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The lattice of integer partitions and its infinite extension

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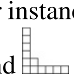
Abstract

In this paper, we use a simple discrete dynamical model to study integer partitions and their lattice. The set of reachable configurations of the model, with the order induced by the transition rule defined on it, is the lattice of all partitions of a positive integer, equipped with a dominance ordering. We first explain how this lattice can be constructed by an algorithm in linear time with respect to its size by showing that it has a self-similar structure. Then, we define a natural extension of the model to infinity, which we compare with the Young lattice. Using a self-similar tree, we obtain an encoding of the obtained lattice which makes it possible to enumerate easily and efficiently all the partitions of a given integer. This approach also gives a recursive formula for the number of partitions of an integer, and some informations on special sets of partitions, such as length bounded partitions.

Keywords: Lattice; Dominance ordering; Integer partitions; Sand Pile model; Young lattice; Discrete dynamical models

1. Preliminaries

A *partially ordered set* (or *poset*) is a set P with a reflexive ($x \leq x$), transitive ($x \leq y$ and $y \leq z$ implies $x \leq z$) and antisymmetric ($x \leq y$ and $y \leq x$ implies $x = y$) binary relation $\leq$. A *lattice* is a partially ordered set such that any two elements a and b have a least upper bound, called *supremum* of a and b and denoted by $\sup(a, b)$, and a greatest lower bound, called *infimum* of a and b and denoted by $\inf(a, b)$. The element $\sup(a, b)$ is the smallest element among the elements greater than both a and b. The element $\inf(a, b)$ is defined dually. A subset L' of a lattice L is called a *sublattice* of L if for any two elements a and b of L', the $\sup(a, b)$ and $\inf(a, b)$ are also elements of L'. Lattices are strongly structured sets, and many general results, for instance efficient encodings and algorithms, are known about them. For more details, see for instance [4].

A *partition* is an integer sequence $a = (a_1, a_2, \dots, a_k)$ such that $a_1 \geq a_2 \geq \cdots \geq a_k > 0$ (by convention, $a_j = 0$ for all $j > k$). We say that a is a partition of n if $\sum_{i=1}^{i=k} a_i = n$. The Ferrers diagram of a partition $a = (a_1, a_2, \dots, a_k)$ is a drawing of a on k adjacent columns such that the ith column is a pile of a_i stacked squares, which we will call *grains* because of the sand pile dynamics we will consider over them. For instance, $p = (4, 3, 3, 2)$ and $q = (6, 2, 1, 1, 1, 1)$ are two partitions of $n = 12$, and their Ferrers diagrams are  and  respectively.

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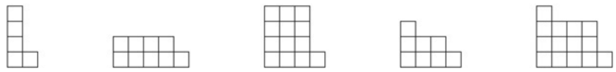

Fig. 1. From left to right: a cliff, a slippery plateau of length 3, a non-slippery plateau of length 2, a slippery step of length 2 and a non-slippery step of length 3.

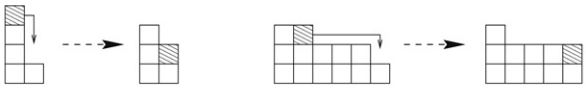

Fig. 2. The two evolution rules of the dynamical model.

The *dominance ordering* is defined in the following way [3]. Consider two partitions of the integer n: $a = (a_1, a_2, \ldots, a_k)$ and $b = (b_1, b_2, \ldots, b_l)$. Then

$$a \geq b \quad \text{if and only if} \quad \sum_{i=1}^{j} a_i \geq \sum_{i=1}^{j} b_i \quad \text{for all } j \geq 1.$$

From [3], it is known that the set of all partitions of an integer n with the dominance ordering is a lattice, denoted by $L_B(n)$. In his paper, Brylawski proposed a dynamical approach to study this lattice. We will introduce some notations to explain it intuitively. For more details about integer partitions, we refer to [2].

Let $a = (a_1, \ldots a_k)$ be a partition. The *height difference of a at i*, denoted by $d_i(a)$, is the integer $a_i - a_{i+1}$. We say that the partition a has a *cliff* at i if $d_i(a) \geq 2$. We say that a has a *slippery plateau* at i if there exists $\ell > i$ such that $d_j(a) = 0$ for all $i \leq j < \ell$ and $d_\ell(a) = 1$. The integer $\ell - i$ is then called the *length* of the slippery plateau at i. Likewise, a has a *non-slippery plateau* at i if $d_j(a) = 0$ for all $i \leq j < \ell$ and it has a cliff at ℓ. The integer $\ell - i$ is called the *length* of the non-slippery plateau at i. The partition a has a *slippery step* at i if the sequence defined by $a' = (a_1, \ldots, a_i - 1, \ldots, a_k)$ is a partition with a slippery plateau at i. Likewise, a has a *non-slippery step* at i if a' is a partition with a non-slippery plateau at i. See Fig. 1 for some illustrations.

Consider now the partition $a = (a_1, a_2, \ldots, a_k)$. Brylawski defined the following two evolution rules: one grain can fall from column i to column $i + 1$ if a has a cliff at i, and one grain can slip from column i to column $i + l + 1$ if a has a slippery step of length l at i. See Fig. 2.

Such a fall or a slip is called a *transition* of the model and is denoted by $a \xrightarrow{i} b$ where i is the column from which the grain falls or slips; we also denote this by $b = a \xrightarrow{i}$. If one starts from the partition (n) and iterates this operation, one obtains all the partitions of n, and the dominance ordering is nothing but the reflexive and transitive closure of the relation induced by the transition rule [3]. See Fig. 3 for illustrations with $n = 7$ and $n = 8$.

Let us recall that one can consider Brylawski's model as a generalization of the so-called Sand Pile Model (SPM), which consists of the first evolution rule only. The SPM was studied in many areas: from physics point of view [13], combinatorics considerations [1,6], and dynamical model theory [7,8,12]. Moreover an infinite extension of this model was studied in [11].

