x_1, x_2, \ldots$ does not converge to 0.

A direct consequence of Theorem 2.2 is the following.

Corollary 3.1. *Let E be a locally convex space in which not every bounded set is a compactoid. Then E has an 'orthogonal' basic sequence.*

The next two results are non-archimedean translations of the Bessaga-Pelczynski Selection Principle (see [1], p. 42).

Corollary 3.2. *Let (E, τ) be a polar locally convex space. Let $x_1, x_2, \ldots$ be a sequence in E such that $x_n \to 0$ weakly but $x_n \overset{\tau}{\nrightarrow} 0$. Then $x_1, x_2, \ldots$ contains an 'orthogonal' basic subsequence.*

Proof. By weak convergence the set $\{x_1, x_2, \ldots\}$ is τ-bounded ([6], 7.7). If $x_1, x_2, \ldots$ had no 'orthogonal' subsequence then $\{x_1, x_2, \ldots\}$ would be a compactoid by Theorem 2.2, so $\tau = \sigma(E, E')$ on $\{x_1, x_2, \ldots\}$ ([6], 5.12) whence $x_n \overset{\tau}{\to} 0$, a contradiction. □

Corollary 3.3. *Let (E, τ) be a metrizable locally convex space. Then the following are equivalent.*

(α) *$(E, \tau)^\wedge$ is dual-separating.*

(β) *Let $x_1, x_2, \ldots$ be a bounded sequence for which $x_n \to 0$ weakly but $x_n \overset{\tau}{\nrightarrow} 0$. Then $x_1, x_2, \ldots$ contains an 'orthogonal' basic subsequence.*

Proof. To prove $(\alpha) \Rightarrow (\beta)$ we may assume that (E, τ) is complete. Suppose $x_1, x_2, \ldots$ has no 'orthogonal' subsequence; we derive a contradiction. By boundedness and Theorem 2.2 the set $\{x_1, x_2, \ldots\}$ is a compactoid hence so is $A = \overline{\text{co}}\{x_1, x_2, \ldots\}$. A is metrizable, absolutely convex, complete and compactoid. By (α), $\sigma(E, E')$ is Hausdorff, so according to [8], 3.2 the topologies τ and $\sigma(E, E')$ coincide on A and therefore $x_n \overset{\tau}{\to} 0$, a contradiction.

To prove $(\beta) \Rightarrow (\alpha)$, let $a \in (E, \tau)^\wedge$, $a \neq 0$ and suppose $f(a) = 0$ for all $f \in ((E, \tau)^\wedge)'$; we derive a contradiction. By metrizability there exist $x_1, x_2, \ldots \in E$ with $x_n \overset{\tau}{\to} a$. Then $x_1, x_2, \ldots$ is Cauchy hence $\{x_1, x_2, \ldots\}$ is compactoid. As $x_n \to 0$ weakly and $x_n \overset{\tau}{\nrightarrow} 0$ we have by (β) that $x_1, x_2, \ldots$ contains an 'orthogonal' subsequence $y_1, y_2, \ldots$. From Theorem 2.2 we obtain $y_n \overset{\tau}{\to} 0$. But also $y_n \overset{\tau}{\to} a$ so $a = 0$, a contradiction. □

Remark 3.4. (i) A locally convex space E is called an O.P. (Orlicz-Pettis) space if each weakly convergent sequence is convergent. It is shown in [3] that if K is spherically complete or E is of countable type, E is an O.P.-space. Obviously, Corollary 3.2 is of interest only for non-O.P. spaces (such as ℓ^∞ over a non-spherically complete K).

(ii) For polar metrizable spaces (E, τ) condition (α) of Corollary 3.3 is satisfied. In such spaces weakly bounded sets are bounded. Hence, in (β) one may drop the condition that $x_1, x_2, \ldots$ be bounded.

(iii) If (E, τ) is a normable space one may also drop boundedness of $x_1, x_2, \ldots$ in (β). In fact, if $x_1, x_2, \ldots$ is unbounded one can select $\lambda_1, \lambda_2, \ldots \in K$, $|\lambda_n| \leq 1$ for all n, such that $\lambda_1 x_1, \lambda_2 x_2, \ldots$ is bounded and not norm-convergent to 0.

About the existence of Schauder bases in Fréchet spaces of countable type we have the following partial result.

Theorem 3.5. *Let E be a metrizable locally convex space of finite type (i.e. for each continuous seminorm p the space $E/\mathrm{Ker}\, p$ is finite-dimensional). Then E has an 'orthogonal' Schauder base.*

Proof. We may assume $\dim E = \infty$. Let $p_1 \leq p_2 \leq \cdots$ be seminorms defining the topology τ. (Observe that $\tau = \sigma(E, E')$.) There exist linearly independent $e_1, \ldots, e_{n_1}$ such that $E = \mathrm{Ker}\, p_1 \oplus [\![e_1, \ldots, e_{n_1}]\!]$. By the same token there exist linearly independent $e_{n_1+1}, \ldots, e_{n_2}$ such that $\mathrm{Ker}\, p_1 = \mathrm{Ker}\, p_2 \oplus [\![e_{n_1+1}, \ldots, e_{n_2}]\!]$, etc.

For each k the formula

$$\sum_{i=1}^{n_k} \lambda_i e_i \mapsto \max\{p_k(\lambda_i e_i) : 1 \leq i \leq n_k\}$$

defines a norm on $[\![e_1, \ldots, e_{n_k}]\!]$, equivalent to p_k. It can by a standard procedure be extended to a seminorm q_k on E that is equivalent to p_k on E. Then $q_1, q_2, \ldots$ induce τ and the sequence $e_1, e_2, \ldots$ is 'orthogonal'. By Proposition 1.4 it is an orthogonal base of $\overline{[\![e_1, e_2, \ldots]\!]}$. To see that $\overline{[\![e_1, e_2, \ldots]\!]} = E$, let $f \in E'$ and $f(e_n) = 0$ for all n. Then $|f| \leq p_k$ for some k so $f = 0$ on $\mathrm{Ker}\, p_k + [\![e_1, \ldots, e_{n_k}]\!] = E$. Thus $[\![e_1, e_2, \ldots]\!]$ is (weakly) dense in E. $\square$

Remark 3.6. Let E be a Fréchet space of countable type with defining seminorms $p_1, p_2, \ldots$. The maps $\pi_n : E \to E_{p_n}^{\wedge}$ (see the preamble to 2.4) yield a homeomorphism of E into $\prod_n E_{p_n}^{\wedge}$. Each $E_{p_n}^{\wedge}$ is either finite-dimensional or linearly homeomorphic to c_0. Thus E is linearly homeomorphic to a closed subspace of $c_0^{\mathbb{N}}$. It is easy to see that $c_0^{\mathbb{N}}$ has an 'orthogonal' base. Thus the question 'does every Fréchet space of countable type have an 'orthogonal' base?' is equivalent to 'If a Fréchet space has an 'orthogonal' base then do closed subspaces have also an 'orthogonal' base?'

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