We will now study the structure of the lattice of the partitions of an integer n and we will show its self-similarity by giving a method to construct $L_B(n+1)$ from $L_B(n)$. Then, we will define an infinite extension of these lattices: the lattice $L_B(\infty)$ of all the partitions of any integer (*i.e.* all finite non-increasing sequences of positive integers). We will compare this lattice with the Young lattice, which also contains all the partitions of any integer, but ordered in a different way. Finally, we will construct an infinite tree based on the construction process described at the beginning of the paper. This tree will make it possible to give a simple and efficient algorithm to enumerate all the partitions of a given integer. It also has a self-similar structure, from which we will obtain a recursive formula for the number of partitions of an integer n and some results about certain classes of partitions.

Before entering the core of the topic, we need one more notation. If the k-tuple $a = (a_1, a_2, \ldots, a_k)$ is a partition, then the k-tuple $(a_1, a_2, \ldots, a_{i-1}, a_i + 1, a_{i+1}, \ldots, a_k)$ is denoted by $a^{\downarrow i}$. In other words, $a^{\downarrow i}$ is obtained from a by adding one grain on its ith column. Notice that the k-tuple obtained in this way is not necessarily a partition. If S is a set of partitions, then $S^{\downarrow i}$ denotes the set $\{a^{\downarrow i} | a \in S\}$. Finally, we denote by dirreach(a) the set of configurations directly reachable from a, *i.e.* the set $\{b \mid a \xrightarrow{i} b \text{ for some } i\}$. Notice that in the context of dynamical model theory,

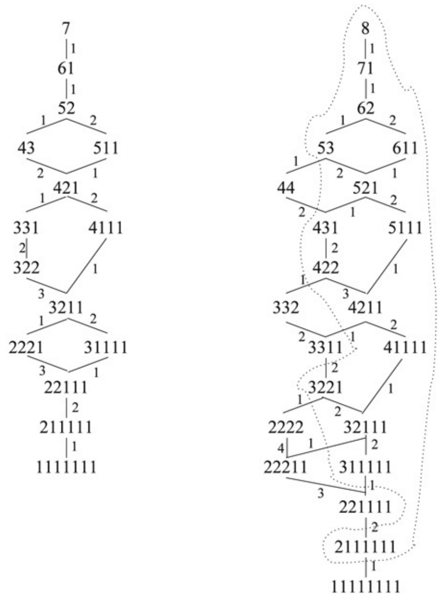

Fig. 3. Diagrams of the lattices $L_B(n)$ for $n = 7$ and $n = 8$. As we will see, the set $L_B(7)$ is isomorphic to a sublattice of $L_B(8)$. On the diagram of $L_B(8)$, we included this sublattice in a dotted line.

those elements are called *the immediate successors* of a. However, since we are concerned here with order theory, we cannot use this term, which takes another meaning in this context.

2. From $L_B(n)$ to $L_B(n+1)$

In this section, our aim is to construct $L_B(n+1)$ from $L_B(n)$, viewed as the graph induced by the dynamical model, with the edges labeled by the number of the column from which the grain falls or slips, as shown in Fig. 3. We will call *construction of a lattice* the computation of this labeled graph. We first show that $L_B(n)^{\downarrow 1}$ is a sublattice of $L_B(n+1)$. For instance, in Fig. 3 we included in a dotted line $L_B(7)^{\downarrow 1}$ within $L_B(8)$. This remark allows us to start the construction of $L_B(n+1)$ from $L_B(n)$ by computing $L_B(n)^{\downarrow 1}$ and then adding the missing elements of $L_B(n+1)$. After characterizing those elements that must be added, we obtain a simple and efficient method to achieve the construction of $L_B(n+1)$ from $L_B(n)$.

Proposition 1. *$L_B(n)^{\downarrow 1}$ is a sublattice of $L_B(n+1)$.*

Proof. We must show that for any two elements a and b of $L_B(n)$, $\inf(a^{\downarrow 1}, b^{\downarrow 1})$ and $\sup(a^{\downarrow 1}, b^{\downarrow 1})$ are in $L_B(n)^{\downarrow 1}$. Let us first consider the element $c = \inf(a, b)$. It is clear that $c^{\downarrow 1}$ is in $L_B(n)^{\downarrow 1}$, and we will show that $c^{\downarrow 1}$ is equal to $\inf(a^{\downarrow 1}, b^{\downarrow 1})$. This statement comes directly from Brylawski's result on dominance ordering [3]:

$$\inf(a, b) = c \quad \text{if and only if, for all } j \geq 1, \quad \text{one has } \sum_{i=1}^{j} c_i = \min\left(\sum_{i=1}^{j} a_i, \sum_{i=1}^{j} b_i\right).$$

Let us consider now $e = \sup(a, b)$. We will show that $e^{\downarrow 1}$ is equal to $d = \sup(a^{\downarrow 1}, b^{\downarrow 1})$. We have $e \geq a$ and $e \geq b$, therefore $e^{\downarrow 1} \geq a^{\downarrow 1}$ and $e^{\downarrow 1} \geq b^{\downarrow 1}$. This implies that $e^{\downarrow 1} \geq d$. To show that $d \geq e^{\downarrow 1}$, let us begin by showing that $d_1 = e_1 + 1$. We can suppose that $a_1 \geq b_1$. The partition $(a_1, a_1, \ldots, a_1, r)$ (with $r = n \text{modulo} a_1$) is greater than or equal to a and b, and so it is greater than or equal to e. Moreover, $e \geq a$ implies $e_1 \geq a_1$ and so $e_1 = a_1$. Since $a^{\downarrow 1} \leq d \leq e^{\downarrow 1}$, we then have $d_1 = a_1 + 1 = e_1 + 1$. Let $f = (d_1 - 1, d_2, d_3, \ldots) = (e_1, d_2, d_3, \ldots)$. Since $d = (e_1 + 1, d_2, d_3, \ldots) \leq e^{\downarrow 1} = (e_1 + 1, e_2, \ldots)$ then $d_2 \leq e_2 \leq e_1$, this implies that f is a partition. Moreover, $d \geq a^{\downarrow 1}$ and $d \geq b^{\downarrow 1}$, and so $f \geq a$ and $f \geq b$. This implies that $f \geq \sup(a, b) = e$ and that $d \geq e^{\downarrow 1}$, which ends the proof. $\square$

This result shows that one can construct the lattice $L_B(n+1)$ from $L_B(n)$ as follows. The first step of this construction is to construct the set $L_B(n)^{\downarrow_1}$ by adding one grain to the first column of each element of $L_B(n)$. Then, one has to add the missing elements and their transitions. Therefore, we will now consider the consequences of the addition of one grain on the first column of a partition, depending on its structure.

It is clear that, for $n \geq 1$, $L_B(n) = C(n) \bigsqcup S(n) \bigsqcup NS(n) \bigsqcup NP(n) \bigsqcup_{n>\ell\geq 1} P_\ell(n)$, where $C(n)$, $S(n)$, $NS(n)$, $P_\ell(n)$ and $NP(n)$ are respectively the set of partitions of n with a cliff at 1, with a slippery step at 1, with a non-slippery step at 1, with a slippery plateau at 1, and with a non-slippery plateau at 1, and where $\sqcup$ denotes the disjoint union.

Proposition 2. *Let a be a partition. Then, we have:*

(1) *if* $a \in C(n)$ *or* $a \in NP(n)$ *then*

$$\mathrm{dirreach}(a^{\downarrow_1}) = \mathrm{dirreach}(a)^{\downarrow_1};$$

(2) *if* $a \in P_\ell(n)$ *then* $a^{\downarrow_1} \xrightarrow{1} a^{\downarrow_{\ell+2}}$ *and*

$$\mathrm{dirreach}(a^{\downarrow_1}) = \mathrm{dirreach}(a)^{\downarrow_1} \cup \{a^{\downarrow_{\ell+2}}\};$$

(3) *if* $a \in S(n)$ *and b is such that* $a \xrightarrow{1} b$, *then we have* $a^{\downarrow_1} \xrightarrow{1} a^{\downarrow_2} \xrightarrow{2} b^{\downarrow_1}$ *and*

$$\mathrm{dirreach}(a^{\downarrow_1}) = (\mathrm{dirreach}(a) \setminus \{b\})^{\downarrow_1} \cup \{a^{\downarrow_2}\};$$

(4) *if* $a \in NS(n)$ *then* $a^{\downarrow_1} \xrightarrow{1} a^{\downarrow_2}$ *and*

$$\mathrm{dirreach}(a^{\downarrow_1}) = \mathrm{dirreach}(a)^{\downarrow_1} \cup \{a^{\downarrow_2}\}.$$

Proof. It is obvious that the right-hand side of each of these equations is a subset of its corresponding left-hand side, thus it is sufficient to prove the converse. So, let us consider an element $c = a^{\downarrow_1} \xrightarrow{i}$ in $\mathrm{dirreach}(a^{\downarrow_1})$.

(1) If $i \neq 1$ then $c = (a \xrightarrow{i})^{\downarrow_1} \in \mathrm{dirreach}(a)^{\downarrow_1}$. If $a \in NP(n)$ then there is no transition at the first column of $a^{\downarrow_1}$. If $a \in C(n)$, and if $c = a^{\downarrow_1} \xrightarrow{1}$ then c is also equal to $(a \xrightarrow{1})^{\downarrow_1}$.
(2) If $i \neq 1$ then $c = (a \xrightarrow{i})^{\downarrow_1} \in \mathrm{dirreach}(a)^{\downarrow_1}$. Otherwise if $i = 1$, it is clear that $c = a^{\downarrow_{\ell+2}}$.
(3) If $i \neq 1$ then $c = (a \xrightarrow{i})^{\downarrow_1} \in \mathrm{dirreach}(a)^{\downarrow_1}$. However, the element $b = a \xrightarrow{1}$ is obtained directly from a, but $b^{\downarrow_1}$ is not obtained directly from $a^{\downarrow_1}$. So we have: $c \in (\mathrm{dirreach}(a) \setminus \{b\})^{\downarrow_1}$. Otherwise if $i = 1$, it is clear that $c = a^{\downarrow_2}$.
(4) If $i \neq 1$ then $c = (a \xrightarrow{i})^{\downarrow_1} \in \mathrm{dirreach}(a)^{\downarrow_1}$. Otherwise if $i = 1$, it is clear that $c = a^{\downarrow_2}$. □

In the following theorem, we will represent the set $L_B(n+1)$ as a disjoint union. In addition, its proof will give the transitions between the elements of this set, which will complete the construction of the lattice $L_B(n+1)$.

Theorem 1. *For all* $n \geq 1$, *we have:*

$$L_B(n+1) = L_B(n)^{\downarrow_1} \bigsqcup S(n)^{\downarrow_2} \bigsqcup NS(n)^{\downarrow_2} \bigsqcup_{n>\ell\geq 1} P_\ell(n)^{\downarrow_{\ell+2}}.$$

Proof. First of all, it is easy to check that this union is a disjoint union.

Let us recall the strategy of the construction of $L_B(n+1)$. First, $L_B(n)^{\downarrow_1}$ is a sublattice of $L_B(n+1)$; we then add to $L_B(n)^{\downarrow_1}$ all directly reachable elements (and transitions) from this set to obtain a new set L^1; finally, we add to L^1 all directly reachable elements (and transitions) from L^1 to obtain a new set L^2, and so on. The key idea of this theorem is to show that the set L^1 is already equal to $L_B(n+1)$, or, equivalently, that all directly reachable elements from L^1 are elements of L^1.

In Proposition 2, we have shown that L^1 is represented as the disjoint union in the right-hand side of the claim. So let $g \in L^1$, and let $h = g \xrightarrow{i}$ be a directly reachable element of g; we shall prove that $h \in L^1$. Several cases are possible.

- $g = a^{\downarrow_1}$ with $a \in L_B(n)$. From Proposition 2, all directly reachable elements from g are in L^1.

- $g = a^{\downarrow 2}$ with $a \in S(n)$. The transition $a \xrightarrow{1} c$ is possible in $L_B(n)$. All transitions $g = a^{\downarrow 2} \xrightarrow{i} h = b^{\downarrow 2}$ are the same as transition $a \xrightarrow{i} b$, except the transition $a \xrightarrow{1} c$. Moreover, it is clear that, if b belongs to $S(n)$, then h belongs to $S(n)^{\downarrow 2}$. Regarding c, we have the transition $a^{\downarrow 2} \xrightarrow{2} c^{\downarrow 1}$, and $h = c^{\downarrow 1}$ belongs to $L_B(n)^{\downarrow 1}$.
- $g = a^{\downarrow 2}$ with $a \in NS(n)$. All transitions $g \xrightarrow{i} h$ are the same as transition $a \xrightarrow{i} b$, and $h = b^{\downarrow 2}$. If $i > 2$ then b belongs to $NS(n)$, and then h belongs to $NS(n)^{\downarrow 2}$. Otherwise, i can be 2 in the case where a has a non-slippery step of length 1 at 1. In this case, b has a cliff at 1, and $b^{\downarrow 2} = c^{\downarrow 1}$ with $c \in L_B(n)$. Hence $h = b^{\downarrow 2} \in L_B(n)^{\downarrow 1}$.
- $g = a^{\downarrow \ell+2}$ with $a \in P_\ell(n)$. This case requires more attention. We distinguish three subcases:
  (1) a has a cliff at $\ell+1$. Then all transitions $g \xrightarrow{i} h$ are the same as transition $a \xrightarrow{i} b$, and $h = b^{\downarrow \ell+2}$. Moreover, b is an element of $P_\ell(n)$ so $h \in P_\ell^{\downarrow \ell+2}$.
  (2) a has a non-slippery step at $\ell + 1$. Then all transitions $g \xrightarrow{i} h$ are the same as transition $a \xrightarrow{i} b$, and $h = b^{\downarrow \ell+2}$. Moreover, b is an element of $P_\ell(n)$ so $h \in P_\ell^{\downarrow \ell+2}$.
  (3) a has a slippery step at $\ell + 1$. The transition $a \xrightarrow{\ell+1} c$ is possible. All transitions $g \xrightarrow{i} h$ are the same as transition $a \xrightarrow{i} b$, except the transition $a \xrightarrow{\ell+1} c$. It is easy to check that if b is in P_ℓ, then $h = b^{\downarrow \ell+2} \in P_\ell^{\downarrow \ell+2}$. Regarding c, we have the transition $a^{\downarrow \ell+2} \xrightarrow{\ell+2} c^{\downarrow \ell+1}$. Moreover c is an element of $P_{\ell-1}$, so $h = c^{\downarrow \ell+1} \in P_{\ell-1}^{\downarrow \ell+1} \in L^1$. This completes the proof. □

This result makes it possible to write an algorithm which constructs the lattice $L_B(n+1)$ from $L_B(n)$ in linear time with respect to the number of added elements and transitions. Notice that we can obtain $L_B(n)$ for an arbitrary integer n by starting from $L_B(0)$ and iterating this algorithm, and so we have an algorithm that constructs $L_B(n)$ in linear time with respect to its size.

3. The infinite lattice $L_B(\infty)$

We will now define $L_B(\infty)$ as the set of all configurations reachable from (∞) (this is the configuration where the first column contains infinitely many grains and all the other columns contain no grain). Therefore, each element a of $L_B(\infty)$ has the form $(\infty, a_2, a_3, \ldots, a_k)$. As in the previous section, the dominance ordering on $L_B(\infty)$ (when the first component is ignored) is equivalent to the order induced by the dynamical model. The first partitions in $L_B(\infty)$ are given in Fig. 4 along with their covering relations (the first component, equal to ∞, is not represented on this diagram).

It is easy to observe that we have a characterization of the order similar to the one given in [3] for the finite case: let a and b be two elements of $L_B(\infty)$, a being of length p and b being of length q. Then,

$$a \geq_{L_B(\infty)} b \quad \text{if and only if for all } j \text{ between 2 and } \max(p,q), \quad \sum_{i \geq j} a_i \leq \sum_{i \geq j} b_i.$$

We will start this section by proving that $L_B(\infty)$ is a lattice and by giving a formula for the infimum in $L_B(\infty)$. After this, we will show that, for any n, there are two different ways to find sublattices of $L_B(\infty)$ isomorphic to $L_B(n)$. We will also give a way to construct some other special sublattices of $L_B(\infty)$, using its self-similarity. Finally, we will compare $L_B(\infty)$ with the Young lattice.

Theorem 2. *The set $L_B(\infty)$ is a lattice. Moreover, if $a = (\infty, a_2, \ldots, a_k)$ and $b = (\infty, b_2, \ldots, b_l)$ are two elements of $L_B(\infty)$, then $\inf_{L_B(\infty)}(a, b) = c$ in $L_B(\infty)$, where c is defined by:*

$$c_i = \max\left(\sum_{j \geq i} a_j, \sum_{j \geq i} b_j\right) - \sum_{j > i} c_j \quad \textit{for all } i \textit{ such that } 2 \leq i \leq \max(k,l).$$

Proof. We shall prove that c is an element of $L_B(\infty)$ and that c is equal to $\inf_{L_B(\infty)}(a, b)$. Let $n = 2(\sum_{i \geq 2} a_i + \sum_{i \geq 2} b_i)$. Let $a' = (n - \sum_{i \geq 2} a_i, a_2, \ldots, a_k)$, $b' = (n - \sum_{i \geq 2} b_i, b_2, \ldots, b_l)$ and $c' = (n - \sum_{i \geq 2} c_i, c_2, \ldots, c_{\max(k,l)})$. It is then obvious that a' and b' are two partitions of n. We show that c' is the infimum of a' and b' by the dominance ordering in $L_B(n)$. From the definition of c and c', it is clear that $\sum_{j \geq i} c'_j = \max(\sum_{j \geq i} a'_j, \sum_{j \geq i} b'_j)$ for all $i \geq 1$.

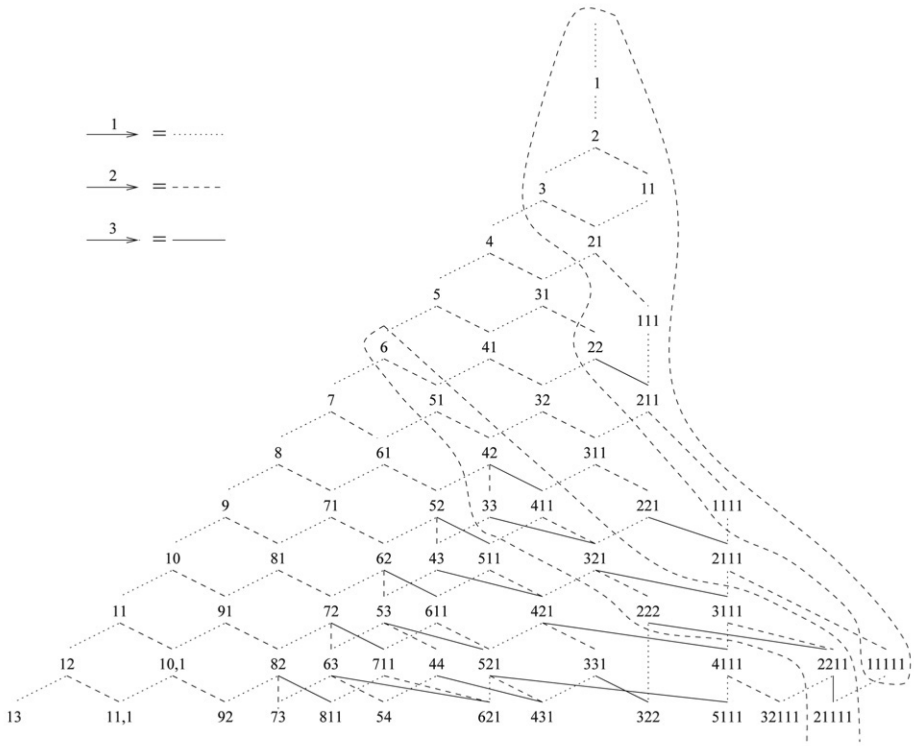

Fig. 4. The first elements and transitions of $L_B(\infty)$. As shown on this figure for $n = 6$, we will discuss two ways to find parts of $L_B(\infty)$ isomorphic to $L_B(n)$ for any n.

Which is equivalent with $\sum_{1\le j\le i} c'_j = \min(\sum_{1\le j\le i} a'_j, \sum_{1\le j\le i} b'_j)$ for all $i \ge 1$. This proves that c' is the infimum of a' and b' in $L_B(n)$. Therefore, c' is a decreasing sequence, and so c is an element of $L_B(\infty)$. Moreover, according to the definition of $\ge_{L_B(\infty)}$, c is the maximal element of $L_B(\infty)$ which is smaller than a and b, and so $c = \inf_{L_B(\infty)}(a, b)$.

By definition, $L_B(\infty)$ has a maximal element. Since it is closed for the infimum, $L_B(\infty)$ is a lattice. $\square$

Let us consider now the injective map

$$\begin{array}{llcl} \pi: & L_B(n) & \longrightarrow & L_B(\infty) \\ & a = (a_1, a_2, \ldots, a_k) & \longrightarrow & \bar{a} = (\infty, a_2, \ldots, a_k). \end{array}$$

One can apply a proof similar to the one of Proposition 1 to show that

$$\inf_{L_B(\infty)}(\pi(a), \pi(b)) = \pi(\inf_{L_B(n)}(a, b)) \quad \text{and}$$
$$\sup_{L_B(\infty)}(\pi(a), \pi(b)) = \pi(\sup_{L_B(n)}(a, b)).$$

This implies that π is a lattice embedding.

Let $\overline{L_B(n)} = \pi(L_B(n))$. We know that $\overline{L_B(n)}$ is a sublattice of $L_B(\infty)$ and from Proposition 1, $L_B(n)^{\downarrow_1}$ is a sublattice of $L_B(n+1)$, therefore, since $\overline{L_B(n)^{\downarrow_1}} = \overline{L_B(n)}$, we have an increasing sequence of sublattices:

$$\overline{L_B(0)} \le \overline{L_B(1)} \le \cdots \le \overline{L_B(n)} \le \overline{L_B(n+1)} \le \cdots \le L_B(\infty)$$

where $\le$ denotes the sublattice relation.

We can say more about this increasing sequence of lattices. Let $a = (\infty, a_2, a_3, \ldots, a_k)$ be an element of $L_B(\infty)$. If one takes $a_1 = a_2 + 1$ and $n = \sum_{i=1}^{k} a_i$, then the partition $a' = (a_1, a_2, \ldots, a_k)$ is an element of $L_B(n)$. Since $a = \pi(a')$, this implies that a is an element of $\overline{L_B(n)}$. Conversely, any element of $L_B(\infty)$ is of the form $a = (\infty, a_2, \ldots, a_k)$. Therefore, $a' = (a_2, \ldots, a_k)$ is a decreasing sequence, and if we put $n = \sum_{i\ge 2} a_i$ then

$a' \in L_B(n)$, *i.e.* $a \in \overline{L_B(n)}$. Finally, we have:

$$\bigcup_{n\geq 0} \overline{L_B(n)} = L_B(\infty).$$

Therefore, $L_B(\infty)$ can be viewed as the limit of $L_B(n)$ when n grows to infinity. On the other hand, we will show that $L_B(\infty)$ can be represented as a disjoint union of $L_B(n)$ for all n.

Let us define the set

$$\widetilde{L_B(\infty)} = \bigsqcup_{n\geq 0} L_B(n).$$

We define the following relations over $\widetilde{L_B(\infty)}$. Let $a \in L_B(m)$ and $b \in L_B(n)$. We have $a \xrightarrow{i} b$ in $\widetilde{L_B(\infty)}$ if and only if one of the following applies: $n = m$ and $a \xrightarrow{i} b$ in $L_B(n)$, or $i = 0$, $n = m + 1$ and $b = a^{\downarrow_1}$. In other terms, the elements of $L_B(n)$ are linked to each other as usual, and each element a of $L_B(n)$ is linked to $a^{\downarrow_1} \in L_B(n + 1)$ by an edge labeled by 0.

From this, one can introduce an order on the set $\widetilde{L_B(\infty)}$ in the usual sense, by defining it as the reflexive and transitive closure of this relation. We now show that $L_B(\infty)$ is isomorphic to $\widetilde{L_B(\infty)}$, and so that $\widetilde{L_B(\infty)}$ is a lattice.

Lemma 1. *The map χ defined by:*

$$\chi : \widetilde{L_B(\infty)} \longrightarrow L_B(\infty)$$
$$a = (a_1, a_2, \ldots, a_k) \mapsto \chi(a) = (\infty, a_1, a_2, \ldots, a_k)$$

is a lattice isomorphism.

Moreover, $a \xrightarrow{i} b$ in $\widetilde{L_B(\infty)}$ if and only if $\chi(a) \xrightarrow{i+1} \chi(b)$ in $L_B(\infty)$.

Proof. χ is clearly bijective. Moreover, it is clear from the definitions that for all a and b in $\widetilde{L_B(\infty)}$, $a \xrightarrow{i} b$ if and only if $\chi(a) \xrightarrow{i+1} \chi(b)$. Therefore, χ is an order isomorphism. Since $L_B(\infty)$ is a lattice, this implies that χ is a lattice isomorphism. □

This lemma means that $\widetilde{L_B(\infty)}$ is nothing but $L_B(\infty)$ when one removes the first component (always equal to ∞) of each element of $L_B(\infty)$ and decreases the label of each edge by 1. We will now see that $L_B(n)$ is a sublattice of $\widetilde{L_B(\infty)}$ for all n, which gives another way to find a part of $L_B(\infty)$ isomorphic to $L_B(n)$.

Theorem 3. *For all integer $n \geq 1$, $L_B(n)$ is a sublattice of $\widetilde{L_B(\infty)}$.*

Proof. Let a and b be two elements of $L_B(n)$, we shall prove that $\inf_{\widetilde{L_B(\infty)}}(a, b)$ and $\sup_{\widetilde{L_B(\infty)}}(a, b)$ belong to $L_B(n)$. Let c be $\inf_{L_B(n)}(a, b)$ and c' be $\inf_{\widetilde{L_B(\infty)}}(a, b)$. We have, $a \geq c' \geq_{\widetilde{L_B(\infty)}} c$, which means that $\sum_{i\geq 1} a_i \leq \sum_{i\geq 1} c'_i \leq \sum_{i\geq 1} c_i$, and so $\sum_{i\geq 1} c'_i = n$. This implies that c' belongs to $L_B(n)$, and we obtain $c' = c$. The proof for the supremum is similar. □

To finish this section, we will discuss the relations between the infinite lattice $\widetilde{L_B(\infty)}$ and the famous Young lattice. These two infinite lattices contain exactly the same elements (all the partitions of all the integers), but ordered in a different way: $a \leq b$ in the Young lattice if for all i we have $a_i \leq b_i$. In other words, the order over the partitions is the componentwise order. This order induces a (distributive) lattice structure over the set of all the integer partitions. It has been widely studied; see for instance [14,5]. It can also be viewed as the set of partitions obtained from the empty one, (), and by iterating the following evolution rule: $a \xrightarrow{i} b$ if b is a partition obtained from the partition a by increasing its ith component. This implies directly that the lattice can be decomposed into levels (the ith level contains the partitions obtained after i applications of the evolution rule), and that level i contains exactly the partitions of n, *i.e.* the elements of $L_B(n)$. Notice moreover that these elements are not comparable in the Young lattice therefore the order in $\widetilde{L_B(\infty)}$ and the one in the Young lattice are very different. However, they are in relation according to the following result:

Proposition 3 (*[10]*). *The map* π *from* $\widetilde{L_B(\infty)}$ *into the Young lattice such that* $\pi(a)_i$ *is equal to* $\sum_{j\geq i} a_j$ *is an order embedding which preserves the infimum.*

Proof. Let a and b be two elements of $\widetilde{L_B(\infty)}$. We must show that $\pi(a)$ and $\pi(b)$ belong to the Young lattice, that $a \geq b$ in $\widetilde{L_B(\infty)}$ is equivalent to $\pi(a) \geq \pi(b)$ in the Young lattice and that $\inf(\pi(a), \pi(b))$ in the Young lattice is equal to $\pi(\inf(a, b))$ in $\widetilde{L_B(\infty)}$. The first two points are easy: $\pi(x)$ is obviously a decreasing sequence of integers for any x, and the order is preserved. Now, let $c = \inf(a, b)$. Then,

$$\begin{aligned}
\pi(c)_i &= \sum_{j\geq i} c_j \\
&= \max\left(\sum_{j\geq i} a_j, \sum_{j\geq i} b_j\right) \quad \text{by Theorem 2} \\
&= \max(\pi(a)_i, \pi(b)_i) \\
&= \inf(\pi(a), \pi(b))_i \quad \text{in the Young lattice}
\end{aligned}$$

which proves the claim. □

Notice that this order embedding is not a *lattice* embedding, since it does not preserve the supremum. For instance, if $a = (2, 2)$ and $b = (1, 1, 1)$, then $\pi(a) = (4, 2)$, $\pi(b) = (3, 2, 1)$, and $c = \sup(a, b) = (2, 1)$ in $\widetilde{L_B(\infty)}$ but $\pi(c) = (3, 1)$ and $\sup((4, 2), (3, 2, 1)) = (3, 2)$ in the Young lattice. There can be no lattice embedding from $\widetilde{L_B(\infty)}$ to the Young lattice since the fact that this one is a *distributive* lattice would imply that $\widetilde{L_B(\infty)}$ would be distributive, which is not true. Finally, notice that a study similar to the one presented in this paper can be found in [9] on another kind of integer partitions, namely b-ary partitions. The Young lattice is a particular case of the lattices introduced in this paper, and the reader interested in the relations between $\widetilde{L_B(\infty)}$ and the Young lattice should refer to it.

4. The infinite binary tree $T_B(\infty)$

As shown in our procedure to construct $L_B(n+1)$ from $L_B(n)$, each element a of $L_B(n+1)$ is obtained from an element a' of $L_B(n)$ by addition of one grain: $a = a'^{\downarrow i}$ for some integer i. We will now represent this relation by a tree where $a \in L_B(n+1)$ is the son of $a' \in L_B(n)$ if and only if $a = a'^{\downarrow i}$ and we label with i the edge $a' \longrightarrow a$ in this tree. We denote this tree by $T_B(\infty)$. The root of this tree is the empty partition (). We will show two ways to find the partitions of a given integer n in $T_B(\infty)$, which will make it possible to give an efficient and simple algorithm to enumerate them. Moreover, the recursive structure of this tree will allow us to obtain a recursive formula for the cardinality of $L_B(n)$ and some special classes of partitions.

From the construction of $L_B(n+1)$ from $L_B(n)$, it follows that the nodes of this tree are the elements of $\bigsqcup_{n\geq 0} L_B(n)$, and that each node a has at least one son, $a^{\downarrow 1}$, and one more if a begins with a slippery plateau of length l: the element $a^{\downarrow \ell+1}$. Therefore, $T_B(\infty)$ is a binary tree. We will call *left son* the first of two sons, and *right son* the other (if it exists). We call *the level n of the tree* the set of elements of depth n. The first levels of $T_B(\infty)$ are shown in Fig. 5.

Like in the case of $L_B(\infty)$, there are two ways to find the elements of $L_B(n)$ in $T_B(\infty)$. From the construction of $L_B(n+1)$ from $L_B(n)$ given above, it is straightforward that:

Proposition 4. *The level n of* $T_B(\infty)$ *is exactly the set of the elements of* $L_B(n)$.

Moreover, it is obvious from the construction of $T_B(\infty)$ that the elements of the set $\overline{L_B(n+1)} \setminus \overline{L_B(n)}$ are sons of elements of $\overline{L_B(n)}$, therefore we deduce the following proposition which can be easily proved by induction:

Proposition 5. *Let* χ^{-1} *be the inverse of the lattice isomorphism defined in Lemma 1. Then, the set* $\chi^{-1}(\overline{L_B(n)})$ *is a subtree of* $T_B(\infty)$ *having the same root.*

This proposition makes it possible to give a simple and efficient algorithm to enumerate all the partitions of a given integer n, using the binary tree structure it gives to the set of all these partitions: Algorithm 1 achieves this in linear time and space with respect to their number, which is optimal.

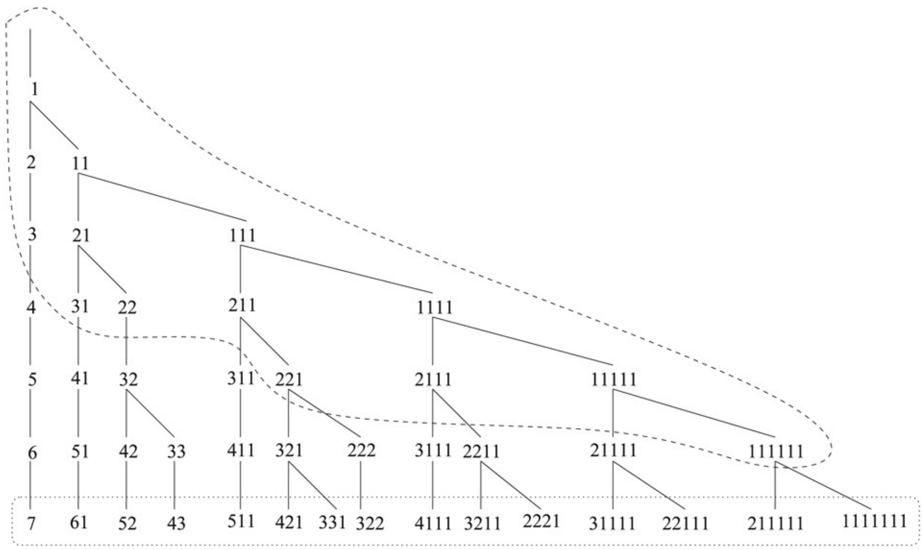

Fig. 5. The first levels of the tree $T_B(\infty)$ (to clarify the picture, the labels are omitted). As shown on this figure for $n = 7$, we will discuss two ways to find the elements of $L_B(n)$ in $T_B(\infty)$ for any n.

Input: An integer n
Output: The partitions of n
begin
  Resu $\leftarrow \emptyset$;
  CurrentLevel $\leftarrow \{()\}$;
  OldLevel $\leftarrow \emptyset$;
  $l \leftarrow 0$;
  while *CurrentLevel* $\neq \emptyset$ **do**
    for each *e in CurrentLevel* **do**
      Compute p such that $p_i = e_{i-1}$ for all $i > 1$ and $p_1 = n - l$;
      Add p to Resu;
    OldLevel $\leftarrow$ CurrentLevel;
    CurrentLevel $\leftarrow \emptyset$;
    $l \leftarrow l + 1$;
    for each *p in OldLevel* **do**
      Add $p^{\downarrow 1}$ to CurrentLevel;
      if *p begins with a slippery plateau of length l* **then**
        Add $p^{\downarrow \ell+1}$ to CurrentLevel;
    for each *p in CurrentLevel* **do**
      if $n - l < p_1$ **then**
        Remove p from CurrentLevel;
  Return(Resu);
end

Algorithm 1: Efficient computation of the partitions of an integer.

We will now give a recursive description of $T_B(\infty)$. We first define a certain kind of subtrees of $T_B(\infty)$. Afterwards, we show how the whole structure of $T_B(\infty)$ can be described in terms of such subtrees.

Fig. 6. Self-referencing structure of X_k subtrees.

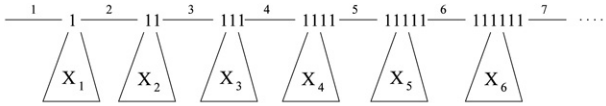

Fig. 7. Representation of $T_B(\infty)$ as a chain.

Definition 1. We will call X_k *subtree* any subtree T of $T_B(\infty)$ which is rooted at an element $a = (\underbrace{i, \dots, i}_{k}, a_{k+1}, \dots)$ with $a_{k+1} \leq i - 1$ and which is either the whole subtree of $T_B(\infty)$ rooted at a in the case a has only one son, or a and its left subtree otherwise. Moreover, we define X_0 as a simple node.

The next proposition shows that all the X_k subtrees are isomorphic.

Proposition 6. *A X_k subtree, with $k \geq 1$, is composed by a chain of $k + 1$ nodes (the rightmost chain) whose edges are labeled by 1, 2, …, k and whose i-th node is the root of a X_{i-1} subtree for all i between 1 and $k + 1$. (See Fig. 6.)*

Proof. The claim is obvious for $k = 1$. Indeed, in this case the root a has the form $(i, a_2, \dots)$ with $a_2 \leq i - 1$, therefore its left son has the form $(i + 1, i - 1, \dots)$, *i.e.* it starts with a cliff, and has only one son. This son also starts with a cliff; we can then deduce that X_1 is simply a chain, which is the claim for $k = 1$.

Suppose now that the claim is proved for any $i < k$ and consider the root a of a X_k subtree: $a = (\underbrace{i, \dots, i}_{k}, a_{k+1}, \dots)$ with $a_{k+1} \leq i-1$. Its left son is $a^{\downarrow_1} = (i+1, i, \dots, i, a_{k+1}, \dots)$ with $a_{k+1} \leq i-1$, therefore it is the root of a X_1 subtree. Moreover, $a^{\downarrow_1}$ has one right son: $a^{\downarrow_1\downarrow_2} = (i+1, i+1, i, \dots, i, a_{k+1}, \dots)$, which by definition is the root of a X_2 subtree. After $k-1$ such stages, we obtain $a^{\downarrow_1\downarrow_2\cdots\downarrow_{k-1}}$, which is equal to $(i+1, \dots, i+1, i, a_{k+1})$. This node is the root of a X_{k-1} subtree and has a right son:

$$a^{\downarrow_1\downarrow_2\cdots\downarrow_{k-1}\downarrow_k}, \textit{i.e.} \left(\underbrace{i+1, \dots, i+1}_{k}, a_{k+1}, \dots\right)$$

and we still have $a_{k+1} \leq i - 1$. Therefore, this node is the root of a X_k subtree, and from the definition of $T_B(\infty)$ we know that it has no other son. This completes the proof. □

This recursive structure and the above propositions allow us to give a compact representation of the tree by a chain:

Theorem 4. *The tree $T_B(\infty)$ can be represented by the infinite chain defined as follows: the i-th node of this chain, $(\underbrace{1, \dots, 1}_{i-1})$, is linked to the following node in the chain by an edge labeled with i and is the root of a X_{i-1} subtree. See Fig. 7.*

Moreover, we can prove a stronger property of each subtree in this chain:

Corollary 1. *The X_k subtree of $T_B(\infty)$ with root $(1, \dots, 1)$ contains exactly the partitions of length k.*

Proof. Owing to their recursive structure shown in Proposition 6, X_k subtrees contain no edge with label greater than k. Therefore, if the root of a X_k subtree is of length k then all its nodes have length k. Moreover, no X_l subtrees with $l \neq k$ and with a root of length l can contain any node of length k. This remark, together with Theorem 4, implies the result. □

We can now state our last result:

Corollary 2. *Let $c(\ell, k)$ denote the number of paths in a X_k tree originating from the root and having length ℓ. We have:*

$$c(\ell, k) = \begin{cases} 1 & \text{if } \ell = 0 \text{ or } k = 1 \\ \displaystyle\sum_{i=1}^{\inf(\ell,k)} c(\ell - i, i) & \text{otherwise.} \end{cases}$$

Moreover, $|L_B(n)| = c(n, n)$ and the number of partitions of n with length exactly k is $c(n - k, k)$.

Proof. The formula for $c(\ell, k)$ is derived directly from the structure of X_k trees (Proposition 6 and Fig. 6). To obtain the formula for $|L_B(n)|$, we recall Proposition 6 and Theorem 4. Applying them, we observe that if we keep the nodes of depth n at most, then the subtrees obtained from $T_B(\infty)$ and X_n turn out to be isomorphic. The last formula is directly derived from Theorem 4 and Corollary 1. □

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